L^2-INDEX FORMULA FOR PROPER COCOMPACT GROUP ACTIONS

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CHAPTER I

INTRODUCTION

An elliptic differential operator on a compact manifold M is *Fredholm*, meaning that it is invertible modulo $\mathscr{K}$, the algebra of compact operators. The *analytical index* of an elliptic operator A defined by

$$\mathrm{ind} A = \dim \mathrm{Ker} A - \dim \mathrm{Ker} A^* \tag{I.1}$$

is invariant under homotopy and stable under compact perturbation. The index of an elliptic operator measures how far it is from being invertible, and depends on the topological information associated to the operator and the manifold. According to the *Atiyah-Singer index theorem*, $\mathrm{ind} A$ depends only on the equivalence class of its *principal symbol* σ_A of A in $K^0(TM)$ [1] and the index formula is obtained by applying the Chern character [2] $\mathrm{ch} : K^0(TM) \to H^{even}(TM)$ to the symbol σ_A, multiplied by the Todd class $\mathrm{Td}(M)$ (a topological invariant of M) and then integrating on the tangent space TM [6],

$$\mathrm{ind} A = \int_{TM} \mathrm{ch}(\sigma_A) \cdot \mathrm{Td}(M). \tag{I.2}$$

The index formula (I.2) indicates significant connections to geometry, topology and complex analysis. For example, (I.2) reduces to the *Gauss-Bonnet-Chern theorem*, the *Hirzebruch signature theorem* or the *Riemann-Roch theorem*, when A is interpreted as a certain Dirac operator. [3] Moreover, (I.2) motivates a careful study of the elliptic operators over M. The abstract elliptic operators on M were made into a group $K_0(M)$ or $K^0(C(M))$, called the *K-homology* of M, by imposing appropriate equivalence relations [3] [21] and the analytical index is reinterpreted as a group homomorphism, called *the assembly map*:

$$\mu : K_0(M) \to K_0(\mathscr{K}) = \mathbb{Z} : [A] \mapsto \mathrm{ind} A. \tag{I.3}$$

[1] The K-theory $K^*(M)$ of a compact manifold M is a cohomology theory. $K^0(M)$ is the Grothendieck group of stably isomorphic vector bundles over M. A vector bundle over BM (the ball bundle of M), which is trivial over the sphere bundle SM, can be viewed as an element in $K^0(TM)$.

[2] The Chern character is a ring homomorphism between two cohomology theories.

[3] The operator A corresponds to the de Rham operator, the signature operator or the Dolbeault (Cauchy-Riemann) operator, respectively.

The link between this analytical index (I.1) and the topological index (I.2) is the *Poincaré duality*

$$K_0(M) \cong K^0(TM) : [A] \mapsto [\sigma_A], \tag{I.4}$$

which is well-interpreted using the product structure invented by Kasparov [23].

We further study (I.3) when there is a symmetry on the space. For example, we obtain an operator $\tilde{A}$ when lifting an elliptic operator A on compact X to its universal cover $\tilde{X}$. The operator $\tilde{A}$ that commutes with the action of the fundamental group $G = \pi_1(X)$ of X, defines an element in the equivariant K-homology $K^0_G(C_0(\tilde{X}))$, and is invertible modulo the algebra of "compact operators" on some Hilbert $C^*(G)$-module, and therefore has an analytical index in the K-theory group $K_0(C^*(G))$:

$$K^0_G(C_0(\tilde{X})) \to K_0(C^*(G)) : [\tilde{A}] \mapsto \mathrm{Ind}\tilde{A}. \tag{I.5}$$

The *higher index* $\mathrm{Ind}\tilde{A}$ in (I.5) relates closely to the representation theory of G, the topological and geometrical properties of the manifold, in particular, to the Novikov conjecture on homotopy invariance of higher signatures [23], and the Gromov-Lawson-Rosenberg conjecture on the existence of positive scaler curvatures [30]. Index theory uses an operator algebra approach to study problems in topology and geometry, some of which have not been solved using the topological and geometrical approach.

