Geometry of Minkowski Planes and Spaces – Selected Topics

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Abstract

The results presented in this dissertation refer to the geometry of Minkowski spaces, i.e., of real finite-dimensional Banach spaces.

First we study geometric properties of radial projections of bisectors in Minkowski spaces, especially the relation between the geometric structure of radial projections and Birkhoff orthogonality. As an application of our results it is shown that for any Minkowski space there exists a number, which plays somehow the role that $\sqrt{2}$ plays in Euclidean space. This number is referred to as the critical number of any Minkowski space. Lower and upper bounds on the critical number are given, and the cases when these bounds are attained are characterized. Moreover, with the help of the properties of bisectors we show that a linear map from a normed linear space X to another normed linear space Y preserves isosceles orthogonality if and only if it is a scalar multiple of a linear isometry.

Further on, we examine the two tangent segments from any exterior point to the unit circle, the relation between the length of a chord of the unit circle and the length of the arc corresponding to it, the distances from the normalization of the sum of two unit vectors to those two vectors, and the extension of the notions of orthocentric systems and orthocenters in Euclidean plane into Minkowski spaces. Also we prove theorems referring to chords of Minkowski circles and balls which are either concurrent or parallel. All these discussions yield many interesting characterizations of the Euclidean spaces among all (strictly convex) Minkowski spaces.

In the final chapter we investigate the relation between the length of a closed curve and the length of its midpoint curve as well as the length of its image under the so-called halving pair transformation. We show that the image curve under the halving pair transformation is convex provided the original curve is convex. Moreover, we obtain several inequalities to show the relation between the halving distance and other quantities well known in convex geometry. It is known that the lower bound for the geometric dilation of rectifiable simple closed curves in the Euclidean plane is $\pi/2$, which can be attained only by circles. We extend this result to Minkowski planes by proving that the lower bound for the geometric dilation of rectifiable simple closed curves in a Minkowski plane X is analogously a quarter of the circumference of the unit circle S_X of X, but can also be attained by curves that are not Minkowskian circles. In addition we show that the lower bound is attained only by Minkowskian circles if the respective norm is strictly convex. Also we give a sufficient condition for the geometric dilation of a closed convex curve to be larger than a quarter of the perimeter of the unit circle.

Keywords: arc length, Birkhoff orthogonality, bisectors, Busemann angular bisector, $\mathcal{C}$ - orthocenter, characterizations of Euclidean planes, characterizations of inner product spaces, chord length, circumradius, convex curve, convex geometry, critical number, detour, Euclidean plane, geometric dilation, geometric inequality, Glogovskij angular bisector, halving distance, halving pair, inner product space, inradius, isometry, isosceles orthogonality, James orthogonality, midpoint curve, minimum width, Minkowski plane, Minkowski plane, normed linear space, normed plane, radial projection, Radon plane, rectification, Singer orthogonality, strictly convex norm, three-circles theorem, Voronoi diagram.

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Contents

Chapter 1

Introduction

Minkowski Geometry, i.e., the geometry of real finite-dimensional Banach spaces, is a geometry "next" to the Euclidean Geometry which is already described in the fourth Hilbert problem (cf. [30]). It is a fascinating mathematical discipline which is closely related to other mathematical fields, such as Functional Analysis, Distance Geometry, Finsler Geometry, and Convex Geometry. Problems from Minkowski Geometry are also studied in Optimization, Combinatorics, Discrete and Computational Geometry, and Operations Research (basic references to the geometry of Minkowski spaces are [48], [46], [47], and the monograph [60]). It is also a research area yielding many interesting open problems. In this dissertation we will focus on four major topics from Minkowski Geometry, namely, radial projections of bisectors, maps preserving isosceles orthogonality, Minkowskian circle geometry, and halving closed curves and the geometric dilation problem. We study mainly, but not only, the two-dimensional case.

1.1 Radial Projections of bisectors

In Minkowski spaces, the bisector of the line segment between two distinct points is the extension of the notion of perpendicular bisector in Euclidean spaces. It is well known that perpendicular bisectors are geometric figures having a simple geometric structure since they are simply straight lines if the space is two-dimensional, and hyperplanes otherwise. This is not true for bisectors in non-Euclidean Minkowski spaces. For example, in the two-dimensional case bisectors are not necessarily straight lines. More precisely, if every bisector is a straight line then the underlying plane is Euclidean, and there exist Minkowski planes such that no bisector is a straight line (cf. [34]); bisectors are even not necessarily one-dimensional, i.e., they may have non-empty interior (cf. [8, Figure 3.3]); moreover, bisectors do not necessarily have asymptotic lines, and pairs of bisectors can exist that intersect each other infinitely many times (cf. [46, Theorem 24]). There is no doubt that the structure of bisectors is very complicated, just as there is no doubt that a detailed study of the structure of bisectors is useful, due to different motivations.

On the one hand, one should notice that in the Minkowskian case we say bisector rather than perpendicular bisector, since there is no natural orthogonality relation in non-Euclidean Minkowski spaces. For discussions of orthogonality types in Minkowski spaces we refer to the survey papers [5] and [6], or to the more original contributions [13], [34], [35], [59], and [7]. However, it is known that the structure of bisectors is fully determined by isosceles orthogonality (cf. [8, P. 26]), which is one of the major generalizations of usual (Euclidean) orthogonality in Minkowski spaces, and it is also somehow connected with another major orthogonality type in Minkowski spaces (to see this relation one can refer to [46, Proposition 22]), namely, the so called Birkhoff orthogonality. These two orthogonality types are in general different, and they coincide if and only if the underlying space is Euclidean (cf. [8] for characterizations of inner product spaces in terms of orthogonality types, and [38] for a discussion of a quantitative characterization of the difference between Birkhoff orthogonality and isosceles orthogonality). Hence a detailed study of the structure of bisectors sheds more light on the relation between those two orthogonality types. Also, we strongly suspect that geometric properties of a Minkowski space (in the sense of isometries) can be determined by the geometric structure of bisectors. This turns out to be true, as one can see in Chapter 4 of this dissertation.

On the other hand, just as in the Euclidean case, the notion of bisectors is connected with notions from Operation Research and Computational Geometry such as Voronoi diagrams. The latter have been intensively studied in Computational Geometry (where most of the results are obtained without the assumption that the unit ball is centrally symmetric, cf. [10] and [11]). So one sees: retrieving geometric properties of bisectors is of theoretical and practical importance.

In this dissertation we study bisectors from a new point of view: we study the projection of the bisector of two points, which are symmetric with respect to the origin, onto the unit circle rather than studying the bisectors themselves. Already from this viewpoint many new and unexpected results are obtained.

1.2 Maps preserving isosceles orthogonality

We say that a linear map T from a normed linear space X to another normed linear space Y preserves isosceles orthogonality if $Tx \perp_I Ty$ holds whenever we have $x \perp_I y$. In this part we show that a linear map preserves isosceles orthogonality if and only if it is a scalar multiple of a linear isometry.

1.3 Minkowskian circle geometry

A geometric figure that determines also the geometric properties of the respective Minkowski plane is the circle since, by definition, the shape of the unit circle of a Minkowski plane determines the norm (cf. Section 2.1). So it is clear: the more we know about the unit cirlce, the better we know the geometric structure of the respective Minkowski plane. On the other hand, since the Euclidean plane is a special Minkowski plane, we always would like to know to what extent its properties remain valid for general normed planes, or by what properties they are replaced then. Our research in this direction is motivated by the two facts mentioned above: First we study the geometry of circles (or of subsets of circles and discs) in Minkowski planes for finding out what properties of circles in the Euclidean plane still hold in general Minkowski planes. Second we look at properites of circles in the Euclidean plane which cannot be carried over to general Minkowski planes, and we ask which "parts of this collection of properties" can at least be extended.

Problems referring to this and covered in this dissertation are related to:

- the lengths of the two tangent segments from any exterior point to the unit circle,
- the relation between the length of a chord of the unit circle and the length of the arc corresponding to it,
- the distances from the normalization of the sum of two unit vectors to those two vectors,
- the extension of the notions of orthocentric systems and orthocenters in Euclidean plane into Minkowski spaces, and
- concurrent and parallel chords of circles and spheres.

1.4 Halving closed curves and the geometric dilation problem

A pair of points halving the Minkowskian length (i.e., the length measured in the norm of the underlying normed plane; a formal definition of the length in Minkowski spaces is given in Subsection 2.1.4) of a closed curve C is called a halving pair, and the line segment connecting these two points is said to be a halving chord of the curve. We consider the image of C obtained by translating each halving chord of C to a segment with the origin as its midpoint. The properties of halving pairs and the corresponding halving pair transformation of closed curves play an essential role in the study of geometric dilation (or detour) problems in the Euclidean plane and in general normed planes, and they deserve to be studied for their own sake, with applications also in Computational Geometry.

We will investigate the relation between the length of a curve and the length of its image under the halving pair transformation. E.g., we try to figure out whether the corresponding image of a closed convex curve is still convex, and we investigate relations between the halving distance (i.e., the length of a halving chord) and other quantities of (convex) curves well known in convex geometry, such as width, inradius, circumradius, and diameter of such curves. These considerations will yield some geometric inequalities, where also the extremal cases will be discussed.

Also we extend geometric dilation problems in the Euclidean plane to general Minkowski planes and figure out the relation between the lower bound of the geometric dilation of simple closed curves in a Minkowski plane and the circumference of the unit circle of the respective plane. In addition we try to provide a sufficient condition for the property that the geometric dilation (detour) of a closed curve is strictly larger than the lower bound of the geometric dilation of the underlying plane.

1.5 Organization of this dissertation

The dissertation is organized as follows. In the next chapter we collect, for later use, some basic definitions and fundamental results, which can be found elsewhere but are not contained in a single monograph or a survey paper. The four topics mentioned above are discussed in Chapter 3, Chapter 4, Chapter 5, and Chapter 6, respectively. Finally, there will be an appendix collecting all new characterizations of Euclidean planes obtained in this dissertation. We note that, since inner product spaces have two-dimensional nature (i.e., a normed linear space is an inner product space if and only if any of its two-dimensional subspaces is Euclidean), our characterizations of Euclidean planes can be extended in a natural way to higher dimensions, i.e., to characterizations of inner product spaces.

Chapter 2

Preliminaries

2.1 Minkowski plane and spaces

A *Minkowski space* is a real finite-dimensional linear space endowed with a *norm* $\|\cdot\|$, that is a functional such that the following properties hold for any elements x and y of the respective linear space:

- $\|x\| \geq 0$,
- $\|x\| = 0$ if and only if $x = o$, i.e., if and only if x is the *origin*,
- $\|\lambda x\| = |\lambda| \cdot \|x\|$ holds for any real number λ,
- $\|x + y\| \leq \|x\| + \|y\|$ (the *triangle inequality*).

In particular, a two-dimensional Minkowski space is also called a *Minkowski plane*. We denote by X a Minkowski space with norm $\|\cdot\|$ and origin o. The subsets $S_X := \{x : \ \|x\| = 1\}$ and $B_X := \{x : \ \|x\| \leq 1\}$ are called the *unit sphere* and *unit ball* of X, respectively. If $x \in S_X$ then x is said to be a *unit vector*. In particular, when X is a Minkowski plane, S_X and B_X are called *unit circle* and *unit disc* of X, respectively. Any homothet of the unit circle S_X, i.e., any set of the form $x + \lambda S_X$, is said to be a *circle* in X and denoted by $S_X(x, \lambda)$.

Another way to introduce the norm is by fixing the unit ball first, i.e., by picking up a *convex body* (i.e., a bounded closed convex set with non-empty interior) D which is symmetric with respect to o, and introducing the norm by the so called *Minkowski functional* $\rho_D(x)$ of D:

$$\|x\| := \rho_D(x) = \inf\{\mu : \ \mu > 0, \frac{x}{\mu} \in D\}.$$

For $x, y \in X$ with $x \neq y$, we denote by

$$[x, y] := \{\lambda x + (1 - \lambda) y : \ \lambda \in [0, 1]\}$$

the (non-trivial) *segment* between x and y (while a trivial segment is just a singleton), by

$$\langle x, y\rangle := \{\lambda x + (1-\lambda)y : \ \lambda \in \mathbb{R}\}$$

the *line* passing through x and y, and by

$$[x, y\rangle := \{(1-\lambda)x + \lambda y : \ \lambda \in [0, +\infty)\}$$

the *ray* with starting point x passing through y. An *angle* $\angle xpy$ is the convex hull of two rays $[p, x\rangle$ and $[p, y\rangle$ not contained in a line, which is called the *sides of the angle* $\angle xpy$, with a common starting point p, called the *apex of the angle* $\angle xpy$. The *distance* between x and y, which is equal to the *length* of $[x, y]$, is measured by $\|x - y\|$. Also we write $\overrightarrow{xy}$ for the *orientation* from x to y, and $\widehat{x}$ for $\frac{x}{\|x\|}$ $(x \neq o)$, i.e., the *normalization* of x. The *convex hull*, *closure*, and *interior* of a set S are denoted by $\mathrm{conv}S$, $\overline{S}$, and $\mathrm{int}S$, respectively. The *distance* from a point x to a set S is denoted by $d(x, S)$.

2.1.1 Strict convexity and smoothness

A line $l = \langle x, y\rangle$ is said to be a *supporting line* of a convex body D at a point z if $z \in l \cap D$ and $l \cap \mathrm{int}D = \varnothing$, in which case we also say that l *supports* D at z. In particular, l is a supporting line of B_X at z if $z \in l \cap S_X$ and $d(o, l) = 1$.

We say that a Minkowsi space X is *strictly convex* if S_X does not contain a non-trivial segment. A Minkowski plane X is *smooth* if there exists precisely one supporting line of B_X at each point $z \in S_X$, and a Minkowski space of dimension greater than 2 is smooth if each of its two-dimensional subspaces is smooth (cf. [53, Chapter 5] for more detailed discussions of smoothness, strict convexity, and some related (infinite-dimensional) topics). The following theorem is a useful characterization of strict convexity:

Theorem 2.1.1. (cf. [48, Proposition 14]) *Each of the following statements are equivalent to strict convexity of a Minkowski plane:*

- *Any vector of norm < 1 is the midpoint of a unique (at most one) chord of the unit circle;*
- *Any vector of norm < 2 is representable as sum of two unit vectors in a unique (at most one) way;*
- *Any two unit circles with centers at distance < 2 intersect in exactly (at most) two points;*
- *Any two circles intersect in at most two points;*
- *Any three points are contained in at most one circle.*

2.1.2 Inner product spaces

An (real) *inner product space* is a special normed linear space (not necessarily finite-dimensional) with the additional structure of an inner product $(\cdot,\cdot)$ which, for any x, y, and $z \in X$, satisfies:

- $(x,y) = (y,x)$;
- $(\alpha x, y) = \alpha(x,y)$, for any $\alpha \in \mathbb{R}$;
- $(x+y,z) = (x,z) + (y,z)$;
- $(x,x) \geq 0$, with equality if and only if $x = o$;
- $(x,x) = \|x\|^2$.

When we say that a Minkowski space is *Euclidean*, we are meaning that it is an inner product space. In particular, a two-dimensional (real) inner product space is identical with the *Euclidean plane*. There are many different ways to characterize inner product spaces (there is even a monograph [8] for this), and one of the best known characterizations is given by the following condition called the *parallelogram law*:

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2\,,\ \forall x, y \in X.$$

It is a basic result that an inner product space is both strictly convex and smooth.

2.1.3 Isometries

Given two Minkowski spaces X and Y, an *isometry* T from X to Y is a map that preserves distances, i.e., for any $x, y \in X$, one has $\|Tx - Ty\| = \|x - y\|$. For the isometry between two Minkowski spaces we have the following theorem.

Theorem 2.1.2. (cf. [60, p. 76]) *If X and Y are two normed linear spaces and if T is an isometry of X onto Y with $T(o) = o$, then T is linear.*

A Minkowski space X is said to be *isometric* to another Minkowski space Y if there exists a surjective linear isometry T from X to Y. We identify isometric Minkowski spaces, since they are affinely equivalent.

2.1.4 Curve and curve length

By a *curve* C in X we mean the range of a continuous function ϕ that maps a closed bounded interval $[\alpha,\beta]$ into X. The curve C defined by $\phi:\ [\alpha,\beta] \mapsto X$ is called *closed* if $[\alpha,\beta]$ is replaced by a Euclidean circle, say, and it is *simple* if it has no self-intersections. Furthermore, C is said to be *rectifiable* if the set of all Riemann sums

$$\left\{ \sum_{i=1}^{n} \|\phi(t_i) - \phi(t_{i-1})\| :\ (t_0, t_1, \cdots, t_n) \text{ is a partition of } [\alpha,\beta] \right\}$$

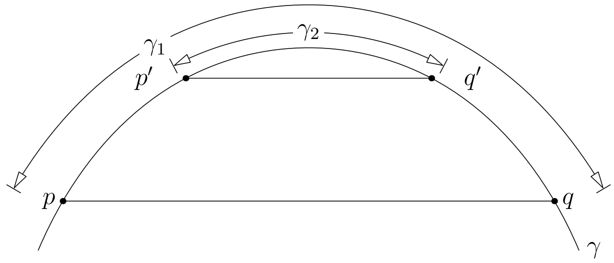

Figure 2.1: Lemma 2.1.3.

with respect to the norm $\|\cdot\|$ of X is bounded from above. If C is a rectifiable curve, then we denote by $|C|$ the *length of* C, i.e.,

$$|C| := \sup \left\{ \sum_{i=1}^{n} \|\phi(t_i) - \phi(t_{i-1})\| : \ (t_0, t_1, \cdots, t_n) \text{ is a partition of } [\alpha, \beta] \right\}.$$

The following results concerning lengths of convex curves (i.e., the boundaries of compact, convex sets in the plane) will be used regularly in this dissertation.

Lemma 2.1.3. (cf. [48, Proposition 29] or [57, 4E]) *Let p, p', q', and q be points in the given order on a convex curve γ such that $\langle p, q\rangle$ and $\langle p', q'\rangle$ are parallel. Let γ_1 be the curve from p to q containing p' and q', and γ_2 be the part of γ_1 from p' to q'. Then*

$$\frac{|\gamma_1|}{\|p-q\|} \geq \frac{|\gamma_2|}{\|p'-q'\|}.$$

Lemma 2.1.4. (cf. [48, Proposition 30]) *For any points p, $q \in S_X$, the length of the circular arc of S_X connecting p and q is not larger than $2\|p-q\|$.*

The following lemma is the two-dimensional case of a result proved by H. S. Witsenhausen for Minkowski spaces (i.e., real finite dimensional Banach spaces) of dimensions $d \geq 2$, where we can replace the half-girth in the higher dimensional cases $d \geq 3$ by $|S_X|/2$ (cf. [62] and [60, p. 291]).

Lemma 2.1.5. (cf. [62] or [60, Theorem 9.4.8]) *Let C be a rectifiable simple closed curve in a Minkowski plane X. Then $|C| \geq (|S_X|/2)h(C)$. Moreover, this inequality is best possible.*

For the definition of $h(C)$ we refer to p. 67.

2.2 Triangle inequality

The triangle inequality and its consequences are also fundamental tools in Minkowski geometry, and they will also be used regularly in this dissertation.

Theorem 2.2.1. *Let x, y, and z be three points in a Minkowski space X. Then $\|x-z\| \leq \|x-y\| + \|y-z\|$, with equality if and only if $[p,q] \subset S_X$, where $p = \widehat{y-x}$ and $q = \widehat{z-y}$.*

One direct consequence of the above theorem is the *generalized Monotonicity Lemma*:

Theorem 2.2.2. (cf. [48, Proposition 31]) *Let x, y, $z \in X \backslash \{o\}$ be points such that $x \neq z$, $[o,y\rangle$ "between" $[o,x\rangle$ and $[o,z\rangle$, and $\|y\| = \|z\|$. Then $\|y-x\| \leq \|z-x\|$, with equality if and only if either*

- *$y = z$,*
- *or o and y are on opposite sides of $\langle x,z\rangle$, and $[\widehat{z-x}, \widehat{y}]$ is a segment on the unit circle,*
- *or o and y are on the same side of $\langle x,z\rangle$, and $[\widehat{z-x}, \widehat{-z}]$ is a segment on the unit circle.*

In particular, if the plane is strictly convex, then we always have strict inequality.

As a corollary, we have the *Monotonicity Lemma*:

Lemma 2.2.3. *For $p \in S_X$ fixed and $x \in S_X$ variable in dimension two, the length $\|p-x\|$ is non-decreasing as x moves on S_X from p to $-p$*

2.3 Orthogonalities

One of the ideas that plays a fundamental role in Euclidean geometry is that of orthogonality, and one of the underlying themes in Minkowski geometry is to look for analogues of this notion. Roberts [55] introduced the *Roberts orthogonality* in 1934: for any x, $y \in X$, x is said to be Roberts orthogonal to y ($x \perp_R y$) if and only if

$$\|x+ty\| = \|x-ty\|\,,\ \forall t \in \mathbb{R}.$$

However, Roberts orthogonality is not "good" enough since there are Minkowski planes where the only point that is orthogonal to an arbitrarily given point $x \neq o$ is the origin o (for such an example, we refer to [34, Example 2.1] or Remark 3.2.3 of this dissertation).

2.3.1 Properties of Birkhoff orthogonality and isosceles orthogonality

Birkhoff [13] introduced the *Birkhoff orthogonality*: x is said to be Birkhoff orthogonal to y ($x \perp_B y$) if and only if

$$\|x+ty\| \geq \|x\|\,,\ \forall t \in \mathbb{R}.$$

James [34] introduced the so-called *isosceles orthogonality* which, as shown by Donghai Ji [37], is closely connected with the nonsquareness of Banach spaces: x is said to be isosceles orthogonal to y ($x \perp_I y$) if and only if

$$\|x + y\| = \|x - y\| .$$

Further two orthogonality types that will appear in this dissertation are *Singer orthogonality* [59] and *area orthogonality* [7]: x is said to be Singer orthogonal to y ($x \perp_S y$) if and only if $\|x\| \cdot \|y\| = 0$ or

$$\|\widehat{x} + \widehat{y}\| = \|\widehat{x} - \widehat{y}\| ;$$

x is said to be area orthogonal to y ($x \perp_A y$) if and only if $\|x\| \cdot \|y\| = 0$ or x and y are linearly independent such that the straight lines spanned by them divide the unit disc of the Minkowski plane spanned by x and y into four parts of equal area.

Now we list some fundamental properties of isosceles orthogonality and Birkhoff orthogonality, which are the two orthogonality types that we discuss in detail in this dissertation.

Theorem 2.3.1. (cf. [34]) *Isosceles orthogonality is*

- *symmetric* ($x \perp_I y \Rightarrow y \perp_I x$),
- *in general not homogeneous* ($x \perp_I y \not\Rightarrow x \perp_I ty$ *for some number* $t \in \mathbb{R}$),
- *and in general not additive* ($x \perp_I y, x \perp_I z \not\Rightarrow x \perp_I (y + z)$).

If isosceles orthogonality is homogeneous or additive, then the respective Minkowski space is Euclidean.

The following two lemmas deal with the uniqueness property of isosceles orthogonality.

Lemma 2.3.2. (cf. [3, Corollary 4]) *For any* $x \in S_X$ *and* $0 \leq \alpha \leq 1$, *there exists a point* $y \in \alpha S_X$ *which is unique up to the sign and satisfies* $x \perp_I y$.

Lemma 2.3.3. (cf. [3]) *Let* X *be a strictly convex Minkowski plane. Then, for any* $x \in X \backslash \{o\}$ *and any number* $\lambda > 0$, *there exists a unique* $y \in \lambda S_X$ *(except for the sign) such that* $x \perp_I y$.

Theorem 2.3.4. (cf. [35]) *Birkhoff orthogonality is*

- *in general not symmetric* ($x \perp_B y \not\Rightarrow y \perp_B x$),
- *homogeneous* ($x \perp_B y \Rightarrow x \perp_B ty$, $\forall t \in \mathbb{R}$),
- *in general not additive on the right* ($x \perp_B y, x \perp_B z \not\Rightarrow x \perp_B (y + z)$),
- *and in general not additive on the left* ($y \perp_B x, z \perp_B x \not\Rightarrow (y + z) \perp_B x$).

Moreover, Birkhoff orthogonality is additive on the right if and only if the respective Minkowski space is smooth.

A Minkowski plane X is said to be a *Radon plane* if Birkhoff orthogonality on X is symmetric; for more information about Radon plane we refer to [47].

Theorem 2.3.5. (cf. [36]) *Let X be a Minkowski space with dimension not less than 3. Then*

- *X is Euclidean if and only if Birkhoff orthogonality on X is additive on the left.*
- *X is Euclidean if and only if Birkhoff orthogonality on X is symmetric.*

Now we discuss the uniqueness of Birkhoff orthogonality.

Theorem 2.3.6. (cf. [35]) *Let X be a Minkowski plane and $x \in S_X$. Then*

- *there exists a unique point (up to the sign) $y \in S_X$ such that $x \perp_B y$ if there exists a unique line that supports B_X at x;*
- *there exists a unique point (up to the sign) $y \in S_X$ such that $y \perp_B x$ if $\langle -x, x\rangle$ is not parallel to any non-trival segment contained in S_X.*

2.3.2 Characterizations of inner product spaces in terms of orthogonalties

Next we collect some characterizations of inner product spaces in terms of the relation between Birkhoff orthogonality and isosceles orthogonality that will be used later.

Lemma 2.3.7. [8, (4.12)] *If for any $x, y \in X$ with $x \perp_I y$ there exists a number $0 < t < 1$ such that $x \perp_I ty$, then X is Euclidean.*

Lemma 2.3.8. *If for any x, $y \in X$ with $x \perp_I y$ there exists a number $t > 1$ such that $x \perp_I ty$, then X is Euclidean.*

Proof. By the assumption of the lemma, for any x, $y \in X$ with $x \perp_I y$ there exists a number $t > 1$ such that $y \perp_I tx$, and therefore $x \perp_I \frac{1}{t}y$. By Lemma 2.3.7, X is Euclidean. $\square$

Lemma 2.3.9. (cf. [8] (4.1)) *If the implication*

$$x \perp_I y \Rightarrow x \perp_B y$$

holds for any $x, y \in X$, then the Minkowski plane X is Euclidean.

Lemma 2.3.10. (cf. [8, 10.2]) *A Minkowski plane X is Euclidean if and only if the implication*

$$x \perp_B y \Rightarrow x \perp_I y$$

holds for any x, $y \in S_X$.

Lemma 2.3.11. (cf. [40] and [8, 10.9]) *A Minkowski plane X is Euclidean if and only if the implication*

$$x \perp_I y \Rightarrow x \perp_B y$$

holds for any $x,\, y \in S_X$.

2.4 Bisectors

The *bisector* $B(p,q)$ of the linear segment $[p,q]$ with endpoints $p \neq q$ in X is defined by

$$B(p,q) := \{x \in X : \ \|x-p\| = \|x-q\|\}.$$

A point z belongs to $B(p,q)$ if and only if $z - \frac{p+q}{2}$ is isosceles orthogonal to $\frac{p-q}{2}$, which means that the geometric structure of bisectors in Minkowski spaces is fully determined by geometric properties of isosceles orthogonality. Later we show that it is also closely related to Birkhoff orthogonality. For the discussion in Chapter 3 we need the following fundamental properties of bisectors.

Theorem 2.4.1. (cf. [8, (3.3)]) *A Minkowski plane X is Euclidean if and only if $B(-x,x)$ is a line for any $x \in X\backslash\{o\}$.*

Theorem 2.4.2. (cf. [31]) *For any point $x \in X$, $B(-x,x)$ is convex in the direction of x, i.e., if a line parallel to $\langle -x,x\rangle$ intersects $B(-x,x)$ in two distinct points then the whole segment with these points as endpoints is contained in $B(-x,x)$.*

Lemma 2.4.3. (cf. [48, Corollary 16]) *For any $x \in X\backslash\{o\}$, any line parallel to $\langle -x,x\rangle$ intersects $B(-x,x)$ in exactly one point if and only if S_X does not contain a non-trivial segment parallel to $\langle -x,x\rangle$.*

2.5 Angular bisectors

For non-collinear rays $[p,x\rangle$ and $[p,y\rangle$, the ray

$$[p, \frac{1}{2}(\frac{x-p}{\|x-p\|} + \frac{y-p}{\|y-p\|}) + p\rangle$$

is called the *Busemann angular bisector* of the angle $\angle xpy$ and denoted by $A_B([p,x\rangle,[p,y\rangle)$ (cf. [15]). It is trivial that when $\|x-p\| = \|y-p\|$, then

$$A_B([p,x\rangle,[p,y\rangle) = [p, \frac{1}{2}(x+y)\rangle.$$

Let z be a point in $\angle xpy$ such that $d(z,[p,x\rangle) = d(z,[p,y\rangle)$. Then the points on the ray $[p,z\rangle$, which is called the *Glogovskij angular bisector* of $\angle xpy$ and denoted by $A_G([p,x\rangle,[p,y\rangle)$, are equidistant to the sides of $\angle xpy$.

