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The paper is concerned with the Nitsche mortaring in the framework of domain decomposition where non-matching meshes and weak continuity of the finite element approximation at the interface are admitted. The approach is applied to singularly perturbed reaction-diffusion problems in 2D. Non-matching meshes of triangles being anisotropic in the boundary layers are applied. Some properties as well as error estimates of the Nitsche mortar finite element schemes are proved. In particular, using a suitable degree of anisotropy of triangles in the boundary layers of a rectangle, we derive convergence rates as known for the conforming finite element method in presence of regular solutions. Numerical examples illustrate the approach and the results.

Key Words. finite element method, non-matching meshes, mortar finite elements, boundary layers, domain decomposition

AMS subject classification. 65N30, 65N55.

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1 Introduction

In the framework of domain decomposition methods, mortar methods are convenient to handle non-matching meshes and discontinuous finite element approximations at the interface of the domain decomposition. There are several approaches to work with mortar methods, the classical one using a saddle point problem and Lagrange multipliers (see e.g. [6, 7, 15, 16]) and the Nitsche mortaring (see e.g. [3, 4, 5, 10, 11, 14]).

In most papers, mortar methods are founded for triangulations $\mathcal{T}_h$ with "shape regular" elements T or even for quasi-uniform triangulations. "Shape regularity" for triangles $T \in \mathcal{T}_h$ means here that the relation $\frac{h_T}{\varrho_T} \leq C < \infty$ is satisfied (h_T: diameter of T, ϱ_T: radius of the largest ball contained in T), where C is independent of h ($h := \max\limits_{T\in\mathcal{T}_h} h_T$) and of some pertubation parameter ε (if h_T, ϱ_T depend on ε). We call such triangles also "isotropic" in contrast to "anisotropic" triangles where $\frac{h_T}{\varrho_T} \to \infty$ as $h \to 0$ or $\varepsilon \to 0$.

In this paper we shall consider the Nitsche mortaring for anisotropic triangulations applied to linear reaction-diffusion problems with diffusion parameter ε^2 ($0 < \varepsilon < 1$). It is well-known that for small values of ε^2 singularly perturbed problems with boundary layers occur, where the solutions show some anisotropic behaviour. It is obvious that isotropic finite elements are not convenient for the efficient treatment of such problems and that anisotropic elements are to be preferred. Therefore, we aim at the application of anisotropic elements within the boundary layers and admit a non-matching coupling with isotropic elements in other parts of the domain.

The paper is organized as follows. First we present a derivation of the Nitsche mortar approach for singularly perturbed reaction-diffusion problems on polygonal domains. Some assumptions on the non-matching anisotropic mesh at the interface of the domain decomposition are stated and the Nitsche mortar finite element approximation is defined. Using an inverse inequality for anisotropic triangles, basic inequalities and the stability of the Nitsche mortar method for some type of anisotropic triangulations are derived. For sufficiently regular solution, here $u \in H^{\frac{3}{2}+\delta}(\Omega)$, $\delta > 0$, and anisotropic meshes a lemma is proved which is a generalization of Céa's lemma to the case of non-matching anisotropic triangulations. In particular, the error $u - u_h$ in some H^1-like norm is bounded by some other discrete norm of the interpolation error $u - I_h u$. Moreover, interpolation error estimates for error functionals of the Nitsche mortaring on anisotropic meshes are given. Furthermore, estimates of the error $u - u_h$ in the H^1-like norm on anisotropic meshes in terms of the meshsize parameter h and the diffusion parameter ε^2 are derived. Since a complete analysis of boundary layers combined with corner singularities in general polygonal domain is rather extensive (cf. [13]) and beyond the scope of this paper, we shall discuss the convergence rate of $u_h \to u$ as $h \to 0$ and uniformly with respect to ε, for simplicity, for boundary layers in rectangular domains only. Finally, some numerical experiments are given which illustrate the diminishing of the error on anisotropic elements and the theoretical rates of convergence.

It should be noted that corner singularities are already treated in [10, 11], and the combination of boundary layers with corner singularities in general polygonal domains (see e.g. [13]) could be investigated by a combination of the methods and results of the present paper and [10, 11] leading to very similar results, if mesh refinement near singular corners is used.

First we take into account analytical preliminaries and derive the Nitsche mortaring approach. Consider the reaction-diffusion model problem

$$\begin{aligned} Lu \;:=\; -\varepsilon^2\Delta u + cu \;&=\; f \qquad \text{in } \Omega\subset\mathbb{R}^2 \\ u \;&=\; 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is assumed to be a polygonal domain with Lipschitz-boundary $\partial\Omega$ consisting of straight segments. Suppose that $0<\varepsilon<1$ and $0<c_0\le c(x)\,(x\in\Omega)$ hold, with some constant c_0, and $c\in L_\infty(\Omega)$. Furthermore, assume $f\in L_2(\Omega)$ at least.

For simplicity the domain Ω is decomposed into two non-overlapping, polygonal subdomains Ω_1 and Ω_2 such that $\overline{\Omega}=\overline{\Omega}_1\cup\overline{\Omega}_2$ and $\Omega_1\cap\Omega_2=\emptyset$ hold, cf. Figure 1. Introduce $\Gamma:=\overline{\Omega}_1\cap\overline{\Omega}_2$ to be the interface of Ω_1 and Ω_2.

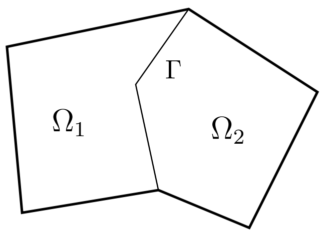

Figure 1: Decomposition of Ω

In this context, we utilize the restrictions $v^i := v|_{\Omega_i}$ of some function v on Ω_i as well as the vectorized form $v=(v^1,v^2)$, i.e., we have $v^i(x)=v(x)$ for $x\in\Omega_i$ $(i=1,2)$. It should be noted that we shall use here the same symbol v for denoting the function v on Ω as well as the vector $v=(v^1,v^2)$, which should not lead to confusion. Moreover, $v|_\Gamma$ denotes the trace of v on Γ, but the symbol $.|_\Gamma$ is often omitted.

Using the domain decomposition, BVP (1) is equivalent to the following problem. Find $u=(u^1,u^2)$ such that

$$\begin{aligned} -\varepsilon^2\Delta u^i + cu^i \;&=\; f^i \quad \text{in } \Omega_i, \qquad u^i \;=\; 0 \quad \text{on } \partial\Omega_i\cap\partial\Omega, \qquad \text{for } i=1,2, \\ \frac{\partial u^1}{\partial n_1}+\frac{\partial u^2}{\partial n_2} \;&=\; 0 \quad \text{on } \Gamma, \qquad u^1 \;=\; u^2 \quad \text{on } \Gamma, \end{aligned} \tag{2}$$

are satisfied, where n_i $(i=1,2)$ denotes the outward normal to $\partial\Omega_i\cap\Gamma$.

We also use the variational equation of (1). Find $u\in H_0^1(\Omega):=\{v\in H^1(\Omega): v|_{\partial\Omega}=0\}$ such that

$$a(u,v) \;=\; f(v) \quad \text{for any } v\in H_0^1(\Omega), \tag{3}$$

$$\text{with}\quad a(u,v) := \varepsilon^2 \int_\Omega (\nabla u, \nabla v)\, dx + \int_\Omega cuv\, dx, \quad f(v) := \int_\Omega fv\, dx.$$

Obviously, there is a unique solution $u \in H^{\frac{3}{2}+\delta}(\Omega)$, with some $\delta > 0$. Due to corner singularities, $u \notin H^2(\Omega)$ holds, and because of small values of ε, the solution u is provided with boundary layers, in general. Nevertheless, we have $\frac{\partial u}{\partial n} \in L_2(\Gamma)$ by the trace theorem and $\Delta u \in L_2(\Omega)$ (see [9]).

Introduce the space $V := V^1 \times V^2$, where V^i $(i = 1, 2)$ is defined by

$$\begin{aligned} V^i &:= \left\{ v^i : v^i \in H^1(\Omega_i),\, v^i\big|_{\partial\Omega\cap\partial\Omega_i} = 0 \right\} && \text{for } \partial\Omega \cap \partial\Omega_i \neq \emptyset, \\ V^i &:= H^1(\Omega_i) && \text{for } \partial\Omega \cap \partial\Omega_i = \emptyset. \end{aligned} \tag{4}$$

Then, the BVP (2) can be formulated in the weak form (see [8]). Clearly, there is weak unique solution $(u^1, u^2) \in V$ of (2), with $Lu^i \in L_2(\Omega_i)$ for $i = 1, 2$. The continuity of the solution u and of its normal derivative $\frac{\partial u^i}{\partial n}$ on Γ ($n = n_1$ or $n = n_2$) is to be required in the sense of $H_*^{\frac{1}{2}}(\Gamma)$ and $H_*^{-\frac{1}{2}}(\Gamma)$ (the dual space of $H_*^{\frac{1}{2}}(\Gamma)$), respectively. Define $H_*^{\frac{1}{2}}(\partial\Omega_i)$ ($H_{00}^{\frac{1}{2}}$) by the range of V^i by the trace operator and to be provided with the quotient norm, see e.g. [7, 9]. So we use $H_*^{\frac{1}{2}}(\partial\Omega_i) \simeq H_{00}^{\frac{1}{2}}(\partial\Omega_i \setminus \partial\Omega)$ for $\partial\Omega \cap \partial\Omega_i \neq \emptyset$, and $H_*^{\frac{1}{2}}(\partial\Omega_i) = H^{\frac{1}{2}}(\partial\Omega_i)$ for $\partial\Omega \cap \partial\Omega_i = \emptyset$. Here $\simeq$ means that we identify the corresponding spaces. In the following, $\langle .\,, . \rangle_\Gamma$ denotes the $H_*^{-\frac{1}{2}}$-$H_*^{\frac{1}{2}}$ duality pairing.