In general the assembly map (I.5) can be formulated for a locally compact group G and a G-invariant elliptic operator A on a proper G-space X with compact quotient. Kasparov [22] constructed a topological index of A using the bivariant K-theory (KK-theory). In fact, on the operator level, there is a duality between a G-invariant elliptic operator $[A]$ in $K^*_G(C_0(X))$ and its symbol $[\sigma_A]$ in $KK_G(C_0(X), C_0(TX))$ via the intersection product with the Dolbeault operator D on TX:

$$[A] = [\sigma_A] \otimes_{C_0(TX)} [D] \in K^*_G(C_0(X)). \tag{I.6}$$

This is the G-equivariant version of (I.4). The topological index of A in $K_*(C^*(G))$ is defined by taking the image of A under the descent map j^G and then by contracting with a projection $[p]$ in

$K_0(C^*(G,C_0(X)))$, constructed from the constant 1 function on the quotient X/G :

$$KK_G(C_0(X),\mathbb{C}) \xrightarrow{j^G} KK(C^*(G,X),C^*(G)) \xrightarrow{[p]\hat{\otimes}} KK(\mathbb{C},C^*(G)) \; .$$

The topological index was proved to be the same as the analytical index in [24], [22].

The difficulty of computing the higher index $\mathrm{Ind}A \in K_0(C^*(G))$ for a general group G motivates the attempt to build a homomorphism from $K_0(C^*(G))$ to a simpler object. **The purpose of this thesis is to find an $\mathbb{R}$-valued index of elliptic operators as above by taking the trace of the** $\mathrm{Ind}A$ **in $K_*(C^*(G))$ and to obtain a cohomological formula concerning the topological information associated to the manifold and the operator.** My work is based on the following two special cases:

- Atiyah investigated the case when X admits a cocompact free action of a discrete group G. The L^2-index is defined using an operator trace on G-invariant elliptic operators on X and coincides with the index of the corresponding operator on the quotient manifold [4].
- When G is a unimodular Lie group and H is a compact subgroup, Connes and Moscovici proved an L^2-index formula for a G-invariant elliptic pseudo-differential operator A acting on sections of homogeneous vector bundles over G/H, considered as an element in some type *II* von Neumann algebra. The trace of the projection onto $\mathrm{Ker}(A)$, called *von Neumann dimension*, is the right concept of the dimension, and an $\mathbb{R}$-valued index formula was obtained [12].[4]

Let G be a locally compact unimodular group, X be a properly cocompact G-manifold with a *cutoff function* $c \in C_c^\infty(X)$,[5] and E be a G-vector bundle over X. Let $A : L^2(X,E) \to L^2(X,E)$ be a G-invariant elliptic operator and let $P_{\mathrm{Ker}A}$ is the projection onto the kernel of A. The L^2-*index* of A is defined as follows:

$$\mathrm{ind}A = \mathrm{tr}_G P_{\mathrm{Ker}A} - \mathrm{tr}_G P_{\mathrm{Ker}A^*}$$

[4]The $\mathbb{R}$ index is the alternating sum of the L^2-*Betti number* when A is the de Rham operator.

[5]A non-negative, compactly supported function $p \in C_c^\infty(X)$ is a *cutoff function* if $\int_G c(g^{-1}x)\mathrm{d}g = 1$ for all $x \in X$.

where G-trace tr_G defined similarly to the definition in [4] and calculated by

$$\mathrm{tr}_G S = \mathrm{tr}(c^{\frac{1}{2}} S c^{\frac{1}{2}}) = \int_X c(x)\,\mathrm{Tr}\,K_S(x,x)\mathrm{d}x, \tag{I.7}$$

where $K_S(x,y) \in \mathrm{End}(E_x, E_y)$ is the Schwartz kernel of S. Here, S is bounded G-invariant pseudo-differential operator with a smooth kernel. In particular, $P_{\mathrm{Ker}A}$, $P_{\mathrm{Ker}A^*}$ are such operators. **The main theorem of this thesis is the following:**

Theorem I.0.1. *Let X be a complete Riemannian manifold where a locally compact unimodular group G acts properly and cocompactly. If A is a zero order properly supported elliptic pseudo-differential operator, then the L^2-index of A defined by taking the "trace" of its K-theoretic index, is given by the formula*

$$\mathrm{ind} A = \int_{TX} c(x)(\hat{A}(TX))^2 \mathrm{ch}(\sigma(A)). \tag{I.8}$$

The formula (I.8) generalizes the L^2-index formula of free cocompact group actions due to Atiyah [4] and the L^2-index formula for the homogeneous space of unimodular a Lie group due to Connes and Moscovici [12]. The study of L^2-index in general has implications in many other areas of mathematics [33], [26]. For example, the non-vanishing of the L^2-index for the signature operator on X indicates the existence of L^2-hormonic forms on X. The L^2-index is of interest to the study of the discrete series representation [12] and has been modified to prove the Novikov conjecture for hyperbolic groups [13].