We need the following result for the discussion in Section 5.1.

Lemma 2.5.1. (cf. [20]) *A Minkowski plane is a Radon plane if and only if Busemann's and Glogovskij's definitions of angular bisectors coincide.*

We refer to [47], [48], and [20] for more information about Radon curves and angular bisectors.

Chapter 3

Bisectors in Minkowski planes and spaces

3.1 Introduction

It is well known that bisectors of Minkowski spaces have, in general, a complicated topological and geometric structure (cf. [31], [32], and the surveys [48] and [46]). It is interesting to observe that, due to this, even their radial projections (onto the unit sphere) have a large variety of properties, still yielding many interesting results. These refer mainly to new characterizations of inner product spaces and to different orthogonality concepts.

In this chapter we mainly study the structure of the *radial projection* $P(x)$ of $B(-x,x)$ for any point $x \in X \backslash \{o\}$, which is defined by

$$P(x) := \{\frac{z}{\|z\|} : \ z \in B(-x,x) \backslash \{o\}\}.$$

It is evident that if X is the Euclidean plane, then $P(x)$ contains precisely two points for any $x \in X \backslash \{o\}$, and when X is an n-dimensional Euclidean space ($n \geq 3$), then $P(x)$ is the unit sphere of an $(n-1)$-dimensional subspace. As we shall see, the geometric properties of $P(x)$ in general Minkowski spaces are much more complicated and worth studying.

In Section 3.2 we study geometric properties of bisectors in Minkowski planes and provide some detailed relation between Birkhoff orthogonality and the geometric structure of bisectors. Moreover, the intersection of radial projections of two bisectors is discussed. We lay special emphasis on planar results, since many of the results in higher dimensions can be directly obtained from their analogues in the planar case. One of the exceptions, namely the connectivity of $P(x)$ in higher dimensions, is presented in Section 3.4.

In Section 3.3 we prove the existence of a critical number $c(X)$ for any Minkowski space X, playing the role that $\sqrt{2}$ plays in Euclidean space. Also we derive lower and upper bounds on $c(X)$ and characterize the situations when $c(X)$ attains these bounds.

And in the last section of this chapter we study maps preserving isosceles orthogonality by applying known results on the structure of bisectors.

The results in this Chapter are contained in [50] and [52].

3.2 Radial projections of bisectors in Minkowski planes

Throughout this section, X is a Minkowski plane with a fixed orientation ω. For any $x \in X\backslash\{o\}$, let H_x^+ and H_x^- be the two open half-planes bounded by $\langle -x, x\rangle$ such that the orientations of $\overrightarrow{(-x)z}$ and $\overrightarrow{zx}$ are given by ω for any point $z \in H_x^+$, and that the orientations of $\overrightarrow{xz}$ and $\overrightarrow{z(-x)}$ are also given by ω for any point $z \in H_x^-$. Set

$$P^+(x) = P(x) \cap H_x^+ \text{ and } P^-(x) = P(x) \cap H_x^-.$$

It is evident that for any $x \in X\backslash\{o\}$ and any number $\alpha > 0$

$$P(\alpha x) = P(x),\ P^+(\alpha x) = P^+(x) = P^-(-\alpha x)$$

and

$$P^-(\alpha x) = P^-(x) = P^+(-\alpha x).$$

Thus it suffices to study the geometric structure of $P(x) = P(-x)$ for each $x \in S_X$.

Theorem 3.2.1. *For any $x \in S_X$, $P^+(x)$ and $P^-(x)$ are two connected subsets of S_X, and $P(x) = P^+(x) \cup P^-(x)$.*

Proof. It is clear that $P(x) = P^+(x) \cup P^-(x)$, since $B(-x, x) \cap \langle -x, x\rangle = \{o\}$.

By $P^-(x) = -P^+(x)$ it suffices to show that $P^+(x)$ is connected. Let $y \in S_X \cap H_x^+$ be a point such that $y \perp_B x$. Then x and y are linearly independent. Set

$$\begin{array}{rccc} T: & X & \longrightarrow & \mathbb{R} \\ & z = \alpha x + \beta y & \longrightarrow & \beta. \end{array}$$

It is clear that T is continuous and $T(H_x^+) = \{t : t > 0\}$.

Now we show that $B(-x, x) \cap H_x^+$ is connected. Suppose the contrary, i.e., that $B(-x, x) \cap H_x^+$ can be partitioned into two disjoint nonempty subsets A_1 and A_2 which are open in the relative topology induced on $B(-x, x) \cap H_x^+$. Assume that there exists a number $t_0 \in T(A_1) \cap T(A_2)$. Then there exist two points $z_1 = \alpha_1 x + t_0 y \in A_1$ and $z_2 = \alpha_2 x + t_0 y \in A_2$. From the convexity of $B(-x, x)$ in the direction of x (see Theorem 2.4.2) it follows that $[z_1, z_2] \subset B(-x, x) \cap H_x^+$. Thus $[z_1, z_2]$ can be partitioned into two disjoint nonempty sets $[z_1, z_2] \cap A_1$ and $[z_1, z_2] \cap A_2$ which are open in the subspace topology of $[z_1, z_2]$.

This is impossible. Thus $T(A_1) \cap T(A_2) = \varnothing$. It is clear that $T(A_1)$ and $T(A_2)$ are open sets, and that

$$T(A_1) \cup T(A_2) = T(B(-x,x) \cap H_x^+) = \{t : t > 0\},$$

a contradiction to the fact that the set $\{t : t > 0\}$ is connected.

Then, as image of $B(-x,x) \cap H_x^+$ under the function $R(X) = \frac{x}{\|x\|}$ which is continuous on $X \backslash \{o\}$, $P^+(x)$ is connected. $\square$

Theorem 3.2.2. *A Minkowski plane X is Euclidean if and only if for any $x \in S_X$ the set $P^+(x)$ is a singleton.*

Proof. The necessity is obvious. Conversely, for any $x \in S_X$ it follows from the assumption of the theorem that $B(-x,x)$ is contained in a line, which, by Theorem 2.4.1, is a characteristic property of Euclidean planes. $\square$

Remark 3.2.3. R. C. James [34] provided the following example. Let X_0 be the normed linear space consisting of all continuous functions of the from $f = ax + bx^2$, where $\|ax + bx^2\|$ is the maximum of $|ax + bx^2|$ in the interval $(0,1)$. If $x \perp_I ty$ holds for any $t \in \mathbb{R}$, then either $x = o$ or $y = o$. In other words, there exists a Minkowski plane X_0 such that $P^+(x)$ contains more than one point for any $x \in S_{X_0}$.

For any $x \in S_X$, we denote by $l(x)$ and $r(x)$ the two points such that $[r(x), l(x)]$ is a maximal segment parallel to $\langle -x, x\rangle$ on $S_X \cap H_x^+$ and that $r(x) - l(x)$ is a positive multiple of x. When there is no non-trivial segment on S_X parallel to $\langle -x, x\rangle$, the points $l(x)$ and $r(x)$ are chosen in such a way that $r(x) = l(x) \in S_X \cap H_x^+$ and $l(x) \perp_B x$ (cf. Figure 3.1 and Figure 3.2 below).

The following lemma, basic for the discussion after it, refers to the shape of bisectors in Minkowski planes.

Lemma 3.2.4. (cf. [46, Proposition 22]) *For any $x \in S_X$, $B(-x,x)$ is fully contained in the bent strip bounded by the rays $[x, x + r(x)\rangle$, $[x, x - l(x)\rangle$, $[-x, -x + l(x)\rangle$, and $[-x, -x - r(x)\rangle$.*

Theorem 3.2.5. *For any x, $y \in S_X$ we have that $y \in \overline{P(x)}$ whenever $y \perp_B x$.*

Proof. Case I: Suppose that there exists a non-trivial maximal segment $[a,b] \subset S_X$ parallel to $\langle -x, x\rangle$. It is trivial that if $y \in S_X$ is a point such that $y \perp_B x$, then either $y \in [a,b]$ or $-y \in [a,b]$. Thus it suffices to show that $[a,b] \subset \overline{P(x)}$.

For any $\lambda \in (0,1)$, let α be an arbitrary number in the open interval $(0, \min\{\lambda, 1-\lambda\})$. Then

$$\|\lambda a + (1-\lambda)b + \alpha(b-a)\| = \|(\lambda - \alpha)a + (1 - \lambda + \alpha)b\| = 1$$

and

$$\|\lambda a + (1-\lambda)b - \alpha(b-a)\| = \|(\lambda + \alpha)a + (1 - \lambda - \alpha)b\| = 1.$$

Thus $\lambda a + (1-\lambda)b \in P(\alpha \|b-a\| x) = P(x)$, and therefore $[a,b] \subset \overline{P(x)}$.

Case II: If there exists a unique point $y \in S_X \cap H_x^+$ such that $y \perp_B x$, then, by Lemma 3.2.4, $B(-x, x)$ is bounded between the lines $\langle x, x+y \rangle$ and $\langle -x, -x+y \rangle$. On the other hand, by Lemma 2.4.3, for any integer $n > 0$ there exists a unique number $\lambda_n \in [0, 1]$ such that $\lambda_n(x+ny)+(1-\lambda_n)(-x+ny) \perp_I x$, i.e.,

$$z_n := (2\lambda_n - 1)x + ny \in B(-x, x).$$

Moreover,

$$\begin{aligned}\left\| \frac{z_n}{\|z_n\|} - y \right\| &= \left\| \frac{(2\lambda_n - 1)x + ny}{\|(2\lambda_n - 1)x + ny\|} - y \right\| \\ &\leq \left\| \frac{(2\lambda_n - 1)x}{\|(2\lambda_n - 1)x + ny\|} \right\| + \left\| \frac{ny}{\|(2\lambda_n - 1)x + ny\|} - y \right\|.\end{aligned}$$

Since

$$\lim_{n\to\infty} \left\| \frac{(2\lambda_n - 1)x}{\|(2\lambda_n - 1)x + ny\|} \right\| = \lim_{n\to\infty} \frac{1}{n} \left\| \frac{(2\lambda_n - 1)x}{\|\frac{2\lambda_n - 1}{n}x + y\|} \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| \frac{ny}{\|(2\lambda_n - 1)x + ny\|} - y \right\| = \lim_{n\to\infty} \left| \frac{1 - \|\frac{2\lambda_n - 1}{n}x + y\|}{\|\frac{2\lambda_n - 1}{n}x + y\|} \right| = 0,$$

we have

$$\lim_{n\to\infty} \left\| \frac{z_n}{\|z_n\|} - y \right\| = 0.$$

It follows that $y \in \overline{P(x)}$, which completes the proof. □

One may expect that those points in S_X, to which x is Birkhoff orthogonal, are all in $\overline{P(x)}$. However, the following example shows that this is not true (see also Remark 3.2.8).

Example 1. Let X be the Minkowski plane on $\mathbb{R}^2$ with the maximum norm $\|(\alpha, \beta)\| = \max\{|\alpha|, |\beta|\}$ and $x = (1, 1)$. Then $B(-x, x) = \langle (-1, 1), (1, -1) \rangle$, and therefore $\overline{P(x)} = \{(1, -1), (-1, 1)\}$. It is clear that $(0, 1) \notin \overline{P(x)}$ and $(1, 0) \notin \overline{P(x)}$, while $x \perp_B (0, 1)$ and $x \perp_B (1, 0)$.

Let $x \in S_X$. By the uniqueness property of isosceles orthogonality, for any $t \in [0, 1]$ there exists a unique point $F_x(t)$ such that

$$F_x(t) \in B(-x, x) \cap tS_X \cap \overline{H_x^+}.$$

For any $t \in (0, 1]$, let

$$T_x(t) = \widehat{F_x(t)}.$$

The proof idea for the following lemma was jointly developed with H. Martini.

Lemma 3.2.6. *Let $\{t_n\} \subset (0, 1]$ be a sequence such that $\lim_{n\to\infty} t_n = 0$ and that $\{T_x(t_n)\}$ is a Cauchy sequence. Then $x \perp_B \lim_{n\to\infty} T_x(t_n)$.*

Proof. From the compactness of S_X and the fact that $\{T_x(t_n)\}$ is a Cauchy sequence it follows that there exists a point $z \in S_X$ such that

$$z = \lim_{n\to\infty} T_x(t_n).$$

We show that $x \perp_B z$, and it suffices to prove that $\inf_{\lambda\in\mathbb{R}} \|x+\lambda z\| = 1$. In fact,

$$\begin{aligned}
\inf_{\lambda\in\mathbb{R}} \|x+\lambda z\| &= \inf_{\lambda\in\mathbb{R}} \left\|x+\lambda \lim_{n\to\infty} T_x(t_n)\right\| \\
&= \lim_{n\to\infty} \inf_{\lambda\in\mathbb{R}} \left\|x+\frac{\lambda}{t_n} t_n T_x(t_n)\right\| \\
&= \lim_{n\to\infty} \inf_{\lambda\in\mathbb{R}} \|x+\lambda F_x(t_n)\| \\
&= \lim_{n\to\infty} \inf_{\lambda\in[-1,1]} \|x+\lambda F_x(t_n)\|,
\end{aligned}$$

where the last equality follows from the fact that $\|x+F_x(t_n)\| = \|x-F_x(t_n)\|$. By the triangle inequality, we have for any $\lambda \in [-1,1]$

$$1-|\lambda|t_n = \|x\| - \|\lambda F_x(t_n)\| \le \|x+\lambda F_x(t_n)\| \le \|x\| + \|\lambda F_x(t_n)\| \le 1+|\lambda|t_n,$$

and therefore

$$\inf_{\lambda\in\mathbb{R}} \|x+\lambda z\| = \lim_{n\to\infty} \inf_{\lambda\in[-1,1]} \|x+\lambda F_x(t_n)\| = 1.$$

This completes the proof. □

Theorem 3.2.7. *Let $x \in S_X$. If there exists a unique point $z \in S_X$ (except for the sign) such that $x \perp_B z$, then $z \in \overline{P(x)}$. And if there exists a point $z \in \overline{P(x)} \backslash P(x)$, then either $z \perp_B x$ or $x \perp_B z$.*

Proof. To prove the first statement, let $\{s_n\} \subset (0,1]$ be an arbitrary sequence such that $\lim_{n\to\infty} s_n = 0$. It is clear that $\{T_x(s_n)\}$ is a bounded subset of S_X, and therefore we can choose a convergent subsequence $\{T_x(s_{n_k})\}$. Let $t_k = s_{n_k}$. From Lemma 3.2.6 it follows that $x \perp_B \lim_{k\to\infty} T_x(t_k)$. Thus either $\lim_{k\to\infty} T_x(t_k) = z$ or $\lim_{k\to\infty} T_x(t_k) = -z$. Since $T_x(t_k) \in P(x)$ for each k, $z \in \overline{P(x)}$.

For proving the second statement, let $z \in \overline{P(x)} \backslash P(x)$. We consider the following two cases.

Case I: The line $\langle -z, z\rangle$ intersects one of the four rays

$$[x, x+r(x)\rangle,\ [x, x-l(x)\rangle,\ [-x, -x+l(x)\rangle,\ \text{and } [-x, -x-r(x)\rangle.$$

Without loss of generality, we can suppose that $[0, z\rangle$ intersects $[x, x+r(x)\rangle$ in some point p; see Figure 3.1. Since $z \in \overline{P(x)}$, there exists a sequence $\{z_n\} \subset P^+(x)$ such that $z_i \neq z_j$ $(i \neq j)$, $\lim_{n\to\infty} z_n = z$, and

$$(\langle -z_n, z_n\rangle \cap [x, x+r(x)\rangle) \in (p + \frac{1}{n} B_X).$$

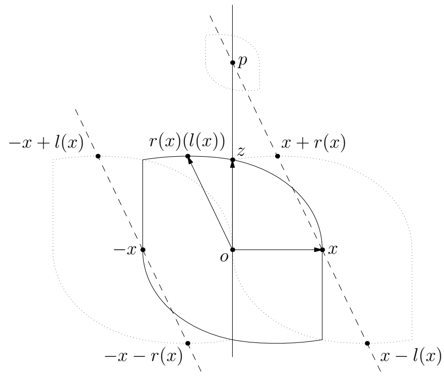

Figure 3.1: The proof of Theorem 3.2.7, Case I

By Lemma 3.2.4, for any number $t > \|p\| + 1$ we have $tz_n \notin B(-x, x)$. Thus, for each z_n there exists a number t_n being the largest positive number such that $t_n z_n \perp_I x$. It is clear that $\{t_n\}_{n=1}^{\infty}$ is bounded. Thus we can choose a subsequence $\{t_{n_k}\}$ such that

$$\lim_{k\to\infty} t_{n_k} = t_0.$$

Hence

$$\|t_0 z + x\| = \lim_{k\to\infty} \|t_{n_k} z_{n_k} + x\| = \lim_{k\to\infty} \|t_{n_k} z_{n_k} - x\| = \|t_0 z - x\|,$$

which means that $t_0 z \perp_I x$. Since $z \in \overline{P(x)} \setminus P(x)$, we see that $t_0 = 0$. Thus we can suppose, without loss of generality, that $\{t_{n_k}\}_{k=1}^{\infty} \subset (0, 1]$. Hence

$$z = \lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} T_x(t_{n_k}).$$

By Lemma 3.2.6, $x \perp_B z$.

Case II: The line $\langle -z, z\rangle$ intersects none of the four rays

$$[x, x + r(x)\rangle,\ [x, x - l(x)\rangle,\ [-x, -x + l(x)\rangle,\ \text{and } [-x, -x - r(x)\rangle;$$

see Figure 3.2. Then it is trivial that the line $\langle -z, z\rangle$ is fully contained in the double cone

$$\{\lambda l(x) + \mu r(x) :\ \lambda\mu \geq 0\}.$$

Thus $\langle -z, z\rangle$ intersects the segment $[l(x), r(x)]$, and therefore $z \perp_B x$. □

Remark 3.2.8. 1. The condition $y \in S_X$ together with $y \perp_B x$ does not imply that in general $y \in P(x)$. For example, take again the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1, 0)$. Then $y = (1, 1)$ is a point such that $y \in S_X$ and $y \perp_B x$. But for any $t > 0$ we have

$$\|x + ty\| - \|x - ty\| = 1 + t - \max\{|1 - t|, t\} > 0,$$

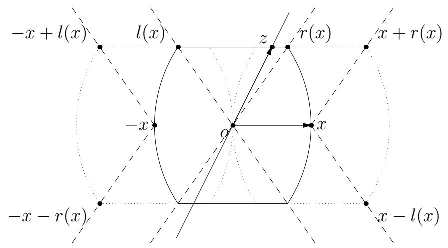

Figure 3.2: The proof of Theorem 3.2.7, Case II

which means that $y \notin P(x)$.

2. In general, the condition that $z \in S_X$ is the unique point (except for the sign) satisfying $x \perp_B z$ does not imply $z \in P(x)$. Let X be the Minkowski plane on $\mathbb{R}^2$ with the norm $\|\cdot\|$, where for any point (α, β)

$$\|(\alpha,\beta)\| := \begin{cases} \sqrt{\alpha^2+\beta^2} & : \ \alpha\beta \geq 0; \\ \max\{|\alpha|,\ |\beta|\} & : \ \alpha\beta < 0. \end{cases}$$

Take $x = (1,0)$ and $z = (0,1)$. Then $x,\ z \in S_X$, and z is the unique point (except for the sign) in S_X such that $x \perp_B z$. But for any $t > 0$ we have

$$\|x+tz\| - \|x-tz\| = \sqrt{1+t^2} - \max\{1,t\} > 0,$$

which implies that $z \notin P(x)$.

3. $\overline{P^+(x)}$ is an arc of S_X (possibly degenerate to a point) since $P^+(x)$ is connected. Theorem 3.2.7 says that if z is one of the endpoints of $\overline{P^+(x)}$ and $z \notin P^+(x)$, then either $x \perp_B z$ or $z \perp_B x$. We remark that, in general, the endpoints of $\overline{P^+(x)}$ have nothing to do with the points that are Birkhoff orthogonal to x or with the points to which x is Birkhoff orthogonal. For example, let X be a Minkowski plane on $\mathbb{R}^2$ (cf. Figure 3.3) with $x, y \in S_X$. Then it can be seen that y is the unique point (except for the sign) in S_X which is Birkhoff orthogonal to x, and it is also the unique point (except for the sign) in S_X to which x is Birkhoff orthogonal. However, y is contained in the arc between y_1 and y_2, which is a subset of $\overline{P^+(x)}$.

Now we study the distance $d(x, P(x))$ from a point x to $P(x)$, and we have to use the following lemma.

Lemma 3.2.9. (cf. [34]) *If x and y are two points such that $x \perp_I y$, then*

(1) $\|x+ky\| \leq |k|\,\|x \pm y\|$ and $\|x \pm y\| \leq \|x+ky\|$, if $|k| \geq 1$.

(2) $\|x+ky\| \leq \|x \pm y\|$ and $|k|\,\|x \pm y\| \leq \|x+ky\|$, if $|k| \leq 1$.

A Minkowski plane X is said to be *rectilinear* if S_X is a parallelogram. One can easily verify that a Minkowski plane is rectilinear if and only if there exist two points $x,\ y \in S_X$ such that $\|x+y\| = \|x-y\| = 2$.

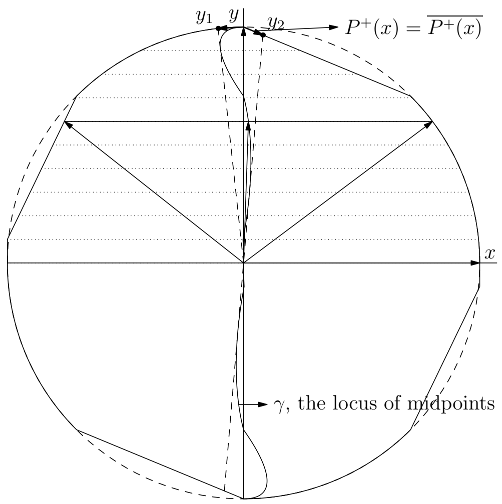

Figure 3.3: $\overline{P(x)}$ is not determined by points which are Birkhoff orthogonal to x or to which x is Birkhoff orthogonal.

Theorem 3.2.10. *For any $x \in S_X$ we have*

$$1 \leq d(x, P(x)) \leq 2,$$

with equality on the right only if X is rectilinear, and with equality on the left only if either there exists a segment parallel to $\langle -x, x\rangle$ on S_X whose length is not less than 1, or there exists a point $z \in S_X$ such that $\|z - x\| = 1$ and $[x, z] \subset S_X$.

Proof. It is trivial that $d(x, P(x)) \leq 2$. If $d(x, P(x)) = 2$, then for any $z \in P(x)$ we have $\|z - x\| = 2$. Let $z_0 \in P(x)$ be a point such that $z_0 \perp_I x$. Then

$$\|z_0 + x\| = \|z_0 - x\| = 2,$$

which implies that X is rectilinear.

For any $x \in S_X$ and $z \in P(x)$ there exists a number $t > 0$ such that $tz \perp_I x$. If $t \geq 1$, then $0 < \frac{1}{t} \leq 1$. By Lemma 3.2.9, we have

$$\|z - x\| = \left\|\frac{1}{t}tz - x\right\| \geq \frac{1}{t}\|tz + x\| = \frac{1}{2t}(\|tz + x\| + \|tz - x\|) \geq 1.$$

If $0 < t < 1$, then $\frac{1}{t} \geq 1$. Again, by Lemma 3.2.9 we have

$$\|z - x\| = \left\|\frac{1}{t}tz - x\right\| \geq \|tz + x\| = \frac{1}{2}(\|tz + x\| + \|tz - x\|) \geq 1.$$

Hence $d(x, P(x)) = \inf\{\|x - z\| : z \in P(x)\} \geq 1$.

Suppose now that $d(x, P(x)) = 1$, and without loss of generality we can assume that $d(x, P^+(x)) = 1$.

Case I: If there exists a point $z \in P^+(x)$ such that $\|z - x\| = 1$, then there exists a number $t > 0$ such that $tz \perp_I x$, which yields

$$\max\{t, 1\} \leq \frac{1}{2}(\|tz + x\| + \|tz - x\|) = \|tz - x\| \,. \tag{3.2.1}$$

If $0 < t < 1$, then it follows from the convexity of the function $f(s) = \|x - sz\|$ and $f(0) = f(1) = 1 \leq f(-t) = f(t)$ that $\|x - \lambda z\| = 1$ for any $\lambda \in [-t, 1]$, which implies that $[x - z, x] \subset S_X$.

If $t \geq 1$, then we have

$$\|tz\| = \|tz - z\| + \|z\| = \|tz - z\| + \|z - x\| \geq \|tz - x\| \,.$$

From (3.2.1) it follows that the convex function $g(s) = \|z - sx\|$ satisfies $g(0) = g(1) = g(-\frac{1}{t}) = g(\frac{1}{t}) = 1$, and then $\|z - \lambda x\| = 1$ for $-\frac{1}{t} \leq \lambda \leq 1$, which implies that $[z - x, z] \subset S_X$.

Case II: If $\|z' - x\| > 1$ for any $z' \in P^+(x)$, then there exists a point $z \in \overline{P^+(x)} \backslash P^+(x)$ such that $\|z - x\| = 1$. By Theorem 3.2.7, either $z \perp_B x$ or $x \perp_B z$. It can be proved in a similar way as in Case I that either $[x - z, x] \subset S_X$ or $[z - x, z] \subset S_X$. The proof is complete. □

Corollary 3.2.11. *For any $x \in S_X$ there exist two points u, $v \in S_X \backslash ((x + d(x, P(x))\mathrm{int}B_X) \cup (-x + d(x, P(x))\mathrm{int}B_X))$ such that*

$$B(-x, x) \subset \{\alpha u + \beta v : \ \alpha\beta \geq 0\}.$$

Theorem 3.2.12. *Let $x \in S_X$. If there exists a segment $[a, b] \subset S_X$ parallel to $\langle -x, x\rangle$ and of length not less than 1, then $d(x, P(x)) = 1$.*

Proof. From Theorem 3.2.5 it follows that $[a, b] \subset \overline{P(x)}$. Since $\|b - a\| \geq 1$, we can assume, without loss of generality, that there exists a point $z \in [a, b]$ such that $z - a = x$. Then $\|z - x\| = \|a\| = 1$, which implies that $d(x, P(x)) = 1$. □

Remark 3.2.13. The fact that there exists a point z with $\|z - x\| = 1$ and $[x, z] \subset S_X$ does in general not imply that $d(x, P(x)) = 1$. Namely, take again the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1, 1)$. Then x is contained in the segment $[(-1, 1), x]$ whose length is 2. But it is clear that $d(x, P(x)) = 2$.

Next, we examine properties of intersections of radial projections of bisectors of two distinct segments, and we start with a characteristic property of the Euclidean plane.

Theorem 3.2.14. *A Minkowski plane X is Euclidean if and only if for any x, $y \in S_X$ with $x \neq \pm y$, $P(x) \cap P(y) = \varnothing$.*

Proof. We only need to show sufficiency. Suppose that X is not Euclidean. Then, by Theorem 3.2.2, there exists a point $x \in S_X$ such that $P^+(x)$ contains more than one point. Let $x' \in S_X \cap H_x^+$ be such that $x \perp_I x'$. Then $x' \in P^+(x)$. Assume that there exists a point $y' \in P^+(x)$, $y' \neq x'$, and let $y \in S_X \backslash \{\pm x\}$ be such that $y \perp_I y'$. Then $y' \in P(y)$ and $P(x) \cap P(y) \neq \varnothing$, a contradiction. $\square$

It is possible that $P(x) = P(y)$ holds for two points x, $y \in S_X$ with $x \neq \pm y$; see the following example.

