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Abstract

Discrete Mathematics is a fundamental subject for Information Technology (IT) majors. For students whose primary orientation is not to mathematics, but computing, it is important to relate fundamental mathematical principles to practical computing realities. At Bond University, Microsoft Excel is the primary practical vehicle used to illustrate these connections, and has proved to be an extremely valuable tool for this purpose. No programming or macro creation is required to implement most of the basic principles taught in Discrete Mathematics. In particular, the topic of Mathematical Induction (MI) poses a considerable challenge to many students, and there are a number of ways in which a spreadsheet such as Excel can assist with the learning process.

1 Introduction

The most fundamental object in mathematics is the *set*, and the most fundamental object in digital computing is the *bit*. Despite the fact that set membership for a given object may be represented by a single bit, very few programing languages, with the notable exceptions of the Pascal family [1] and ISETL [2], have direct support for sets. Why? Surely it is the duty of mathematics educators of IT students to make these connections plain. Certainly, in a traditional discrete mathematics course for tertiary IT students, the topics of binary forms, truth-tables, algebra of Boolean expressions, set theory, the binomial theorem and mathematical induction are commonly found. What is perhaps not so common is for the teacher to take the time to point out the many intimate connections between all of these topics, along with the associated elementary algorithms, and then have the students implement the algorithms in practical computer laboratory sessions or assignments. If one does this, the hope of course, is not only to establish these links in the minds of the students, but also to deepen and enrich their understanding of each of the original fundamental topics. There is an added bonus if the techniques used are seen to have recognizable, direct application to concrete IT examples.

In this paper, we consider the application of Excel to support the following inductive proof investigations: firstly that the *Catalan numbers* (defined below) eventually exceed the exponential function 3^n, and secondly, that a set of cardinality n has powerset of cardinality 2^n. Just as an inductive proof consists of two parts: the base and the inductive step, our Catalan example uses Excel to assist with the base, while the powerset example uses Excel to assist with the inductive step. But first, we give a brief summary of relevant Excel capabilities.

2 Capabilities of modern spreadsheets

We assume that the elementary capabilities of spreadsheet software are well-known, and therefore confine our attention to features of special benefit to the teaching of mathematics. Considering Microsoft Excel 2000, the following fundamental capabilities/features are seen to directly support what we might term *basic mathematical applications*. We do not consider *macros*.

1. A vast array of intrinsic functions plus the function wizard are provided.

2. Filling cells with formulae. This simple mouse action allows easy generation of sequences.

3. Powerful intrinsic graphics capability allows instant graphical display of sequences.

4. Conditional formatting: easy formatting and highlighting of cells which satisfy some condition.

5. Named cells and arrays, so that mathematical formulae may be entered in terms which are very close to the usual algebraic form, rather than obscure references such as E3*F1/A5.

The advent of the *graphical user interface* (GUI) and the power of mouse-based selection has enormously extended both the power and ease of use of the spreadsheet program. For more advanced applications, Excel has a module named *Solver*. Stone's paper [3] has an example of the use of Excel's *Solver* for Mixed-Integer Linear Programming. Other useful references to uses of spreadsheets in mathematical education are [4]—[11].

3   Diversity and flexibility

Recall that we are not considering Computer Science (CS) but IT students, who often have a more modest mathematical background, particularly with reference to algebraic skills. What are some of the characteristics of our undergraduate IT student population at Bond?

- Many students are severely challenged by basic algebra.

- A significant percentage are also challenged by elementary programming concepts.

- Almost all are doing maths only because they must (two compulsory units in IT degree).

- Until very recently, lecture classes were very small (less than 40).

- We have always had great diversity: a given class will typically be bimodal and contain students very proficient in algebra (and calculus), while many others are struggling with the basics.