With this notation, (2) implies formally

$$\sum_{i=1}^{2} \left(\int_{\Omega_i} \varepsilon^2 \left(\nabla u^i, \nabla v^i\right) dx + \int_{\Omega_i} cuv\, dx - \left\langle \varepsilon^2 \frac{\partial u^i}{\partial n_i}, v^i \right\rangle_\Gamma \right) = \sum_{i=1}^{2} \int_{\Omega_i} f^i v^i\, dx \quad \forall v \in V, \tag{5}$$

or equivalently, owing to $\frac{\partial u^1}{\partial n_1} = \alpha_1 \frac{\partial u^1}{\partial n_1} - \alpha_2 \frac{\partial u^2}{\partial n_2} = -\frac{\partial u^2}{\partial n_2}$ for any $\alpha_i \geq 0$ $(i = 1, 2)$ such that $\alpha_1 + \alpha_2 = 1$,

$$\begin{aligned} &\sum_{i=1}^{2} \left(\int_{\Omega_i} \varepsilon^2 \left(\nabla u^i, \nabla v^i\right) dx + \int_{\Omega_i} cuv\, dx \right) - \left\langle \alpha_1 \varepsilon^2 \frac{\partial u^1}{\partial n_1} - \alpha_2 \varepsilon^2 \frac{\partial u^2}{\partial n_2}, v^1 - v^2 \right\rangle_\Gamma \\ &- \left\langle \alpha_1 \varepsilon^2 \frac{\partial v^1}{\partial n_1} - \alpha_2 \varepsilon^2 \frac{\partial v^2}{\partial n_2}, u^1 - u^2 \right\rangle_\Gamma + \int_\Gamma \sigma \left(u^1 - u^2\right)\left(v^1 - v^2\right) ds = \sum_{i=1}^{2} \int_{\Omega_i} f^i v^i\, dx. \end{aligned} \tag{6}$$

Note that the two additional terms (both equal to zero) containing $u^1 - u^2$ and introduced artificially have the following purpose. The first one ensures the symmetry (in u, v) of the left-hand side, the second one penalizes (after the discretization) the jump of the trace of the approximate solution and guarantees the stability for appropriately chosen weighting function $\sigma > 0$. The Nitsche mortar finite element method is the discretization of equation (6) in the sense of (13) given subsequently, using a finite element subspace V_h of V allowing non-matching meshes and discontinuity of the finite element approximation along Γ. The function σ is taken as $\gamma\varepsilon^2 h^{-1}(x)$, where $\gamma > 0$ is a sufficiently large constant and $h(x)$ is a mesh parameter function on Γ. The non-matching property arises naturally since elements of V are not continuous across Γ, in general.

2 Non-matching mesh finite element discretization

Let $\mathcal{T}_h^i$ be a triangulation of $\overline{\Omega}_i$ $(i = 1, 2)$ consisting of triangles T $(T = \overline{T})$. The triangulations $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are independent of each other, in general, i.e. the nodes of $T \in \mathcal{T}_h^i$ $(i = 1, 2)$ do not match along $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$. Let h denote the mesh parameter of the triangulation $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$, with $0 < h \leq h_0$ and sufficiently small h_0. Take e.g. $h := \max_{T\in\mathcal{T}_h} h_T$. Admit that h_T and ϱ_T may depend on some parameter $\varepsilon \in (0, 1)$ (cf. (41)). Furthermore, employ F $(F = \overline{F})$ for denoting any side of a triangle, h_F its length. Sometimes we use T_F in order to indicate that F is a side of $T = T_F$. Throughout the paper let the following assumption on the geometrical conformity of $\mathcal{T}_h^i$ $(i = 1, 2)$ be satisfied.

Assumption 2.1. *For $i = 1, 2$, it holds $\overline{\Omega}_i = \bigcup_{T\in\mathcal{T}_h^i} T$, and two arbitrary triangles $T, T' \in \mathcal{T}_h^i$ $(T \neq T')$ are either disjoint or have a common vertex, or a common side.*

Since anisotropic triangles (see [1]) will be applied, they are not shape regular and, therefore, the mesh is not quasi-uniform (with respect to ε).

Consider further some triangulation $\mathcal{E}_h$ of the interface Γ by intervals E $(E = \overline{E})$, i.e., $\Gamma = \bigcup_{E\in\mathcal{E}_h} E$, where h_E denotes the diameter of E. Here, two segments $E, E' \in \mathcal{E}_h$ are either disjoint or have a common endpoint. A natural choice for the triangulation $\mathcal{E}_h$ of Γ is $\mathcal{E}_h := \mathcal{E}_h^1$ or $\mathcal{E}_h := \mathcal{E}_h^2$ (cf. Figure 2), where $\mathcal{E}_h^1$ and $\mathcal{E}_h^2$ denote the triangulations of Γ defined by the traces of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ on Γ, respectively, i.e.,

$$\mathcal{E}_h^i := \{E : E = \partial T \cap \Gamma, \text{ if } E \text{ is a segment}, \ T \in \mathcal{T}_h^i\} \quad \text{for } i = 1, 2, \tag{7}$$

i.e., here $E = F = T_F \cap \Gamma$ for some $T_F \in \mathcal{T}_h^i$ holds.

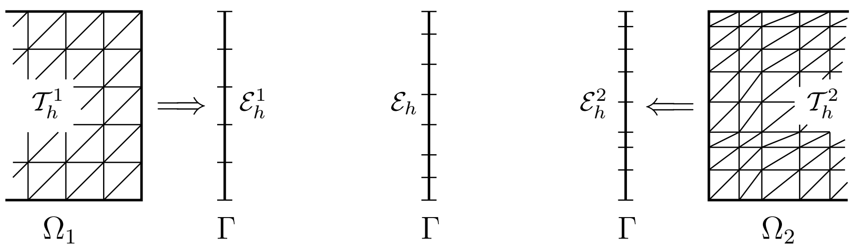

Figure 2: Choice of $\mathcal{E}_h$

Subsequently we use real parameters α_1, α_2 with

$$0 \leq \alpha_i \leq 1 \ (i = 1, 2), \quad \alpha_1 + \alpha_2 = 1, \tag{8}$$

and require the asymptotic behaviour of the triangulations $\mathcal{T}_h^1, \mathcal{T}_h^2$ and of $\mathcal{E}_h$ to be consistent on Γ in the sense of the following assumption. For $\Lambda \in \{E, F\}$ apply the notation $\mathring{\Lambda} := \operatorname{int} \Lambda$.

Assumption 2.2. *According to different cases of chosing $\mathcal{E}_h$ and α_1, α_2 from* (8) *assume that there are constants C_1, C_2 independent of $h \in (0, h_0]$ and $\varepsilon \in (0,1)$ such that, uniformly with respect to E and F, the following relations hold,*

(i) case $\mathcal{E}_h$ arbitrary:
for any $E \in \mathcal{E}_h$ and $F \in \mathcal{E}_h^i$ $(i = 1,2)$ with $\mathring{E} \cap \mathring{F} \neq \emptyset$, we have

$$C_1 h_F \;\leq\; h_E \;\leq\; C_2 h_F,$$

(ii) case $\mathcal{E}_h := \mathcal{E}_h^i$ and $\alpha_i = 1$ $(i = 1$ or $i = 2)$:
for any $E \in \mathcal{E}_h^i$ and $F \in \mathcal{E}_h^{3-i}$ with $\mathring{E} \cap \mathring{F} \neq \emptyset$, we have

$$C_1 h_F \;\leq\; h_E.$$

Assumption 2.2 ensures that the asymptotics of segments E and sides F which touch each other is locally the same, uniformly with respect to h and ε, in case (ii) with some weakening which admits different asymptotics of triangles $T_1 \in \mathcal{T}_h^1$ and $T_2 \in \mathcal{T}_h^2$, with $T_1 \cap T_2 \neq \emptyset$.

For getting stability of the method in the case of anisotropic meshes we require that the following assumption is satisfied, which restricts the orientation of anisotropy of the triangles T at Γ.

Assumption 2.3. *If $h_F^{\perp}$ denotes the height of the triangle $T_F \in \mathcal{T}_h^i$ over the side $F \in \mathcal{E}_h^i$, then for $i \in \{1,2\}$ with $0 < \alpha_i \leq 1$ assume that the quotient $\frac{h_F}{h_F^{\perp}}$ satisfies*

$$\frac{h_F}{h_F^{\perp}} \;\leq\; C_3 \qquad \forall\, F \in \mathcal{E}_h^i, \tag{9}$$

where C_3 is independent of $h \in (0, h_0]$ and $\varepsilon \in (0,1)$.

This assumption guarantees that anisotropic triangles $T = T_F \in \mathcal{T}_h^i$ touching Γ along the whole side F and being "active" (for $i : \alpha_i \neq 0$) in the approximation have their "short side" F on Γ.

For $i = 1,2$, introduce the finite element space V_h^i (subspace of V^i from (4)) of functions v^i on Ω_i by

$$V_h^i \;:=\; \{v^i \in H^1(\Omega_i) : v^i\big|_T \in \mathbb{P}_k(T) \;\; \forall\, T \in \mathcal{T}_h^i,\;\; v^i\big|_{\partial\Omega_i \cap \partial\Omega} = 0\}, \tag{10}$$

where $\mathbb{P}_k(T)$ denotes the set of all polynomials on T with degree $\leq k$. The finite element space V_h of functions v_h with components v_h^i on Ω_i is given by

$$V_h \;:=\; V_h^1 \times V_h^2 \;=\; \left\{v_h = \left(v_h^1, v_h^2\right) : v_h^1 \in V_h^1, v_h^2 \in V_h^2\right\}. \tag{11}$$

In general, $v_h \in V_h$ is not continuous across Γ. For the approximation of (6) on V_h let us fix a positive constant γ (to be specified subsequently) and real parameters α_1, α_2 from

(8), and introduce the bilinear form $\mathcal{B}_h(.\,,.)$ on $V_h \times V_h$ and the linear form $\mathcal{F}_h(.)$ on V_h as follows:

$$\begin{aligned}\mathcal{B}_h(u_h, v_h) \;:=\; & \sum_{i=1}^{2} \Big(\varepsilon^2 (\nabla u_h^i, \nabla v_h^i)_{\Omega_i} + \big(c u_h^i, v_h^i\big)_{\Omega_i}\Big) - \Big\langle \alpha_1 \varepsilon^2 \frac{\partial u_h^1}{\partial n_1} - \alpha_2 \varepsilon^2 \frac{\partial u_h^2}{\partial n_2}, v_h^1 - v_h^2 \Big\rangle_\Gamma \\ & - \Big\langle \alpha_1 \varepsilon^2 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \varepsilon^2 \frac{\partial v_h^2}{\partial n_2}, u_h^1 - u_h^2 \Big\rangle_\Gamma + \varepsilon^2 \gamma \sum_{E \in \mathcal{E}_h} h_E^{-1} \left(u_h^1 - u_h^2, v_h^1 - v_h^2\right)_E , \\ \mathcal{F}_h(v_h) \;:=\; & \sum_{i=1}^{2} \left(f, v_h^i\right)_{\Omega_i} . \end{aligned} \tag{12}$$

Here, $(.\,,.)_\Lambda$ denotes the scalar product in $L_2(\Lambda)$ for $\Lambda \in \{\Omega_i, E\}$ and $\langle .\,,. \rangle_\Gamma$ is taken from (6). The weights in the fourth term of $\mathcal{B}_h$ are introduced in correspondence to $\sigma = \gamma \varepsilon^2 h^{-1}(x)$ at (6) and ensure the stability of method, if γ is a sufficiently large positive constant (cf. Theorem 2.11 below).