Example 2. Let X be the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1, \frac{1}{2})$, and $x' = (1, \frac{1}{3})$. We show that

$$P(x) = P(x') = [(-1,1),(0,1)] \cup [(0,-1),(1,-1)] \backslash \{(-1,1),(1,-1)\}. \quad (3.2.2)$$

On the one hand, we have

$$\left\| \frac{1}{2}(0,1) + x \right\| = \left\| \frac{1}{2}(0,1) - x \right\| \text{ and } \left\| \frac{2}{3}(0,1) + x' \right\| = \left\| \frac{2}{3}(0,1) - x' \right\|,$$

and therefore $\{(0,1),(0,-1)\} \subseteq P(x) \cap P(x')$ and

$$d(x, P(x)) = \|(0,1) - x\| = \|(0,1) - x'\| = d(x', P(x')) = 1.$$

On the other hand, it is evident that $\|z - x\| < 1$ for any point $z \in S_X$ strictly between $(0,1)$ and x, and that $\|z - x'\| < 1$ for any point $z \in S_X$ strictly between $(0,1)$ and x'.

Now we show that

$$\{(-1,1),(1,-1)\} \subseteq (\overline{P(x)} \backslash P(x)) \cap (\overline{P(x')} \backslash P(x')). \quad (3.2.3)$$

For any $t > 0$ we have

$$\|t(-1,1) + x\| - \|t(-1,1) - x\| = \left\| (-t+1, t+\frac{1}{2}) \right\| - \left\| (-t-1, t-\frac{1}{2}) \right\| \neq 0$$

and

$$\|t(-1,1) + x'\| - \|t(-1,1) - x'\| = \left\| (-t+1, t+\frac{1}{3}) \right\| - \left\| (-t-1, t-\frac{1}{3}) \right\| \neq 0.$$

On the other hand, for any integer $n > 0$ we have

$$\left\| (1-n, n-\frac{1}{2}) + x \right\| = \|(2-n, n)\| = n = \|(-n, n-1)\| = \left\| (1-n, n-\frac{1}{2}) - x \right\|$$

and

$$\left\| (1-n, n-\frac{1}{3}) + x' \right\| = \|(2-n, n)\| = n = \left\| (-n, n-\frac{2}{3}) \right\| = \left\| (1-n, n-\frac{1}{3}) - x' \right\|.$$

It is evident that

$$\lim_{n\to\infty} \frac{(1-n, n-\frac{1}{2})}{\|(1-n, n-\frac{1}{2})\|} = \lim_{n\to\infty} \frac{(1-n, n-\frac{1}{3})}{\|(1-n, n-\frac{1}{3})\|} = (-1,1).$$

Thus (3.2.3) holds, and therefore (3.2.2) holds.

Remark 3.2.15. This example shows also that $d(x, P^+(x))$ is not necessarily equal to $d(-x, P^+(x))$.

Next, we derive a sufficient condition for the property that two radial projections satisfy $P(x) \cap P(y) = \varnothing$.

Lemma 3.2.16. *Let $x, y \in S_X$. If $x \perp_I y$, then for any number $t > 1$ the inequality*

$$\|x + ty\| > \|x + y\|$$

holds.

Proof. Suppose the contrary, i.e., that there exists a number $t_0 > 1$ such that $\|x + t_0 y\| \leq \|x + y\|$. Then, the obvious convexity of the function $f(t) = \|x + ty\|$ implies

$$\|x + y\| = \|x - y\| = \|x + t_0 y\| = f(0) = 1.$$

This implies that $[x + t_0 y, x - y]$ is a segment on S_X having length larger than 2, which is impossible. □

Theorem 3.2.17. *For any x, $y \in S_X$ with $x \perp_I y$, $P(x) \cap P(y) = \varnothing$.*

Proof. First we show that $\frac{x+y}{\|x+y\|} \notin P(x)$. Suppose that there exists a number $t > 0$ such that $\|t(x + y) + x\| = \|t(x + y) - x\|$. Then

$$\left\|(1 + \frac{1}{t})x + y\right\| = \left\|(1 - \frac{1}{t})x + y\right\|.$$

If $t \geq \frac{1}{2}$, then $|1 - \frac{1}{t}| \leq 1$. Thus, from Lemma 3.2.16 and the convexity of the function $\lambda \to \|\lambda x + y\|$ we get

$$\left\|(1 + \frac{1}{t})x + y\right\| > \|x + y\| \geq \left\|(1 - \frac{1}{t})x + y\right\|,$$

a contradiction. Hence $0 < t < \frac{1}{2}$. Then we have

$$\left\|(1 - \frac{1}{t})x + y\right\| = \left\|(\frac{1}{t} - 1)x - y\right\| = \frac{1}{2}(\left\|(1 + \frac{1}{t})x + y\right\| + \left\|(\frac{1}{t} - 1)x - y\right\|) \geq \frac{1}{t}.$$

On the other hand, we have

$$\left\|(\frac{1}{t} - 1)x - y\right\| \leq \frac{1}{t} - 1 + 1 = \frac{1}{t}$$

and, therefore,

$$\left\|(1 + \frac{1}{t})x + y\right\| = \left\|(\frac{1}{t} - 1)x - y\right\| = \frac{1}{t}.$$

Then the convex function $f(\lambda) = \|x + \lambda(x + y)\|$ satisfies $f(-1) = f(-t) = f(0) = f(t) = 1$ with $-1 < -t < 0 < t$. Therefore $f(\lambda) = 1$ for $-1 \leq \lambda \leq t$. In particular, we have $f(-\frac{1}{2}) = 1$ and then $\|x + y\| = \|x - y\| = 2$.

This implies that S_X is a parallelogram with $\pm x$ and $\pm y$ as vertices. Then $[x, y] \subseteq S_X$, and therefore $\left\|(1+\frac{1}{t})x + y\right\| = 2 + \frac{1}{t}$, again a contradiction.

Since x and y are arbitrary, we also have $\frac{x-y}{\|x-y\|} \notin P(x)$.

Without loss of generality, we can suppose that $y \in H_x^+$. Then, since $P^+(x)$, $P^-(x)$, $P^+(y)$, and $P^-(y)$ are all connected sets, $P^+(x)$ lies strictly between $\frac{x+y}{\|x+y\|}$ and $\frac{y-x}{\|x+y\|}$, $P^-(x)$ lies strictly between $\frac{-x-y}{\|x+y\|}$ and $\frac{x-y}{\|x+y\|}$, $P^+(y)$ lies strictly between $\frac{-x-y}{\|x+y\|}$ and $\frac{y-x}{\|x+y\|}$, and $P^-(y)$ lies strictly between $\frac{x+y}{\|x+y\|}$ and $\frac{x-y}{\|x+y\|}$. Thus $P(x) \cap P(y) = \varnothing$, and this completes the proof. □

Remark 3.2.18. It is possible that there exist two points $x, y \in S_X$ with $x \perp_I y$ such that $\overline{P(x)} \cap \overline{P(y)} \neq \varnothing$. For example, let X be the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1, 0)$ and $y = (0, 1)$. Then $(1, 1) \in \overline{P(x)} \cap \overline{P(y)}$ (cf. [34, Example 4.1]).

3.3 A critical number for Minkowski spaces

The discussion in this section arises from the following natural problem: Determine the sign of the difference

$$\|x + y\| - \|x - y\| \tag{3.3.1}$$

when only the directions of the vectors x and y are known. We exclude the trivial case where one of the two vectors is o. In Euclidean case, this problem can be solved in different ways. For example, we know that the difference (3.3.1) is positive if and only if the angle between x and y is less than $\pi/2$. Equivalently, (3.3.1) is positive if and only if

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| < \sqrt{2}. \tag{3.3.2}$$

From the discussion in the foregoing sections it can be seen that for general Minkowski spaces we cannot determine the sign of (3.3.1). The only thing we can probably do in this direction is to provide a sufficient condition for guaranteing that (3.3.1) is positive. As there is no natural definition of angular measure in Minkowski spaces, we would like to find a number which plays a role as the number $\sqrt{2}$ does in (3.3.2).

For the discussion in the sequel, we need to introduce the so called *non-square constants*

$$J(X) := \sup\{\min\{\|x + y\|, \|x - y\|\} : \ x, y \in S_X\}$$

and

$$S(X) := \inf\{\max\{\|x + y\|, \|x - y\|\} : \ x, y \in S_X\}.$$

Also we shall use the following equivalent representations of these two constants, which were provided in [37]:

$$J(X) = \sup\{\|x - y\| : \ x, y \in S_X, \ x \perp_I y\}$$

and

$$S(X) = \inf\{\|x - y\| : \ x, y \in S_X, \ x \perp_I y\}.$$

It has been shown (cf. [16], [23], and [29, Theorem 10]) that

$$1 \leq S(X) \leq \sqrt{2} \leq J(X) \leq 2$$

and

$$J(X) \cdot S(X) = 2.$$

Now we are going to define, for any Minkowski space, the so-called *critical number* $c(X) := \inf_{x \in S_X} d(x, P(x))$. Our first result on $c(X)$ is given by

Theorem 3.3.1. *For any Minkowski space X we have*

$$1 \leq c(X) \leq \sqrt{2},$$

with equality on the left if and only if there exists a segment contained in S_X whose length is not less than 1, and with equality on the right if and only if X is Euclidean.

Proof. By Theorem 3.2.10, for any $x \in S_X$ the inequality $d(x, P(x)) \geq 1$ holds. Thus it is trivial that $\inf_{x \in S_X} \{d(x, P(x))\} \geq 1$. When $\inf_{x \in S_X} \{d(x, P(x))\} = 1$, by the compactness of the unit sphere there exists a point $x_0 \in S_X$ such that $d(x_0, P(x_0)) = 1$. Then, by Theorem 3.2.10, there exists a segment in S_X having length not less than 1.

Conversely, suppose that there exists a segment $[a, b] \subset S_X$ with $\|a - b\| \geq 1$. Then it follows from Theorem 3.2.12 that $d(\widehat{a-b}, P(\widehat{a-b})) = 1$.

On the other hand, for any x, $y \in S_X$ with $x \perp_I y$ we have

$$\|x + y\| = \|x - y\| \geq d(x, P(x)) \geq \inf_{x \in S_X} \{d(x, P(x))\}.$$

Thus

$$\sqrt{2} \geq S(X) = \inf\{\|x - y\| : \ x, \ y \in S_X, \ x \perp_I y\} \geq \inf_{x \in S_X} \{d(x, P(x))\}.$$

If $\inf_{x \in S_X} \{d(x, P(x))\} = \sqrt{2}$, then

$$\sup\{\|x - y\| : \ x, \ y \in S_X, \ x \perp_I y\} = \inf_{x \in S_X} \{d(x, P(x))\} = \sqrt{2}.$$

To prove that X is Euclidean, it suffices to show that each two-dimensional subspace of X is Euclidean, and therefore we can assume, without loss of generality, that $\dim X = 2$. Then, by Theorem 3.2.2, we only have to show that $P(x) = \{y, -y\}$ for any x, $y \in S_X$ with $x \perp_I y$. Suppose the contrary, i.e., that there exist some points x, y, $z \in S_X$ with $x \perp_I y$ such that $z \in P(x) \backslash \{y, -y\}$

and, without loss of generality, that z and y lie in the same half-plane bounded by $\langle -x, x\rangle$. It is clear that

$$\|z-x\| \geq d(x,P(x)) \geq \inf_{x\in S_X}\{d(x,P(x))\} = \sqrt{2} = \|y-x\|$$

and

$$\|z+x\| \geq d(-x,P(-x)) \geq \inf_{x\in S_X}\{d(x,P(x))\} = \sqrt{2} = \|y+x\|.$$

If one of $\|z+x\|$ and $\|z-x\|$ is $\sqrt{2}$ then, since $J(X) = S(X) = \sqrt{2}$, it follows from [4, Proposition 1] that $\|z+x\| = \|z-x\|$, which contradicts the uniqueness property of isosceles orthogonality (see Lemma 2.3.2). Thus we have $\min\{\|z+x\|, \|z-x\|\} > \sqrt{2}$, which contradicts the fact that $J(X) = S(X) = \sqrt{2}$. This completes the proof. $\square$

Theorem 3.3.2. *For any Minkowski space X we have that*

$$c(X) = \sup\{c>0 :\ x,y\in X\backslash\{o\}, \|\hat{x}-\hat{y}\| < c \textit{ implies } \|x-y\| < \|x+y\|\}.$$

Proof. Let x and y be arbitrary points from $X\backslash\{o\}$ and $\|\widehat{x}-\widehat{y}\| < c(X)$. We show that $\|x-y\| < \|x+y\|$. Suppose the contrary, i.e., that $\|x-y\| \geq \|x+y\|$. Let

$$f(t) = \|(tx+y)+x\| - \|(tx+y)-x\|.$$

Then $f(0)\leq 0$ and, by [34, Lemma 4.4],

$$\begin{aligned}
\lim_{t\to+\infty} f(t) &= \lim_{t\to+\infty}(\|(tx+y)+x\| - \|(tx+y)-x\|)\\
&= \lim_{t\to+\infty}(\|((t-1)+2)x+y\| - \|(t-1)x+y\|) = 2\|x\|.
\end{aligned}$$

Thus, by the continuity of $\|\cdot\|$, there exists a number $t_0\geq 0$ such that $f(t_0)=0$, and therefore $\widehat{t_0x+y}\in P(x)$. It is clear that $\widehat{t_0x+y}$ lies between $\widehat{x}$ and $\widehat{y}$. From the Monotonicity Lemma it follows that

$$c(X) \leq d(\widehat{x}, P(\widehat{x})) \leq \left\|\widehat{t_0x+y}-\widehat{x}\right\| \leq \|\widehat{y}-\widehat{x}\| < c(X),$$

which is impossible.

It is then sufficient to show that $c(X)$ is the largest number having the required properties. Suppose the contrary, i.e., that there exists a number $\alpha_0 > c(X)$ having the required properties. By the compactness of S_X, there exists a point $x_0\in S_X$ such that $d(x_0, P(x_0)) = c(X)$. Since $P(x_0)$ is not empty, there exists a number $\varepsilon\geq 0$ such that $\varepsilon + c(X) < \alpha_0$ and that there exists a point $y\in P(x_0)$ with $\|y-x_0\| = \varepsilon + c(X)$. Then there exists a number $t>0$ such that $\|ty+x_0\| = \|ty-x_0\|$, which is in contradiction to the assumption that $\alpha_0 > c(X)$ is a number having the required properties. $\square$

3.4 Higher dimensions

In this short section one important property of radial projections of bisectors in dimensions $d \geq 3$ is proved. It should be noticed that the following theorem is basically due to H. Martini; see again [50].

Theorem 3.4.1. *Let X be a Minkowski space with* $\dim X \geq 3$. *Then for any $x \in S_X$, $P(x)$ is a connected subset of S_X.*

Proof. For any $x \in S_X$, let H_x be a hyperplane through o such that $x \perp_B H_x$. We show first that $B(-x,x)\backslash\{o\}$ is connected. Let

$$\begin{array}{cccc} T: & X & \longrightarrow & H_x \\ & z = \alpha x + \beta y & \longrightarrow & \beta y. \end{array}$$

It is clear that T is continuous, $T(z) = o$ if and only if $z \in \langle -x, x\rangle$, and $T(B(-x,x)\backslash\{o\}) \subset H_x\backslash\{o\}$. On the other hand, from [34, Theorem 4.4] it follows that for any $y \in H_x\backslash\{o\}$ there exists a number α such that $\alpha x + y \in B(-x,x)\backslash\{o\}$. Thus $T(B(-x,x)\backslash\{o\}) = H_x\backslash\{o\}$.

Suppose that $B(-x,x)\backslash\{o\}$ can be partitioned into two disjoint nonempty subsets A_1 and A_2, which are open in the relative topology induced on $B(-x,x)\backslash\{o\}$. We show that $T(A_1) \cap T(A_2) = \varnothing$. Suppose the contrary, i.e., that there exists a point $y \in T(A_1) \cap T(A_2)$. Then it is evident that $y \neq o$. Let $\alpha_1 \neq \alpha_2$ be two numbers such that $\alpha_1 x + y \in A_1$ and $\alpha_2 x + y \in A_2$. Then, from the convexity of $B(-x,x)$ in the direction of x (see Theorem 2.4.2) it follows that $[\alpha_1 x + y, \alpha_2 x + y] \subset B(-x,x)\backslash\{o\}$, and therefore $[\alpha_1 x + y, \alpha_2 x + y]$ can be partitioned into two disjoint nonempty sets $[\alpha_1 x + y, \alpha_2 x + y] \cap A_1$ and $[\alpha_1 x + y, \alpha_2 x + y] \cap A_2$ which are open in the subspace topology of $[\alpha_1 x + y, \alpha_2 x + y]$. This is impossible. Thus $T(A_1) \cap T(A_2) = \varnothing$ and $T(A_1) \cup T(A_2) = H_x\backslash\{o\}$, which contradicts the fact that $H_x\backslash\{o\}$ is connected. Thus $B(-x,x)\backslash\{o\}$ is connected.

Then, as image of $B(-x,x)\backslash\{o\}$ under the continuous function $R(z) = \frac{z}{\|z\|}$ on $X\backslash\{o\}$, $P(x)$ is connected. $\square$

Chapter 4

Maps preserving isosceles orthogonality

4.1 Introduction

The study of distance-preserving maps in normed linear spaces is based on the Mazur-Ulam theorem (see Theorem 2.1.2), and regarding such isometries also maps preserving orthogonality types are interesting. In this chapter a result in this direction is proved.

One can easily verify that in inner product spaces, Birkhoff orthogonality, isosceles orthogonality, area orthogonality, and Singer orthogonality yield usual orthogonality. Therefore they can be considered as natural extensions of usual (Euclidean) orthogonality to normed linear spaces. It is common to ask what properties of Euclidean orthogonality can be extended to normed linear spaces. For example, one can check whether the following result in inner product spaces (cf. [17, Theorem 1]) can be extended to normed linear spaces, in view of generalized orthogonality types:

> An orthogonality preserving linear map between two inner product spaces is necessarily a scalar multiple of a linear isometry, where a map T is said to be orthogonality preserving if the property that x is orthogonal to y implies that $T(x)$ is orthogonal to $T(y)$.

In [14] this result has been extended to (real or complex) normed linear spaces for the case of Birkhoff orthogonality, namely by the following

Theorem 4.1.1. *Let X and Y be two normed linear spaces. A linear map $T : X \mapsto Y$ preserves Birkhoff orthogonality if and only if it is a scalar multiple of a linear isometry.*

A special case of Theorem 4.1.1, namely when $X = Y$ and X is real, was obtained in [42].

The aim of this chapter is to prove a similar result for isosceles orthogonality. We only consider the case when X is real and non-trivial, i.e., the dimension of X is at least 2.

4.2 Main results

Lemma 4.2.1. *Let X and Y be two real normed linear spaces. If a linear map $T : X \mapsto Y$ preserves isosceles orthogonality, then it also preserves Birkhoff orthogonality.*

Proof. Let x and y be two points such that $x \perp_B y$. We show that $T(x) \perp_B T(y)$. The case that one of the points x and y is the origin is trivial, and since Birkhoff orthogonality is homogeneous, we can assume, without loss of generality, that $x, y \in S_X$. Then it is clear that x and y are linearly independent. Let X_0 be the two-dimensional subspace of X spanned by x and y. We consider the following three cases:

Case I: There exists a non-trivial segment $[a, b]$ parallel to $\langle -y, y\rangle$ and contained in S_{X_0} such that x is a relative interior point of $[a, b]$.

Let $\alpha = \min\{\|x - a\|, \|x - b\|\}$. Then $\alpha > 0$ and, for any $t \in (0, \alpha)$,

$$\|x + ty\| = \|x - ty\| = 1,$$

which means that $x \perp_I ty$ holds for any $t \in (0, \alpha)$. It follows from our assumption on T that

$$T(x) \perp_I tT(y) \quad \forall t \in (0, \alpha).$$

From the convexity of the function $f(t) = \|T(x) + tT(y)\|$ (which is implied by the convexity of the norm) it follows that, for any $t \in (0, \alpha)$, there exists a number $t_0 \in [-t, t]$ such that $f(t)$ attains its minimum at t_0. Hence $f(t)$ attains its minimum at 0, which implies that $T(x) \perp_B T(y)$.

Case II: There exists a non-trivial maximal segment $[a, b]$ parallel to $\langle -y, y\rangle$ and contained in S_{X_0} such that $x = a$.

Let $\{x_n\} \subset [a, b]\backslash\{a, b\}$ be a sequence such that $\lim_{n\to\infty} x_n = x$. It is clear that, for each n, $x_n \perp_B y$. Then it follows from what we have proved in Case I that $T(x_n) \perp_B T(y)$. Since T is a continuous map from X_0 to $T(X_0)$, we have $T(x) = \lim_{n\to\infty} T(x_n)$ and therefore $T(x) \perp_B T(y)$.

Case III: There is no non-trivial segment contained in S_{X_0} and parallel to $\langle -y, y\rangle$. Then, by Lemma 3.2.4, $B(-y, y) \cap X_0$ is bounded between the lines $\langle y, y + x\rangle$ and $\langle -y, -y + x\rangle$. On the other hand, Lemma 2.4.3 implies that for any integer $n > 0$ there exists a unique number $\lambda_n \in [0, 1]$ such that $\lambda_n(y + nx) + (1 - \lambda_n)(-y + nx) \in B(-y, y)$, which implies that

$$(2\lambda_n - 1)T(y) + nT(x) \perp_I T(y).$$

Thus

$$\left\|T(x) + \frac{2\lambda_n}{n}T(y)\right\| = \left\|T(x) - \frac{2(1 - \lambda_n)}{n}T(y)\right\|.$$

From the convexity of the function $g(t) = \|T(x) + tT(y)\|$ it follows that, for any integer $n > 0$, $g(t)$ attains its minimum at some number $t_0 \in [-\frac{2(1-\lambda_n)}{n}, \frac{2\lambda_n}{n}]$, which implies that $T(x) \perp_B T(y)$. The proof is complete. □

It should be noticed that the idea (not the proof) of the following theorem is due to H. Martini.

Theorem 4.2.2. *Let X and Y be two real normed linear spaces. A linear map $T : X \mapsto Y$ preserves isosceles orthogonality if and only if T is a scalar multiple of a linear isometry.*

Proof. If T is linear and preserves isosceles orthogonality, Lemma 4.2.1 implies that T preserves Birkhoff orthogonality, and therefore T is a scalar multiple of a linear isometry.

Conversely, if T is a scalar multiple of a linear isometry, then there exists a number $t > 0$ such that tT is a linear isometry. For any $x, y \in X$ with $x \perp_I y$ we have

$$t\,\|T(x) - T(y)\| = \|tT(x) - tT(y)\| = \|x - y\|$$

and

$$t\,\|T(x) + T(y)\| = \|tT(x) + tT(y)\| = \|x + y\|\,.$$

Hence $T(x) \perp_I T(y)$. □

Remark 4.2.3. We know that the geometric structure of bisectors in a Minkowski space X is determined by the property of isosceles orthogonality on X. Thus a map preserves isosceles orthogonality if and only if it preserves the geometric structure of bisectors. In this sense we can say that the property of X is determined by the geometric structure of bisectors in X.

Remark 4.2.4. In Theorem 4.2.2, isosceles orthogonality cannot be replaced by Singer orthogonality or area orthogonality, Take, e.g., X and Y as the two normed planes with norm $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively; this means that for any point $x = (x_1, x_2) \in \mathbb{R}^2$

$$\|x\|_1 := |x_1| + |x_2| \quad \text{and} \quad \|x\|_\infty := \max\{|x_1|, |x_2|\}$$

hold. Let I be the identity map from X to Y that maps each point in $\mathbb{R}^2$ onto itself. It is clear that I is linear, and it can be easily verified that I preserves both Singer orthogonality and area orthogonality. However, I is apparently not a scalar multiple of a linear isometry from X to Y.

Remark 4.2.5. We note that Theorem 4.2.2 can also be proved by using another approach which was used to obtain Theorem 3 of [58] (the two-dimensional case is not covered in this paper). This approach can be summarized as follows: first one shows that a map preserving isosceles orthogonality also preserves isosceles triangles, which will in turn imply that this map preserves "equality of distance". Then, by the main result of [61], the map has to be a scalar multiple of a linear isometry.

It is clear that, in contrast to this, our approach is based on the recent result obtained in [14] and the relation between Birkhoff orthogonality and isosceles orthogonality.

Chapter 5

Minkowskian circle geometry

This chapter contains five independent sections, each one of which deals with a single topic in Minkowskian circle geometry, and the results in this chapter are contained in [63], [51], and [50].

5.1 Tangent segments in Minkowski Planes

In Euclidean planar geometry, there is a number of interesting theorems referring to tangent segments and secant segments of circles. E.g., the circle is the only closed convex curve in the Euclidean plane with the property that its two tangent segments from any exterior point have equal lengths (see Theorem 5.1.2). This is an easy consequence of the fact that only the circle has an axis of symmetry in any direction, which is possibly due to Hermann Brunn (see [8, (3.5')]). A similar result for higher dimensions was recently derived in [26]. It is interesting to ask whether such results have analogues in normed linear spaces. For dimension two we will prove a related characterization of the Euclidean plane; see also the paper [63].

We continue by defining tangent segments and secant segments in Minkowski planes.

Definition 5.1.1. Let C be a closed convex curve in X, x be an exterior point with respect to C, l_1 be a supporting line of $\mathrm{conv}(C)$ through x, and l_2 be a line that intersects (but does not support) C. Then the segment $[x, y]$ is called a *tangent segment* (from x to C) if

$$y \in l_1 \cap C \text{ and } \|x - y\| = \inf\{\|x - w\| : \ w \in l_1 \cap C\},$$

and the segments $[x, z]$, $[x, z']$ are called the *secant segment* and *external secant segment* (from x to C along l_2), respectively, if

$$z \in l_2 \cap C \text{ and } \|x - z\| = \sup\{\|x - w\| : \ w \in l_2 \cap C\},$$

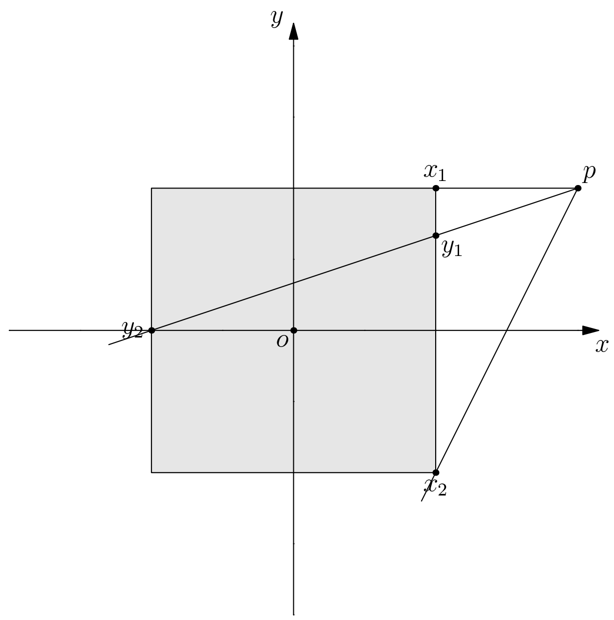

Figure 5.1: Tangents and secants in l_∞ plane.

$$z' \in l_2 \cap C \text{ and } \|x - z'\| = \inf\{\|x - w\| : \ w \in l_2 \cap C\}.$$

5.1.1 Main results

We begin with a characterization of Euclidean circles in the Euclidean plane. We are sure that this statement was proved already in ancient times; a related paper is [64].

Theorem 5.1.2. *The circle is the only closed convex curve in the Euclidean plane with the property that its two tangent segments from any exterior point have equal lengths.*

Proof. First we show that C has to be strictly convex. Suppose the contrary. Then there is a nontrivial maximal segment $[a, b] \subset C$, and we can suppose, without loss of generality, that o is an interior point with respect to C. Therefore for any $t > 1$ the point $p_t = t(\frac{1}{3}a + \frac{2}{3}b)$ is an exterior point of the convex region bounded by C, and as $t \to 1$, tangent segments from p_t to C cannot have equal lengths.

Second we show that the curve C has to be smooth. Suppose to the contrary, i.e., there is a point p on C such that there are two different supporting lines l_1 and l_2 supporting C at p. Let l_3 be a line supporting C at another point q and intersecting l_1 and l_2 at m and n (we can require that $m \neq n$), respectively. Thus, by the assumption of the theorem we have $\|m - p\| = \|m - q\|$ and

$\|n-p\| = \|n-q\|$. Therefore $(m-\frac{1}{2}(p+q)) \perp (p-q)$ and $(n-\frac{1}{2}(p+q)) \perp (p-q)$, which is impossible.