Most of this should not be surprising; indeed many papers from Delta '99 [12] dealt with the last point: the problem of student diversity. How do we get basic mathematical ideas across to such a group? My answer: forget algebra and use Excel. This does *not* mean that algebra is ignored, but that extensive and long-winded algebraic manipulations are kept to a minimum. Instead, structures and patterns, which reinforce and extend the basic algebraic theory, may be observed in many ways within the spreadsheet environment. Once students see that Excel can be made to do all the boring computation, they become quite interested in looking at structures, ideas, algorithms, *what-if* scenarios etc. In short, it is argued here that Excel offers us *flexibility*, both in terms of accommodating the needs of students of widely-differing mathematical backgrounds, and also in terms of fostering an empirical, investigative approach to mathematics learning. Proofs are definitely there, and algebra is used, but the computer is also used to motivate or suggest a promising line of proof. Some examples are briefly considered below.

4   Mathematical Induction (MI)

Excel is ideal for looking at many properties of sequences of integers or real numbers, including comparison of direct and recursive formulae, limiting behaviour, and very importantly, investigation of conjectures for later inductive proof. Consider the proposition $2^n > n^2$. It is true for $n = 0, 1$; false for $n = 2, 3, 4$; and true thereafter. This behavior may be seen very easily in Excel

by use of simple tabulation, conditional formatting, graphing, differencing etc. If one wishes to then construct an inductive proof, then the folly of starting at $n = 0$ may be avoided, as it is obvious from the virtually instant tabulation and graphing of the sequences in Excel that such a proof is doomed to failure. Students may then be asked a more interesting question: where and why does the "proof" fail? In this manner, it is possible to gain insight into the structure of an inductive proof, without too much algebra getting in the way. However, some may say: "Is Excel really a useful mathematics teaching tool? Or does it discourage development of algebraic skills?" The answers to these questions surely depend on parameters such as:

- Diversity: What are the mathematical backgrounds of the students?

- Time: How much time does the teacher have to cover the required material?

- IT: What are the skills of the students in basic use of spreadsheet software?

4.1 Problems with induction—why is it tough for students?

- It is a *meta-strategy*: a widely-applicable proof technique. Potentially, at least, any proposition $P(n)$ about a natural number n is a candidate for inductive proof.

- The main, difficult part of an inductive proof, i.e., showing that $P(k) \implies P(k + 1)$, always requires *domain knowledge*, i.e., some specific facts or properties of the underlying algebraic system. This usually means fluency in manipulation of algebraic expressions is needed, and often requires the use of index laws, simplification and factorization of expressions, and possibly working with inequalities and radicals.

- It requires a clear understanding of what the *goal* (last line of proof) is, and an ability to detect when this goal has been reached. Comprehension of this vital piece of information: "Where are we going?", seems surprisingly elusive for a significant percentage of students.

MI is a very powerful and general technique, and this fact, together with the points listed above, conspire to present a formidable challenge to many students, especially those with modest algebraic skills. Quite apart from these considerations, the whole *conceptual framework* of MI also presents a challenge to many students.

4.2 Examples

4.2.1 Catalan numbers

These have many applications in IT; for example, they enumerate distinct ordered binary trees. The sequence may be defined by the initial value $x_0 = 1$ and thereafter by the recurrence $n \geq 1 : (n + 1)\, x_n = (4n - 2)\, x_{n-1}$. We use a 4–step procedure for MI to show that $x_n > 3^n$ if $n \geq 17$. Note the use of a *slack variable* α in steps 2 and 4.

1. Show true for $n = 17$. This is not trivial, given the recursive definition above, and just pencil and paper! However, we can make short work of it with Excel (see Figure 1).

2. Suppose $k \geq 17$ and that $x_k > 3^k$. That is, there exists $\alpha > 0$ such that $x_k = 3^k + \alpha$.

3. State GOAL: Given the information in Step 2, to show that $x_{k+1} > 3^{k+1}$.

4. Establish GOAL by making use of the assumption in Step 2 and domain information. Consider $x_{k+1} - 3^{k+1}$, and put $n = k+1$ in the defining recurrence for Catalan, to obtain

$(k+2)x_{k+1} = (4k+2)x_k$. Then:

$$\begin{aligned}
x_{k+1} - 3^{k+1} &= \left(\frac{4k+2}{k+2}\right)x_k - 3^{k+1} \\
&= \left(\frac{4k+2}{k+2}\right)\left(3^k + \alpha\right) - 3^{k+1} \\
&= \left(\frac{4k+2}{k+2}\right)\alpha + \left(\frac{k-4}{k+2}\right)3^k \\
&> 0
\end{aligned}$$