The **proof** of the Theorem I.0.1 uses the proof structure in [12] and concerns the following three steps:

1. To prove (I.8), consider A as an element in the K-homology $K_G^0(C_0(X))$, which has a "higher" index in $K_0(C^*(G))$. The L^2-index of A depends only on the equivalent class of its higher index in $K_0(C^*(G))$. This is proved by defining a trace on a dense holomorphic closed ideal $\mathscr{S}(\mathscr{E})$ in $\mathscr{K}(\mathscr{E})$, which has the same K-theory with $C^*(G)$. The trace is the von Neumann trace of a type II von Neumann algebra in the sense of [11]. A general discussion on the link between the L^2-index and the "higher" index" may be found in [31].

2. We reduce the problem of finding $\mathrm{ind} A$ into finding $\mathrm{ind} D_{V(\sigma_A)}$ for some Dirac type operator $D_{V(\sigma_A)}$ which has the same "higher index" as A. Here D is the Dolbeault operator on ΣX and

$V(\sigma_A)$ is a vector bundle over ΣX obtained by a gluing construction using the fact that $\sigma_A|_{SX}$ is elliptic. [6] The Kasparov's K-theoretic index formula [22] is essential in the argument. The formulation of Dirac type operators out of elliptic operators is related to the vector bundle modification construction in the definition of geometric K-homology [7].

3. Calculate $\operatorname{ind} D_{V(\sigma_A)}$ using the heat kernel method. When D is the first order operator of Dirac type, there is the *Mckean-Singer formula* for the L^2-index:

$$\operatorname{ind} D = \operatorname{tr}_G e^{-tD^*D} - \operatorname{tr}_G e^{-tDD^*}, t > 0. \tag{I.9}$$

   In the case of the compact manifold without group action, a cohomological formula was obtained by studying the local invariants of metrics and connections [5], [16]. The proof for the local index formula was greatly simplified by a rescaling argument of Getzler [15] on the asymptotic expansion of the heat kernel e^{-tD^2} around $t = 0$, [7] Since the index $\operatorname{ind} D$ in (I.9) is local when $t \to 0+$, [8] the group action is not going to affect the calculation. Instead of approximating the heat kernel on a compact manifold, I approximate the heat kernel $k(x,x)$ timed by cutoff function $c(x)$ for the non-compact X. The proof is based on a modification of the proofs in [29], [8].

The **structure** of the thesis goes as follows:

Chapter II is to introduce the elliptic theory and to formulate the problem. G-trace and L^2-index are defined in the last section of this chapter. Chapter III is to formulate Kasparov's K-theoretic index formula and to relate it with the L^2-index. The first two steps of the proof are finished in section III.3. Chapter IV is devoted to the heat kernel method in sections IV.1 and IV.2 and to complete the third step of the proof in section IV.3. The reductions and applications are in section IV.4. The Appendices serves as some background information to the thesis.

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[6] ΣX is the manifold glued by two copies of ball bundle $BX \subset TX$ along the boundary $SX \in TX$, the sphere bundle.

[7] Here D means $\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ and it acts on a graded Hilbert space.

[8] We compute the coefficients in the asymptotic expansion by localizing at a point.

CHAPTER II

ELLIPTIC OPERATORS

This chapter introduces the G-invariant elliptic operator on a manifold having properly cocompact group action and its L^2-index. Section 1 focuses on relevant properties of elliptic operators in the classical definition. Second 2 is about Dirac operators, the important examples of elliptic operators to be referred later. The elliptic operator in the context of the group action is discussed in section 3. The relation between the Von Neumann trace and the L^2-index are presented in the last section.

II.1 Elliptic pseudo-differential operators

This section serves as the motivation, construction and canonical knowledges of elliptic pseudo-differential operators. The section is summarized mainly from [18] and [1].