According to (6), but with the discrete forms $\mathcal{B}_h$ and $\mathcal{F}_h$ from (12), the Nitsche mortar finite element approximation u_h of the solution u of equation (3), with respect to the space V_h, is defined by $u_h = (u_h^1, u_h^2) \in V_h^1 \times V_h^2$ being the solution of

$$\mathcal{B}_h(u_h, v_h) \;=\; \mathcal{F}_h(v_h) \quad \forall\, v_h \in V_h. \tag{13}$$

In the following, we derive important properties of the discretization (13). First we show that the solution u of (1) is consistent with it.

Theorem 2.4. *Let u be the weak solution of* (1)*. Then $u = (u^1, u^2)$ satisfies*

$$\mathcal{B}_h(u, v_h) \;=\; \mathcal{F}_h(v_h) \quad \forall\, v_h \in V_h\,. \tag{14}$$

Proof. The functional $\mathcal{B}_h(u, v_h)$ taken from (12) by writing u instead of u_h is well defined for the weak solution u of (1) and any $v_h \in V_h$. Since $u^1|_\Gamma - u^2|_\Gamma = 0$ in $H_*^{\frac{1}{2}}(\Gamma)$, the third and fourth term of $\mathcal{B}_h(u, v_h)$ vanish. Using $\alpha_1 + \alpha_2 = 1$ and $\frac{\partial u^1}{\partial n_1}\Big|_\Gamma + \frac{\partial u^2}{\partial n_2}\Big|_\Gamma = 0$ in $H_*^{-\frac{1}{2}}(\Gamma)$, we finally obtain relation (5) with V_h instead of V. Because of $\Delta u^i \in L_2(\Omega_i)$, $u^i \in H^1(\Omega_i)$ and $u^i|_{\partial\Omega \cap \partial\Omega_i} = 0$, Green's first formula yields $(-\Delta u^i, v_h^i)_{\Omega_i} = (\nabla u^i, \nabla v_h^i)_{\Omega_i} - \Big\langle \frac{\partial u^i}{\partial n_i}, v_h^i \Big\rangle_{\partial\Omega_i}$ (cf. [9]) and, therefore, also (14). □

Remark 2.5. Note that owing to (13) and (14) we get the $\mathcal{B}_h$-orthogonality of the error $u - u_h$ on V_h, i.e.

$$\mathcal{B}_h(u - u_h, v_h) \;=\; 0 \quad \forall\, v_h \in V_h. \tag{15}$$

For further results on stability and convergence of the method, we need bounds of the term $\sum_{E \in \mathcal{E}_h} h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2$ for $v_h^i \in V_h^i$ in terms of the square of L_2-norms of ∇v_h^i on triangles touching Γ. This will be done by the following lemmas. At first we consider some type of inverse inequality on triangles T which are, in general, not shape regular.

Lemma 2.6. *Let T be a triangle of $\mathcal{T}_h$, F a side of $T = T_F$ and $h_F^\perp$ the height of T over F. Then, for any $v_h \in \mathbb{P}_k(T)$ and $T \in \mathcal{T}_h$ the following inequality holds,*

$$\|v_h\|_{0,F}^2 \;\le\; c_{SI} \frac{1}{h_F^\perp} \|v_h\|_{0,T}^2 \,. \tag{16}$$

Here, c_{SI} is a positive constant independent of T, h and ε.

Proof. Let $\hat{T}$ be the reference triangle with nodes $(0,0)$, $(1,0)$ and $(0,1)$, with $\operatorname{meas}\hat{T} = \frac{1}{2}$, $x = B\hat{x} + b$ the affine-linear mapping $\hat{T} \to T$ such that $\operatorname{meas}\hat{F} = 1$. Then $|\det B| = h_F h_F^\perp$. For $\hat{v}_h(\hat{x}) = v_h(x)$, $x \in T$, $\hat{v}_h \in \mathbb{P}_k(\hat{T})$, obviously the inequalities $\|\hat{v}_h\|_{0,\hat{F}} \;\le\; c_S\Big(\|\hat{v}_h\|_{0,\hat{T}} + \big\|\hat{\nabla}\hat{v}_h\big\|_{0,\hat{T}} \Big)$ and $\big\|\hat{\nabla}\hat{v}_h\big\|_{0,\hat{T}} \;\le\; c_I \|\hat{v}_h\|_{0,\hat{T}}$ hold. Combining these inequalities and applying the inverse transformation, we get (16). Clearly, c_{SI} is independent of T, h and ε. □

Remark 2.7. If v_h is piecewise constant on the triangles T, inequality (16) holds and c_{SI} can be easily specified. Starting from $\|\hat{v}_h\|_{0,\hat{F}}^2 \;\le\; c_{SI} \|\hat{v}_h\|_{0,\hat{T}}^2$ with $\hat{v}_h := a_0$ we get $a_0^2 \operatorname{meas}\hat{F} \le c_{SI} a_0^2 \operatorname{meas}\hat{T}$ and, hence, the choice $c_{SI} = 2$ is possible.

We often need inequalities with broken L_2-norms of $v_h \in L_2(\Gamma)$, where weights $h_E^{\pm\frac{1}{2}}$ and $h_F^{\pm\frac{1}{2}}$ are applied. In particular, under the assumption $C_1 h_F \le h_E$ and $h_E \le C_2 h_F$ for all $E \in \mathcal{E}_h$ and $F \in \mathcal{E}_h^i$ with $\mathring{E} \cap \mathring{F} \neq \emptyset$, for $v_h^i \in V_h^i$ $(i = 1,2)$, the inequalities

$$\sum_{E\in\mathcal{E}_h} h_E^{-1} \|v_h^i\|_{0,E}^2 \;\le\; C_1^{-1} \sum_{F\in\mathcal{E}_h^i} h_F^{-1} \|v_h^i\|_{0,F}^2 \,, \quad \sum_{E\in\mathcal{E}_h} h_E \|v_h^i\|_{0,E}^2 \;\le\; C_2 \sum_{F\in\mathcal{E}_h^i} h_F \|v_h^i\|_{0,F}^2 \tag{17}$$

hold. For the proof, we consider the set $\mathcal{E}_\cap^i := \{D : D := \mathring{E} \cap \mathring{F} \neq \emptyset, \text{ with } E \in \mathcal{E}_h, F \in \mathcal{E}_h^i\}$, rearrange the sums at (17) using the identity $\bigcup_{E\in\mathcal{E}_h} E = \bigcup_{F\in\mathcal{E}_h^i} F = \bigcup_{D\in\mathcal{E}_\cap^i} \overline{D} = \Gamma$, and apply the inequalities $h_E^{-1} \le C_1^{-1} h_F^{-1}$ as well as $h_E \le C_2 h_F$.

Lemma 2.8. *Under the assumption $h_E \le C_2 h_F$ for all $E \in \mathcal{E}_h$ and $F \in \mathcal{E}_h^i$ with $\mathring{E} \cap \mathring{F} \neq \emptyset$ and for $v_h^i \in V_h^i$ $(i = 1,2)$, the following estimate holds, with c_{SI} from (16):*

$$\sum_{E\in\mathcal{E}_h} h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 \;\le\; C_2 c_{SI} \sum_{F\in\mathcal{E}_h^i} \frac{h_F}{h_F^\perp} \|\nabla v_h^i\|_{0,T_F}^2 \,. \tag{18}$$

Proof. Relation $v_h^i \in V_h^i$ implies $\frac{\partial v_h^i}{\partial x_s}\Big|_\Gamma \in L_2(\Gamma)$, $(s = 1,2)$. Clearly, $\left\|\frac{\partial v_h^i}{\partial n_i}\right\|_{0,E} \le \|\nabla v_h^i\|_{0,E}$ holds for any $E \in \mathcal{E}_h$, too. By summation and application of (17) as well as of (16), we get

$$\sum_{E\in\mathcal{E}_h} h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 \;\le\; C_2 \sum_{F\in\mathcal{E}_h^i} h_F \|\nabla v_h^i\|_{0,F}^2 \;\le\; C_2 c_{SI} \sum_{F\in\mathcal{E}_h^i} \frac{h_F}{h_F^\perp} \|\nabla v_h^i\|_{0,T_F}^2 \,.$$

□

Theorem 2.9. *Let Assumption 2.2 and Assumption 2.3 be satisfied. Then, the inequalities*

$$\sum_{E\in\mathcal{E}_h} h_E \left\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 \le \sum_{i=1}^{2} \sum_{F\in\mathcal{E}_h^i} C_I^i \left\| \nabla v_h^i \right\|_{0,T_F}^2 \le C_I \sum_{i=1}^{2} \left\| \nabla v_h^i \right\|_{0,\Omega_i}^2, \tag{19}$$

hold, where C_I^i and C_I are independent of h ($h \le h_0$) and of ε ($\varepsilon < 1$). They are given by $C_I^i = c_\alpha^i C_2 C_3 c_{SI}$ ($i = 1,2$) and $C_I = \max\{C_I^1, C_I^2\}$, where C_2, C_3 and c_{SI} are taken from Assumption 2.2, Assumption 2.3, and Lemma 2.6, resp., and c_α^i is defined by

$$c_\alpha^i = \begin{cases} 0 & \text{for } \alpha_i = 0 \\ 1 & \text{for } \alpha_i = 1 \\ 2\alpha_i^2 & \text{for } 0 < \alpha_i < 1. \end{cases}$$

Proof. Consider $\alpha_i \in (0,1)$ ($i = 1,2$). Clearly, inequality

$$\sum_{E\in\mathcal{E}_h} h_E \left\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 \le \sum_{i=1}^{2} 2\alpha_i^2 \sum_{E\in\mathcal{E}_h} h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2$$

holds. Applying Lemma 2.8 as well as some simple estimates, we also get (19). For $\alpha_i = 1$ ($\alpha_{3-i} = 0$) or $\alpha_{3-i} = 1$ ($\alpha_i = 0$), the proofs are quite analogous. □

Remark 2.10. For example, consider the estimation of the constant C_I on triangles for $\mathcal{E}_h = \mathcal{E}_h^1$ according to (7), $\alpha_1 = 1$ and $k = 1$, i.e., $v_h^1|_T \in \mathbb{P}_1(T)$. Then, we have $C_I = C_I^1 = 2\sup_{h,\varepsilon} \max_{F\in\mathcal{E}_h^1} \left(\frac{h_F}{h_F^\perp}\right)$, since $c_\alpha^1 = 1$, $C_2 = 1$, $C_3 = \sup_{h,\varepsilon} \max_{F\in\mathcal{E}_h^1} \left(\frac{h_F}{h_F^\perp}\right)$, and $c_{SI} = 2$ (see Remark 2.7). For $i = 2$ ($\mathcal{E}_h = \mathcal{E}_h^2$, $\alpha_2 = 1$) instead of $i = 1$, we get an analogous result.