What did we achieve by using Excel here? It certainly did not help us with the algebra, but it allowed us to very quickly tabulate and graph the sequence, enabling a rapid formulation of the hypothesis that $x_n > 3^n$ for $n \geq 17$. It also takes care of the first step of the inductive proof (base case). Further, many other hypotheses may be formulated by inspection of an extended Excel model; for example the conjecture that $x_n < 4^n$ for $n > 0$ may be formulated and then proved by induction.
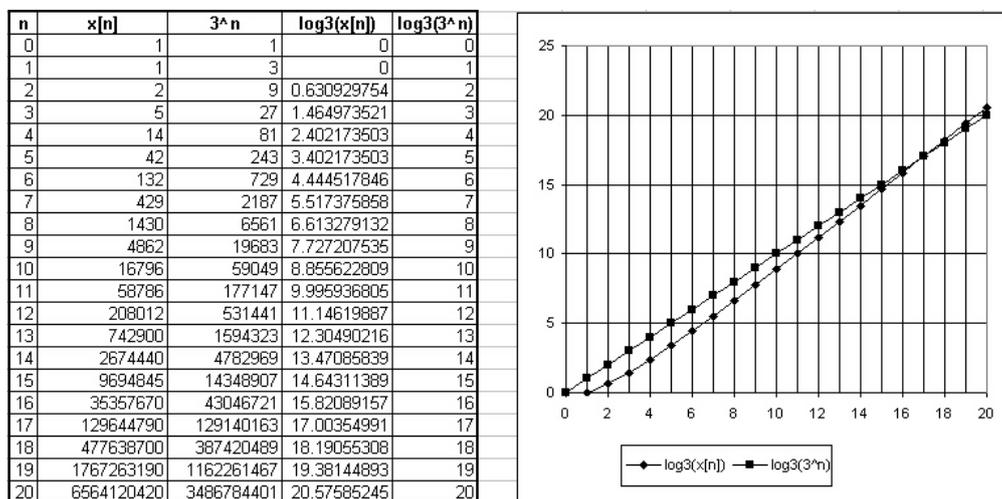
n	x[n]	3^n	log3(x[n])	log3(3^n)
0	1	1	0	0
1	1	3	0	1
2	2	9	0.630929754	2
3	5	27	1.464973521	3
4	14	81	2.402173503	4
5	42	243	3.402173503	5
6	132	729	4.444517846	6
7	429	2187	5.517375858	7
8	1430	6561	6.613279132	8
9	4862	19683	7.727207535	9
10	16796	59049	8.855622809	10
11	58786	177147	9.995936805	11
12	208012	531441	11.14619887	12
13	742900	1594323	12.30490216	13
14	2674440	4782969	13.47085839	14
15	9694845	14348907	14.64311389	15
16	35357670	43046721	15.82089157	16
17	129644790	129140163	17.00354991	17
18	477638700	387420489	18.19055308	18
19	1767263190	1162261467	19.38144893	19
20	6564120420	3486784401	20.57585245	20

Figure 1: Catalan Numbers in Excel

4.2.2 Counting subsets

Let S be a set of cardinality n. We wish to count the number of subsets of S that have cardinality r, where $0 \leq r \leq n$. This may be done quite conveniently by using a *truth table*—bread and butter to IT students. The number of such subsets is given by the so-called *binomial coefficient* $C(n,r)$. Figure 2 shows an example in Excel for $n = 4$.