II.1.1 Elliptic theory on $\mathbb{R}^n$.

A *differential operator* on $\mathbb{R}^n$ of order m is a linear map

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n),$$

where $a_\alpha \in C^\infty(\mathbb{R}^n)$ and D^α is the $(\alpha_1, \cdots, \alpha_n)$ partial derivative with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Differential operators can be made into multiplication, which is less complicated via Fourier transformation, [1] which motivates the definition of symbol. The *symbol* $p(x,\xi)$ of P is

$$p(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)(i\xi)^\alpha.$$

Note that $\widehat{Pu}(\xi) = p(x,\xi)\hat{u}(\xi)$ where $\hat{u}(\xi) = \int e^{-i<x,\xi>} u(x) \mathrm{d}x$, with $<x,\xi> = \sum_{i=1}^n x_i \xi_i$ and P may be reproduced from its symbol by

[1] Fourier transformation can be defined on the tempered distribution $\mathscr{S}'$ i.e. the set of linear forms on $\mathscr{S}$, which are continuous with respect to the semi-norm defined by $\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta u(x)|$. It is not hard to see that $\mathscr{S}'$ contains all L^2-functions.

$$Pu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} p(x,\xi)\hat{u}(\xi)\mathrm{d}\xi = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i<x-y,\xi>} p(x,\xi)u(y)\mathrm{d}y\mathrm{d}\xi. \qquad \text{(II.1)}$$

The *principal symbol* of P is the ξ-homogenous part with the highest degree:

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

We say that P is *elliptic* if its principal symbol σ_P is nonzero when $\xi \neq 0$. Ellipticity is an important property in the sense that such a operator has an "almost inverse" Q. Since Q may not be a differential operator, it is necessary to investigate a broader class of operators – "Pseudo-differential operators".

A *pseudo-differential operator* A is of order m if its symbol $a(x,\xi)$ belongs to the Hörmander's $1,0$-class $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n) =$

$$\{a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \forall \alpha,\beta \in \mathbb{N}, x \in K \subset \mathbb{R}^n \text{ compact}\}. \qquad \text{(II.2)}$$

The topology on S^m is defined by $|a|^m_{\alpha,\beta} = \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n}\{(1+|\xi|^{-(m-|\beta|)})|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|\}$, which is a semi-norm. We define A to be the integration formula out of its symbol $a(x,\xi) \in S^m$:

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} a(x,\xi)\hat{u}(\xi)\mathrm{d}\xi = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i<x-y,\xi>} a(x,\xi)u(y)\mathrm{d}y\mathrm{d}\xi. \qquad \text{(II.3)}$$

Denote the set of order m pseudo-differential operator by Ψ^m. We say that A is a *smoothing operator* if $a(x,\xi) \in S^{-\infty} \doteq \cap_{m\in\mathbb{R}} S^m$. It is trivial to check that $\cup_{m\in\mathbb{R}} S^m$ forms a graded algebra, i.e.

- If $m < n$ then $S^m \subset S^n$; If $a,b \in S^m$, then $a+b \in S^m$; If $a \in S^m, b \in S^n$ then $ab \in S^{m+n}$.

Therefore given $A \in \Psi^m$, we define the *principle symbol* $\sigma_A(x,\xi)$ to be the class of the symbol $a(x,\xi)$ in S^m/S^{m-1}. One reason that principle symbols are more convenient in use is that they preserve algebraic operation of operators, i.e.

$$\sigma_{AB} = \sigma_A\sigma_B, \sigma_{A^*} = \overline{\sigma_A}.$$

The algebraic operations for symbols, however, are more complicated:

Proposition II.1.1. *1. (*Symbol of adjoint*) Let $a(x,\xi) \in S^m$ where A is the corresponding operator and let $a^*(x,\xi)$ be the symbol of the adjoint A^*. Then $a^*(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \bar{a}$. Recall that an* asymptotic sum *of $a(x,\xi)$ denoted by*

$$a \sim \sum_{j=0}^{\infty} a_j \tag{II.4}$$

is a series of symbols $\sum_{j=0}^{\infty} a_j$ satisfying $a_j \in S^{m_j}$ and $a - \sum_{j=0}^{k} a_j \in S^{m_{k+1}}, \forall k \geq 0$, where $m_j \to -\infty$ is a decreasing sequence.