For deriving stability of the discrete bilinear form $\mathcal{B}_h(.\,,.)$, for $v_h \in V_h$ we introduce a discrete energy-like norm $\|\,.\,\|_{1,h}$ depending on ε^2, $c(x)$ and on the mesh, viz.

$$\|v_h\|_{1,h}^2 = \sum_{i=1}^{2} \left(\varepsilon^2 \left\| \nabla v_h^i \right\|_{0,\Omega_i}^2 + \left\| \sqrt{c} v_h^i \right\|_{0,\Omega_i}^2 \right) + \varepsilon^2 \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\| v_h^1 - v_h^2 \right\|_{0,E}^2. \tag{20}$$

Theorem 2.11. *Let Assumption 2.2 and Assumption 2.3 be satisfied. If the constant γ in* (12) *is chosen (independently of h and ε) such that $\gamma > C_I$ is valid, C_I from Theorem 2.9, then*

$$\mathcal{B}_h(v_h, v_h) \ge \mu_1 \|v_h\|_{1,h}^2 \qquad \forall\, v_h \in V_h \tag{21}$$

holds, with a positive constant μ_1 independent of h ($h \le h_0$) and ε ($\varepsilon < 1$).

Proof. Take $\mathcal{B}_h(.\,,.)$ from (12), put $v_h = u_h$, choose a real $\zeta > 0$ and get, using Cauchy's and Young's inequality, the estimate

$$\begin{aligned} \mathcal{B}_h(v_h, v_h) \ge{}& \sum_{i=1}^{2} \left(\varepsilon^2 \left\| \nabla v_h^i \right\|_{0,\Omega_i}^2 + \left\| \sqrt{c} \nabla v_h^i \right\|_{0,\Omega_i}^2 \right) + \varepsilon^2 \gamma \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\| v_h^1 - v_h^2 \right\|_{0,E}^2 \\ &+ \sum_{E\in\mathcal{E}_h} \left(-\frac{h_E}{\zeta\varepsilon^2} \left\| \alpha_1 \varepsilon^2 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \varepsilon^2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 - \frac{\zeta\varepsilon^2}{h_E} \left\| v_h^1 - v_h^2 \right\|_{0,E}^2 \right). \end{aligned}$$

Fix constants γ and ζ such that $\gamma > \zeta > C_I$, apply inequalities (19) and get obviously

$$\begin{aligned}\mathcal{B}_h(v_h, v_h) \;\geq\; & \sum_{i=1}^{2}\left(\left(1-\frac{C_I^i}{\zeta}\right)\varepsilon^2\left\|\nabla v_h^i\right\|_{0,\Omega_i}^2+\left\|\sqrt{c}\nabla v_h^i\right\|_{0,\Omega_i}^2\right)\\ &+(\gamma-\zeta)\varepsilon^2\sum_{E\in\mathcal{E}_h}h_E^{-1}\left\|v_h^1-v_h^2\right\|_{0,E}^2 \;\geq\; \mu_1\left\|v_h\right\|_{1,h}^2,\end{aligned}$$

with some constant $\mu_1 = \min\left\{\left(1-\frac{C_I}{\zeta}\right), (\gamma-\zeta)\right\} > 0$ being independent of h and ε. □

Remark 2.12. Under the Assumption 2.2 and Assumption 2.3, $\mathcal{B}_h(.\,,.)$ is continuous on $V_h \times V_h$ in the sense of

$$|\mathcal{B}_h(u_h, v_h)| \;\leq\; \mu_2\left\|u_h\right\|_{1,h}\left\|v_h\right\|_{1,h} \qquad \forall\, u_h, v_h \in V_h, \tag{22}$$

with some constant μ_2 independent of h and ε. For the proof we apply Cauchy's inequality as well as inequalities (19).

3 Error estimates

Let u be the solution of (1) and u_h its finite element approximation defined by (13). We shall estimate the error $u - u_h$ in the norm $\|\,.\,\|_{1,h}$ from (20). For $\varepsilon = 1$, $c \equiv 0$ (Poisson equation) and regular solutions u such estimates can be found in [4, 5, 14], for problems with singularities in [11]. For transmission problems with jumping coefficients p_1 and p_2 assigned to Ω_1 and Ω_2, respectively, as well as for low regularity of u due to interface corners, see [10].

For functions v satisfying $v^i \in H^1(\Omega_i)$ and $\frac{\partial v^i}{\partial n_i} \in L_2(\Gamma)$ $(i = 1, 2)$, introduce a second discrete norm $\|\,.\,\|_{h,\Omega}$ being related with the bilinear form $\mathcal{B}_h(.\,,.)$, viz.

$$\begin{aligned}\|v\|_{h,\Omega}^2 \;=\; & \sum_{i=1}^{2}\left(\varepsilon^2\left\|\nabla v^i\right\|_{0,\Omega_i}^2+\left\|\sqrt{c}v^i\right\|_{0,\Omega_i}^2+\varepsilon^2\sum_{E\in\mathcal{E}_h}h_E\left\|\alpha_i\frac{\partial v^i}{\partial n_i}\right\|_{0,E}^2\right)\\ &+\varepsilon^2\sum_{E\in\mathcal{E}_h}h_E^{-1}\left\|v^1-v^2\right\|_{0,E}^2.\end{aligned} \tag{23}$$

We now prove a lemma of Céa's type bounding $\|u-u_h\|_{1,h}$ and $\|u-u_h\|_{h,\Omega}$ by the norm $\|\,.\,\|_{h,\Omega}$ of the interpolation error $u - I_h u$. Here, $I_h u := (I_h u^1, I_h u^2)$ denotes the generalized Lagrange interpolant of u in the sense that $I_h u^i \in V_h^i$ is the standard Lagrange interpolant of u^i in the space V_h^i, $i = 1, 2$. The proof uses a mixed continuity property of $\mathcal{B}_h(.\,,.)$ instead of the full continuity given at (22).

Lemma 3.1. *Let $u \in H^{\frac{3}{2}+\delta}(\Omega)$ $(\delta > 0)$ be the solution of* (1), *$u_h \in V_h$ its approximation defined by* (13). *Further, let Assumption 2.2, Assumption 2.3 and $\gamma > C_I$ be satisfied. Then, the estimate*

$$\|u - u_h\|_{1,h} \;\leq\; \mu_3 \, \|u - I_h u\|_{h,\Omega} \tag{24}$$

holds, with $\mu_3 > 0$ independent of $h \in (0, h_0]$ and $\varepsilon \in (0, 1)$.

Proof. Obviously, $I_h u \in V_h$ holds, and the triangle inequality yields

$$\|u - u_h\|_{1,h} \;\leq\; \|u - I_h u\|_{1,h} + \|u_h - I_h u\|_{1,h} \,. \tag{25}$$

Owing to $u_h - I_h u \in V_h$, to the V_h-ellipticity of $\mathcal{B}_h(.\,,.)$ given by (21) and to the $\mathcal{B}_h$-orthogonality relation (15), we get

$$\|u_h - I_h u\|_{1,h}^2 \;\leq\; \mu_1^{-1}\, \mathcal{B}_h(u - I_h u, u_h - I_h u). \tag{26}$$

Using the triangle inequality, Cauchy's inequality and Theorem 2.9, we estimate the term on the right-hand side of (26) by

$$|\mathcal{B}_h\,(u - I_h u, u_h - I_h u)| \;\leq\; \mu_4 \, \|u - I_h u\|_{h,\Omega} \, \|u_h - I_h u\|_{1,h} \,, \tag{27}$$

where $\mu_4 > 0$ depends on γ, but it is independent of h and ε. The combination of (26) and (27) yields

$$\|u_h - I_h u\|_{1,h} \;\leq\; \frac{\mu_4}{\mu_1} \, \|u - I_h u\|_{h,\Omega} \,,$$

which implies, together with (25), relation (24), where $\mu_3 = 1 + \frac{\mu_4}{\mu_1}$. □

In correspondence to the anisotropic behaviour of the solution u in the boundary layers, we shall apply anisotropic triangular meshes like treated e.g. in [1]. Introduce vectors $\underline{h}_{T,i}$ with length $h_{T,i} := \left|\underline{h}_{T,i}\right|$ $(i = 1, 2)$ as follows:

$$\begin{array}{ll} \underline{h}_{T,1} : & \text{vector of the longest side of } T, \\ \underline{h}_{T,2} : & \text{vector of the height of } T \text{ over the longest side of } T. \end{array} \tag{28}$$

Apply the multiindex $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$, with $|\beta| = \beta_1 + \beta_2$, $\beta_i \geq 0$ $(i = 1, 2)$, and write shortly

$$D^\beta \;:=\; \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}}, \qquad h_T^\beta \;:=\; h_{T,1}^{\beta_1} h_{T,2}^{\beta_2}. \tag{29}$$

For the estimation of the interpolation error on anisotropic triangles we need the so-called maximal angle condition and the coordinate system condition (according to [1]) given subsequently, cf. also Fig. 3. Employ various constants independent of T, $h \in (0, h_0]$ and $\varepsilon \in (0, 1)$, e.g. C as a generic constant.