#	d in X	c in X	b in X	a in X	d	c	b	a	X	cardinality	r	C(4,r)
0	0	0	0	0					{}	0	0	1
1	0	0	0	1				a	{a}	1	1	4
2	0	0	1	0			b		{b}	1	2	6
3	0	0	1	1			b	a	{ba}	2	3	4
4	0	1	0	0		c			{c}	1	4	1
5	0	1	0	1		c		a	{ca}	2		
6	0	1	1	0		c	b		{cb}	2		
7	0	1	1	1		c	b	a	{cba}	3		
8	1	0	0	0	d				{d}	1		
9	1	0	0	1	d			a	{da}	2		
10	1	0	1	0	d		b		{db}	2		
11	1	0	1	1	d		b	a	{dba}	3		
12	1	1	0	0	d	c			{dc}	2		
13	1	1	0	1	d	c		a	{dca}	3		
14	1	1	1	0	d	c	b		{dcb}	3		
15	1	1	1	1	d	c	b	a	{dcba}	4		

Figure 2: Powerset in Excel

An assignment was set to construct the powerset in Excel for a set of cardinality 8. This powerset has cardinality $2^8 = 256$. A model similar to that of Figure 2 was given. Thus, the students were required to perform several inductive steps to increase cardinality of the underlying set by one each time. As this is done, it is clear from the truth-table model that the cardinality of the powerset doubles at each inductive step. This is essentially the inductive proof. The students actually see the powerset automatically generated by Excel, and the exponential nature of its size becomes obvious. Many side-benefits accrue: the Excel intrinsic COUNTIF can be used to count the frequency of each subset cardinality, and *presto*, the binomial coefficients emerge. These may be compared to those obtained via the Excel intrinsic COMBIN, and then summed to check that 2^n is obtained. In hindsight, this was possibly a rather easy assignment, but the potential for learning is huge!

5 Conclusions

5.1 Student difficulties with Excel

While using Excel to support the teaching of elementary discrete and continuous mathematics, the following difficulties were met.

- Some students (a small minority) had problems with basic operations such as *fill-down*, and the fact that each formula must begin with the "=" character.

- Several students had trouble constructing simple *arithmetic expressions* in Excel.

- Many students found it difficult to relate consecutive sequence terms, i.e., the concept that a_{n-1} is the term just before a_n seemed curiously difficult for some.

5.2 Claimed benefits of Excel

How is Excel able to help students who are having conceptual problems with the relationship between a_{n-1} and a_n? This is almost too simple to describe, but when these two consecutive sequence elements are shown in Excel and it is pointed out that they correspond to adjacent cells, the difficulty seems to largely vanish. It is conjectured that this is due to the students' seemingly innate ability to relate to modern computer software such as a graphical spreadsheet rather than to algebraic formalisms! An example may serve to illustrate this point. Suppose we are constructing a superannuation model in which the recurrence is $a_n = 1.01a_{n-1} + 500$. In Excel, we simply type a formula such as = 1.01*B2 + 500. In fact, the cell-reference B2 is not needed either, and *students are encouraged to use the mouse to click in the relevant cell,*

rather than typing hard cell references. This process in itself emphasizes dependence on the preceding term in the sequence, and it should be noted that, here, there is no direct algebraic reference to any sequence element at all!

For MI, it is argued that by seeing an inductive step in action, as in the powerset example of the previous section, students are better able to understand the *principle* of inductive proof. There is no claim here that algebraic skills needed for constructing the algebraic steps will benefit from use of Excel, and no attempt has been made by the author to conduct the difficult task of measuring any improvement in skills at MI proofs by students.

5.3 Student reaction

This has been universally positive, with most students very enthusiastic. Consider the following anonymous written responses to the question "Was the Excel material a help to you, or did it make the work harder?"

> "*Made it a lot easier!*" "*I found Excel made it easy to see how things worked and what was going on.*" "*Helped to check answers.*" "*Helped me to understand.*" "*Helpful, more user-friendly [than] I thought.*" "*Easier, not as much syntax to understand. Made things very clear, and quick to implement.*" "*I found Excel made it easy to see how things worked and what was going on.*" "*Definitely more helpful.*" "*Made it a lot easier!*" "*Good help! Can see the answers and understand concepts better.*" "*It helped. Should continue using Excel!*"

It should be noted that there were no negative responses to this question at all (all other responses were blank, with one "NA"). By contrast, there were several negative comments (and no positive ones) about a small 3GL programming language component of the course.

Of course, these reactions are very encouraging. Similar approaches have been used for a wide variety of other mathematical topics, both in the discrete and continuous realms, with similar positive reactions from students. New uses of Excel are always popping up, and we recommend its use to other mathematics educators.

6 Acknowledgment

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