For any direction u, let l_1 and l_1' be two parallel supporting lines of C perpendicular to u and supporting C at p and p', respectively. For any chord $[a, b]$ parallel to l_1 of C between p and the affine diameter $[a_0, b_0]$ parallel to l_1, let l_2 and l_3 be the lines supporting C at a and b, respectively. Then l_2 and l_3 will intersect at some point q and meet l_1 at e and f, respectively. By the assumption of the theorem we have the following equalities:

$$\|q-a\| = \|q-b\|\,,\ \ \|e-a\| = \|e-p\|\,, \|f-b\| = \|f-p\|\,,$$

and therefore

$$\|e-p\| = \|e-a\| = \|f-b\| = \|f-p\|\,.$$

Thus, the midpoint of $[a, b]$ will lie in the line through p and perpendicular to l_1. Since $[a, b]$ is an arbitrary chord parallel to l_1 of C between p and $[a_0, b_0]$, the midpoint of $[a_0, b_0]$ will lie in the line through p and perpendicular to l_1, which implies that the midpoint of any chord of C parallel to l_1 will lie in the line through p and perpendicular to l_1. Therefore C has an axis of symmetry in the direction of u. Since the direction u is arbitrary, C has an axis of symmetry in any direction, which completes the proof. □

We have shown that among all closed convex curves in Euclidean planes only the circle has the property that the two tangent segments from any exterior point have equal lengths. Now we will show that among all Minkowski planes only the Euclidean plane has the property that the two tangent segments from any exterior point of the unit (Minkowskian) circle have equal (Minkowskian) lengths.

Theorem 5.1.3. *A Minkowski plane X is Euclidean if and only if for any exterior point x of S_X the lengths of the two corresponding tangent segments from x to S_X are equal.*

Proof. It is obvious that we only have to prove sufficiency. First we show that by the assumption of equal lengths of the tangent segments the plane is strictly convex.

Suppose that X is not strictly convex. Then there will be a nontrivial maximal segment $[a, b]$ on S_X. Let $x = t(\frac{1}{3}a + \frac{2}{3}b)$ for $t > 1$. Then x is an exterior point of S_X. As $t \to 1$, tangent segments from x to S_X cannot have equal lengths.

Second we show that the plane has to be Radon. (For the geometry of Radon planes we refer to [48] and [47].)

For any $x \in S_X$ let $y \in S_X$ be a point with $x \perp_B y$. Then $[x+y, x]$ will be a tangent segment from $x+y$ to S_X. Since X is strictly convex by the statement above, any unit vector of norm < 2 is the sum of two unit vectors in a unique way, which will imply, together with the fact that the length of the segment $[x+y, y]$ is 1, that $[x+y, y]$ is also a tangent segment from $x+y$ to S_X. Hence $y \perp_B x$. Since x is arbitrary, the Birkhoff orthogonality is symmetric.

For any $x, y \in X\backslash\{o\}$ with the property that $x \perp_I y$, the triangle $(-x)xy$ is an isosceles one. Let o' be the center of the circle inscribed to the triangle $(-x)xy$, and let $a \in [x, y]$, $b \in [-x, y]$, $c \in [-x, x]$ be the points such that $o' - a \perp_B y - x$, $o' - b \perp_B y + x$, $o' - c \perp_B x$. By strict convexity this means that $[y, a]$, $[y, b]$, $[-x, b]$, $[-x, c]$, $[x, a]$, $[x, c]$ are tangent segments from $x, y, -x$ to the inscribed circle, respectively. Thus

$$\|y - a\| = \|y - b\|, \|x - a\| = \|x - c\|, \|-x - b\| = \|-x - c\|.$$

These equalities together with the fact that

$$\|y - a\| + \|x - a\| = \|x - y\| = \|x + y\| = \|y - b\| + \|-x - b\|$$

imply $\|-x - c\| = \|x - c\|$, which means that $[y, c]$ is the median (i.e., the segment joining one vertex and the midpoint of the opposite side of a triangle) of $[-x, x]$.

Now, since Busemann's and Glogovskij's definitions of angular bisectors coincide in Radon planes (by Lemma 2.5.1), o' must lie on the segment $[c, y]$ or, equivalently, on $[o, y]$, and hence $y \perp_B x$ as well as $x \perp_B y$.

Now we have proved that the implication $x \perp_I y \Rightarrow x \perp_B y$ holds for all $x, y \in X\backslash\{o\}$. This implication is trivial when either $x = o$ or $y = o$. Therefore $x \perp_I y \Rightarrow x \perp_B y$ holds for all $x, y \in X$. By Lemma 2.3.9, X is Euclidean and the proof is complete. □

Corollary 5.1.4. *Let X be a Minkowski plane. If for any exterior point x of S_X the squared length of the tangent segment from x to S_X equals the product of the lengths of the secant segment and the external secant segment, then X is Euclidean.*

The following corollary is an immediate consequence of the well known result that a d-dimensional Minkowski space ($d \geq 2$) is Euclidean if and only if all its 2-dimensional subspaces are Euclidean.

Theorem 5.1.5. *Let X be a d-dimensional Minkowski space ($d \geq 2$). The space X is Euclidean if and only if for any exterior point x of S_X and any 2-dimensional subspace X_0 through x and o the lengths of the two corresponding tangent segments from x to $S(X_0)$ are equal.*

5.2 Halving circular arcs in normed planes

In the recent paper [21] N. Düvelmeyer provided various results concerning different types of angular bisectors in Minkowski spaces, with the help of which the following characterizations of the Euclidean plane can be easily derived.

(I) A Minkowski plane is Euclidean if and only if the midpoint of any *minor arc* (i.e., an arc which does not contain any Minkowskian semicircle) of the Minkowskian unit circle is equidistant from the two rays starting at the origin and passing through the two endpoints of that arc, respectively.

(II) A Minkowski plane is Euclidean if and only if the radius of the Minkowskian unit circle meeting the midpoint of any minor arc of that unit circle bisects the chord between the two endpoints of that arc.

The aim of this section is to prove a similar characterization of the Euclidean plane, which can be formulated as follows:

(III) A Minkowski plane is Euclidean if and only if the midpoint of any arc (not necessarily a minor one) of the unit circle is equidistant to the endpoints of that arc.

For any two distinct points p, $q \in S_X$, the part of the unit circle connecting p to q in the positive orientation is called the *directed arc* from p to q and denoted by $\overrightarrow{S_X}(p,q)$. If $p \neq -q$, the intersection of the cone $\{\lambda p + \mu q : \ \lambda, \mu \geq 0\}$ and S_X is called the *(undirected) arc* between p and q and denoted by $S_X(p,q)$. We write $\overrightarrow{\delta_X}(p,q)$ for the Minkowskian length of $\overrightarrow{S_X}(p,q)$, and $\delta_X(p,q)$ for the Minkowskian length of the arc between p and q (when $p = -q$, $\delta_X(p,q)$ is defined to be $\frac{1}{2}|S_X|$). For any given point $x \in S_X$ and a number $\alpha \in (0,1)$, we denote by $A_\alpha(x)$ and $A_\alpha^{-1}(x)$ the two points on S_X such that

$$\overrightarrow{\delta_X}(A_\alpha^{-1}(x), x) = \overrightarrow{\delta_X}(x, A_\alpha(x)) = \alpha |S_X|.$$

Moreover, we define $A_\alpha^0(x) := x$ and, for an integer $n \geq 1$,

$$A_\alpha^n(x) := A_\alpha(A_\alpha^{n-1}(x)) \text{ and } A_\alpha^{-n}(x) := A_\alpha^{-1}(A_\alpha^{-n+1}(x)).$$

Clearly, if X is the Euclidean plane, then for any point $x \in S_X$ and $\alpha \in (0,1)$ the equality

$$\|x - A_\alpha(x)\| = \left\|x - A_\alpha^{-1}(x)\right\| \tag{5.2.1}$$

holds. The main result of this section is the following

Theorem 5.2.1. *A Minkowski plane X is Euclidean if and only if equality (5.2.1) holds for any point $x \in S_X$ and any irrational number $\alpha \in (0,1)$.*

5.2.1 A characterization of the Euclidean plane

In this subsection we derive, as an intermediate step, a new characterization of the Euclidean plane by studying the relation between chord length and corresponding arc length of the unit circle. Set

$$T = \{2\cos(\frac{k\pi}{2n}) : \ n = 2, 3, \dots, \ k = 1, 2, \dots, n-1\}.$$

For the proof of Theorem 5.2.4 we need the following two lemmas.

Lemma 5.2.2. (cf. [5, Corollary]) *If there exists a number $\varepsilon \in (0,2)\backslash T$ such that the implication*

$$\|x - y\| = \varepsilon \Longrightarrow \|x + y\| = \sqrt{4 - \varepsilon^2} \tag{5.2.2}$$

holds for any points x, $y \in S_X$, then X is Euclidean.

The next lemma can be proved easily with the help of the lemma obtained in [54] (see also [8, Lemma 8.1]).

Lemma 5.2.3. *Let X be a Minkowski plane. For any $0 \le \varepsilon \le 2$ there exist two points x, $y \in S_X$ such that $\|x - y\| = \varepsilon$ and $\|x + y\| = \sqrt{4 - \varepsilon^2}$.*

Proof. The cases when $\varepsilon = 0$ and $\varepsilon = 2$ are trivial.

The lemma in [54] says that the locus S_ε of all midpoints of chords of length ε, which is a centrally symmetric simple closed curve when $0 < \varepsilon < 2$, encloses the same area as the curve $\frac{\sqrt{4-\varepsilon^2}}{2} S_X$ does, and therefore these two curves have to intersect in some point. Consequently, there exist two points x, $y \in S_X$ such that $\|x - y\| = \varepsilon$ and $\|x + y\| = \sqrt{4 - \varepsilon^2}$. $\square$

Theorem 5.2.4. *A Minkowski plane X is Euclidean if there exists a number $\varepsilon \in (0,2)\backslash T$ such that $\delta_X(u,v) = \delta(\varepsilon_1)$ ($= \delta(\varepsilon_2)$, resp.) whenever $u, v \in S_X$ and $\|u - v\| = \varepsilon_1$ ($=\varepsilon_2$, resp.), where $\varepsilon_1 = \sqrt{4 - \varepsilon^2}$, $\varepsilon_2 = \varepsilon$, and $\delta(\varepsilon_1)$ and $\delta(\varepsilon_2)$ are constants determined by ε_1 and ε_2, respectively.*

Proof. It suffices to prove that for any x, $y \in S_X$ the implication (5.2.2) holds.

First we show that for any $u, v \in S_X$ the equality $\|u - v\| = \varepsilon_i$ holds whenever $\delta_X(u,v) = \delta(\varepsilon_i)$, $i = 1, 2$. Suppose the contrary, namely that there exist two points u, $v \in S_X$ such that $\delta_X(u,v) = \delta(\varepsilon_i)$ and $\|u - v\| \ne \varepsilon_i$. If $\|u - v\| > \varepsilon_i$, the Monotonicity Lemma (cf. [48, Proposition 31]) implies that there exists a point $v' \in S_X(u,v)$ such that $\|u - v'\| = \varepsilon_i$. Then, by the assumption of the lemma,

$$\delta_X(u,v') = \delta_X(u,v) - \delta_X(v',v) = \delta(\varepsilon_i),$$

which is impossible. If $\|u - v\| < \varepsilon_i$, then it follows from $0 < \varepsilon_i < 2$ and the Monotonicity Lemma that there exists a point $v' \in S_X(v,-u)$ such that $\|u - v'\| = \varepsilon_i$. Then

$$\delta_X(u,v') = \delta_X(u,v) + \delta_X(v,v') = \delta(\varepsilon_i),$$

which is also impossible.

Now we show that $\delta(\varepsilon_1) + \delta(\varepsilon_2) = \frac{1}{2}|S_X|$. By Lemma 5.2.3, there exist points x_0, $y_0 \in S_X$ such that $\|x_0 - y_0\| = \varepsilon$ and $\|x_0 + y_0\| = \sqrt{4 - \varepsilon^2}$. Thus

$$\delta(\varepsilon_1) + \delta(\varepsilon_2) = \delta_X(x_0,y_0) + \delta_X(x_0,-y_0) = \frac{1}{2}|S_X|.$$

Let x and y be any two points in S_X such that $\|x - y\| = \varepsilon$. Then

$$\begin{aligned} \delta_X(y,-x) &= \delta_X(x,y) + \delta_X(y,-x) - \delta_X(x,y) \\ &= \frac{1}{2}|S_X| - \delta(\varepsilon_2) = \delta(\varepsilon_1). \end{aligned}$$

From the foregoing discussion it follows that $\|x + y\| = \sqrt{4 - \varepsilon^2}$, which, by Lemma 5.2.2, completes the proof. $\square$

Remark 5.2.5. Theorem 5.2.4 is sharp in the following sense:

1. The assumption $\varepsilon \in (0,2)\backslash T$ in Theorem 5.2.4 cannot be replaced by $\varepsilon \in (0,2)$. For example, Let X be the Minkowski plane on $\mathbb{R}^2$ with the (Euclidean) regular octagon centered at the origin as unit circle and $\varepsilon = \sqrt{2}$. Then $\varepsilon \in T$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, and $\delta_X(u,v) = \frac{1}{4}|S_X|$ whenever $u, v \in S_X$ and $\|u - v\| = \varepsilon$. But it is clear that X is not Euclidean.

2. The implication $\|u - v\| = \varepsilon \Rightarrow \delta_X(u,v) = \delta(\varepsilon)$ may hold for one single $\varepsilon \in (0,2)\backslash T$, or may hold for two distinct numbers $\varepsilon_1, \varepsilon_2 \in (0,2)\backslash T$ (no further assumption about the relation between these two numbers) in a non-Euclidean Minkowski plane. Here is an example: Let X be a Minkowski plane on $\mathbb{R}^2$ endowed with the following norm:

$$\|x\| = \begin{cases} \|x\|_\infty, & \text{if } x_1x_2 \geq 0, \\ \|x\|_1, & \text{if } x_1x_2 < 0, \end{cases} \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Clearly, $|S_X| = 6$. From a result proved in [41] (see also [18, Proposition 2.1]), saying that each point $x \in S_X$ is a vertex of an equilateral Minkowskian hexagon of side length 1 inscribed in S_X, it follows that $\delta_X(u,v) = 1$ whenever $u, v \in S_X$ and $\|u - v\| = 1$. Now, applying Proposition 29 in [48], we conclude that for any $\alpha \in (0, 1/6)$ and any points $u, v \in S_X$ with $\|u - v\| = 6\alpha$ we have $\delta_X(u,v) = 6\alpha$.

Corollary 5.2.6. *Let X be a Minkowski plane. If there exists a function $\varphi : [0,2] \to [0,4]$ such that for any $u, v \in S_X$ we have $\delta_X(u,v) = \varphi(\|u - v\|)$, then X is Euclidean.*

5.2.2 Proof of Theorem 5.2.1

To prove Theorem 5.2.1, we need to consider several lemmas, the first one being well known from number theory.

Lemma 5.2.7. (Kronecker's Theorem, cf. [28, Theorem 439, p. 376]) *If α is an irrational number, then the set of all numbers of the form*

$$\{n\alpha\} := n\alpha - [n\alpha],$$

where $[n\alpha]$ is the largest integer which does not exceed $n\alpha$, is dense in the interval $(0,1)$.

Some of the proof ideas with respect to the following two lemmas are due to H. Martini.

Lemma 5.2.8. *For any point $x \in S_X$ and any irrational number $\alpha \in (0,1)$, the set $\{A_\alpha^n(x)\}_{n=-\infty}^{+\infty}$ is dense in S_X.*

Proof. For any $y \in S_X$, set

$$\alpha_0 = \frac{\overrightarrow{\delta_X}(x,y)}{|S_X|}.$$

Then $\alpha_0 \in (0,1)$ (without loss of generality we can suppose that $y \neq x$). For any number $\varepsilon > 0$, Lemma 5.2.7 implies that there exists an integer n_0 such that
$$|\alpha_0 - \{n_0\alpha\}| < \frac{\varepsilon}{|S_X|}.$$
We show that $\|y - A_\alpha^{n_0}(x)\| < \varepsilon$. Clearly, we have
$$\overrightarrow{\delta_X}(x, A_\alpha^{n_0}(x)) = (n_0\alpha - [n_0\alpha])|S_X| = \{n_0\alpha\}|S_X|$$
and
$$\begin{aligned}
\|y - A_\alpha^{n_0}(x)\| &\leq \delta_X(y, A_\alpha^{n_0}(x)) \\
&\leq |\overrightarrow{\delta_X}(x,y) - \overrightarrow{\delta_X}(x, A_\alpha^{n_0}(x))| \\
&= |\alpha_0 - \{n_0\alpha\}||S_X| < \varepsilon.
\end{aligned}$$
□

Lemma 5.2.9. *The function $f(x,\alpha) = \|x - A_\alpha(x)\|$, which is defined on $S_X \times (0,1)$, is continuous with respect to both x and α.*

Proof. First we show that the functional $\|x - A_\alpha(x)\|$ is continuous with respect to x.

For any $\varepsilon > 0$, let $\delta = \min\{\frac{1}{3}\varepsilon, (1/4)\alpha|S_X|\}$. For any point $y \in S_X$ with $\|x - y\| < \delta$ we only consider the case that $\delta_X(x,y) = \overrightarrow{\delta_X}(x,y)$. (The other case, when $\delta_X(x,y) = \overrightarrow{\delta_X}(y,x)$, can be proved in a similar way.)

Note that
$$\overrightarrow{\delta_X}(x,y) = \delta_X(x,y) \leq 2\,\|x-y\| \leq \frac{1}{2}\alpha|S_X| \leq \overrightarrow{\delta_X}(x, A_\alpha(x)),$$
which means "passing through y" when moving from x toward $A_\alpha(x)$ with positive orientation. Thus
$$\overrightarrow{\delta_X}(y, A_\alpha(y)) = \overrightarrow{\delta_X}(x, A_\alpha(x)) = \overrightarrow{\delta_X}(x,y) + \overrightarrow{\delta_X}(y, A_\alpha(x)),$$
which means that
$$\overrightarrow{\delta_X}(x,y) = \overrightarrow{\delta_X}(A_\alpha(x), A_\alpha(y)).$$
Putting all things together, we have
$$\begin{aligned}
|\,\|x - A_\alpha(x)\| - \|y - A_\alpha(y)\|\,| &\leq \|x - y + A_\alpha(y) - A_\alpha(x)\| \\
&\leq \|x-y\| + \|A_\alpha(y) - A_\alpha(x)\| \\
&\leq \|x-y\| + \overrightarrow{\delta_X}(A_\alpha(x), A_\alpha(y)) \\
&= \|x-y\| + \delta_X(x,y) \\
&\leq \|x-y\| + 2\,\|x-y\| \leq \varepsilon.
\end{aligned}$$
Next we show the continuity of the function $\|x - A_\alpha(x)\|$ with respect to α.

For any $\varepsilon > 0$, let $\delta = \min\{(1/|S_X|)\varepsilon, \frac{1}{4}\}$. Then, for any α_0 with $|\alpha - \alpha_0| < \delta$ we have

$$\begin{aligned} |\,\|x - A_\alpha(x)\| - \|x - A_{\alpha_0}(x)\|\,| &\leq \|A_{\alpha_0}(x) - A_\alpha(x)\| \\ &\leq \delta_X(A_{\alpha_0}(x), A_\alpha(x)) \\ &= |\alpha - \alpha_0||S_X| < \varepsilon. \end{aligned}$$

This completes the proof. □

Proof of Theorem 5.2.1:

CLAIM 1. *Under the assumption of Theorem 5.2.1, for any number $\alpha \in (0,1)$ and any points x, $y \in S_X$ the following equality holds:*

$$\|x - A_\alpha(x)\| = \|y - A_\alpha(y)\|.$$

Proof of CLAIM 1: First we deal with the case when α is an irrational number.

Suppose first that $y \in \{A_\alpha^n(x)\}_{n=-\infty}^{+\infty}$, which means that there exists an integer n_0 such that $A_\alpha^{n_0}(x) = y$. Without loss of generality we can assume that $n_0 > 0$. By the assumption of Theorem 5.2.1 we have

$$\begin{aligned} \|y - A_\alpha(y)\| &= \|A_\alpha^{n_0}(x) - A_\alpha(A_\alpha^{n_0}(x))\| \\ &= \left\|A_\alpha^{n_0}(x) - A_\alpha^{n_0-1}(x)\right\| \\ &\cdots \\ &= \|x - A_\alpha(x)\|. \end{aligned}$$

Now suppose that $y \notin \{A_\alpha^n(x)\}_{n=-\infty}^{+\infty}$. By Lemma 5.2.8, the set $\{A_\alpha^n(x)\}_{n=-\infty}^{+\infty}$ is dense in S_X, and hence there exists a sequence of integers $\{n_k\}_{k=1}^{+\infty}$ such that

$$y = \lim_{k\to\infty} A_\alpha^{n_k}(x).$$

By Lemma 5.2.9,

$$\|y - A_\alpha(y)\| = \lim_{k\to\infty} \|A_\alpha^{n_k}(x) - A_\alpha(A_\alpha^{n_k}(x))\| = \|x - A_\alpha(x)\|.$$

Next we consider the case when α is a rational number.

Let $\{\alpha_n\} \subset (0,1)$ be a sequence of irrational numbers such that $\lim_{n\to\infty} \alpha_n = \alpha$. By Lemma 5.2.9, for any $x, y \in S_X$ we have

$$\|x - A_\alpha(x)\| = \lim_{n\to\infty} \|x - A_{\alpha_n}(x)\| = \lim_{n\to\infty} \|y - A_{\alpha_n}(y)\| = \|y - A_\alpha(y)\|.$$

This completes the proof of CLAIM 1. □

CLAIM 2. *Under the assumption of Theorem 5.2.1, for any numbers $0 < \alpha_1 < \alpha_2 < \frac{1}{2}$ and any x, $y \in S_X$ the inequality*

$$\|x - A_{\alpha_1}(x)\| < \|y - A_{\alpha_2}(y)\|$$

holds.

Proof of CLAIM 2: By CLAIM 1 we can suppose, without loss of generality, that $(x - A_{\alpha_1}(x))/(\|x - A_{\alpha_1}(x)\|)$ is an extreme point of the unit ball. Clearly, we can find a point $u \in S_X$ such that $u - A_{\alpha_2}(u)$ is a positive multiple of $x - A_{\alpha_1}(x)$. Since $(x - A_{\alpha_1}(x))/(\|x - A_{\alpha_1}(x)\|)$ is an extreme point, we have

$$\|x - A_{\alpha_1}(x)\| < \|u - A_{\alpha_2}(u)\|,$$

which implies that

$$\|x - A_{\alpha_1}(x)\| < \|y - A_{\alpha_2}(y)\|.$$

This completes the proof of CLAIM 2. $\square$

Now we are ready to prove Theorem 5.2.1, and we only need to show sufficiency.

For any $u, v \in S_X$ we show that $\delta_X(u,v)$ is constant, determined by $\|u-v\|$. The case when $u = -v$ is trivial. Suppose now that $u, v, u', v' \in S_X$ be four points such that $u \neq \pm v$, $u' \neq \pm v'$, and $\|u - v\| = \|u' - v'\|$. By interchanging u and v, u' and v' if necessary, we can assume that $\delta_X(u,v) = \overrightarrow{\delta_X}(u,v)$ and $\delta_X(u',v') = \overrightarrow{\delta_X}(u',v')$. Then there exist two numbers $\alpha_1, \alpha_2 \in (0, \frac{1}{2})$ such that $v = A_{\alpha_1}(u)$ and $v' = A_{\alpha_2}(u')$. It suffices to show that $\alpha_1 = \alpha_2$. If this is not the case, another point $v'' \in S_X$ with $v \neq v''$ has to exist such that $v'' = A_{\alpha_2}(u)$. By CLAIM 2 we have

$$\|u - A_{\alpha_1}(u)\| \neq \|u - A_{\alpha_2}(u)\|,$$

which means that $\|u - v\| \neq \|u - v''\|$. Note that by CLAIM 1

$$\|u - A_{\alpha_2}(u)\| = \|u' - A_{\alpha_2}(u')\|,$$

which is equivalent to $\|u - v''\| = \|u' - v'\|$. This is a contradiction. Thus there exists a function $\phi :\ [0,2] \to [0,4]$ such that for any $u, v \in S_X$ we have $\delta_X(u,v) = \phi(\|u - v\|)$. By Corollary 5.2.6 this implies that X is Euclidean. $\square$

Remark 5.2.10. From the example in Remark 5.2.5 it can be easily seen that we cannot replace the assumption of Theorem 5.2.1 by "there exists an (irrational) number $\alpha \in (0,1)$ such that (5.2.1) holds", or by "there exist some (irrational) numbers $\alpha_1, \alpha_2 \in (0,1)$ such that (5.2.1) holds". However, it might be true that the assumption can be replaced by the following one: "There exists an irrational number $\alpha \in (0,1/2)$ such that (5.2.1) holds for α and $1/2 - \alpha$." This replacement can be done if the following conjecture, which is due to J. Alonso and C. Benítez and slightly related to the main result in [5], is true.

Conjecture: If there exist two integers k, n with $1 \leq k \leq n-1$ and a real number $\lambda = \tan(k\pi/2n)$ such that the implication

$$\|x + \lambda y\| = \|x - \lambda y\| \Rightarrow \|x + \lambda y\|^2 = 1 + \lambda^2$$

holds for any points $x, y \in S_X$, then X is linearly isometric to a Minkowski plane on $\mathbb{R}^2$ whose unit circle is invariant under rotations of angles $\pi/2n$.

5.3 Normalization of the sum of unit vectors

Theorem 2 in [21] implies that if for any two points x, $y \in S_X$ with $x \neq -y$, the Busemann angular bisector $A_B([o,x\rangle, [o,y\rangle)$ of the angle $\angle xoy$ bisects the minor arc connecting x and y, then the respective Minkowski plane is Euclidean. In this section we prove a similar result, which can be interpreted as follows: If for any two points x, $y \in S_X$ with $x \neq -y$ the point of intersection of $A_B([o,x\rangle, [o,y\rangle)$ and S_X is equidistant to x and y, then the respective plane is Euclidean. This result strengthens some characterizations of inner product spaces collected in [8].

Theorem 5.3.1. *A Minkowski plane X is Euclidean if for any u, $v \in S_X$ with $u \neq -v$*

$$\left\|\widehat{u+v} - u\right\| = \left\|\widehat{u+v} - v\right\|.$$

Proof. For any $x \in S_X$ and $z \in B(-x,x)\backslash\{o\}$, let

$$G_x(z) = \frac{\|x+z\| - \|z\|}{\|z\|} z.$$

We show first that $x \perp_I G_x(z)$. Let

$$u = \frac{x+z}{\|x+z\|} \text{ and } v = \frac{z-x}{\|z-x\|}.$$

Then

$$\left\|\widehat{u+v} - u\right\| = \left\|\frac{z}{\|z\|} - \frac{x+z}{\|x+z\|}\right\| = \left\|(\frac{1}{\|z\|} - \frac{1}{\|x+z\|})z - \frac{1}{\|x+z\|}x\right\|, \quad (5.3.1)$$

$$\left\|\widehat{u+v} - v\right\| = \left\|\frac{z}{\|z\|} - \frac{z-x}{\|x+z\|}\right\| = \left\|(\frac{1}{\|z\|} - \frac{1}{\|x+z\|})z + \frac{1}{\|x+z\|}x\right\|. \quad (5.3.2)$$

By the assumption of the theorem, $\left\|\widehat{u+v} - u\right\| = \left\|\widehat{u+v} - v\right\|$. Hence

$$\left\|\frac{\|x+z\| - \|z\|}{\|z\|} z - x\right\| = \left\|\frac{\|x+z\| - \|z\|}{\|z\|} z + x\right\|,$$

which means that $x \perp_I G_x(z)$. It is clear that

$$\|G_x(z)\| = \|x+z\| - \|z\| \leq \|x\| = 1.$$

Let $y \in S_X$ be a point such that $x \perp_I y$. Next we show that $x \perp_B y$.

Let $\{s_n\} \subset (0,1]$ be an arbitrary sequence such that $\lim_{n\to\infty} s_n = 0$. It is clear that $\{T_x(s_n)\}$ is a bounded subset of S_X, and therefore we can choose a convergent subsequence $\{T_x(s_{n_k})\}$. Let $t_k = s_{n_k}$. Then $\lim_{k\to\infty} t_k = 0$, and $\{T_x(t_k)\}$ is a Cauchy sequence. From Lemma 3.2.6 it follows that $x \perp_B \lim_{k\to\infty} T_x(t_k)$.