By definition, for $a \in S^{m_0}$, there exists $a_{m_j} \in S^{m_j}$, where $m_j \to -\infty$ is decreasing, so that $a \sim \sum_{j=0}^{\infty} a_j$.

*2. (*Symbol of composition*) Let $a(x,\xi) \in S^m, b(x,\xi) \in S^n$, where A, B are the corresponding operators, and let $c(x,\xi)$ be the symbol of AB. Then $c \in S^{m+n}$ and $c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\xi^\alpha a)(D_x^\alpha b)$.*

The pseudo-differential operator $A \in \Psi^m$ extends to a densely defined operator on $L^2(\mathbb{R}^n)$ in the sense of distribution, i.e.

$$< Au(x), \phi(x) > = < u(x), A^t\phi(x) > \forall u(x) \in L^2(\mathbb{R}^n), \forall \phi(x) \in C_c^\infty(\mathbb{R}^n),$$

where if $A = \sum a_\alpha D^\alpha$ then $A^t\phi = \sum(-1)^{|\alpha|}D^\alpha(a_\alpha\phi)$. When $m > 0$, A is not bounded. But A can be made into a bounded operator (Proposition II.1.3) if we choose to use *Sobolev space*. Recall that *Sobolev space H^s* is the completion of

$$\{u \in \mathscr{S} : \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \mathrm{d}\xi < \infty\} \tag{II.5}$$

under the Sobolev norm

$$\|u\|_s^2 = (2\pi)^{-n} \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \mathrm{d}\xi.$$

Clearly, $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n) \subset H^t(\mathbb{R}^n)$ if $s > t$. A remarkable fact about the Sobolev space is *the Sobolev embedding theorem*:

Theorem II.1.2. *For each real number $s > \frac{n}{2} + k$, there is a continuous embedding $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$, where $C^k(\mathbb{R}^n)$ are the set of the k-times differentiable functions on $\mathbb{R}^n$.*

The following proposition is a consequence of Theorem II.1.2.

Proposition II.1.3. *If $a(x,\xi) \in S^m$, then $A = a(x,D) : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ extends to a bounded operator $H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. In particular, a pseudo-differential operator with a non-positive order is bounded on $L^2(\mathbb{R}^n)$.*

Ellipticity extends to pseudo-differential operators. A pseudo differential operator $A \in \Psi^m$ is *elliptic* if

$$|\sigma_A(x,\xi)| \geq c_K|\xi|^m, \text{ for any } |\xi| \geq c_K, \text{ and } x \in K \subset \mathbb{R}^n,$$

where K is any compact subset in $\mathbb{R}^n$ and c_K is a constant depending on K. The following equivalent definitions suggest a key property of elliptic operators.

Proposition II.1.4. *Let $a(x,\xi) \in S^m$ be the symbol of a pseudo-differential operator. Then the following statements are equivalent:*

1. *The operator A corresponds to $a(x,\xi)$ is elliptic.*
2. *There exists $b(x,\xi) \in S^{-n}$ (let B be the corresponding operator), such that $AB - Id$ and $BA - Id$ are smoothing operators.*

II.1.2 Elliptic theory on manifold

At this point, we investigate pseudo-differential operators acting on smooth functions on a manifold, or smooth sections of a vector bundle, instead of on $C_c^\infty(\mathbb{R}^n, \mathbb{C}^k)$. In order to conveniently represent a pseudo-differential operator on a manifold, we introduce the *amplitude*, a concept slightly extending the symbol. Set $S^m(\mathbb{R}^n \times \mathbb{R}^n) =$

$$\{a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) : |\partial_{x,y}^\alpha \partial_\xi^\beta a(x,y,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \forall \alpha, \beta, a(x,y,\xi) = 0, |x-y| \text{ is large}\}. \tag{II.6}$$

$a(x,y,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is an *amplitude* and defines a pseudo-differential operator of order m via

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i<x-y,\xi>} a(x,y,\xi) u(y) \mathrm{d}y \mathrm{d}\xi.^2 \tag{II.7}$$

Locally, for an open set $U \subset \mathbb{R}^n$, a continuous linear operator

$$A : C_c^\infty(U, \mathbb{R}^k) \to C^\infty(U, \mathbb{R}^k)$$

is a *pseudo-differential operator* of order m if for all $\phi, \psi \in C_0^\infty(\mathbb{R}^n), \operatorname{supp}\phi, \operatorname{supp}\psi \subset U, \phi A \psi$ is an order less or equal to m pseudo-differential operator and there exists ψ_0, ϕ_0 so that $\psi_0 A \psi_0$ is of order m.