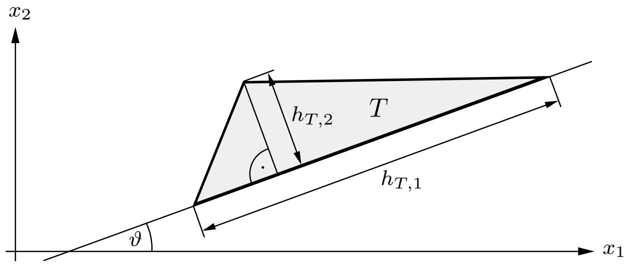

Figure 3: Anisotropic triangle T

Assumption 3.2. *The interior angles θ of any triangle $T \in \mathcal{T}_h$ satisfy the condition*

$$0 \; < \; \theta \; \leq \; \pi - \theta_0$$

where the constant $\theta_0 > 0$ is independent of T, $h \in (0, h_0]$ and $\varepsilon \in (0,1)$.

Assumption 3.3. *The position of the triangle T with respect to the x_1-x_2-coordinate system is such that the angle ϑ between $\underline{h}_{T,1}$ and the x_1-axis is bounded as given by*

$$|\sin \vartheta| \; \leq \; C_4 \frac{h_{T,1}}{h_{T,2}},$$

where C_4 is independent of T, $h \in (0, h_0]$ and $\varepsilon \in (0,1)$.

Lemma 3.4. *Let Assumption 3.3 be satisfied and let T be an arbitrary triangle, with F as a face of it. Then, for $v \in H^1(T)$ the trace inequality*

$$\|v\|_{0,F}^2 \; \leq \; C \left(h_F^{\perp}\right)^{-1} \left(\|v\|_{0,T}^2 + \|v\|_{0,T} \sum_{|\beta|=1} h_T^{\beta} \left\|D^{\beta} v\right\|_{0,T} \right) \tag{30}$$

holds, where $h_F^{\perp}$ denotes the length of the height of T over the side F.

Proof. Define the matrix $H = \left(\underline{h}_{T,1}, \underline{h}_{T,2}\right)$, with $\underline{h}_{T,i}$ $(i = 1, 2)$ from (28), and get like proved in [12] the inequality

$$\|v\|_{0,F}^2 \; \leq \; C \, \frac{1}{h_F^{\perp}} \left(\|v\|_{0,T}^2 + \|v\|_{0,T} \left\|H^T \nabla v\right\|_{0,T} \right).$$

Assumption 3.3 and some simple estimates yield $\left\|H^T \nabla v\right\|_{0,T} \leq C \sum_{|\beta|=1} h_T^{\beta} \left\|D^{\beta} v\right\|_{0,T}$ and, therefore, (30). □

An estimate of the interpolation error on anisotropic triangles is given by the following theorem.

Theorem 3.5. *Let the triangle $T \subset \Omega_i$ $(i = 1, 2)$ satisfy Assumption 3.2 and Assumption 3.3. Then, for the Lagrange interpolate $I_h v^i \in V_h^i$ $(i = 1, 2)$ of $v^i \in H^l(\Omega_i)$ $(l \geq 2)$ the following error estimate holds:*

$$\left|v^i - I_h v^i\right|_{m,T} \leq C \sum_{|\beta|=l-m} h_T^\beta \left|D^\beta v^i\right|_{m,T} \tag{31}$$

for $i = 1, 2$, where $2 \leq l \leq k + 1$ and $m = 0, \ldots, l - 1$ $(l, m \in \mathbb{N})$.

The proof is given by the proof of Theorem 2.1 in [1].

We now combine Lemma 3.1 with Theorem 3.5, in particular estimate (24) with (31). First, using the abbreviation $w := u - I_h u$, we get

$$\begin{aligned} \|u - u_h\|_{1,h} &\leq \mu_3 \|u - I_h u\|_{h,\Omega} \\ &= \mu_3 \left\{ \sum_{i=1}^{2} \left(\varepsilon^2 \left\|\nabla w^i\right\|_{0,\Omega_i}^2 + \left\|\sqrt{c} w^i\right\|_{0,\Omega_i}^2 + \varepsilon^2 \sum_{E \in \mathcal{E}_h} h_E \left\| \alpha_i \frac{\partial w^i}{\partial n_i} \right\|_{0,E}^2 \right) \right. \\ &\quad \left. + \varepsilon^2 \sum_{E \in \mathcal{E}_h} h_E^{-1} \left\|w^1 - w^2\right\|_{0,E}^2 \right\}. \end{aligned} \tag{32}$$

Since in this paper error estimates will be derived under the smoothness assumption $u \in H^2(\Omega)$, it suffices to formulate the following lemmas and theorems for $u^i \in H^2(\Omega_i)$, $i = 1, 2$. Moreover, in the remaining part of the paper we shall consider piecewise linear finite elements, i.e. $k = 1$ in V_h. Owing to Theorem 3.5, the estimation of the interpolation error $w^i := u^i - I_h u^i$ on anisotropic triangles $T \in \mathcal{T}_h^i$ $(i = 1, 2)$ located in Ω_i is obvious. Thus, for the terms $\|\nabla w^i\|_{0,\Omega_i}$ and $\|\sqrt{c} w^i\|_{0,\Omega_i}$ in (32) we get the following result where the proof is given by the application of (31).

Lemma 3.6. *Let $u^i \in H^2(\Omega_i)$ $(i = 1, 2)$ and the Assumption 3.2 as well as Assumption 3.3 be satisfied. Then, for $w^i := u^i - I_h u^i$ the following estimate holds,*

$$\varepsilon^2 \left\|\nabla w^i\right\|_{0,\Omega_i}^2 + \left\|\sqrt{c} w^i\right\|_{0,\Omega_i}^2 \leq C \sum_{T \in \mathcal{T}_h^i} \sum_{|\alpha|=1} \sum_{|\beta|=1} \left(\varepsilon^2 h_T^{2\beta} + \|c\|_{L_\infty(T)} h_T^{2(\alpha+\beta)} \right) \left\|D^{\alpha+\beta} u^i\right\|_{0,T}^2 . \tag{33}$$

In the following, we shall estimate the weighted L_2-norms of $\frac{\partial w^i}{\partial n_i}$ $(i = 1, 2)$ and $w^1 - w^2$ defined on the interface Γ, cf. the right-hand side of (32).

Lemma 3.7. *Let $u^i \in H^2(\Omega_i)$ $(i = 1, 2)$ as well as Assumptions 2.2, 2.3, 3.2 and 3.3 be satisfied. Then, the following interpolation error estimate holds,*

$$\begin{aligned} &\sum_{i=1}^{2} \sum_{E \in \mathcal{E}_h} h_E \left\| \alpha_i \frac{\partial \left(u^i - I_h u^i\right)}{\partial n_i} \right\|_{0,E}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \left\| \left(u^1 - I_h u^1\right) - \left(u^2 - I_h u^2\right) \right\|_{0,E}^2 \\ &\leq C \sum_{i=1}^{2} \left(\sum_{F \in \mathcal{E}_h^i} \sum_{|\beta|=1} \sum_{|\delta|=1} \left(\alpha_i^2 C_2 \frac{h_F}{h_F^\perp} h_{T_F}^{2\beta} + 2 C_1^{-1} \frac{1}{h_F h_F^\perp} h_{T_F}^{2(\beta+\delta)} \right) \left\|D^{\beta+\delta} u^i\right\|_{0,T_F}^2 \right), \end{aligned} \tag{34}$$

with C_1, C_2 from Assumption 2.2.

Proof. Taking $w^i := u^i - I_h u^i$ and using Assumption 2.2 we obtain by analogy to (17)

$$\begin{aligned}\sum_{E\in\mathcal{E}_h} h_E \left\|\alpha_i \frac{\partial w^i}{\partial n_i}\right\|_{0,E}^2 &\leq C_2 \alpha_i^2 \sum_{F\in\mathcal{E}_h^i} h_F \left\|\nabla w^i\right\|_{0,F}^2 , \qquad i=1,2,\\ \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\|w^1 - w^2\right\|_{0,E}^2 &\leq 2C_1^{-1} \sum_{i=1}^{2} \sum_{F\in\mathcal{E}_h^i} h_F^{-1} \left\|w^i\right\|_{0,F}^2 . \end{aligned} \tag{35}$$

In order to estimate $\|\nabla w^i\|_{0,F}$ and $\|w^i\|_{0,F}$ by interpolation error norms on the corresponding triangle $T = T_F$, we apply Lemma 3.4 (trace theorem) for $v = \frac{\partial w^i}{\partial x_j}$ as well as for $v = w^i$ and get

$$\begin{aligned}\left\|\nabla w^i\right\|_{0,F}^2 &\leq c \frac{1}{h_F^\perp} \left(\left|w^i\right|_{1,T}^2 + \left|w^i\right|_{1,T} \sum_{|\beta|=1} \sum_{|\delta|=1} h_T^\beta \left\|D^{\beta+\delta} w^i\right\|_{0,T} \right),\\ \left\|w^i\right\|_{0,F}^2 &\leq C \frac{1}{h_F^\perp} \left(\left\|w^i\right\|_{0,T}^2 + \left\|w^i\right\|_{0,T} \sum_{|\beta|=1} h_T^\beta \left\|D^\beta w^i\right\|_{0,T} \right), \end{aligned} \tag{36}$$

for $i = 1,2$. Using Theorem 3.5 and relation (31) for $i = 1,2$, we deduce the following estimates of the error terms of the right-hand side of (36):

$$\begin{aligned}\left\|u^i - I_h u^i\right\|_{0,T} &\leq C \sum_{|\beta|=2} h_T^\beta \left\|D^\beta u^i\right\|_{0,T} ,\\ \left|u^i - I_h u^i\right|_{1,T} &\leq C \sum_{|\beta|=1} \sum_{|\delta|=1} h_T^\beta \left\|D^{\beta+\delta} u^i\right\|_{0,T} ,\\ \sum_{|\beta|=1} h_T^\beta \left\|D^\beta \left(u^i - I_h u^i\right)\right\|_{0,T} &\leq C \sum_{|\beta|=2} h_T^\beta \left\|D^\beta u^i\right\|_{0,T} ,\\ \sum_{|\beta|=1} \sum_{|\delta|=1} h_T^\beta \left\|D^{\beta+\delta} \left(u^i - I_h u^i\right)\right\|_{0,T} &\leq C \sum_{|\beta|=1} \sum_{|\delta|=1} h_T^\beta \left\|D^{\beta+\delta} u^i\right\|_{0,T} . \end{aligned} \tag{37}$$