On the other hand, we have

$$\begin{aligned}\lim_{k\to\infty} G_x(t_kT_x(t_k)) &= \lim_{k\to\infty} \frac{\|x+t_kT_x(t_k)\|-\|t_kT_x(t_k)\|}{\|t_kT_x(t_k)\|}t_kT_x(t_k) \\ &= \lim_{k\to\infty} (\|x+t_kT_x(t_k)\|-\|t_kT_x(t_k)\|)T_x(t_k) = \lim_{k\to\infty} T_x(t_k).\end{aligned}$$

Then $\lim_{k\to\infty} T_x(t_k) \perp_I x$, since $G_x(t_kT_x(t_k)) \perp_I x$ for each t_k. From the uniqueness property of isosceles orthogonality it follows that either $\lim_{k\to\infty} T_x(t_k) = y$ or $\lim_{k\to\infty} T_x(t_k) = -y$. Thus $x \perp_B y$ and, by Lemma 2.3.11, X is Euclidean. □

In particular, Theorem 5.3.1 strengthens the following statement, which is used to derive many characterizations of inner product spaces in Chapter 3, Chapter 4, and Chapter 5 of the book [8].

Corollary 5.3.2. (cf. [34, Theorem 4.7]) *A Minkowski plane X is Euclidean if and only if the implication*

$$x \perp_I y \Rightarrow x \perp_I \alpha y \quad \forall \alpha \in \mathbb{R}$$

holds for any $x, y \in X$.

Proof. For any u, $v \in S_X$ with $u \neq -v$ it is clear that $(u+v) \perp_I (u-v)$. Then, by the assumption, we have

$$\frac{1}{2}(u-v) \perp_I (\frac{1}{\|u+v\|}-\frac{1}{2})(u+v),$$

which implies that

$$\left\|\frac{1}{2}(u-v)+(\frac{1}{\|u+v\|}-\frac{1}{2})(u+v)\right\| = \left\|\frac{1}{2}(u-v)-(\frac{1}{\|u+v\|}-\frac{1}{2})(u+v)\right\|$$

or, equivalently,

$$\left\|\widehat{u+v}-u\right\| = \left\|\widehat{u+v}-v\right\|.$$

This completes the proof. □

5.4 S_X-orthocenters in strictly convex Minkowski planes

It has been shown by E. Asplund and B. Grünbaum in [9] that the following theorem, which is the extension of the classical three-circles theorem in the Euclidean plane (see the survey [43] and the monograph [39]), holds also in strictly convex and smooth Minkowski planes.

Theorem 5.4.1. *If three circles $S_X(x_1,\lambda)$, $S_X(x_2,\lambda)$, and $S_X(x_3,\lambda)$ pass through a common point p_4 and intersect pairwise in the points p_1, p_2, and p_3, then there exists a circle $S_X(x_4,\lambda)$ such that $\{p_1,p_2,p_3\} \subseteq S_X(x_4,\lambda)$.*

Actually, the smoothness condition can be relaxed, and Theorem 5.4.1 holds also in strictly convex Minkowski planes (cf. [60, Theorem 4.14, p. 104] and [44]). This theorem is also basic for extensions of Clifford's chain of theorems to strictly convex normed planes; see [45].

The point p_4 in Theorem 5.4.1 is called the S_X*-orthocenter* of the triangle $p_1p_2p_3$. By Theorem 5.4.1 it is evident that p_i is the S_X-orthocenter of the triangle $p_jp_kp_l$, where $\{i,j,k,l\} = \{1,2,3,4\}$. A set of four points, each of which is the S_X-orthocenter of the triangle formed by the other three points, is called a S_X*-orthocentric system.*

The proof of the above theorem for strictly convex Minkowski planes is based on the following properties of strictly convex Minkowski planes (cf. [9] and [48]), which will be used throughout the section:

Lemma 5.4.2. *Let X be a strictly convex Minkowski plane. If $x_1 \neq x_2$ and $\{y_1, y_2\} \subseteq S_X(x_1, \lambda) \cap S_X(x_2, \lambda)$, then $x_1 + x_2 = y_1 + y_2$.*

Lemma 5.4.3. *Any three non-collinear points in a strictly convex Minkowski plane are contained in at most one circle.*

The following facts are well known in Euclidean geometry:

1. The three altitudes of a triangle intersect in the orthocenter of that triangle.
2. The altitude to the base of an isosceles triangle bisects the corresponding vertex angle.
3. If one of the altitudes of a triangle is also an angular bisector, then the triangle is isosceles.
4. The altitude to the base of an isosceles triangle bisects its base.
5. If one of the altitudes of a triangle is also a median (i.e., a segment from a vertex to the midpoint of the opposite side), then the triangle is isosceles.

As the notion of S_X-orthocenter can be viewed as a natural extension of that of orthocenter in Euclidean geometry, one may ask whether results in Euclidean geometry related to orthocenters still hold in Minkowski geometry. It is our aim to continue the investigations from [44] in this spirit.

5.4.1 Some lemmas

The following lemmas are needed for our investigations.

Lemma 5.4.4. *Let $S_X(x_1, \lambda)$ and $S_X(x_2, \lambda)$ be two circles in a strictly convex Mink-owski plane X. If $\{w, z\} \subseteq S_X(x_1, \lambda)$, $\{w', z'\} \subseteq S_X(x_2, \lambda)$, and $w - z = w' - z'$, then*

$$d(x_1, \langle w, z\rangle) = d(x_2, \langle w', z'\rangle).$$

Proof. Without loss of generality, we can suppose that $x_1 = x_2 = o$ and $\lambda = 1$. By the assumed strict convexity, $\|w - z\| = \|w' - z'\| = 2$ implies that $w = -z$ and $w' = -z'$, yielding $d(x_1, \langle w, z\rangle) = d(x_2, \langle w', z'\rangle) = 0$. Now we consider the case when $\|w - z\| < 2$. Again by strict convexity of X, any vector with norm < 2 is the sum of two unit vectors in a unique way (cf. [48, Proposition 14]). Thus, either $w = w'$ and $z = z'$ or $w = -z'$ and $z = -w'$ hold, and each of these two cases clearly gives that $d(x_1, \langle w, z\rangle) = d(x_2, \langle w', z'\rangle)$. $\square$

Lemma 5.4.5. (cf. [44]) *Let $\{p_1, p_2, p_3, p_4\}$ be a S_X-orthocentric system in a strictly convex Minkowski plane X. If x_i is the circumcenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then $\{x_1, x_2, x_3, x_4\}$ is also a S_X-orthocentric system and*

$$p_i - p_j = x_j - x_i.$$

Lemma 5.4.6. *Let X be a strictly convex Minkowski plane. For any x, $y \in X\backslash\{o\}$ with $x \perp_I y$, let $p_3 = y$, $p_4 = -y$, $x_1 = x$, $x_2 = -x$, and $\lambda = \|x + y\|$. Then there exist two points $p_1 \in S_X(x_2, \lambda)$ and $p_2 \in S_X(x_1, \lambda)$ such that $\{p_1, p_2, p_3, p_4\}$ is a S_X-orthocentric system, and that one of the following conditions is satisfied:*

(1) $\|p_3 - p_1\| = \|p_3 - p_2\|$, and p_3 and the line $\langle p_1, p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2\rangle$,

(2) $p_4 \in [p_3, \frac{p_1+p_2}{2}]$,

(3) p_3 and the line $\langle p_1, p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2\rangle$, and $p_4 \in A_B([p_3, p_1\rangle, [p_3, p_2\rangle)$,

(4) p_3 and the line $\langle p_1, p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2\rangle$, and $\langle p_1, p_2\rangle$ is a common supporting line of the circle containing $S_X(x_2, \lambda)$ and the circle containing $S_X(x_1, \lambda)$.

Proof. (1) By the assumed strict convexity and the fact that $x \perp_I y$, one can easily verify that the circles $S_X(x_1, \lambda)$ and $S_X(x_2, \lambda)$ intersect in exactly two points, which are p_3 and p_4. Also, one can easily verify that $2x_2 - y$ lies in the circle $S_X(x_2, \lambda)$ and $2x_1 - y$ in $S_X(x_1, \lambda)$, and that the point p_4 lies in the segment $[2x_2 - y, 2x_1 - y]$. Denote by H^- the closed half plane bounded by $\langle 2x_2 - y, 2x_1 - y\rangle$ that does not contain p_3, and by $S_X(p_4, \lambda)^-$ the intersection of H^- and $S_X(p_4, \lambda)$. Then, since the point p_3 and the line $\langle x_2 - 2y, x_1 - 2y\rangle$ are separated by the line $\langle 2x_2 - y, 2x_1 - y\rangle$, the points $x_1 - 2y$ and $x_2 - 2y$ lie in the semicircle $S_X(p_4, \lambda)^-$.

Let

$$w = -y - \frac{2}{3}x \text{ and } z = -y + \frac{2}{3}x.$$

Then simple calculation shows that $[p_3, x_2 - 2y]$ intersects $[2x_2 - y, 2x_1 - y]$ in w, and $[p_3, x_1 - 2y]$ intersects $[2x_2 - y, 2x_1 - y]$ in z.

Since

$$\|w - p_4\| = \|z - p_4\| = \frac{2}{3}\|x\| \leq \frac{1}{3}(\|x + y\| + \|x - y\|) = \frac{2}{3}\lambda < \lambda,$$

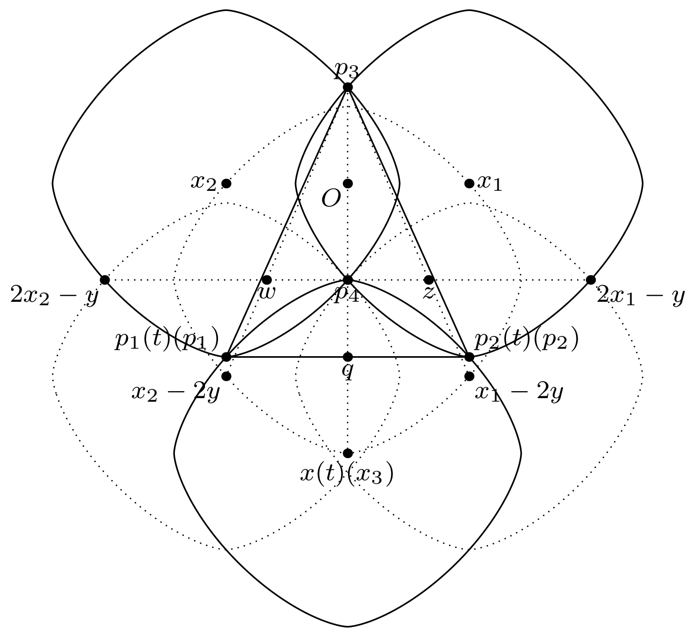

Figure 5.2: The proof of Lemma 5.4.6

for any $t \in (0,1)$ there exists a unique point $x(t)$ such that the line $\langle p_3, tw + (1-t)z\rangle$ intersects the semicircle $S_X(p_4, \lambda)^-$ in a point $x(t)$. From the fact that the segment $[x_1, x_2 - 2y]$ intersects $[x_2, x_1 - 2y]$ in p_4 it follows that $\|x(t) - x_1\| < 2\lambda$ and $\|x(t) - x_2\| < 2\lambda$ hold for any $t \in (0,1)$. Thus there exist two points $p_1(t)$ and $p_2(t)$ such that $S_X(x(t), \lambda)$ intersects $S_X(x_2, \lambda)$ exactly in $p_1(t)$ and p_4, and $S_X(x(t), \lambda)$ intersects $S_X(x_1, \lambda)$ exactly in $p_2(t)$ and p_4. Then for any $t \in (0,1)$, $\{p_1(t), p_2(t), p_3, p_4\}$ is a S_X-orthocentric system; see Figure 5.2.

Moreover, for any $t \in (0,1)$ we have by Lemma 5.4.5 that $p_2(t) - p_1(t) = x_1 - x_2$. Then, by Lemma 5.4.4,

$$d(x(t), \langle p_1(t), p_2(t)\rangle) = d(x_2 - 2y, \langle 2x_2 - y, 2x_1 - y\rangle),$$

and therefore

$$d(x(t), \langle p_1(t), p_2(t)\rangle) < d(x(t), \langle 2x_2 - y, 2x_1 - y\rangle),$$

which implies that p_3 and the line $\langle p_1(t), p_2(t)\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1(t), p_2(t)\rangle$.

Now we show the existence of the points p_1 and p_2 with the desired properties. It is trivial that the functions $x(t)$, $p_1(t)$, and $p_2(t)$ as well as the function

$$f(t) = \|p_3 - p_2(t)\| - \|p_3 - p_1(t)\|$$

are continuous. So

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(\|p_3 - p_2(t)\| - \|p_3 - p_1(t)\|) = \|p_3 - (2x_1 - y)\| - \|p_3 - p_4\| > 0,$$

and

$$\lim_{t\to 1} f(t) = \lim_{t\to 1}(\|p_3 - p_2(t)\| - \|p_3 - p_1(t)\|) = \|p_3 - p_4\| - \|p_3 - (2x_2 - y)\| < 0.$$

Hence there exists a number $t_0 \in (0,1)$ such that $f(t_0) = 0$. Let $x_3 = x(t_0)$, $p_1 = p_1(t_0)$, and $p_2 = p_2(t_0)$. Then p_1 and p_2 are two points having the desired properties.

(2) For any $t \in (0,1)$, let the functions $x(t)$, $p_1(t)$, and $p_2(t)$ be defined as in (1), and $w(t)$, $z(t)$ be the points where the line $\langle 2x_2 - y, 2x_1 - y\rangle$ meets $\langle p_3, p_1(t)\rangle$ and $\langle p_3, p_2(t)\rangle$, respectively. It is clear that when t is sufficiently close to 0, the midpoint of $[w(t), z(t)]$ has to lie strictly between p_4 and $2x_1 - y$, and when t is sufficiently close to 1, the midpoint of $[w(t), z(t)]$ has to lie strictly between p_4 and $2x_2 - y$. Thus there exists a number $t_0 \in (0,1)$ such that $\frac{1}{2}(w(t_0) + z(t_0)) = p_4$. Then $p_1 = p_1(t_0)$ and $p_2 = p_2(t_0)$ are two points having the desired properties.

(3) For any $t \in (0,1)$, let the functions $x(t)$, $p_1(t)$, and $p_2(t)$ be defined as in (1). It is clear that when t moves from 0 to 1, the ray $A_B([p_3, p_1(t)\rangle, [p_3, p_2(t)\rangle)$ turns continuously from $A_B([p_3, 2x_1 - y\rangle, [p_3, p_4\rangle)$ to $A_B([p_3, 2x_2 - y\rangle, [p_3, p_4\rangle)$. Thus there exists a number $t_0 \in (0,1)$ such that $A_B([p_3, p_1(t_0)\rangle, [p_3, p_2(t_0)\rangle) = [p_3, p_4\rangle$. Let $p_1 = p_1(t_0)$, $p_2 = p_2(t_0)$. Then p_1 and p_2 are two points having the desired property.

(4) Let $y' \in S_X$ be a point such that $y' \perp_B x$ and $\{x_2 + \lambda y', x_1 + \lambda y'\} \subseteq H^-$. Then $\langle x_2 + \lambda y', x_1 + \lambda y'\rangle$ is a common supporting line of the circles $S_X(x_2, \lambda)$ and $S_X(x_1, \lambda)$.

Let $p_1 = x_2 + \lambda y'$, $p_2 = x_1 + \lambda y'$, and $x_3 = p_1 + p_4 - x_2$. Then one can easily verify that $\{p_1, p_2, p_4\} \subseteq S_X(x_3, \lambda)$, and therefore p_1 and p_2 are the two points with the desired properties. □

5.4.2 Main results on S_X-orthocentric systems

Now we will present our main results which are new characterizations of the Euclidean plane among all strictly convex normed planes via properties of S_X-orthocentric systems.

Theorem 5.4.7. *A strictly convex Minkowski plane is Euclidean if and only if for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the relation*

$$p_i - p_j \perp_B (p_k - p_l)$$

holds, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. If X is Euclidean then, for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$, p_i is the orthocenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Thus

$$p_i - p_j \perp_B (p_k - p_l).$$

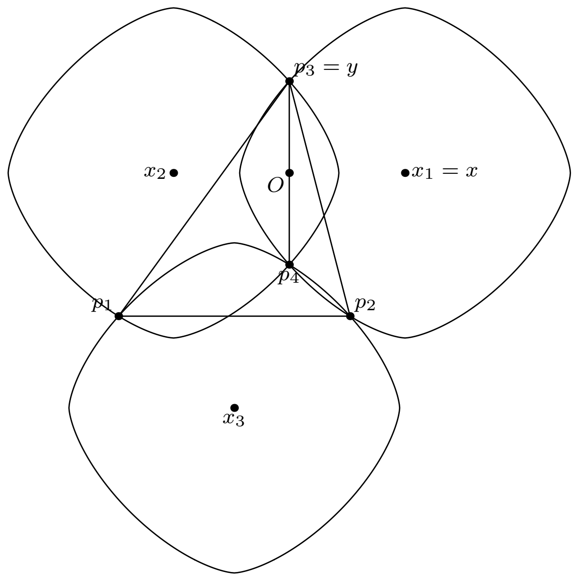

Figure 5.3: The proof of Theorem 5.4.7

Conversely, for any $x,\ y \in S_X$ with $x \perp_I y$ let

$$p_3 = y,\ p_4 = -y,\ x_1 = x,\ \text{and}\ x_2 = -x.$$

By Lemma 5.4.6, there exist two points p_1 and p_2 such that $\{p_1, p_2, p_3, p_4\}$ is a S_X-orthocentric system; see Figure 5.3. By Lemma 5.4.5, $p_2 - p_1 = x_1 - x_2 = 2x$. On the other hand, by the assumption of the theorem we have

$$(p_2 - p_1) \perp_B (p_3 - p_4)$$

or, equivalently, $x \perp_B y$. By Lemma 2.3.11, X is Euclidean. $\square$

Theorem 5.4.8. *A strictly convex Minkowski plane X is Euclidean if and only if for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the point p_4 belongs to the line $\langle p_3, \frac{p_1+p_2}{2}\rangle$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$.*

Proof. We only have to prove sufficiency. By Lemma 2.3.8, we just need to show that for any $x,\ y \in X$ with $x \perp_I y$ there exists a number $t > 1$ such that $x \perp_I ty$, and it is trivial in the case where at least one of x and y is o.

For any $x,\ y \in X \backslash \{o\}$ with $x \perp_I y$, let

$$p_3 = y,\ p_4 = -y,\ x_1 = x,\ \text{and}\ x_2 = -x.$$

By (1) of Lemma 5.4.6, there exist two points p_1 and p_2 such that $\{p_1, p_2, p_3, p_4\}$ is a S_X-orthocentric system, $\|p_3 - p_1\| = \|p_3 - p_2\|$, and that p_3 and the line

$\langle p_1, p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2\rangle$. Then, by the assumption of the theorem,

$$p_4 \in \langle p_3, \frac{p_1+p_2}{2}\rangle.$$

Since p_3 and the line $\langle p_1, p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2\rangle$, we have

$$p_4 \in [p_3, \frac{p_1+p_2}{2}].$$

Thus there exists a number $t > 2$ such that

$$p_3 - \frac{p_1+p_2}{2} = \frac{t}{2}(p_3 - p_4) = ty.$$

On the other hand, by Lemma 5.4.5 we have

$$\begin{aligned} \|x+ty\| &= \left\| \frac{x_1-x_2}{2} + (p_3 - \frac{p_1+p_2}{2}) \right\| \\ &= \|p_3 - p_1\| \\ &= \left\| \frac{p_2-p_1}{2} - (p_3 - \frac{p_1+p_2}{2}) \right\| \\ &= \|x-ty\|. \end{aligned} \tag{5.4.1}$$

Hence there exists a number $t > 2$ such that $x \perp_I ty$, which completes the proof. □

Theorem 5.4.9. *A strictly convex Minkowski plane X is Euclidean if and only if for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the equality $\|p_3 - p_1\| = \|p_3 - p_2\|$ holds whenever $p_4 \in \langle p_3, \frac{p_1+p_2}{2}\rangle$.*

The proof of Theorem 5.4.9 makes use of (2) of Lemma 5.4.6 and is very similar to that of Theorem 5.4.8, and so we omit it.

Theorem 5.4.10. *A strictly convex Minkowski plane X is Euclidean if and only if for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$, p_4 lies on the line containing $A_B([p_3, p_1\rangle, [p_3, p_2\rangle)$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$.*

Proof. We only have to prove sufficiency. By Theorem 5.4.8, it is sufficient to show that for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$, $p_4 \in \langle p_3, \frac{p_1+p_2}{2}\rangle$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$.

By the assumption of the theorem, for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ with $\|p_3 - p_1\| = \|p_3 - p_2\|$, p_4 lies on the line containing $A_B([p_3, p_1\rangle, [p_3, p_2\rangle)$. By the definition of Busemann angular bisectors and the fact that $\|p_3 - p_1\| = \|p_3 - p_2\|$, we have

$$A_B([p_3, p_1\rangle, [p_3, p_2\rangle) = [p_3, \frac{p_1+p_2}{2}\rangle.$$

Thus $\langle p_3, \frac{p_1+p_2}{2}\rangle$ is the line containing $A_B([p_3,p_1\rangle,[p_3,p_2\rangle)$, and therefore

$$p_4 \in \langle p_3, \frac{p_1+p_2}{2}\rangle.$$

The proof is complete. $\square$

Theorem 5.4.11. *A strictly convex Minkowski plane X is Euclidean if and only if for any S_X-orthocentric system $\{p_1,p_2,p_3,p_4\}$ the equality $\|p_3-p_1\| = \|p_3-p_2\|$ holds whenever p_4 lies on the line containing $A_B([p_3,p_1\rangle,[p_3,p_2\rangle)$.*

Proof. We only have to prove sufficiency. By Lemma 2.3.8, we just need to show that for any x, $y \in X$ with $x \perp_I y$ there exists a number $t > 1$ such that $x \perp_I ty$.

For any x, $y \in X\backslash\{o\}$ with $x \perp_I y$, let

$$p_3 = y,\ p_4 = -y,\ x_1 = x,\ \text{and}\ x_2 = -x.$$

By (3) of Lemma 5.4.6, there exist two points p_1 and p_2 such that $\{p_1,p_2,p_3,p_4\}$ is a S_X-orthocentric system, p_3 and the line $\langle p_1,p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1,p_2\rangle$, and that

$$p_4 \in A_B([p_3,p_1\rangle,[p_3,p_2\rangle).$$

By the assumption of the theorem we have $\|p_3-p_1\| = \|p_3-p_2\|$, and then

$$A_B([p_3,p_1\rangle,[p_3,p_2\rangle) = [p_3, \frac{p_1+p_2}{2}\rangle.$$

Since p_3 and the line $\langle p_1,p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1,p_2\rangle$, we have

$$p_4 \in [p_3, \frac{p_1+p_2}{2}].$$

Hence there exists a number $t > 2$ such that

$$p_3 - \frac{p_1+p_2}{2} = \frac{t}{2}(p_3-p_4) = ty.$$

In a way analogous to that referring to Theorem 5.4.8 it can be proved that $x \perp_I ty$, which completes the proof. $\square$

Theorem 5.4.12. *A strictly convex Minkowski plane is Euclidean if and only if for any S_X-orthocentric system $\{p_1,p_2,p_3,p_4\}$ the equality $\|p_3-p_1\| = \|p_3-p_2\|$ holds whenever $\langle p_1,p_2\rangle$ is a common supporting line of the circle containing $\{p_1,p_3,p_4\}$ and the circle containing $\{p_2,p_3,p_4\}$.*

Proof. Suppose first that X is Euclidean. For any S_X-orthocentric system $\{p_1, p_2,p_3,p_4\}$, let $S_X(x_i,\lambda)$ be the circumcircle of $\{p_jp_kp_l\}$, where $\{i,j,k,l\} = \{1,2,3,4\}$. Then $\langle p_3,p_4\rangle$ is the radical axis of $S_X(x_1,\lambda)$ and $S_X(x_2,\lambda)$. (Note that the radical axis of two circles is the locus of points of equal circle power with respect to two non-concentric circles, where the power of a point with

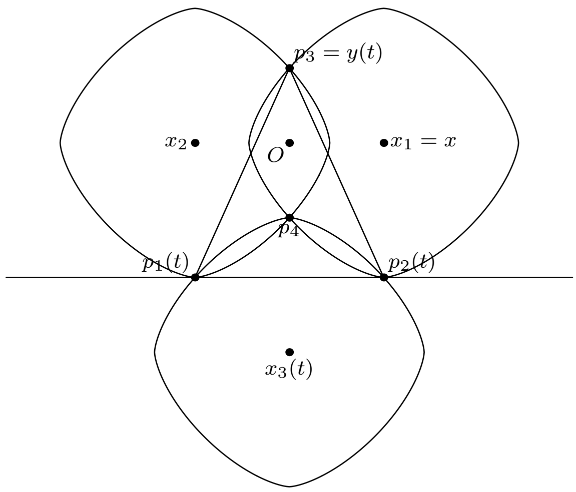

Figure 5.4: The proof of Theorem 5.4.12

respect to a circle is equal to the squared distance from the point to the center of the circle minus the square of the radius of the circle; cf. [39].) If $\langle p_1, p_2 \rangle$ is a common supporting line of $S_X(x_1, \lambda)$ and $S_X(x_2, \lambda)$, then $\langle p_3, p_4 \rangle$ will be the perpendicular bisector of $[p_1, p_2]$, and therefore $\|p_3 - p_1\| = \|p_3 - p_2\|$.

Now we prove sufficiency. By Lemma 2.3.10 we only have to show that for any x, $z \in S_X$, $z \perp_B x \Rightarrow z \perp_I x$. Let ω be a fixed orientation on X. Since $z \perp_B x$ if and only if $(-z) \perp_B x$, it will be sufficient to prove that for any x, $z \in S_X$ with $\overrightarrow{(-x)z} = \overrightarrow{zx} = \omega$, $z \perp_B x$ implies $z \perp_I x$.

By strict convexity of X, for any $x \in S_X$ there exists a unique point $z \in S_X$ such that $z \perp_B x$ and $\overrightarrow{(-x)z} = \overrightarrow{zx} = \omega$. On the other hand, by Lemma 2.3.3 there is, for any number $t > 0$, a unique point $y(t) \in tS_X$ such that $x \perp_I y(t)$ and $\overrightarrow{(-x)y} = \overrightarrow{yx} = \omega$. Let

$$x_1 = x,\ x_2 = -x,\ p_3(t) = y(t),\ \text{and } p_4(t) = -y(t).$$

Then, by (4) of Lemma 5.4.6, there exist two points $p_1(t)$ and $p_2(t)$ such that the set $\{p_1(t), p_2(t), p_3(t), p_4(t)\}$ is a S_X-orthocentric system, $p_3(t)$ and the line $\langle p_1(t), p_2(t) \rangle$ are separated by the line passing through $p_4(t)$ which is parallel to $\langle p_1(t), p_2(t) \rangle$, and that the line $\langle p_1(t), p_2(t) \rangle$ is the common supporting line of $S_X(x_1, \|x + y(t)\|)$ and $S_X(x_2, \|x + y(t)\|)$; see Figure 5.4. From Lemma 5.4.5 it follows that

$$p_1(t) - p_2(t) = x_2 - x_1.$$

Thus $x_2 - p_1(t) \perp_B x$ and $x_1 - p_2(t) \perp_B x$, and therefore

$$\|x + y(t)\|\, z = x_2 - p_1(t) = x_1 - p_2(t).$$

Let $z(t) = \|x + y(t)\| \, z$. Then

$$\begin{aligned} \|p_3(t) - p_2(t)\| &= \|p_3(t) - x_1 + x_1 - p_2(t)\| \\ &= \|p_3(t) - x_1 + z(t)\| \end{aligned}$$

and

$$\begin{aligned} \|p_3(t) - p_1(t)\| &= \|p_3(t) - x_2 + x_2 - p_1(t)\| \\ &= \|p_3(t) - x_2 + z(t)\| \, . \end{aligned}$$

By assumption $\|p_3(t) - p_1(t)\| = \|p_3(t) - p_2(t)\|$, and therefore

$$\|p_3(t) - x_2 + z(t)\| = \|p_3(t) - p_1(t)\| = \|p_3(t) - p_2(t)\| = \|p_3(t) - x_1 + z(t)\| \, ,$$

i.e.,

$$\|(y(t) + z(t)) + x\| = \|(y(t) + z(t)) - x\| \, .$$

It is evident that

$$\lim_{t \to 0} y(t) = o,$$

and therefore

$$\lim_{t \to 0} \|x + y(t)\| = \|x\| = 1.$$

It follows that

$$\lim_{t \to 0} (y(t) + z(t)) = \lim_{t \to 0} \|x + y(t)\| \, z = z,$$

and then

$$\begin{aligned} \|z + x\| &= \left\| \lim_{t \to 0} (y(t) + z(t)) + x \right\| \\ &= \left\| \lim_{t \to 0} (y(t) + z(t)) - x \right\| \\ &= \|z - x\| \, . \end{aligned}$$

Hence $z \perp_I x$.