Up to the difference of a smoothing operator on U, A has a unique *symbol a* belonging to the set of the the *local symbols* $S^m_{loc}(U \times \mathbb{R}^n)$,

i.e. $a \in C^\infty(U \times \mathbb{R}^n)$ and $\phi a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\phi \in C_c(U)$.

When A extends to $\mathscr{S}'(U) \to \mathscr{D}(U)$, where $\mathscr{D}(U)$ is the set of distributions on U, A^* is no longer a pseudo-differential operator. To obtain a "closed set of operators", we require A to have proper support. Recall that A is *properly sopported* if for any compact subset $K \subset U$, there is a compact subset L with

$$\operatorname{supp} u \subset K \Rightarrow \operatorname{supp} Au \in L \text{ and } u = 0 \text{ on } L \Rightarrow Au = 0 \text{ on } K. \tag{II.8}$$

A properly supported A maps $C_c(U, \mathbb{R}^m)$ to itself. There are "plenty of" properly supported pseudo-differential operators because of the following proposition. Therefore we may assume properly supportness without loss of generality.

Proposition II.1.5. *If $a(x,\xi) \in S^m_{loc}(U \times \mathbb{R}^n)$, then there exists an operator R with smooth kernel in $C^\infty(U \times U)$ such that $a(x,D) + R$ is properly supported.*

Pseudo-differential operators can be defined on $C_c^\infty(X,E)$, smooth sections of vector bundle

[2] In the case of manifold, the exponent $i < x - y, \xi >$ turns out to be a general function of three variable, and is called *phase function.*

E over manifold X with compact support [3]. Recall that a *smooth manifold* M of dimension n is described by an open cover $\{U_i\}$ patched by smooth transition functions, i.e. homeomorphism $\phi_i : V_i \to U_i$, where V_is are open set of a n dimensional vector space and $\phi_j^{-1} \circ \phi_i$ is smooth on where it is defined. Recall that a *vector bundle* is a pair $(E, p : E \to X)$, where p is a continuous onto map between two topological spaces E and X, satisfying the following conditions:

1. The set $p^{-1}(x)$ is homeomorphic to a vector space V of finite dimemsion for any $x \in X$. We call V the fiber and X the base space.

2. (*Local trivialization.*) For any $x \in X$, there exists a open set U containing x such that there is a homeomorphism ϕ from $U \times V$ to $p^{-1}(U)$.

3. If $\phi_1 : U_1 \times V \to p^{-1}(U_1)$ and $\phi_2 : U_2 \times V \to p^{-1}(U_2)$ are two trivializations and $U_1 \cap U_2$ is non empty, then the composition function $\phi_1^{-1} \circ \phi_2$ is continuous on $(U_1 \cap U_2) \times V$.

Example II.1.6. The *cotangent bundle* T^*X is the set of all points $(x, v) \in T^*X$ consisting of a point $x \in X$ and a covector $v \in T_x^*X$, and the basis at x is $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$, where $\mathrm{d}x_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. The projection map is $p(x, v) = x$. Let $(x, v) \in T^*X$ and open set $V \subset M$ containing x has local coordinates $(x_1, \cdots, x_n)$, so the one form v is denoted by $\sum_{i=1}^n \xi_i dx_i$, and

$$V \times \mathbb{R}^n \to p^{-1}(V) : (x_1, \cdots, x_n, \xi_1, \cdots, \xi_n) \mapsto (x, v)$$

is the local trivialization. If $(y_1, \cdots, y_n)$ is another coordinates on V, then it can be written into functions of x: $(y_1(x_1, \cdots, x_n), \cdots, y_n(x_1, \cdots, x_n))$ and let $y'(x)$ be the Jacobi matrix (i.e. the derivative of y_is with respect to x_js), then by simple calculation we have that $(x, {}^t y'(x)\eta)$ and $(y(x), \eta)$ represents the same point.