Inserting the estimates (37) in the right-hand side of (36), we are led to

$$\begin{aligned}\left\|\nabla \left(u^i - I_h u^i\right)\right\|_{0,F}^2 &\leq C \frac{1}{h_F^\perp} \sum_{|\beta|=1} \sum_{|\delta|=1} h_T^{2\beta} \left\|D^{\beta+\delta} u^i\right\|_{0,T}^2 ,\\ \left\|u^i - I_h u^i\right\|_{0,F}^2 &\leq C \frac{1}{h_F^\perp} \sum_{|\beta|=2} h_T^{2\beta} \left\|D^\beta u^i\right\|_{0,T}^2 , \quad i = 1,2. \end{aligned} \tag{38}$$

Finally, the combination of (35) with (38) yields estimate (34). □

In order to investigate the interpolation error $u^i - I_h u^i$ $(i = 1, 2)$ for a solution u with boundary layers and for anisotropic meshes, we consider for simplicity the BVP (1) on a rectangle, w.l.o.g. for $\Omega = (0,1)^2$. For describing the behaviour of the solution u and the mesh adapted to this solution, we split the domain as given by Fig. 4 into an interior part Ω^{i}, boundary layers $\Omega^{\mathrm{b},j}$ $(j = 1, 2, 3, 4)$ of width a and corner neighbourhoods $\Omega^{\mathrm{c},j}$ $(j = 1, 2, 3, 4)$. Employ also $\Omega^{\mathrm{b}} = \bigcup_{j=1}^{4} \Omega^{\mathrm{b},j}$ and $\Omega^{\mathrm{c}} = \bigcup_{j=1}^{4} \Omega^{\mathrm{c},j}$. It should be noted that depending on the data and according to the solution properties some of the boundary layer parts $\Omega^{\mathrm{b},j}$ may be empty (cf. numerical example); then Ω^{i} is extended to the corresponding part of the boundary. Take $\mathrm{dist}(x, \partial\Omega)$ and $\mathrm{dist}(x, \Sigma)$ for the distance of an interior point $x \in \Omega$ to the boundary $\partial\Omega$ and to corner vertices Σ, respectively. In $\Omega^{\mathrm{b},j}$ choose a local coordinate system (x_1, x_2), where x_1 goes with the tangent of the boundary $\partial\Omega$ and x_2 with its interior normal such that $x_2 = \mathrm{dist}(x, \partial\Omega)$ holds. The derivatives $D^\beta u$, $\beta = (\beta_1, \beta_2)$, in the boundary layers are taken with respect to these coordinates.

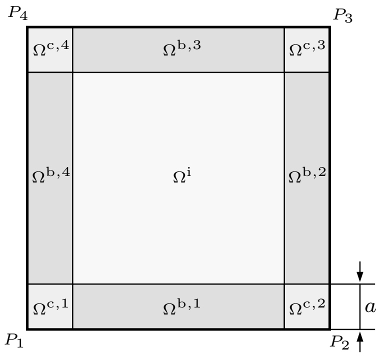

Figure 4: Subdomains of Ω : Ω^{i} and four parts $\Omega^{\mathrm{b},j}$ and $\Omega^{\mathrm{c},j}$

Lemma 3.8 ([2, Lemma 2]). *Let u be the solution of the BVP* (1) *with sufficiently smooth f and $c(x) = c_0 = const > 0$. Then, for $|\beta| = 2$ the following estimate holds:*

$$\left|D^\beta u(x)\right| \;\le\; C\left(1 + \varepsilon^{-\beta_2} e^{-\underline{c}_0 \frac{\mathrm{dist}(x,\partial\Omega)}{\varepsilon}} + \left(1 + \left|\ln \frac{\mathrm{dist}\,(x,\Sigma)}{\varepsilon}\right|\right) e^{-\underline{c}_0 \frac{\mathrm{dist}(x,\Sigma)}{\varepsilon}}\right), \qquad (39)$$

with $0 < \underline{c}_0 < c_0$.

According to (39), the L_2-norms of the second order derivatives $D^\beta u$ $(|\beta| = 2)$ considered in the subdomains Ω^{i}, Ω^{b} and Ω^{c} satisfy at least the estimates

$$\left\|D^\beta u\right\|_{0,\Omega^{\mathrm{i}}}^2 \;\le\; C\,, \qquad \left\|D^\beta u\right\|_{0,\Omega^{\mathrm{b}}}^2 \;\le\; C\,a\,\varepsilon^{-2\beta_2}\,, \qquad \left\|D^\beta u\right\|_{0,\Omega^{\mathrm{c}}}^2 \;\le\; C\,a^2 \varepsilon^{-2|\beta|}\,, \qquad (40)$$

with $a := \frac{a_0}{\underline{c}_0}\varepsilon\,|\ln\varepsilon|$ and $a_0 \ge 2$.

Introduce triangulations $\mathcal{T}_h(\Omega^{\mathrm{i}})$, $\mathcal{T}_h(\Omega^{\mathrm{c}})$ and $\mathcal{T}_h(\Omega^{\mathrm{b}})$ of the subdomains Ω^{i}, Ω^{b} and Ω^{c}, respectively. For each of these subdomains employ $\mathcal{O}(h^{-1}) \times \mathcal{O}(h^{-1})$ triangles T with mesh sizes $h_{T,1}$ and $h_{T,2}$ according to the asymptotics (41) of the subtriangulations given by

$$\begin{aligned} &h_{T,1} \sim h_{T,2} \sim h && \text{for } T \in \mathcal{T}_h(\Omega^{\mathrm{i}}), \\ &h_{T,1} \sim h, \quad h_{T,2} \sim ah && \text{for } T \in \mathcal{T}_h(\Omega^{\mathrm{b}}), \\ &h_{T,1} \sim h_{T,2} \sim ah && \text{for } T \in \mathcal{T}_h(\Omega^{\mathrm{c}}). \end{aligned} \tag{41}$$

Here, for brevity the symbol $\sim$ is used for equivalent mesh asymptotics (see e.g. [1]). In particular, assumption (41) means that we apply isotropic triangles in Ω^{i}, Ω^{c}, and anisotropic triangles in Ω^{b}. It is also supposed that Assumptions 2.1, 3.1 and 3.2 are satisfied. If non-matching meshes at the interface Γ are used, we additionally require Assumptions 2.2 and 2.3 to be valid. According to the domain decomposition of Ω we shall have subdomains $\Omega_i^{\mathrm{i}} = \Omega_i \cap \Omega^{\mathrm{i}}$, $\Omega_i^{\mathrm{b},j} = \Omega_i \cap \Omega^{\mathrm{b},j}$ and $\Omega_i^{\mathrm{c},j} = \Omega_i \cap \Omega^{\mathrm{c},j}$ for $i = 1, 2$. Of course, depending on the choice of Γ, some of them may be empty.

An important example satisfying all these mesh assumptions is given as follows. We decompose Ω into Ω_1, Ω_2 such that the interface Γ is formed by the interior boundary part of the boundary layer. Then, cover Ω_1 and Ω_2 by axiparallel rectangles. They are formed by putting $\mathcal{O}(h^{-1})$ points on the axiparallel edges of the subdomains defining axiparallel mesh lines. Finally, we obtain rectangular triangles by dividing the rectangles in the usual way, see e.g. Figure 6.

In the following, consider the interpolation error estimates (33) of Lemma 3.6 as well as (34) of Lemma 3.7 and shall bound the right-hand sides of (33) and (34) in presence of boundary layers of the solution u. We shall always tacitly suppose that Assumptions 2.1, 2.2, 2.3, 3.2 and 3.3 as well as (41) are satisfied.

Lemma 3.9. *Under the assumptions $a = \frac{a_0}{\underline{c}_0}\varepsilon\,|\ln\varepsilon|$ with $a_0 \geq 2$, $c(x) = c_0 = const > 0$, $0 < \underline{c}_0 < c_0$, and with the bounds of the L_2-norms of the solution at* (40)*, the following estimate holds for $i = 1, 2$:*

$$\mathbf{I}_{\mathcal{T}_h^i} := \sum_{T\in\mathcal{T}_h^i} \sum_{|\alpha|=1} \sum_{|\beta|=1} \left(\varepsilon^2 h_T^{2\beta} + \|c\|_{L_\infty(T)} h_T^{2(\alpha+\beta)} \right) \left\| D^{\alpha+\beta} u^i \right\|_{0,T}^2 \;\leq\; C\left(\varepsilon\,|\ln\varepsilon|^3\, h^2 + h^4\right). \tag{42}$$

Proof. For brevity we employ the symbol $\mathbf{I}_{\mathcal{T}_h^i}$ denoting the sum given by (42) (cf. (33)) and define by analogy the corresponding parts $\mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{i}})}$, $\mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{b}})}$ and $\mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{c}})}$ of the full sum $\mathbf{I}_{\mathcal{T}_h^i}$ assigned to the subsets of triangulations $\mathcal{T}_h(\Omega_i^{\mathrm{i}})$, $\mathcal{T}_h(\Omega_i^{\mathrm{b}})$ and $\mathcal{T}_h(\Omega_i^{\mathrm{c}})$, respectively.