Now we show that x is the unique point in S_X such that $z \perp_B x$ and $\overrightarrow{(-x)z} = \overrightarrow{zx} = \omega$. Suppose the contrary, i.e., that there exists a point $x' \in S_X$, $x' \neq x$, such that $z \perp_B x'$ and $\overrightarrow{(-x')z} = \overrightarrow{zx'} = \omega$. Then, by the foregoing discussion, $z \perp_I x'$. Hence there exist two points x, $x' \in S_X$, $x \neq \pm x'$, such that $z \perp_I x$ and $z \perp_I x'$, a contradiction to the uniqueness property of isosceles orthogonality.

Thus for any x, $z \in S_X$ with $\overrightarrow{(-x)z} = \overrightarrow{zx} = \omega$, $z \perp_B x$ implies that $z \perp_I x$. This completes the proof. □

The theorem above says that a Minkowski plane is Euclidean if and only if the implication

$$\langle p_1, p_2 \rangle \text{ supports the circles } S_X(x_1, \lambda) \text{ and } S_X(x_2, \lambda) \implies \|p_3 - p_1\| = \|p_3 - p_2\|$$

holds for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$. In the next theorem we show that the reverse implication also characterizes the Euclidean plane.

Theorem 5.4.13. *A strictly convex Minkowski plane is Euclidean if and only if for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$, $\langle p_1, p_2\rangle$ is a common supporting line of the circle containing $\{p_1, p_3, p_4\}$ and the circle containing $\{p_2, p_3, p_4\}$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$.*

Proof. For any x, $z \in S_X$ with $z \perp_B x$ and any number $t > 0$, we can define $y(t)$, x_1, x_2, $p_3(t)$, and $p_4(t)$ as in the proof of Theorem 5.4.12. Then, by (1) of Lemma 5.4.6, there exist two points $p_1(t)$ and $p_2(t)$ such that the set $\{p_1(t), p_2(t), p_3(t), p_4(t)\}$ is a S_X-orthocentric system, $p_3(t)$ and the line $\langle p_1(t), p_2(t)\rangle$ are separated by the line passing through $p_4(t)$ parallel to $\langle p_1(t), p_2(t)\rangle$, and $\|p_3(t) - p_1(t)\| = \|p_3(t) - p_2(t)\|$. By assumption, the line $\langle p_1(t), p_2(t)\rangle$ is the common supporting line of $S_X(x_1, \|x + y(t)\|)$ and $S_X(x_2, \|x + y(t)\|)$. Then, as in the proof of Theorem 5.4.12, it can be shown that $z \perp_I x$, which completes the proof. □

5.5 Concurrent and parallel chords of spheres in Minkowski spaces

In this section a segment $[p, q]$ is said to be a (maximal) *chord* of the unit circle (or unit disc) if $\langle p, q\rangle \cap B_X = [p, q]$.

5.5.1 Concurrent chords of spheres in Minkowski spaces

For any point $x \in B_X$ we denote by $M(x)$ the set of midpoints of chords of S_X passing through x, i.e.,

$$M(x) := \{\frac{1}{2}(p + q) : x \in [p, q], \text{ and } [p, q] \text{ is a chord of } S_X\}.$$

If X is the Euclidean plane, then $M(x)$ is a *circle* centered at $\frac{1}{2}x$ and having radius $\frac{1}{2}\|x\|$ (i.e., $M(x) = \frac{1}{2}x + \frac{1}{2}\|x\| S_X$, which is clearly a homothet of S_X). It is natural to ask whether this result still holds in general Minkowski planes, and one can easily obtain the following theorem.

Theorem 5.5.1. *Let X be a strictly convex Minkowski plane and x be an arbitrary point on S_X. Then $M(x)$ is a circle.*

Proof. For any point $z \in M(x)$, let $[p, x]$ be the chord of S_X having z as its midpoint, and $o' = \frac{1}{2}x$. Then

$$\|o' - z\| = \frac{1}{2}\|o - p\| = \frac{1}{2},$$

which means that z lies on the circle $\frac{1}{2}x + \frac{1}{2}S_X$. □

We note that this theorem still holds for Minkowski planes which are not strictly convex if we define a chord of S_X to be a segment between two points lying on S_X.

On the other hand, however, for interior points of B_X different to o the situation is much more complicated: $M(x)$ may not be a circle, or even may not be convex. Actually, we will show in this subsection that any set $M(x)$ is a circle only in the Euclidean case, and that this result can be easily extended to higher dimensions.

First we consider the case when X is a Minkowski plane, needing the following lemmas.

Lemma 5.5.2. *Let X be a Minkowski plane, $x \in S_X$, and $o' \in [-x, x]$ be a point such that $\|x - o'\| \leq \|x + o'\|$. Then for any point $z \in S_X \backslash \{x\}$ and $y \in S_X(x, z) \backslash \{z\}$ the inequality*

$$\|y - o'\| \leq \|z - o'\|$$

holds. Moreover, if X is strictly convex and $o' \neq o$, we have strict inequality.

Proof. When $o' = o$, it is clear that we always have

$$\|o' - y\| = \|o' - z\| = 1.$$

And if $o' \neq o$ and $y = x$, we have

$$\|o' - y\| = \|o' - x\| = 1 - \|o'\| = \|o - z\| - \|o - o'\| \leq \|o' - z\|.$$

When X is strictly convex, the right-most inequality becomes an equality only if $z = x$, which is impossible. Thus strict inequality holds. The case when $o' \neq o$ and $y \neq x$ follows directly from the Monotonicity Lemma (cf. [48, Proposition 31]). □

Lemma 5.5.3. *Let X be a Minkowski plane, and $[p, q]$ be a chord of S_X bisected by a point x with $0 < \|x\| < 1$. Then $M(x)$ is contained in the closed half plane H bounded by $\langle p, q \rangle$ and containing o.*

Proof. Suppose the contrary, namely that $[s, t]$ is a chord of S_X passing through x such that $\{s, \frac{1}{2}(s + t)\} \subset B_X \backslash H$. Then it follows from Lemma 5.5.2 that

$$\|x - s\| \leq \|x - q\| = \|x - p\| \leq \|x - t\|,$$

a contradiction to the fact that the midpoint of $[s, t]$ lies strictly between x and s. □

Lemma 5.5.4. *Let X be a Minkowski space. For any $x \in B_X$ we have*

$$\|x - z\| \leq \|x\|, \ \forall z \in M(x).$$

Proof. The case $x = o$ is trivial. If $x \in S_X$, then

$$\|x - z\| = \frac{1}{2} \|x - z'\| \leq 1 = \|x\|,$$

where $z' \in S_X$ is chosen in such a way that z is the midpoint of the chord $[x, z']$.

Now assume that $0 < \|x\| < 1$. For any point $z \in M(x)$ there exists a chord $[p, q]$ of S_X such that $z = \frac{1}{2}(p+q)$. Without loss of generality we may assume that $\|p - x\| \leq \|q - x\|$. Suppose the contrary, namely that $\|x - z\| > \|x\|$. Then

$$\|z - q\| = \|x - z\| + \|x - p\| > \|x\| + \|x - p\| \geq \|p\| = 1,$$

meaning that $\|p - q\| > 2$, which is impossible. $\square$

Lemma 5.5.5. *Let X be a Minkowski plane, and $x \in X$ be a point with $0 < \|x\| < 1$ such that $M(x)$ is a circle. Then $\frac{x}{\|x\|}$ is not contained in the interior of a chord contained in S_X.*

Proof. To the contrary, suppose that $\frac{x}{\|x\|}$ is contained in the interior of a chord $[p, q] \subset S_X$. By the assumption of the theorem and Lemma 5.5.4 we have that the length of a diameter chord of the circle $M(x)$ equals $\|x\|$. Let $[p', q']$ be the diameter chord of S_X parallel to $\langle p, q\rangle$, and p'' and q'' be two points such that

$$\{p''\} = \langle -p, x\rangle \cap [p', q'] \text{ and } \{q''\} = \langle -q, x\rangle \cap [p', q'].$$

Then one can easily verify that $[p'', q'']$ is a chord of $M(x)$ parallel to $\langle p, q\rangle$, and therefore it should have length $\frac{1}{2}\|x\|\,\|p - q\|$.

On the other hand we have

$$\frac{\|x\|}{1 + \|x\|} = \frac{\|p'' - q''\|}{\|p - q\|} = \frac{1}{2}\|x\|,$$

which implies that $\|x\| = 1$, a contradiction. $\square$

Theorem 5.5.6. *Let X be a Minkowski plane. If $M(x)$ is a circle (possibly degenerate to a point) for any $x \in B_X$, then X is Euclidean.*

Proof. We only need to show that for any $x, y \in S_X$ with $x \perp_I y$ we have $x \perp_B y$.

First we show that X is strictly convex. If this is not true, a point x with $0 < \|x\| < 1$ has to exist such that $\frac{x}{\|x\|}$ is an interior point of a segment contained in S_X. Then, by Lemma 5.5.5, $M(x)$ cannot be a circle, a contradiction to the assumption of the theorem.

Let $x, y \in S_X$ be an arbitrary pair of points with $x \perp_I y$,

$$u = \frac{x + y}{\|x + y\|},\ v = \frac{x - y}{\|x + y\|}, \text{ and } o' = \frac{1}{2}(u + v).$$

We consider the circle $M(o')$. By Lemma 5.5.3 and the fact that $[u, v]$ is bisected by o', we have that $\langle u, v\rangle$ is a line supporting the convex hull of $M(o')$ at o'. Moreover, $\langle u, v\rangle \cap M(o') = \{o'\}$ since X is strictly convex. Again, by the strict convexity of X we know that $[o, o']$ is a diameter chord of $M(o')$, since a chord of $M(o')$ having the length of its diameter is itself a diameter chord. Thus $M(o')$ is a circle centered at $\frac{1}{2}o'$ and having radius $\frac{1}{2}\|o'\|$. Summarizing this, we have

$$(o' - \frac{1}{2}o') \perp_B (u - v),$$

which implies that $x \perp_B y$ and completes the proof. $\square$

Finally we extend Theorem 5.5.6 to higher dimensions.

Theorem 5.5.7. *Let X be a Minkowski space of dimension at least 2. If $M(x)$ is a homothet of S_X (possibly degenerate to a point) for any point $x \in B_X$, then X is Euclidean.*

Proof. First we need to ensure that $M(x)$ is centered at $\frac{1}{2}x$ for any point x with $0 < \|x\| < 1$. If this is not true, then there exists an $x' \in B_X \backslash \langle -x, x \rangle$ such that $M(x)$ is centered at x'. Clearly, x and x' are linearly independent and therefore span a two-dimensional subspace X_0 of X. Let $M_0(x) := M(x) \cap X_0$. Since $M(x)$ is a homothet of S_X and the center x' of this sphere is contained in X_0, $M_0(x)$ is a homothet of S_{X_0}. Clearly, $M_0(x)$ is the set of midpoints of all chords of S_{X_0} passing through x. By Lemma 5.5.4 we have that the diameter of $M_0(x)$ equals $\|x\|$, and therefore

$$\|x - x'\| + \|x' - o\| = \|x - o\|\,,$$

implying by the triangle inequality (cf. [48, Proposition 1]) that

$$[\frac{x - x'}{\|x - x'\|}, \frac{x'}{\|x'\|}] \subset S_{X_0}.$$

By the choice of x' we have $\|x - x'\| = \|x'\|$, and therefore

$$x = \frac{1}{2}\frac{x}{\|x'\|} = \frac{1}{2}(\frac{x - x'}{\|x'\|} + \frac{x'}{\|x'\|}) \in [\frac{x - x'}{\|x - x'\|}, \frac{x'}{\|x'\|}] \subset S_{X_0}.$$

By Lemma 5.5.5 this is impossible.

Then for any two-dimensional subspace X_0 of X and any point $x \in B_{X_0}$ the set of midpoints of chords of S_{X_0} passing through x is exactly the set $M(x) \cap X_0$, which is a circle. By Theorem 5.5.6, X_0 is Euclidean. This completes the proof, since X is Euclidean if and only if each of its two-dimensional subspaces is Euclidean (cf. [8, (1.4')]). $\square$

5.5.2 Parallel chords and characterizations of the Euclidean plane

In Euclidean geometry it is known that parallel chords of a circle always intersect this circle so that the two arcs between them have the same arc-length. Also, the two chords corresponding to these arcs are of the same length; see Figure 5.5. In this subsection we show that these properties characterize the Euclidean plane among all normed planes.

Theorem 5.5.8. *A Minkowski plane X is Euclidean if and only if for any two chords $[p, q]$ and $[p', q']$ of S_X with $\widehat{p - q} = \widehat{p' - q'}$ we have*

$$\|p - p'\| = \|q - q'\|\,.$$

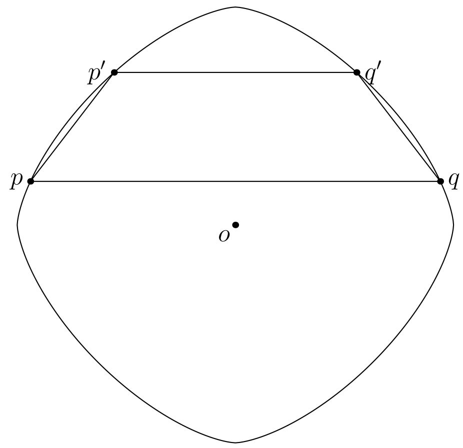

Figure 5.5: Parallel chords of circles.

Proof. For any u, $v \in S_X$ with $u \neq -v$, let $p = u$, $q = -v$, $p' = \widehat{u+v}$, and $q' = -\widehat{u+v}$. Clearly,

$$\widehat{p-q} = \widehat{p'-q'} = \widehat{u+v}.$$

Then, by the assumption of the theorem, we have

$$\left\|\widehat{u+v} - u\right\| = \|p - p'\| = \|q - q'\| = \left\|\widehat{u+v} - v\right\|.$$

By Theorem 5.3.1, X is Euclidean as claimed. $\square$

To prove the last theorem in this section we need the following lemma, which follows directly from Theorem 2 in the recent paper [21].

Lemma 5.5.9. *A Minkowski plane X is Euclidean if and only if for any points u, $v \in S_X$, with $u \neq -v$, the arc $S_X(u, v)$ is bisected by $\widehat{u+v}$.*

Theorem 5.5.10. *A Minkowski plane X is Euclidean if and only if for any two chords $[p, q]$ and $[p', q']$ of S_X with $\widehat{p-q} = \widehat{p'-q'}$, $S_X(p, p')$ and $S_X(q, q')$ are of equal length.*

Proof. For any u, $v \in S_X$ with $u \neq -v$, let $p = u$, $q = -v$, $p' = \widehat{u+v}$, and $q' = -\widehat{u+v}$. Clearly,

$$\widehat{p-q} = \widehat{p'-q'} = \widehat{u+v}.$$

Then, by the assumption of the theorem, we have that the arcs $S_X(u, \widehat{u+v})$ and $S_X(-v, -\widehat{u+v})$ are of equal length. Then the central symmetry of S_X implies that the arc $S_X(u, v)$ is bisected by $\widehat{u+v}$. By Lemma 5.5.9, X is Euclidean as claimed. $\square$

Final remark: Since a finite dimensional real Banach space is an inner product space if and only if each of its two-dimensional subspaces is isometric to the Euclidean plane, it is clear that all our characterization theorems in this chapter can be interpreted (in the spirit of the monograph [8]) as characterizations of inner product spaces among all (strictly convex and) finite dimensional real Banach spaces.

Chapter 6

Halving closed curves and the geometric dilation problem

6.1 Introduction

Let C be a simple planar closed curve. A pair of points $p, q \in C$ is said to be a halving pair of C if the length of each part of C connecting p and q is one half of the perimeter of C. In the Euclidean plane, the property of halving pairs of simple planar closed curves plays an important role in recent investigations of the geometric dilation problem; see [19], [22], and [27]. Also, the relations between the halving distance (the distance between a halving pair) and some further important quantities of a closed curve yield many interesting results; see [19] and [27, Chapter 4]. In this chapter we study further properties of halving pairs, the halving pair transformation and the halving distance in arbitrary Minkowski planes, deriving also related inequalities. In Section 6.3, the first attempt was made to extend the geometric dilation problem from the Euclidean plane to Minkowski planes. General lower bounds on the geometric dilation of closed planar curves in Minkowski planes were obtained by applying basic properties of halving pairs and the so called halving pair transformation.

Throughout this chapter we consider simple, rectifiable, closed curves in an arbitrary Minkowski plane X. We shall frequently use the *arc-length parametrization* $c : [0, |C|) \to C$ of a rectifiable closed curve C, which is continuous, bijective, and has the property that $\|\dot{c}(t)\| = 1$ whenever the derivative exists. Two points $p = c(t)$ and $\hat{p} = c(t + |C|/2)$ on C that split C regarding its length into two equal parts form a *halving pair* of C, and the segment $[p, \hat{p}]$ is said to be a *halving chord*. For any $v \in S_X$, the *v-halving distance* in direction v, denoted by $h_C(v)$, is the length of the halving chord of C having direction v (note that this quantity is defined only for convex curves); the *v-length*, denoted by $l_C(v)$, is the maximum distance between pairs of points on C whose difference vector is of direction v. The *minimum width* w of a closed convex curve C is the minimum distance between two parallel supporting lines of $\mathrm{conv}(C)$. The *diameter* of C, denoted by $D(C)$, is the maximum of all possible v-lengths. The *inradius* r and *circumradius* R of C is the radius of the maximum inscribed circle and the minimum circumscribed circle of C, respectively. The *maximum halving distance* and *minimum halving distance* of C are defined by

$$H = H(C) = \max_{t\in[0,|C|)} \{\|c(t) - c(t + |C|/2)\|\}$$

and

$$h = h(C) = \min_{t\in[0,|C|)} \{\|c(t) - c(t + |C|/2)\|\},$$

respectively. The *midpoint curve* M of the curve C is formed by the midpoints of halving chords of C, and it is parameterized by

$$m(t) := \frac{1}{2}(c(t) + c(t + \frac{|C|}{2})).$$

The image C^* of C under the *halving pair transformation* is given by the parametrization

$$c^*(t) := \frac{1}{2}(c(t) - c(t + \frac{|C|}{2})).$$

Parts of the results in this section are contained in [49].

6.2 Halving closed curves

6.2.1 The halving pair transformation

Let C be a simple rectifiable closed curve. By definition, the halving pair transformation translates the midpoints of the halving chords to the origin,

$$c^*(t) = -c^*(t + |C|/2),$$

and hence C^* is centrally symmetric. Moreover,

$$h(C^*) = h(C) = h \text{ and } H(C^*) = H(C) = H.$$

First we would like to give an upper bound on $|M|$. The following theorem is due to H. Martini.

Theorem 6.2.1. $|C| \geq \max\{2|M|, |C^*|\}$.

Proof. We deal only with the case when C is piecewise continuously differentiable, and the proof of this case can be extended to arbitrary rectifiable curves. By definitions and the triangle inequality we have

$$2|M| = \int_0^{|C|/2} \left\|\dot{c}(t) + \dot{c}(t + \frac{|C|}{2})\right\| \mathrm{d}t \leq \int_0^{|C|/2} (\|\dot{c}(t)\| + \left\|\dot{c}(t + \frac{|C|}{2})\right\|)\mathrm{d}t = |C|$$

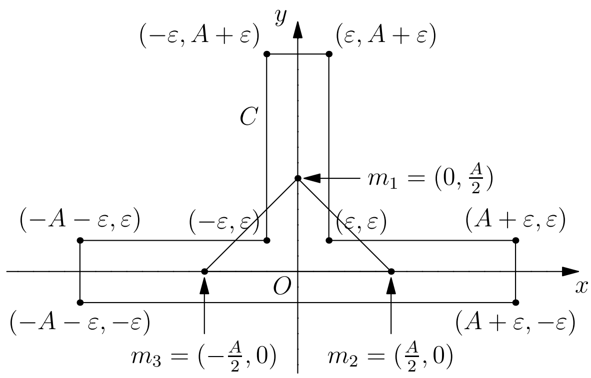

Figure 6.1: $2|M|$ can be arbitrarily close to $|C|$.

and

$$|C^*| = \int_0^{|C|} \frac{1}{2} \left\| \dot{c}(t) - \dot{c}(t + \frac{|C|}{2}) \right\| \mathrm{d}t \le \frac{1}{2} \int_0^{|C|/2} (\|\dot{c}(t)\| + \left\| \dot{c}(t + \frac{|C|}{2}) \right\|) \mathrm{d}t = |C|.$$

The proof is complete. $\square$

Remark 6.2.2. Dumitrescu et al. [19] showed that the inequality $4|M|^2 + |C^*|^2 \le |C|^2$ holds in the Euclidean plane, which means that the number $2|M|$ cannot be too large since we have the inequality $|C^*| \ge \pi h$. However, this is not true in general Minkowski planes. Consider the closed curve C depicted in Figure 6.1, in the Minkowski plane $\mathbb{R}^2$ with norm $\|(\alpha, \beta)\| = |\alpha| + |\beta|$. Calculations show that

$$|C| = 6A + 8\varepsilon, h = A + 2\varepsilon, \text{ and } H = A + 4\varepsilon,$$

where A is a constant positive number. Note that $|M|$ is not smaller than the perimeter of the triangle formed by m_1, m_2, and m_3, that is, $|M| \ge 3A$. By the symmetry of C^*, any two points p and $-p$ on C^* form a halving pair of distance $2\|p\| \ge h$. Hence C^* contains the disc $(h/2)B_X$, and then

$$|C^*| \ge (h/2)|S_X| = 4h = 4(A + 2\varepsilon).$$

Clearly, $2|M|$ tends to $|C|$ as ε tends to zero while $|C^*| > 4A$, and therefore $\sqrt{|C|^2 - |C^*|^2}/2$ is not an upper bound on $|M|$ in general Minkowski planes.

Remark 6.2.3. It is also interesting to observe that $|C| = 2|M|$ may hold for some closed convex curves in a metric space on $\mathbb{R}^2$, where the metric is induced by a certain convex distance function (gauge) as in the following example, i.e., the corresponding metric is not centrally symmetric. Let C be a triangle in the metric space on $\mathbb{R}^2$ with unit circle S_X (see Figure 6.2), where a point

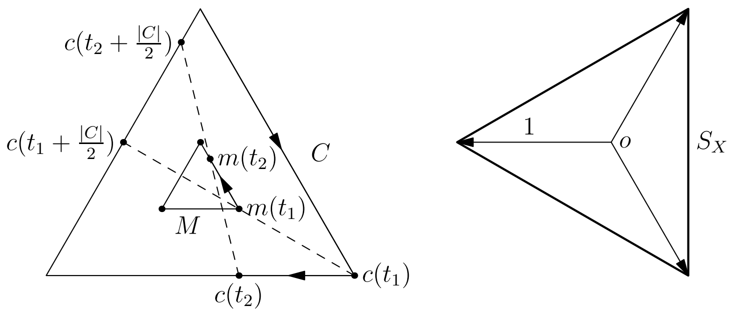

Figure 6.2: The case where $|C| = 2|M|$.

is moving on C clockwise. Then the point on the midpoint curve M moves counter-clockwise, and simple calculations show that $|C| = 2|M|$.

Ebbers-Baumann et al. [22] proved that, in the Euclidean plane, the image of a closed convex curve under the halving pair transformation is also convex. We show that this result still holds in general Minkowski planes.

Lemma 6.2.4. *For any $u, v \in S_X$ and $\lambda \in [0, 1)$ we have $u \not\perp_B (u + \lambda v)$ and $v \not\perp_B (v + \lambda u)$. Moreover, $u \perp_B (u + v)$ if and only if $[u, -v] \subset S_X$.*

Proof. The case $u = \pm v$ is trivial. Suppose that $u \neq \pm v$ and that there exists a number $\lambda_0 \in [0, 1)$ such that $u \perp_B (u + \lambda_0 v)$. Then, by the definition of Birkhoff orthogonality, the inequality

$$\|u + t(u + \lambda_0 v)\| \geq \|u\| = 1$$

holds for any $t \in \mathbb{R}$. By setting $t = -1$ we have $|\lambda_0| \geq 1$, a contradiction.

Suppose that $u \perp_B (u + v)$. Then

$$\left\| u - \frac{1}{2}(u + v) \right\| = \frac{1}{2} \|u - v\| \geq 1,$$

which implies that $\|u - v\| = 2$. Thus $[u, -v] \subset S_X$. □

Lemma 6.2.5. *Suppose that $C \subset X$ is a continuously differentiable, closed, convex curve, and p, q be two points on C such that a pair of parallel supporting lines of* conv(C) *contains p and q, respectively. Then $d_C(p, q) > \|p - q\|$.*

Proof. Let C_0 be that part of C connecting p and q which has minimum length, and l_p and l_q be the supporting lines of $\operatorname{conv}(C)$ at p and q, respectively (see Figure 6.3). Since C is continuously differentiable, there exists a point $q_0 \in C_0$ such that the supporting line of $\operatorname{conv}(C)$ at q_0 (which intersects l_p and l_q in p_1 and q_1, respectively) is parallel to the line $\langle p, q \rangle$.

For any number $0 < \varepsilon < d_C(p, q_0)$, let $q_\varepsilon \in C_0$ be the point such that $d_C(p, q_\varepsilon) = \varepsilon$; p_ε, q''_ε, and q'_ε be the points where the line passing through q_ε

parallel to $\langle p, q\rangle$ intersects l_p, l_q, and the arc on C_0 between q_0 and q, respectively. Since

$$d_C(p, q_\varepsilon) \geq \|p - q_\varepsilon\|\,,$$

$$d_C(q_\varepsilon, q'_\varepsilon) \geq \|q_\varepsilon - q'_\varepsilon\|\,,$$

and

$$d_C(q'_\varepsilon, q) \geq \|q'_\varepsilon - q\|\,,$$

it suffices to show that

$$\|p - q_\varepsilon\| + \|q'_\varepsilon - q\| > \|p_\varepsilon - q_\varepsilon\| + \|q'_\varepsilon - q''_\varepsilon\|$$

for some sufficiently small ε.

Suppose that the line $\langle p, q_\varepsilon\rangle$ intersects $[p_1, q_1]$ in q'_1. Then

$$\frac{\|p - p_\varepsilon\|}{\|p_\varepsilon - q_\varepsilon\|} = \frac{\|p - p_1\|}{\|p_1 - q'_1\|}.$$

Since $\|p - p_1\|$ is fixed and C is differentiable at p, we have $\lim_{\varepsilon\to 0} \|p_1 - q'_1\| = 0$, and therefore

$$\lim_{\varepsilon\to 0} \frac{\|p_\varepsilon - q_\varepsilon\|}{\|p - p_\varepsilon\|} = 0.$$

Thus

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\|p - q_\varepsilon\|}{\|p_\varepsilon - q_\varepsilon\|} &\geq \lim_{\varepsilon\to 0} \frac{\|p - p_\varepsilon\| - \|p_\varepsilon - q_\varepsilon\|}{\|p_\varepsilon - q_\varepsilon\|} \\
&= \lim_{\varepsilon\to 0} \frac{\|p - p_\varepsilon\|}{\|p_\varepsilon - q_\varepsilon\|} - 1 = +\infty,
\end{aligned}$$

and then

$$\lim_{\varepsilon\to 0} \frac{\|p_\varepsilon - q_\varepsilon\|}{\|p - q_\varepsilon\|} = 0,$$

which implies $\|p_\varepsilon - q_\varepsilon\| < \|p - q_\varepsilon\|$ for sufficiently small ε. In a similar way we can prove that $\|q'_\varepsilon - q\| > \|q'_\varepsilon - q''_\varepsilon\|$ when ε is sufficiently small. □

Theorem 6.2.6. *If C is convex, then C^* is convex.*

Proof. First we assume that C is smooth. Then the derivative $\dot{c}(\cdot)$ is a continuous function mapping $[0, |C|)$ into the unit circle S_X. Due to convexity, the derivative vectors $\dot{c}(t)$ and $-\dot{c}(t + |C|/2)$ always turn into the same direction, say ω_0.

Note that $\dot{c}(t) - \dot{c}(t + |C|/2) = 0$ cannot occur, since this would imply that $\dot{c}(\tau) = \dot{c}(t) = \dot{c}(t + |C|/2)$ holds for each τ in $[t, t + |C|/2]$ or $[t + |C|/2, t + |C|]$, due to convexity. Then C would contain a line segment of length $|C|/2$. On the other hand, it follows from the assumption that the supporting lines of $\mathrm{conv}(C)$ at $c(t)$ and $c(t + |C|/2)$ are parallel to each other. By Lemma 6.2.5 we have $\|c(t) - c(t + |C|/2)\| < |C|/2$, a contradiction.

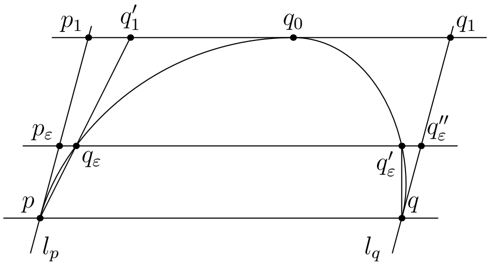

Figure 6.3: Proof of Lemma 6.2.5.

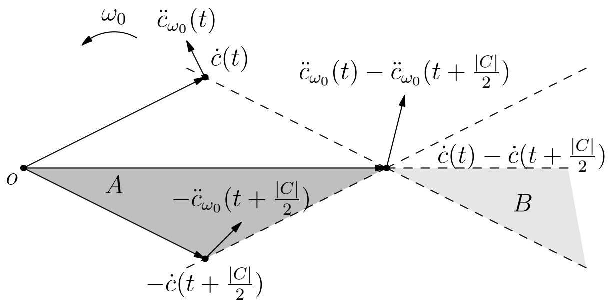

Figure 6.4: $\dot{c}(t) - \dot{c}(t + |C|/2)$ turns into the same direction as $\dot{c}(t)$ and $\dot{c}(t + |C|/2)$.

Furthermore, by Lemma 6.2.4 we have for any $\lambda \in (0, 1)$

$$\dot{c}(t) \not\angle_B (\dot{c}(t) - \lambda \dot{c}(t + \frac{|C|}{2})) \tag{6.2.1}$$

and

$$-\dot{c}(t + \frac{|C|}{2}) \not\angle_B (-\dot{c}(t + \frac{|C|}{2}) + \lambda \dot{c}(t)). \tag{6.2.2}$$

Denote by $\ddot{c}_{\omega_0}(t)$ the derivative of $\dot{c}(t)$ in direction ω_0, i.e., the one-side derivative turns $\dot{c}(t)$ in the direction ω_0 (see Figure 6.4). (6.2.1) and (6.2.2) imply that $\ddot{c}_{\omega_0}(t) + \dot{c}(t) - \dot{c}(t + |C|/2)$ and $-\ddot{c}_{\omega_0}(t + |C|/2) + \dot{c}(t) - \dot{c}(t + |C|/2)$ cannot lie in the domains A and B, respectively. Therefore, $\ddot{c}_{\omega_0}(t)$ and $-\ddot{c}_{\omega_0}(t + |C|/2)$ turn the vector $\dot{c}(t) - \dot{c}(t + |C|/2)$ into the direction ω_0. Hence C^* is convex.

This result can be extended to closed convex curves, approximating them by smooth closed convex curves. □

6.2.2 On the halving distance

The relations between different geometric quantities of convex bodies yield interesting (geometric) inequalities. In this subsection we relate the minimum and maximun halving distance h and H to other geometric quantities, such as, for example, the minimum width w. The results in the following theorem can be derived immediately from the definitions of the corresponding quantities.

Theorem 6.2.7. *Let $C \subset X$ be a closed convex curve. Then the following inequalities hold:*

1. $h \leq w$,
2. $H \leq |C|/2$,
3. $2r \leq w$,
4. $h \leq H \leq D$,
5. $H \leq 2R$.

Lemma 6.2.8. *Let $C \subset X$ be a closed convex curve. Then there exists a point $p_0 \in C$ such that* $\operatorname{conv}(C)$ *has parallel supporting lines at p_0 and $\hat{p_0}$.*

Proof. First we assume that C is a smooth curve. Note that

$$\int_0^{|C|/2} \dot{c}(t)\mathrm{d}t = c(\frac{|C|}{2}) - c(0) = -\int_0^{|C|/2} \dot{c}(t + \frac{|C|}{2})\mathrm{d}t.$$

By the intermediate value theorem of integration, there exists a number $t_0 \in (0, |C|/2)$ such that $\dot{c}(t_0) + \dot{c}(t_0 + |C|/2) = 0$. Let $p_0 = c(t_0)$. Then $\operatorname{conv}(C)$ has parallel supporting lines at p_0 and $\hat{p_0}$.

Again this result can be generalized to closed convex curves, approximating them by smooth closed convex curves. □

Theorem 6.2.9. *Let $C \subset X$ be a closed convex curve. Then $H \geq w$.*

Proof. By Lemma 6.2.8 there exists a point $p_0 \in C$ such that $[p_0, \hat{p_0}]$ is a halving chord and $\operatorname{conv}(C)$ has parallel supporting lines at p_0 and $\hat{p_0}$. Then the distance between the supporting lines at p_0 and $\hat{p_0}$ is not smaller than the minimum width w of C. Since $(p_0, \hat{p_0})$ is a halving pair, it follows that $H \geq \|p_0 - \hat{p_0}\| \geq w$. The inequality is tight, since circles attain the equality case. □

The following corollary follows from Theorem 6.2.7 and Theorem 6.2.9.

Corollary 6.2.10. *Let $C \subset X$ be a closed convex curve. Then $H \geq 2r$. This inequality is tight, because equality holds for circles.*

Lemma 6.2.11. (cf. [12, Theorem 3]) *If $C \subset X$ is a closed convex curve, then $w = \min\limits_{v \in S_X} l_C(v)$.*

Lemma 6.2.12. *Let $C \subset X$ be a closed convex curve. Then the inequality $h_C(v) > l_C(v)/2$ holds for every direction $v \in S_X$. This inequality cannot be improved.*

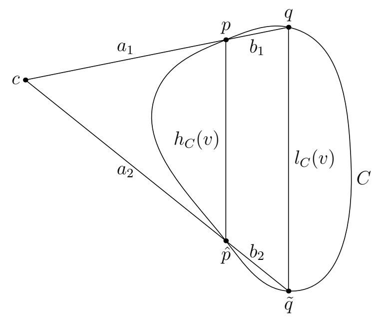

Figure 6.5: Proof of Lemma 6.2.12.

Proof. For any $v \in S_X$, let p and $\hat{p}$ be the halving pair in the direction v; $[q, \tilde{q}]$ be the longest chord of C in the direction v, i.e., $l_C(v) = \|q - \tilde{q}\|$. Without loss of generality, we can assume that $p - \hat{p}$ is a positive multiple of $q - \tilde{q}$. The following proof is similar to the proof of Lemma 4.12 in [27].

If $l_C(v) \leq h_C(v)$, then the proof is complete. If $l_C(v) > h_C(v)$, then the line $\langle p, q\rangle$ has to intersect the line $\langle \hat{p}, \tilde{q}\rangle$ at some point c which is seperated from the segment $[q, \tilde{q}]$ by the line $\langle p, \hat{p}\rangle$ (see Figure 6.5). Let

$$a_1 = \|c - p\|\,, a_2 = \|c - \hat{p}\|\,, b_1 = \|p - q\|\,, \text{ and } b_2 = \|\hat{p} - \tilde{q}\|\,.$$

Since $[p, \hat{p}]$ is a halving chord and C is convex, we have

$$b_1 + b_2 + l_C(v) \leq |C|/2 \leq a_1 + a_2.$$

Note that both the chords $[p, \hat{p}]$ and $[q, \tilde{q}]$ have direction v. It follows that

$$\frac{a_1}{a_1 + b_1} = \frac{a_2}{a_2 + b_2} = \frac{h_C(v)}{l_C(v)},$$

and then

$$h_C(v) = l_C(v)\frac{a_1 + a_2}{a_1 + a_2 + b_1 + b_2} \geq l_C(v)\frac{b_1 + b_2 + l_C(v)}{2(b_1 + b_2) + l_C(v)} > \frac{1}{2}l_C(v).$$

The second inequality cannot become an equality, but the two numbers can come arbitrarily close to each other if $l_C(v)$, compared with $b_1 + b_2$, is small enough. $\square$

Corollary 6.2.13. *Let $C \subset X$ be a closed convex curve. Then $h > w/2$, and this inequality is tight.*

Proof. The first part of this result follows from Lemma 6.2.11 and Lemma 6.2.12.

In order to see that the bound is tight, we consider an isosceles triangle in $\mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$, as shown in Figure 6.6. One can

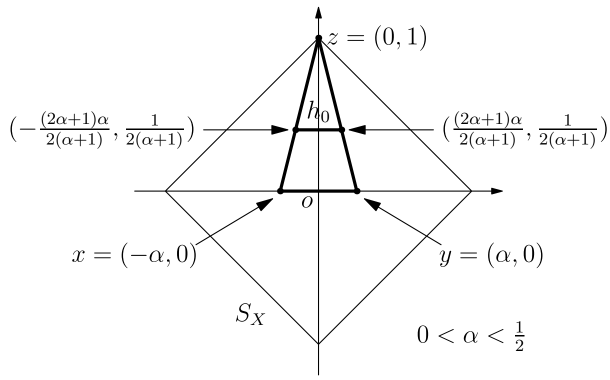

Figure 6.6: Proof of Corollary 6.2.13.

easily show that $w = 2\alpha$ and the halving distance in direction $(x-y)/\|x-y\|$ is $h_0 = (2\alpha+1)\alpha/(\alpha+1)$. Then

$$\frac{1}{2} < \frac{h}{w} \le \frac{h_0}{w} = \frac{2\alpha+1}{2(\alpha+1)},$$

which implies that h/w tends to $1/2$ when α tends to 0. $\square$

From Theorem 6.2.7 and Corollary 6.2.13, the relation between h and r can be derived immediately in the following corollary.

Corollary 6.2.14. *Let $C \subset X$ be a closed convex curve. Then $h > r$.*

In order to obtain the upper bound for h in terms of r, we need the following lemma.

Lemma 6.2.15. *For any triangle in a Minkowski plane X, there exists a height (i.e., the distance from a vertex to the line containing its opposite side) which is not larger than three times the radius of the incircle of that triangle.*

Proof. Suppose that the vertices of the triangle are p_1, p_2, and p_3, and the incirle of the triangle is $c_0 + rS_X$ with radius r and center c_0. Let $c = (p_1+p_2+p_3)/3$, and $p_4 \in [p_1, p_3]$, $p_5 \in [p_1, p_2]$, and $p_6 \in [p_2, p_3]$ be points such that the lines $\langle c, p_4\rangle$,$\langle c, p_5\rangle$, and $\langle c, p_6\rangle$ are parallel to $\langle p_2, p_3\rangle$,$\langle p_1, p_3\rangle$, and $\langle p_1, p_2\rangle$, respectively (see Figure 6.7). Then

$$\frac{\|p_1 - p_4\|}{\|p_1 - p_3\|} = \frac{\|p_2 - p_5\|}{\|p_2 - p_1\|} = \frac{\|p_3 - p_6\|}{\|p_3 - p_2\|} = \frac{2}{3},$$

and the segments $[c, p_4]$,$[c, p_5]$, and $[c, p_6]$ divide the triangle into three regions. The center c_0 of the incircle should lie in one of these three regions. Suppose, without loss of generality, that c_0 lies in the convex hull of the points c, p_4, p_1 and p_5. Let l be the line passing through c_0 parallel to $\langle p_2, p_3\rangle$, and d be the

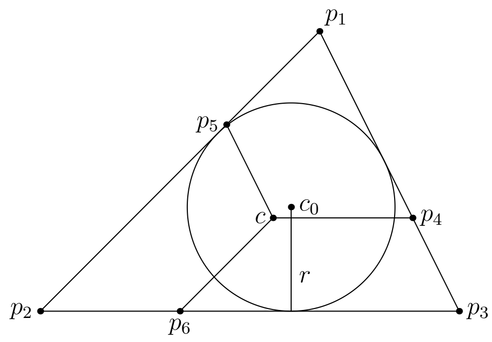

Figure 6.7: Proof of Lemma 6.2.15.

distance from p_1 to l. Then, since the distance between l and $\langle p_2, p_3\rangle$ is r, we have $d/(d+r) \leq 2/3$, which yields $d \leq 2r$. Hence the height from p_1 to the side $[p_2, p_3]$ is not larger than $3r$. □

Theorem 6.2.16. *Let $C \subset X$ be a closed convex curve. Then $h \leq 3r$.*

Proof. Suppose that the incirle of C is $c_0 + rS_X$. Then $c_0 + rS_X$ should touch C at more than one point. We consider the following two cases:

Case 1: $c_0 + rS_X$ touches C at exactly two points, say p and q.

In this case, there should be a pair of parallel supporting lines of $\mathrm{conv}(C)$ at p and q. Suppose the contrary, namely that any supporting line of $\mathrm{conv}(C)$ at p is not parallel to each supporting line of $\mathrm{conv}(C)$ at q. Let l_p and l_q be supporting lines of $\mathrm{conv}(C)$ at p and q, respectively, intersecting each other at a point c, and l be a line supporting $c_0 + rB_X$ at a point c_1 and parallel to $\langle p, q\rangle$, where c_1 is separated from c by $\langle p, q\rangle$. Note that l_p and l_q are also supporting lines of $c_0 + rB_X$ at p and q, respectively.

If the chord $[p, q]$ is not a diameter of $c_0 + rS_X$, then there exist a diameter $[p', q']$ and $u \in S_X$ such that $p' - q'$ is a positive multiple of $p - q$ and $p' - q' \perp_B u$ (see Figure 6.8). Since c_1, p', and q' are interior points of $\mathrm{conv}(C)$, there exists a number $\delta_1 > 0$ such that the points $c_1 + \delta_1 u$, $p' + \delta_1 u$, and $q' + \delta_1 u$ are still interior points of $\mathrm{conv}(C)$. Thus we can obtain a translate $c_0 + \delta_1 u + rS_X$ of the incirle which is contained in $\mathrm{conv}(C)$ and does not touch C. This is a contradiction.

Suppose that $[p, q]$ is a diameter of $c_0 + rS_X$. Suppose that l_p, l_q and the line passing through q parallel to l_p intersects l in p_1, q_1, and q', respectively. Let q_0 be the point on $[q', q_1]$ nearest to q' such that the line $\langle q, q_0\rangle$ supports $\mathrm{conv}(C)$; $p_0 \in [p_1, q_1]$ be a point such that the line $\langle p, p_0\rangle$ is parallel to $\langle q, q_0\rangle$ (see Figure 6.9). Then $\langle p, p_0\rangle$ does not support $\mathrm{conv}(C)$ because of the assumption. Let $p_2 \in [p_1, p_0]$ be a point sufficiently close to p_0 such that $p_2 \neq p_0$ and the line $\langle p, p_2\rangle$ still does not support $\mathrm{conv}(C)$, and q_2 be the point on l such that $\langle q, q_2\rangle$ is parallel to $\langle p, p_2\rangle$. Then $q_2 \in [q', q_0]$, $q_2 \neq q_0$, and $\langle q, q_2\rangle$ is not a supporting

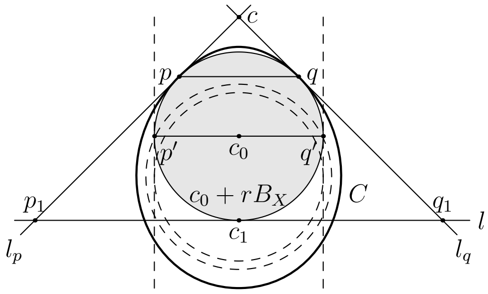

Figure 6.8: The chord $[p, q]$ is not a diameter of $c_0 + rS_X$.

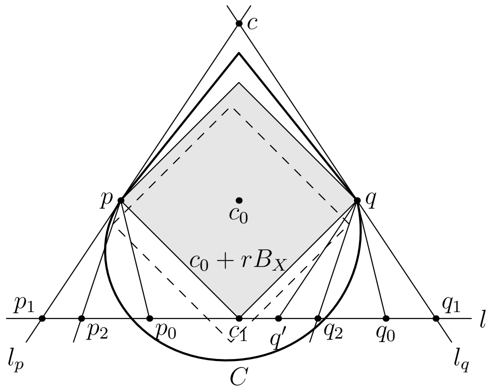

Figure 6.9: The chord $[p, q]$ is a diameter of $c_0 + rS_X$.

line of $\operatorname{conv}(C)$. We note that l_p, l_q, $\langle p, p_0\rangle$, $\langle q, q_0\rangle$, $\langle p, p_2\rangle$, and $\langle q, q_2\rangle$ are all supporting lines of c_0+rB_X. Since c_1 is a interior point of $\operatorname{conv}(C)$, there exists a number $\delta_2 > 0$ such that the points $c_1+\delta_2(p_2-p)$, $p+\delta_2(p_2-p)$, and $q+\delta_2(p_2-p)$ are still interior points of $\operatorname{conv}(C)$. Thus the translate $c_0 + \delta_2(p_2 - p) + rS_X$ of the incirle is contained in $\operatorname{conv}(C)$ and does not touch C, a contradiction to the fact that $c_0 + rS_X$ is the incircle of C.

The minimum width w of C cannot be larger than the distance between the parallel supporting lines of $\operatorname{conv}(C)$ at p and q, which is the minimum width of the incircle $c_0 + rS_X$, that is, $w \leq 2r$. By Theorem 6.2.7, it follows that $h \leq w < 3r$.

Case 2: The set $c_0 + rS_X$ touches C at more than two points. Then there exist at least three such points that are not collinear. Otherwise, we could obtain a translate of c_0+rS_X which is contained in $\operatorname{conv}(C)$ and does not touch C, in a similar way as in Case 1, a contradiction.

Let q_1, q_2, and q_3 be three points in the intersection of $c_0 + rS_X$ and C with supporting lines l_1, l_2, and l_3, respectively. Suppose that any two of these

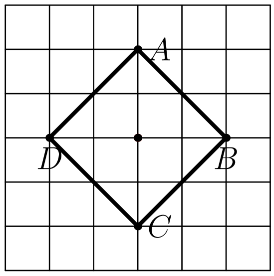

Figure 6.10: The dilation shall be calculated by using taxicab norm.

lines are not parallel to each other, and $\{p_1\} = l_1 \cap l_3$, $\{p_2\} = l_1 \cap l_2$, and $\{p_3\} = l_2 \cap l_3$. Thus the triangle formed by p_1, p_2, and p_3 contains the curve C, and then the minimum width of C is not larger than the minimum height of the triangle. By Lemma 6.2.15 we have $w \leq 3r$, and then $h \leq w \leq 3r$. $\square$

6.3 The geometric dilation problem

6.3.1 Introduction

For any rectifiable simple closed curve C in the Euclidean plane, the *geometric dilation* $\delta_E(C)$ of C (see [22]) is the number defined by

$$\delta_E(C) := \sup_{p,q \in C, p \neq q} \frac{d_C(p,q)}{|pq|},$$

where $d_C(p,q)$ is the minimum of the lengths of the two curve parts of C connecting p and q, and $|pq|$ is the Euclidean distance between p and q. Ebbers-Baumann et al. [22] proved that the lower bound for the geometric dilation of closed curves is $\pi/2$ (this bound was already obtained by Gromov; see [24] and [25]), and that the circle is the only closed curve achieving this lower bound.

It is necessary to study the geometric dilation problem in a much wider framework since, in many cases, it is more meaningful to replace the Euclidean distance in the definition above by a different one. Here is an example. Let G be a grid on $\mathbb{R}^2$ as shown in Figure 6.10. When points are allowed to move only on the grid, the distance between two points should be given by the taxicab norm (i.e., the unit circle of this norm is a parallelogram with vertices $(0,1)$, $(1,0)$, $(0,-1)$, and $(-1,0)$), and therefore it makes more sense to calculate the dilation of a closed curve on this grid by using the taxicab norm. And the aim of this section is to calculate lower bounds on the geometric dilation of closed curves not only for this, but for all norms. Therefore this section refers to arbitrary Minkowski planes.

Definition 6.3.1. For any rectifiable simple closed curve $C \subset X$, the *geometric* *dilation* $\delta_X(C)$ of C is defined by

$$\delta_X(C) := \sup_{p,q \in C, p \neq q} \frac{d_C(p,q)}{\|p-q\|},$$

where $d_C(p,q)$ is the minimum of the lengths of the two curve arcs of C connecting p and q. The dilation $\delta_C(p,q)$ of p and q in C is the ratio of $d_C(p,q)$ and the Minkowskian distance between p and q.

The geometric dilation was already studied for open curves, polygonal chains, graphs, and point sets; see, e.g., [2], [33], [19] and, for computational approaches, [1]. The results presented here are also contained in [49]. I wish to thank Prof. H. Martini for inspiring discussions on this topic.

6.3.2 Lower bounds in Minkowski planes

Actually, Ebbers-Baumann et al. [22] presented two different ways to show that the lower bound for the geometric dilation in the Euclidean plane is $\pi/2$. The first one is based on the two-dimensional version of Cauchy's surface area formula, which cannot be carried over directly to general Minkowski planes. (To see this, consider the Minkowskian unit circle of the Minkowski plane on $\mathbb{R}^2$ with the taxicab norm. Clearly, it is of constant Minkowskian width 2 and has length 8. However, the curve length obtained by applying the two-dimensional Cauchy's surface area formula is 2π, which is apparently not true.) For the extension of Cauchy's surface area formula we refer to Section 6.3 in [60]. The other way in [22] is by processing a series of simplifications of the original problem, yielding again the result that the lower bound is attained only by circles. Both these ways are based on the Euclidean version of the following lemma.

Lemma 6.3.2. *Let C be a rectifiable simple closed curve in the Minkowski plane X, and $\partial\mathrm{conv}(C)$ be the boundary of its convex hull $\mathrm{conv}(C)$. Then $|C| \geq |\partial\mathrm{conv}(C)|$.*

This lemma is obvious by considering a "very good" polygonal approximation of $\partial\mathrm{conv}(C)$, which then is "very close" to a polygonal approximation of C. Since an "even better" polygonal approximation of C (whose vertices are along C) could be larger, in the limit we get this inequality. (More generally, this statement is even true in any projective metric defined on a convex set, i.e., a metric which behaves normally on straight lines regarding segment addition together with the triangle inequality.) Thus, in details the proof of Lemma 6.3.2 is analogous to the proof given in [56] for the Euclidean case, and so we omit these details here. Schaer [56] also proved that $|C| = |\partial\mathrm{conv}(C)|$ if and only if $C = \partial\mathrm{conv}(C)$, which is not true in general Minkowski planes. For example, let X be the Minkowski plane induced on $\mathbb{R}^2$ by the taxicab norm (i.e., S_X is the parallelogram with vertices $(\pm1, 0)$, $(0, \pm1)$), and C be the polygon which linearly connects $(0, 1)$, $(0, 0)$, $(1, 0)$, $(0, -1)$, $(-1, 0)$, $(0, 1)$ in that order. Then $\partial\mathrm{conv}(C) = S_X \neq C$, and $|C| = |\partial\mathrm{conv}(C)| = 8$.

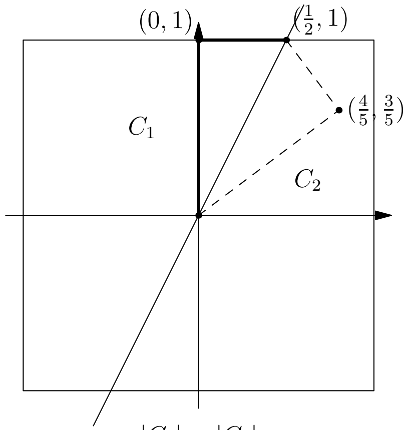

Figure 6.11: The length of the reflection of a curve is not equal to the length of that curve

As a direct corollary of Lemma 6.3.2 we have the following:

Corollary 6.3.3. *Let p and q be two points lying on a rectifiable simple closed curve C in the Minkowski plane X such that $\{p,q\} \subset (C \cap \partial\mathrm{conv}(C))$. Then $d_C(p,q) \geq d_{\partial\mathrm{conv}(C)}(p,q)$.*

Similar to the approach in [22], our first step to obtain the general lower bound for geometric dilations of closed curves in a Minkowski plane is to show that for any rectifiable simple closed curve C we can find a closed convex curve with a geometric dilation not larger than $\delta_X(C)$. Then we show that the maximum dilation of a closed convex curve is attained by a halving pair of points. Finally, a result from Minkowski geometry will be applied to obtain the lower bound on the geometric dilation of rectifiable simple closed curves in Minkowski planes.

Lemma 6.3.4. *Let $C \subset X$ be a rectifiable simple closed curve. Then*

$$\delta_X(\partial\mathrm{conv}(C)) \leq \delta_X(C).$$

Proof. The proof of Lemma 9 in [22] can be carried over to prove this lemma here, except for one detail only: To prove $|p'q| \leq |pq|$ in one subcase of Case 2 there, the authors used the facts that the bisector of two points in the Euclidean plane is a line and that the mirror of a curve with respect to a given line has the same length as the original curve, which does not hold in general Minkowski planes. So we have to provide a new method to show that $\|p'-q\| \leq \|p-q\|$ in this subcase. We include a complete proof to make this paper self-contained.

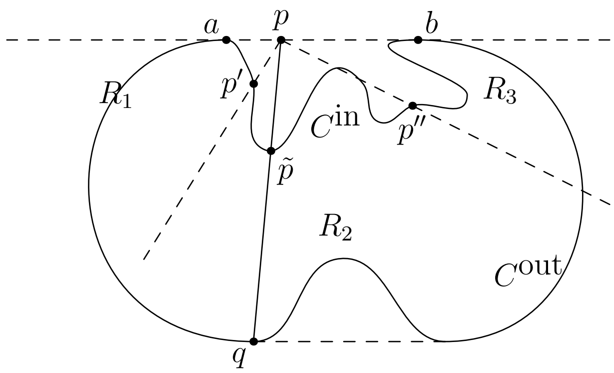

Figure 6.12: Case 2: $p \in \partial\text{conv}(C)$ and $q \in \partial\text{conv}(C) \cap C$.

We will prove that for any pair of points $\{p, q\} \subset \partial\text{conv}(C)$ we can find a corresponding pair of points $\{\tilde{p}, \tilde{q}\} \subset C$ not having smaller dilation, i.e. $\delta_C(\tilde{p}, \tilde{q}) \geq \delta_{\partial\text{conv}(C)}(p, q)$. We distinguish three cases:

Case 1: $p, q \in \partial\text{conv}(C) \cap C$. In this case we pick $\tilde{p} := p$ and $\tilde{q} := q$. By Corollary 6.3.3, $d_{\partial\text{conv}(C)}(p, q) \leq d_C(p, q)$ holds and this implies $\delta_{\partial\text{conv}(C)}(p, q) \leq \delta_C(p, q) = \delta_C(\tilde{p}, \tilde{q})$.

Case 2: $p \in \partial\text{conv}(C)\backslash C$ and $q \in \partial\text{conv}(C) \cap C$. Let $[a, b]$ be the line segment of $\partial\text{conv}(C)$ so that $p \in [a, b]$ and $[a, b] \cap C = \{a, b\}$ (see Fig. 6.3.2). Let C^{in} denote the path on C connecting a and b that is contained in the interior of the convex hull, and let $C^{\text{out}} := C\backslash C^{\text{in}}$ be the other path on C connecting a and b. Clearly, C lies in one of the half-planes H bounded by the line passing through a and b. Then q is contained in H. By the conditions of Case 2 the point q cannot be part of $C^{\text{in}} \subset C\backslash\partial\text{conv}(C)$. Hence $q \in C^{\text{out}}$. Since C is simple, C^{out} cannot intersect C^{in}, and by the definition of $[a, b]$ it cannot intersect $[a, b]$. Thus C^{out} cannot enter the region bounded by the closed curve formed as the union of $[a, b]$ and C^{in}. Let p' (p'', resp.) be the point on C^{in} satisfying $d_C(a, p') = \|a - p\|$ ($d_C(b, p'') = \|p - b\|$, resp.). Clearly, p' (p'', resp.) is contained in the Minkowskian ball centered at a (b, resp.) and having radius $\|p - a\|$ ($\|p - b\|$, resp.). Hence the two rays emanating from p through p' (p'', resp.) divide H into three parts. If we remove the closed region bounded by $C_b^a \oplus [a, b]$, we get three regions (from left to right) R_1, R_2 and R_3 whose union contains q (see Fig. 6.3.2). We note that the closure of R_2 could degenerate to a ray, since in a non-strictly convex Minkowski plane it is possible that $p' = p''$.