(i) $T \in \mathcal{T}_h(\Omega_i^{\mathrm{i}})$: Here we have isotropic triangles with $h_{T,1} \sim h_{T,2} \sim h_T \sim h$. This yields, together with $\left\| D^\beta u^i \right\|_{0,\Omega_i^{\mathrm{i}}}^2 \leq C$ ($|\beta| = 2$), the estimate

$$\mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{i}})} \;\leq\; C \sum_{|\alpha|=1} \sum_{|\beta|=1} \left(\varepsilon^2 h^{2\beta} + h^{2(\alpha+\beta)}\right) \;\leq\; C\left(\varepsilon^2 h^2 + h^4\right).$$

(ii) $T \in \mathcal{T}_h(\Omega_i^{\mathrm{b}})$: In Ω^{b} anisotropic triangles T with $h_{T,1} \sim h$ and $h_{T,2} \sim ah$ are employed. Since $\left\|D^\beta u^i\right\|^2_{0,\Omega_i^{\mathrm{b}}} \le C\, a\, \varepsilon^{-2\beta_2}$ $(|\beta| = 2)$ holds, where $a = \frac{a_0}{\underline{c}_0}\varepsilon\,|\ln\varepsilon|$, we easily derive the estimates

$$\begin{aligned}\mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{b}})} &\le C \sum_{T\in\mathcal{T}_h(\Omega_i^{\mathrm{b}})} \sum_{|\alpha|=1} \sum_{|\beta|=1} \left(\varepsilon^2 h_{T,1}^{2\beta_1} h_{T,2}^{2\beta_2} + h_{T,1}^{2(\alpha_1+\beta_1)} h_{T,2}^{2(\alpha_2+\beta_2)}\right) \left\|D^{\alpha+\beta} u^i\right\|^2_{0,T} \\ &\le C\left(\varepsilon\,|\ln\varepsilon|^3\, h^2 + \varepsilon\,|\ln\varepsilon|^5\, h^4\right).\end{aligned}$$

(iii) $T \in \mathcal{T}_h(\Omega_i^{\mathrm{c}})$: In the subset Ω^{c} we apply isotropic triangles with $h_T \sim ah$ and the relation $\left\|D^\beta u^i\right\|^2_{0,\Omega_i^{\mathrm{c}}} \le C a^2 \varepsilon^{-2|\beta|}$, $|\beta| = 2$. This yields

$$\mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{c}})} \le C\left(\varepsilon^2\,(ah)^2 + (ah)^4\right) a^2\varepsilon^{-4} \le C\left(\varepsilon^2\,|\ln\varepsilon|^4\, h^2 + \varepsilon^2\,|\ln\varepsilon|^6\, h^4\right).$$

Taking the sum over all bounds we get the estimate

$$\mathbf{I}_{\mathcal{T}_h^i} = \mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{i}})} + \mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{b}})} + \mathbf{I}_{\mathcal{T}_h(\Omega_i^{\mathrm{c}})} \le C\left(\varepsilon\,|\ln\varepsilon|^3\, h^2 + h^4\right),$$

which verifies the assertion of Lemma 3.9. □

Lemma 3.10. *Under the assumptions $a = \frac{a_0}{\underline{c}_0}\varepsilon\,|\ln\varepsilon|$, $a_0 \ge 2$, $0 < \underline{c}_0 < c_0$, and with the bounds of the L_2-norms of the solution at* (40)*, the following estimate holds for $i = 1, 2$:*

$$\mathbf{II}_{\mathcal{E}_h^i} := \sum_{F\in\mathcal{E}_h^i} \sum_{|\alpha|=1} \sum_{|\beta|=1} \left(\varepsilon^2\alpha_i^2 C_2 \frac{h_F h_T^{2\beta}}{h_F^{\perp}} + 2\varepsilon^2 C_1^{-1} \frac{h_T^{2(\alpha+\beta)}}{h_F h_F^{\perp}}\right) \left\|D^{\alpha+\beta} u^i\right\|^2_{0,T} \le C\varepsilon^2\,|\ln\varepsilon|^4\, h^2. \tag{43}$$

Proof. For abbreviation, we use the symbol $\mathbf{II}_{\mathcal{E}_h^i}$ denoting the sum defined by (43) (cf. (34)). The summation index $F \in \mathcal{E}_h^i$ indicates the triangle $T := T_F \in \mathcal{T}_h^i$ to be taken in the sum of L_2-norm squares $\left\|D^{\alpha+\beta} u\right\|^2_{0,T}$, $|\alpha| = |\beta| = 1$. According to the position of the interface Γ, triangles T belonging to $\mathcal{T}_h(\Omega_i^{\mathrm{i}})$, $\mathcal{T}_h(\Omega_i^{\mathrm{b}})$ and $\mathcal{T}_h(\Omega_i^{\mathrm{c}})$ may occur and again apply the symbols $\mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{i}})}$, $\mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{b}})}$ and $\mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{c}})}$ for the corresponding contributions, like in the proof given subsequently.

(i) $T \in \mathcal{T}_h(\Omega_i^{\mathrm{i}})$: Since T is isotropic with $h_T \sim h$, we have $h_F \sim h$ as well as $h_F^{\perp} \sim h$. Using $\left\|D^\beta u^i\right\|^2_{0,\Omega_i^{\mathrm{i}}} \le C$ $(|\beta| = 2)$ we have

$$\mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{i}})} \le C\left(\alpha_i^2 C_2\varepsilon^2 h^2 + 2C_1^{-1}\varepsilon^2 h^{-2} h^4\right) \sum_{|\beta|=2} \sum_{T\in\mathcal{T}_h(\Omega_i^{\mathrm{i}})} \left\|D^\beta u^i\right\|^2_{0,T} \le C\varepsilon^2 h^2 \quad (i = 1, 2).$$

(ii) $T \in \mathcal{T}_h(\Omega_i^{\mathrm{b}})$: In the boundary layer strips Ω_i^{b} the triangles T are anisotropic with $h_{T,1} \sim h$ and $h_{T,2} \sim ah$; it holds $\left\|D^\beta u^i\right\|^2_{0,\Omega_i^{\mathrm{b}}} \leq C\, a\, \varepsilon^{-2\beta_2}$ for $a = \frac{a_0}{\underline{c}_0}\varepsilon\,|\ln\varepsilon|$ and $|\beta| = 2$. There are two possibilities of the asymptotics of h_F: $h_F \sim h$ (the side F is a long side touching Γ) or $h_F \sim ah$ (the side F is a short side touching Γ).

The first term in the brackets of (43) appears only if $h_F \sim ah$ for each $F \in \mathcal{E}_h^i(\Omega_i^{\mathrm{b}})$ ($\alpha_i \neq 0$), otherwise $\alpha_i = 0$ holds. In the second term in the brackets of (43) we apply $h_F h_F^{\perp} \sim ah^2$. Then, if $h_F \sim ah$ for each $F \in \mathcal{E}_h^i(\Omega_i^{\mathrm{b}})$, we have the estimate

$$\begin{aligned}\mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{b}})} \;&\leq\; C \sum_{F\in\mathcal{E}_h^i(\Omega_i^{\mathrm{b}})} \sum_{|\alpha|=1} \sum_{|\beta|=1} \left(\varepsilon^2 \alpha_i^2 C_2 \frac{ah}{h} h_T^{2\beta} + 2\varepsilon^2 C_1^{-1} \frac{1}{ah^2} h_T^{2(\alpha+\beta)}\right) \left\|D^{\alpha+\beta} u^i\right\|^2_{0,T} \\ &\leq\; C\varepsilon^2 \,|\ln\varepsilon|^4\, h^2,\end{aligned}$$

otherwise we get

$$\mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{b}})} \;\leq\; C \sum_{F\in\mathcal{E}_h^i(\Omega_i^{\mathrm{b}})} \sum_{|\alpha|=1} \sum_{|\beta|=1} 2\varepsilon^2 C_1^{-1} \frac{1}{ah^2} h_T^{2(\alpha+\beta)} \left\|D^{\alpha+\beta} u^i\right\|^2_{0,T} \;\leq\; C\varepsilon^2\,|\ln\varepsilon|^4\, h^2.$$

(iii) $T \in \mathcal{T}_h(\Omega_i^{\mathrm{c}})$: In the corner neighbourhood Ω^{c} the triangels T are assumed to be isotropic with $h_T \sim ah$, i.e. $h_F \sim ah$. Owing to $\left\|D^\beta u^i\right\|^2_{0,\Omega_i^{\mathrm{c}}} \leq Ca^2\varepsilon^{-2|\beta|}$ ($|\beta| = 2$) we derive by analogy to the previous cases

$$\mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{c}})} \;\leq\; C\varepsilon^2\,(ah)^2 \sum_{|\beta|=2} \left\|D^\beta u^i\right\|^2_{0,\Omega_i^{\mathrm{c}}} \;\leq\; C\varepsilon^2\,|\ln\varepsilon|^4\, h^2.$$

Finally, we get a bound of $\mathbf{II}_{\mathcal{E}_h^i} = \mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{i}})} + \mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{b}})} + \mathbf{II}_{\mathcal{E}_h^i(\Omega_i^{\mathrm{c}})}$ by summing up the previous estimates, viz.

$$\mathbf{II}_{\mathcal{E}_h^i} \;\leq\; C\varepsilon^2\,|\ln\varepsilon|^4\, h^2 \qquad (i = 1, 2).$$

□

Owing to the interpolation error estimates of Lemma 3.9 and 3.10 as well as to Lemma 3.1, we are able derive the following theorem.

Theorem 3.11. *Assume that u is the solution of BVP* (1) *over the domain $\Omega = (0,1)^2$, with smoothness assumptions* (40)*. Furthermore, suppose that all mesh assumptions quoted previously are satisfied (in particular,* (41) *and Ass. 2.1, 2.2, 2.3, 3.2, 3.3, with $a = \frac{a_0}{\underline{c}_0}\varepsilon\,|\ln\varepsilon|$, $a_0 \geq 2$ and $0 < \underline{c}_0 < c_0 = c(x) = const$) and that u_h denotes the Nitsche mortar finite element approximation according to* (13)*, with $\gamma > C_I$. Then, the error $u - u_h$ can be bounded in the norm $\|\,.\,\|_{1,h}$* (20) *as follows:*

$$\|u - u_h\|^2_{1,h} \;\leq\; C\left(\varepsilon\,|\ln\varepsilon|^3\, h^2 + h^4\right),$$

where C is independent of $h \in (0, h_0]$ and $\varepsilon \in (0,1)$.

Proof. According to the assumptions, we can combine the estimates of Lemma 3.9 and 3.10 with the estimates of Lemma 3.6 and 3.7, respectively, as well as with (32). This proves the assertion. □

4 Numerical experiments

In the following, we consider the BVP

$$-\varepsilon^2 \Delta u + u \;=\; 0 \quad \text{in } \Omega, \qquad u \;=\; -e^{-\frac{x}{\varepsilon}} - e^{-\frac{y}{\varepsilon}} \quad \text{on } \partial\Omega, \tag{44}$$

where Ω is given by $\Omega = (0,1)^2 \subset \mathbb{R}^2$. Obviously, problem (44) can be written as a problem of type (1), the solution u of (44) is $u = -e^{-\frac{x}{\varepsilon}} - e^{-\frac{y}{\varepsilon}}$. For small values of $\varepsilon \in (0,1)$, boundary layers near $x = 0$ and $y = 0$ occur. The L_2-norms of the second order derivatives of u satisfy the inequalities at (40). Here, according to the position of the boundary layers the definition of Ω^{i}, Ω^{b} and Ω^{c} is to be modified, cf. Figs. 4 and 6.