If $q \in R_2$, then $[p, q]$ intersects C^{in} in a point $\tilde{p}$ between p' and p''. It follows that $\|\tilde{p}-q\| \le \|p-q\|$ and

$$\begin{aligned} d_C(\tilde{p},q) &= \min\{d_C(\tilde{p},a)+d_C(a,q), d_C(\tilde{p},b)+d_C(b,q)\} \\ &\ge \min\{d_C(p',a)+d_C(a,q), d_C(p'',b)+d_C(b,q)\} \\ &\ge \min\{d_{\partial\mathrm{conv}(C)}(p,a)+d_{\partial\mathrm{conv}(C)}(a,q), \\ &\quad\ d_{\partial\mathrm{conv}(C)}(p,b)+d_{\partial\mathrm{conv}(C)}(b,q)\} \\ &= d_{\partial\mathrm{conv}(C)}(p,q), \end{aligned} \tag{6.3.1}$$

and we conclude that $\delta_{\partial\mathrm{conv}(C)}(p,q) \le \delta_C(\tilde{p},q)$. Choosing $\tilde{q} := q$ completes the proof of this subcase of Case 2.

If $q \in R_1$, we have $d_C(p',q) \ge d_{\partial\mathrm{conv}(C)}(p,q)$ which follows analogously to (6.3.1). We will show that $\|p'-q\| \le \|p-q\|$. Namely, let $\bar{p}$ be a point of intersection of the segment between p and q and the curve $C_a^{p'}$. Then

$$\|\bar{p}-p'\| + \|\bar{p}-a\| \le d_C(a,p') = \|p-a\|\,,$$

and therefore

$$\|\bar{p}-p'\| \le \|p-a\| - \|\bar{p}-a\| \le \|p-\bar{p}\|\,.$$

Hence

$$\|p'-q\| \le \|\bar{p}-p'\| + \|\bar{p}-q\| \le \|p-\bar{p}\| + \|\bar{p}-q\| = \|p-q\|\,.$$

Finally, $\delta_{\partial\mathrm{conv}(C)}(p,q) \le \delta_C(p',q)$, and we can choose $\tilde{p} := p'$ and $\tilde{q} := q$ to complete the proof.

In the last subcase of Case 2 the point q is contained in R_3. But then we can argue analogously to the case $q \in R_1$.

Case 3: $p,q \in \partial\mathrm{conv}(C)\backslash C$. If p and q are located on the same line segment of $\partial\mathrm{conv}(C)$, we have

$$\delta_{\partial\mathrm{conv}(C)}(p,q) = 1 \le \delta_C(p',q')$$

for any $\{p',q'\} \subset \partial\mathrm{conv}(C)$.

In the remaining case, we can apply the step of Case 2 twice. First, consider the cycle $C' := \partial\mathrm{conv}(C)\backslash[a,b]\cup C^{\mathrm{in}}$, where $[a,b]$ is replaced by C^{in} in $\partial\mathrm{conv}(C)$ and everything is defined as in Case 2. Again we can find a point $\tilde{p} \in C^{\mathrm{in}} \subset C'$ so that $\delta_{C'}(\tilde{p},q) \ge \delta_{\partial\mathrm{conv}(C)}(p,q)$. Next, we can apply the arguments of Case 2 to the pair $(q,\tilde{p})$ instead of (p,q) and C' instead of $\partial\mathrm{conv}(C)$. We get a point $\tilde{q} \in C$ so that $\delta_C(\tilde{p},\tilde{q}) \ge \delta_{C'}(\tilde{p},q) \ge \delta_{\partial\mathrm{conv}(C)}(p,q)$.

□

Next we show that for a closed convex curve C, $\delta_X(C)$ is attained by a halving pair. Our proof is valid for any Minkowski plane and relatively short. But first we need to show the existence of a halving pair for any direction $v \in S_X$.

Lemma 6.3.5. *Let C be a rectifiable simple closed convex curve in the Minkowski plane X. Then for every direction $v \in S_X$ there exists a unique halving pair $(p,\hat{p})$ (i.e., $p-\hat{p} = \|p-\hat{p}\|\,v$).*

Proof. First we show the existence of the halving pair for every direction.

For every direction $v \in S_X$ there exist two lines of direction v supporting C. We denote them by l_1, l_2 and suppose that $l_1 \cap C = [a,b]$ and $l_2 \cap C = [c,d]$. Then $\|b-a\| \leq \frac{|C|}{2}$ and $\|d-c\| \leq \frac{|C|}{2}$. In case that either $\|b-a\| = \frac{|C|}{2}$ or $\|d-c\| = \frac{|C|}{2}$, we can choose $p = a$ and $\hat{p} = b$, or $p = c$ and $\hat{p} = d$.

If $\|b-a\| < \frac{|C|}{2}$ and $\|d-c\| < \frac{|C|}{2}$, then choose $z_1 \in [a,b]$ and $z_2 \in [c,d]$. Note that, for any $\lambda \in (0,1)$, the line through $(1-\lambda)z_1 + \lambda z_2$ of direction v must intersect C in two points p_λ and $\hat{p}_\lambda$. Let $f(\lambda)$ be the function defined as the difference between the length of the curve part of C which contains z_1 and connects p_λ to $\hat{p}_\lambda$, and the length of the curve part of C which contains z_2 and connects p_λ to $\hat{p}_\lambda$. Then $f(\lambda)$ is continuous, $\lim\limits_{\lambda\to 0} f(\lambda) = 2\|b-a\| - |C| < 0$, and $\lim\limits_{\lambda\to 1} f(\lambda) = |C| - 2\|d-c\| > 0$. Hence there exists a number $\lambda_0 \in (0,1)$ such that $f(\lambda_0) = 0$, which gives $d_C(p_{\lambda_0}, \hat{p}_{\lambda_0}) = |C|/2$. With $p = p_{\lambda_0}$ and $\hat{p} = \hat{p}_{\lambda_0}$, $(p,\hat{p})$ is a halving pair of C.

To show uniqueness we assume that there exists a direction $v \in S_X$ such that we have two distinct halving pairs $(p,\hat{p})$ and $(p',\hat{p}')$ with respect to v. Since C is convex, the intersection of the line l passing through p and $\hat{p}$ and $\mathrm{conv}(C)$ is exactly the segment $[p,\hat{p}]$. Then p' and $\hat{p}'$ have to lie in the same half plane bounded by l. Then, since the arc lengths connecting p and p', as well as $\hat{p}$ and $\hat{p}'$, are larger than 0, $(p',\hat{p}')$ cannot be a halving pair, a contradiction. Hence the halving pair for a given direction is unique.

$\square$

Remark 6.3.6. In Minkowski planes it is possible that there exists a rectifiable simple closed curve C containing a segment having length $\frac{|C|}{2}$ on it, and that is why our proof of Lemma 6.3.5 is slightly diffenent from that of Lemma 3 in [22]. Take, e.g., the Minkowski plane on $\mathbb{R}^2$ with unit ball $conv\{(\pm 1, \pm 1)\}$ (maximum norm), and consider the points $x = (1,1)$, $y = (-1,1)$. Then the length of the curve C formed by the segments connecting o with x, x with y, and y with o is 4, while the length of the segment connecting x and y, which is contained in C, is 2.

Lemma 6.3.7. *Let C be a rectifiable simple closed convex curve. Then its maximum dilation is attained by a halving pair. This implies that $\delta_X(C) = \frac{|C|}{2h(C)}$, where*

$$h(C) := \inf_{v\in S_X} \{\|p-\hat{p}\| : \ (p,\hat{p}) \textit{ is the halving pair of direction } v\}.$$

Proof. Since there exists a halving pair $(p_v, \hat{p}_v)$ for any direction v, we have

$$\delta_X(C) \geq \frac{d_C(p_v,\hat{p}_v)}{\|p_v - \hat{p}_v\|} = \frac{|C|}{2\,\|p_v - \hat{p}_v\|}.$$

Therefore,

$$\delta_X(C) \geq \sup_{v\in S_X} \frac{|C|}{2\,\|p_v - \hat{p}_v\|} = \frac{|C|}{2 \inf\limits_{v\in S_X} \|p_v - \hat{p}_v\|} = \frac{|C|}{2h(C)}.$$

For any different $p_1, p_2 \in C$ there exists a halving pair $(p, \hat{p})$ of direction $\frac{p_1 - p_2}{\|p_1 - p_2\|}$. By Lemma 2.1.3 we have

$$\frac{d_C(p_1, p_2)}{\|p_1 - p_2\|} \leq \frac{d_C(p, \hat{p})}{\|p - \hat{p}\|} \leq \frac{|C|}{2h(C)}.$$

Thus $\delta_X(C) \leq \frac{|C|}{2h(C)}$, since p_1 and p_2 are chosen arbitrarily from C.

On the other hand, by compactness of C there exists a point $p \in C$ such that $\|p - \hat{p}\| = h(C)$, and therefore

$$\frac{d_C(p, \hat{p})}{\|p - \hat{p}\|} = \delta_X(C),$$

which completes the proof. □

Theorem 6.3.8. *For the dilation of any rectifiable simple closed curve C in X we have $\delta_X(C) \geq |S_X|/4$.*

Proof. From the foregoing discussion we have

$$\delta_X(C) \overset{\text{Lemma 6.3.4}}{\geq} \delta_X(\partial\text{conv}(C)) \overset{\text{Lemma 6.3.7}}{=} \frac{|\partial\text{conv}(C)|}{2h(\partial\text{conv}(C))} \overset{\text{Lemma 2.1.5}}{\geq} |S_X|/4,$$

yielding the proof. □

Ebbers-Baumann et al. [22] also showed that the circle is the only closed curve achieving a dilation of $\pi/2$ in the Euclidean plane. However, the analogue of this characterization of circles does not hold in general Minkowski planes.

For example, let X be the Minkowski plane whose unit circle S_X is the convex polygon with vertices $(-1, -1)$, $(-1, \pi - 3)$, $(3 - \pi, 1)$, $(1, 1)$, $(1, 3 - \pi)$, $(\pi - 3, -1)$. Then one can easily verify $|S_X| = 2\pi$. Now we add two new vertices $(0, \pi - 2)$ and $(0, 2 - \pi)$ to this polygon, denoting by C the convex hull of all eight points, i.e., the new convex polygon with eight vertices. Then $|C| = |S_X| = 2\pi$ and $h(C) = h(S_X) = 2$. Hence $\delta_X(C) = \pi/2$.

However, with the help of the halving pair transformation we can show that the lower bound for geometric dilations in strictly convex Minkowski planes can only be attained by circles.

The proof of the following lemma is similar to that of Lemma 19 and Lemma 20 in [22], since strict convexity of the norm is assumed. The proof is due to H. Martini.

Lemma 6.3.9. *Let C be a rectifiable simple closed convex curve in a strictly convex Minkowski plane. Then $|C| \geq |C^*|$, with equality if and only if C is centrally symmetric.*

Proof. It follows from the uniqueness of halving pair in a given direction that C^* is simple. Let $\bar{c}(\cdot)$ be an arc-length parameterization of C. Then

$$\begin{aligned}|C| &= \int_0^{\frac{|C|}{2}} \|\dot{\bar{c}}(t)\| + \left\|\dot{\bar{c}}(t+\frac{|C|}{2})\right\| dt \geq \int_0^{\frac{|C|}{2}} \left\|\dot{\bar{c}}(t) - \dot{\bar{c}}(t+\frac{|C|}{2})\right\| dt \\ &= 2\int_0^{\frac{|C|}{2}} \left\|\dot{c^*}(t)\right\| dt = \int_0^{|C|} \left\|\dot{c^*}(t)\right\| dt = |C^*|.\end{aligned}$$

If equality holds, then the triangle inequality implies that

$$\|\dot{\bar{c}}(t)\| + \left\|\dot{\bar{c}}(t+\frac{|C|}{2})\right\| = \left\|\dot{\bar{c}}(t) - \dot{\bar{c}}(t+\frac{|C|}{2})\right\|$$

for almost every $t \in [0, |C|/2]$. Since X is strictly convex and $\|\dot{\bar{c}}(t)\| = 1$ for almost every $t \in [0, |C|/2]$, $\dot{\bar{c}}(t) = -\dot{\bar{c}}(t+\frac{|C|}{2})$ holds for almost every $t \in [0, |C|/2]$. Without loss of generality we suppose that $\bar{c}(0) = -\bar{c}(\frac{|C|}{2})$. Then

$$\bar{c}(t) = \bar{c}(0) + \int_0^t \dot{\bar{c}}(\tau)d\tau = -\bar{c}(\frac{|C|}{2}) + \int_0^t -\dot{\bar{c}}(\tau + \frac{|C|}{2})d\tau = -\bar{c}(t+\frac{|C|}{2}),$$

which means that C is centrally symmetric. $\square$

Theorem 6.3.10. *Let X be a strictly convex Minkowski plane. Then the circles of X are the only rectifiable simple closed curves that can attain $|S_X|/4$, i.e., a quarter of their Minkowskian circumference as their geometric dilation.*

Proof. Suppose that C is a rectifiable simple closed curve with a geometric dilation $|S_X|/4$ which is not a circle. First we consider the case when C is convex and centrally symmetric. Without loss of generality we can assume that C is symmetric with respect to the origin. Then it is trivial that the circle $\frac{1}{2}h(C)S_X$ is inscribed in C, and that $\mathrm{conv}(\frac{1}{2}h(C)S_X)$ is a proper subset of $\mathrm{conv}(C)$. Hence there exists a point $p \in \frac{1}{2}h(C)S_X \backslash C$. It is clear that p is an interior point of $\mathrm{conv}(C)$, and therefore any line passing through p intersects C at exactly two points. Let $\hat{p}$ be the point of intersection of C and the ray starting from the origin and passing through p; p_1 and p_2 be the points of intersection of C and the line supporting $\frac{1}{2}h(C)S_X$ at p. Then there will be an arc C_0 of C connecting p_1 and p_2 which does not pass through $\hat{p}$. Let C_1 be the closed convex curve formed by C_0 and the segment between p_1 and p_2, and C_2 be the closed convex curve formed by C_0, the segment between $\hat{p}$ and p_1, and the segment between $\hat{p}$ and p_2. Since X is strictly convex, we have

$$|\frac{1}{2}h(C)S_X| \leq |C_1| < |C_2| \leq |C|.$$

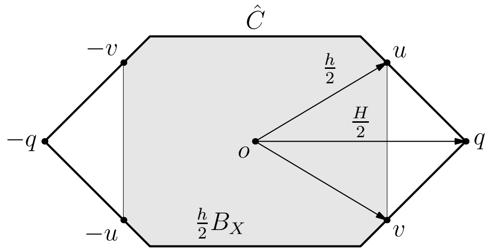

Figure 6.13: The curve $\hat{C}$ is the shortest curve containing the disc $(h/2)B_X$ and connecting q and $-q$.

Hence

$$\delta_X(C) = \frac{|C|}{2h(C)} > \frac{|\frac{1}{2}h(C)S_X|}{2h(C)} \geq \frac{|S_X|}{4}.$$

If C is centrally symmetric but not convex, then $\partial\text{conv}(C)$ is centrally symmetric but not strictly convex, and therefore not a circle. Hence

$$\delta_X(C) \geq \delta_X(\partial\text{conv}(C)) > \frac{|S_X|}{4}.$$

If C is convex but not centrally symmetric, we have $|C^*| < |C|$. Thus

$$\delta_X(C) = \frac{|C|}{2h(C)} > \frac{|C^*|}{2h(C^*)} \geq \frac{|\partial\ \text{conv}(C^*)|}{2h(\partial\text{conv}(C^*))} = \delta_X(\partial\text{conv}(C^*)) \geq \frac{|S_X|}{4}.$$

Suppose now that C is neither centrally symmetric nor convex. If $\partial\text{conv}(C)$ is centrally symmetric, then (since $\partial\text{conv}(C)$ is not strictly convex) it cannot be a circle. Hence

$$\delta_X(C) \geq \delta_X(\partial\text{conv}(C)) > \frac{|S_X|}{4}.$$

If $\partial\text{conv}(C)$ is not centrally symmetric, again we have

$$\delta_X(C) \geq \delta_X(\partial\text{conv}(C)) > \frac{|S_X|}{4},$$

and the proof is complete. □

In the following theorem we present a sufficient condition for the geometric dilation of a curve to be larger than $|S_X|/4$.

Theorem 6.3.11. *Let C be a closed convex curve with $H/h > 2$. Then $\delta_X(C) > |S_X|/4$.*

Proof. Suppose that $\delta_X(C) = |S_X|/4$. By Theorem 6.2.6 and the convexity of C we have

$$\delta_X(C) = \frac{|C|}{2h} \geq \frac{|C^*|}{2h} = \delta_X(C^*) \geq \frac{|S_X|}{4},$$

which yields $|C| = |C^*|$. As stated in Remark 6.2.2, C^* contains the disc $(h/2)B_X$. On the other hand, C^* has to connect some halving pair q and $-q$ having maximum halving distance H.

Suppose that the supporting lines of $(h/2)B_X$ passing through q support $(h/2)B_X$ at u and v, respectively (see Figure 6.13: if one of the lines supports $(h/2)B_X$ at a segment, then we choose the point nearest to q on that segment). Thus the supporting lines of $(h/2)B_X$ passing through $-q$ support $(h/2)B_X$ at $-u$ and $-v$, respectively. Let $\hat{C}$ be the closed convex curve depicted in Figure 6.13. Then

$$\frac{h}{2}|S_X| \leq |\hat{C}| \leq |C^*| = \frac{h}{2}|S_X|,$$

which implies that $|\hat{C}| = (h/2)|S_X|$. It follows that

$$\|q-u\| + \|q-v\| = d_{\frac{h}{2}S_X}(u,v).$$

Thus $\|q-u\| + \|q-v\| = \|u-v\|$. Since $\|q\| = H/2 > h$ and $\|u\| = \|v\| = h/2$, we have $\|q-u\| \geq \|q\| - \|u\| > h/2$ and $\|q-v\| > h/2$. Therefore $h \geq \|u-v\| = \|q-u\| + \|q-v\| > h$, a contradiction. $\square$

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Appendix. Characterizations of Euclidean planes

In this appendix, we collect all the new characterizations that we obtained in this dissertation.

A Minkowski plane X is Euclidean if and only if:

- for any $x \in S_X$ the set $P^+(x)$ is a singleton (Theorem 3.2.2);
- for any x, $y \in S_X$ with $x \neq \pm y$, $P(x) \cap P(y) = \varnothing$ (Theorem 3.2.14);
- $c(X) = \sqrt{2}$ (Theorem 3.3.1);
- for any exterior point x of S_X the lengths of the two corresponding tangent segments from x to S_X are equal (Theorem 5.1.3);
- for any exterior point x of S_X the squared length of the tangent segment from x to S_X equals the product of the lengths of the secant segment and the external secant segment (Corollary 5.1.4);
- there exists a number $\varepsilon \in (0,2)\backslash T$ such that $\delta_X(u,v) = \delta(\varepsilon_1)$ $(= \delta(\varepsilon_2)$, resp.) whenever $u, v \in S_X$ and $\|u - v\| = \varepsilon_1$ $(=\varepsilon_2$, resp.), where $\varepsilon_1 = \sqrt{4-\varepsilon^2}$, $\varepsilon_2 = \varepsilon$, and $\delta(\varepsilon_1)$ and $\delta(\varepsilon_2)$ are constants determined by ε_1 and ε_2, respectively (Theorem 5.2.4);
- there exists a function $\varphi : [0,2] \to [0,4]$ such that for any $u, v \in S_X$ we have $\delta_X(u,v) = \varphi(\|u - v\|)$ (Corollary 5.2.6);
- equality $\|x - A_\alpha(x)\| = \left\|x - A_\alpha^{-1}(x)\right\|$ holds for any point $x \in S_X$ and any irrational number $\alpha \in (0,1)$ (Theorem 5.2.1);
- for any u, $v \in S_X$ with $u \neq -v$ $\left\|\widehat{u+v} - u\right\| = \left\|\widehat{u+v} - v\right\|$ (Theorem 5.3.1);
- for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the relation $p_i - p_j \perp_B (p_k - p_l)$ holds, where $\{i,j,k,l\} = \{1,2,3,4\}$ (Theorem 5.4.7);

- for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$, $p_4 \in \langle p_3, \frac{p_1+p_2}{2}\rangle$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$ (Theorem 5.4.8);

- for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the equality $\|p_3 - p_1\| = \|p_3 - p_2\|$ holds whenever $p_4 \in \langle p_3, \frac{p_1+p_2}{2}\rangle$ (Theorem 5.4.9);

- for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$, p_4 lies on the line containing $A_B([p_3, p_1\rangle, [p_3, p_2\rangle)$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$ (Theorem 5.4.10);

- for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the equality $\|p_3 - p_1\| = \|p_3 - p_2\|$ holds whenever p_4 lies on the line containing $A_B([p_3, p_1\rangle, [p_3, p_2\rangle)$ (Theorem 5.4.11);

- for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the equality $\|p_3 - p_1\| = \|p_3 - p_2\|$ holds whenever $\langle p_1, p_2\rangle$ is a common supporting line of the circle containing $\{p_1, p_3, p_4\}$ and the circle containing $\{p_2, p_3, p_4\}$ (Theorem 5.4.12);

- for any S_X-orthocentric system $\{p_1, p_2, p_3, p_4\}$, $\langle p_1, p_2\rangle$ is a common supporting line of the circle containing $\{p_1, p_3, p_4\}$ and the circle containing $\{p_2, p_3, p_4\}$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$ (Theorem 5.4.13);

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Theses for the dissertation
"Geometry of Minkowski Planes and Spaces – Selected Topics",
submitted by M. Sc. Senlin Wu,
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1. A Minkowski space X is a real, finite dimensional Banach space (or normed linear space of finite dimension). In particular, a two-dimensional Minkowski space is called a Minkowski plane (or normed plane).

2. For any point $x \in X \backslash \{o\}$, the bisector $B(-x, x)$ is defined as the set of points which are equidistant to $-x$ and x, and the radial projection $P(x)$ of $B(-x, x)$ is the set of normalizations $z/\|z\|$ of non-zero points z in $B(-x, x)$. Basic properties of radial projections of $B(-x, x)$ for any point $x \in X \backslash \{o\}$ are studied. For the planar case the following properties of such radial projections are proved:

   (a) $P(x)$ is the union of two connected subsets of the unit sphere S_X.

   (b) A Minkowski plane X is Euclidean if and only if for any $x \in S_X$ the set $P(x)$ contains precisely two points.

   (c) For any $x, y \in S_X$ we have that y lies in the closure of $P(x)$ whenever y is Birkhoff orthogonal to x, which is denoted by $y \perp_B x$, i.e., whenever $\|y + tx\| \geq \|y\|$ holds for any real number t.

   (d) Let $x \in S_X$. If there exists a point $z \in S_X$ (unique except for the sign) such that $x \perp_B z$, then z lies in the closure of $P(x)$. And if there exists a point z that lies in the closure of $P(x)$ but not in $P(x)$ itself, then either $z \perp_B x$ or $x \perp_B z$.

   (e) In general, $P(x)$ cannot be determined by the points that are Birkhoff orthogonal to x and the points to which x is Birkhoff orthogonal.

   (f) For any point $x \in S_X$, the distance from x to $P(x)$ is not less than 1 and not greater than 2, and the cases when these bounds are attained are discussed.

   (g) A Minkowski plane X is Euclidean if and only if for any points x, y in the unit circle S_X with $x \neq \pm y$, $P(x) \cap P(y) = \varnothing$. A sufficient condition is provided for the general case to ensure that $P(x) \cap P(y) = \varnothing$.

   In case that the dimension of the space is not less than 3, it is shown that $P(x)$ is a connected subset of S_X.

3. There exists a number for any Minkowski space, which is the infimum of the distances from x to $P(x)$ for points in the unit sphere S_X and plays somehow the role that $\sqrt{2}$ plays in Euclidean space. This number is referred to as the critical number of any Minkowski space. Lower and upper bounds on the critical number are given, and corresponding characterization theorems are derived.

4. Let X and Y be two real normed linear spaces. A linear map $T : X \mapsto Y$ preserves isosceles orthogonality if and only if T is a scalar multiple of a linear isometry. (For x, $y \in X$, x is said to be isosceles orthogonal to y if and only if x is equidistant to y and $-y$.)

5. We show that a Minkowski plane X is Euclidean if and only if for any exterior point x of the unit circle S_X the lengths of the two corresponding tangent segments from x to S_X are equal.

6. Furthermore, a Minkowski plane X is Euclidean if there exists a number $\varepsilon \in (0, 2)\backslash T$ such that the minimum of the lengths of the two circular arcs connecting two points u and v from S_X is determined by the length of the chord $[u, v]$ whenever $\|u - v\| = \varepsilon_1$ or $\|u - v\| = \varepsilon_2$, where $\varepsilon_1 = \sqrt{4 - \varepsilon^2}$, $\varepsilon_2 = \varepsilon$, and the set T is defined by
$$T = \{2\cos(\frac{k\pi}{2n}) : \; n = 2, 3, \ldots, \; k = 1, 2, \ldots, n - 1\}.$$

7. Having angular bisectors in mind, we prove that a Minkowski plane X is Euclidean if and only if the equality $\|x - A_\alpha(x)\| = \left\|x - A_\alpha^{-1}(x)\right\|$ holds for any point $x \in S_X$ and any irrational number $\alpha \in (0, 1)$. Here $A_\alpha(x)$ and $A_\alpha^{-1}(x)$ are two points in S_X such that the length of the circular arc connecting $A_\alpha^{-1}(x)$ to x and the length of the circular arc connecting x to $A_\alpha(x)$ are both equal to α times the circumference of S_X.

8. Also we show that a Minkowski plane X is Euclidean if for any u, $v \in S_X$ with $u \neq -v$ the normalization $(u + v)/\|u + v\|$ of $u + v$ is equidistant to u and v.

9. The notions of S_X-orthocenter and S_X-orthocentric system, which extend the notion of orthocenter and orthocentric systems in Euclidean Geometry, are studied. We investigate whether the "Euclidean results" related to orthocenters can be carried over to Minkowski geometry and obtain by this various new characterizations of the Euclidean plane.

10. Let x be an arbitrary interior point of the unit ball B_X of a Minkowski space X. If the locus of the midpoints of all chords of B_X passing through x is a homothetical copy of S_X, then the space X is Euclidean.

11. Referring to chords of (unit) circles, we verify that a Minkowski plane X is Euclidean if and only if the two chords between the "left" as well as "right" endpoints of any two parallel chords of S_X are of the same length.

12. Similarly, we prove that a Minkowski plane X is Euclidean if and only if the arcs between any two parallel chords of S_X have the same length.

13. It is known that the lower bound for the geometric dilation, which is defined for rectifiable simple closed curves in the Euclidean plane as the supremum of the ratio of the minimum length of curve arcs connecting

two points on the curve to the distance between them, is $\pi/2$. This lower bound can be attained only by circles in the Euclidean plane. We extend this result to Minkowski planes by proving that the lower bound for the geometric dilation of rectifiable simple closed curves in a Minkowski plane X is analogously a quarter of the circumference of S_X, but can also be attained by curves that are not Minkowskian circles. In addition we show that the lower bound is attained only by Minkowskian circles if and only if the respective norm is strictly convex.

14. A chord of a simple closed curve is called a halving chord if it halves the circumference of the curve. A transformation of the curve is said to be a halving pair transformation if it translate each halving chord into a segment symmetric with respect to the origin. In an arbitrary Minkowski plane we study the relation between the length of a closed curve and the length of its midpoint curve, which is the locus of the midpoints of halving chords of the curve, as well as the length of its image under the halving pair transformation. We show that the image curve under the halving pair transformation is convex provided the original curve is convex. We give a sufficient condition for the geometric dilation of a closed convex curve to be larger than a quarter of the perimeter of the unit circle. Moreover, we obtain several inequalities to show the relation between the halving distance and other quantities well known in convex geometry.

Erklärung

Ich erkläre an Eides Statt, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Chemnitz, am 2. November 2008 Senlin Wu