In correspondence to the boundary layers, we subdivide Ω into subdomains $\Omega_1 = (a,1) \times (a,1)$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$, define Γ by $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. The subdomains $\overline{\Omega}_1$, $\overline{\Omega}_2$ are meshed by triangles independent from each other. Choose an initial mesh like in Figure 6, where the nodes of triangles $T \in \mathcal{T}_h(\Omega_1)$ and $T \in \mathcal{T}_h(\Omega_2)$ must not coincide on Γ. The initial mesh is refined by subdividing a triangle T into four equal triangles such that new vertices coincide with the old ones or with the midpoints of the old triangle sides. Therefore, the mesh sequence parameters $\{h_1, h_2, h_3, \ldots\}$ are given by $\left\{h_1, \frac{h_1}{2}, \frac{h_1}{4}, \ldots\right\}$.

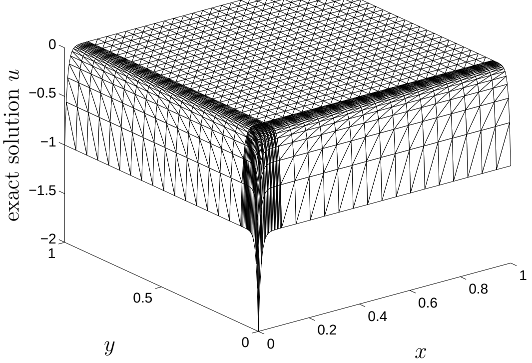

Figure 5: Solution u on the h_3-mesh for $\varepsilon = 10^{-2}$ and $a = 2\varepsilon\,|\ln\varepsilon|$

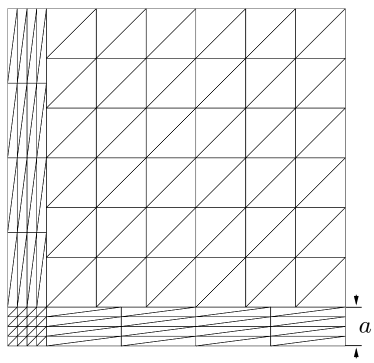

Figure 6: h_1-mesh with layer thickness a, for $a = \frac{1}{2}\varepsilon\,|\ln\varepsilon|$ and $\varepsilon = 10^{-1}$

Let u_h be the Nitsche mortar finite element approximation defined by (13) as an approximation of u from (44). Since u is known, the error $u - u_h$ in the $\|\,.\,\|_{1,h}$-norm can be calculated. Then, the convergence rates with respect to h will be estimated as follows. We fix ε and assume that the constant C in the relation $\|u - u_h\|_{1,h} \;\approx\; Ch^{\beta}$ is nearly the same for a pair of mesh sizes $h_i = h$ and $h_{i+1} = \frac{h}{2}$. Under this assumption we have

$$\beta \;\approx\; \log_2 \frac{\|u - u_h\|_{1,h}}{\left\|u - u_{\frac{h}{2}}\right\|_{1,\frac{h}{2}}}. \tag{45}$$

In Table 1 the errors $\|u - u_h\|_{1,h}$ and the convergence rates according to (45) are given, with the settings $\mathcal{E}_h = \mathcal{E}_h^1$, $\alpha_1 = 1$ and $\gamma = 2.5$. The results of Table 1 show that for

appropriate choice of the mesh layer parameter a, here e.g. for $a = \varepsilon|\ln\varepsilon|$ and $a = 2\varepsilon\,|\ln\varepsilon|$, optimal convergence rates $\mathcal{O}(h)$ like in Theorem 3.11 stated can be observed for a wide range of mesh parameters h. In Figure 7, for $\varepsilon = 10^{-2}$ and on a mesh of level h_3, the local error $u_h - u$ for two different values of the parameter a is represented. The influence of the parameter a is visible, in particular, the local error is significantly smaller for the value $a = 2\varepsilon\,|\ln\varepsilon|$ compared with that of $a = \varepsilon\,|\ln\varepsilon|$. For constant $a = 0.5$ the observed rate is far from $\mathcal{O}(h)$ (cf. also Figure 8, left-hand side) and, for $a = \frac{1}{2}\varepsilon\,|\ln\varepsilon|$, the rates are not appropriate for small h. In a second series of experiments, the following choice is made: $\mathcal{E}_h = \mathcal{E}_h^2$, $\alpha_2 = 1$, $\gamma = 2.5$ for $a = 0.5$ and $\gamma \approx 2\frac{1-a}{a}$ otherwise. Although the definition of $\mathcal{E}_h$ and α_1, α_2 compared with the first series was completely changed, nearly the same convergence rates β have been obtained (coincidence within the first and second digits).
In particular, the experiments show that non-matching meshes can be applied to singularly perturbed problems without loss of the optimal convergence rate. The appropriate choice of the width a of the strip with anisotropic triangles is important for diminishing the error and getting optimal convergence rates.

Parameter a	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$	
	$\|u - u_h\|_{1,h}$	β	$\|u - u_h\|_{1,h}$	β	$\|u - u_h\|_{1,h}$	β
0.5	7.132e-03 3.566e-03 1.783e-03	1.0000 1.0000	5.260e-02 3.154e-02 1.718e-02	0.7380 0.8760	6.738e-02 4.757e-02 3.361e-02	0.5023 0.5011
$\frac{1}{2}\varepsilon\,|\ln\varepsilon|$	3.656e-03 1.733e-03 8.397e-04	1.0774 1.0449	1.660e-03 1.500e-03 1.127e-03	0.1465 0.4124	8.472e-05 4.394e-05 2.483e-05	0.9471 0.8233
$\varepsilon\,|\ln\varepsilon|$	3.396e-03 1.692e-03 8.446e-04	1.0050 1.0026	9.863e-04 4.948e-04 2.488e-04	0.9951 0.9917	1.653e-04 8.271e-05 4.136e-05	0.9994 0.9998
$2\varepsilon\,|\ln\varepsilon|$	6.569e-03 3.284e-03 1.642e-03	1.0000 1.0000	1.969e-03 9.851e-04 4.926e-04	0.9993 0.9998	3.275e-04 1.639e-04 8.200e-05	0.9984 0.9995

Table 1: Observed errors in the $\|\,.\,\|_{1,h}$-norm on the level h_i $(i = 5, 6, 7)$ and the convergence rate β assigned to the levelpair (h_i, h_{i+1}) $(i = 5, 6)$; case $\mathcal{E}_h = \mathcal{E}_h^1$, $\alpha_1 = 1$ and $\gamma = 2.5$.

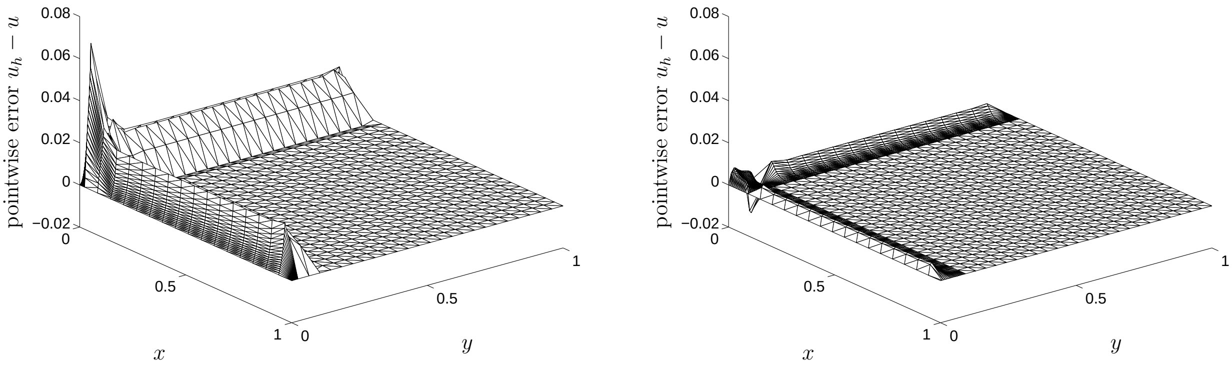

Figure 7: Pointwise error $u_h - u$ for $\varepsilon = 10^{-2}$ and for $\mathcal{E}_h = \mathcal{E}_h^2$ ($\alpha_2 = 1$) on meshes with $a = \varepsilon\,|\ln\varepsilon|$ (left) and $a = 2\varepsilon\,|\ln\varepsilon|$ (right)

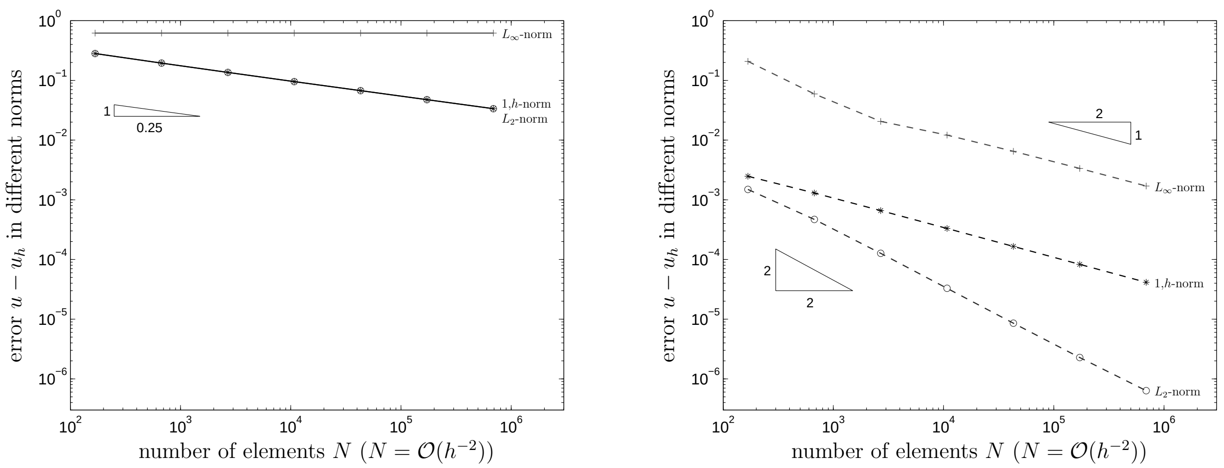

Figure 8: Observed error $u - u_h$ in the L_2-, L_∞- and $\|\,.\,\|_{1,h}$-norm for $\varepsilon = 10^{-5}$, $a = 0.5$ (left) and $a = \varepsilon\,|\ln\varepsilon|$ (right)

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