Definition II.1.7. Let X be a manifold of dimension n and let E, F be complex vector bundles over X. We say that

$$A : C_c^\infty(X, E) \to C^\infty(X, F)$$

is a *pseudo-differential operator* of order m if for each diffeomorphism f from a coordinate neigh-

[3] A *section* s of a vector bundle $(E, p : E \to X)$, is a continuous map from X to E such that $p \circ i = Id_X$

borhood $X_f \subset X$ to an open set $U_f \in \mathbb{R}^n$, there exists $A_f \in \Psi^m_{loc}(U_f)$ such that

$$(A_f u) \circ f = A(u \circ f), \forall u \in C_c^\infty(U_f).$$

Alternatively, A can be expressed as a finite sum $\sum A_\alpha$ module infinite smoothing operators, where $\phi_\alpha A_\alpha \psi_\beta^{-1}$ is a pseudo-differential operator of order k with compact support, and where $\phi_\alpha : \mathbb{R}^{\dim E} \to U_i, \psi_\beta : \mathbb{R}^{\dim F} \to V_j$ are local coordinates. The class of all order n pseudo-differential operator from section of E to that of F is denoted by $\Psi^n(X;E,F)$.

We define $a(x,\xi)$ to be a *symbol* in class $S^m(X;E,F)$ if the following are satisfied,

1. In the local chart presentations or in coordinates of T^*X, $a(x,\xi)$ is a symbol of order m.

2. Globally, $a(x,\xi)$ is a smooth section of the bundle $\mathrm{End}(\pi^*E,\pi^*F)$, the linear transformation from fiber of π^*E to the fiber of π^*F over same base point in T^*X depending continuously on the coordinate on X, over T^*X. Recall that $\pi : T^*X \to X$ is a projection and that π^*E, π^*F are the pull-back vector bundle over T^*X [4]

$$\begin{array}{ccc} \pi^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ T^*X & \xrightarrow{\pi} & X. \end{array}$$

The notion of the symbol **cannot** be extended to the case of the manifold because it is not compatible under coordinate change. But the difference of the same local symbol on two coordinate systems are of lower order. In fact, let $A \in \Psi^m(X;E,F)$, and let $f_i : X_{f_i} \to U_{f_i}, i = 1,2$ be the diffeomorphism from coordinate neighborhood X_{f_i} to an open set in $\mathbb{R}^n$. If $X_{f_1} \cap X_{f_2} \neq \emptyset$ and denote $f_{12} = f_1 \circ f_2^{-1} : f_2(X_{f_1} \cap X_{f_2}) \to f_2(X_{f_1} \cap X_{f_2})$, in the definition $(A_{f_1} u) \circ f_1 = A(u \circ f_1) = A_{f_2}(u \circ f_1 \circ f_2^{-1}) \circ f_2$. Then $(A_{f_1} u) \circ f_{12} = A_{f_2}(u \circ f_{12})$. If the symbol of A_{f_i} is $\sigma_{A_{fi}}$, then

$$\sigma_{A_{f_1}}(f_{12}(x),\xi) - \sigma_{A_{f_2}}(x, {f'}^t_{12}\xi) \in S^{m-1}(f_2(X_{f_1} \cap X_{f_2})).$$

Hence, a good replacement is the principle symbol which only cares the highest order part. The

[4] If $\pi : E \to X$ is a vector bundle and $f : Y \to X$ is a continuous map, then the *pull back bundle* $\pi^*(E)$ over T^*X is defined as $\{(x,v) : f(x) = \pi(v), \forall x \in Y, \forall v \in E\}$.

following proposition shows that the *principal symbol* is globally defined on the cotangent bundle T^*X.[5]

Proposition II.1.8. *There is a well-defined element* $\sigma \in S^m(T^*X)$ *such that* $\sigma - \sigma_{A_f} \in S^{m-1}(T^*X_f)$ *for any coordinate system* $f : X_f \to U_f$. *This element is the* **principal symbol** *of* A.

Remark II.1.9. The symbol

$$\sigma_A : \pi^*E \to \pi^*F \tag{II.9}$$

of A is a globally defined linear map and σ_A is a matrix-valued function over T^*X, i.e. $\sigma_A(x,\xi) \in \mathrm{End}(E_x, F_x)$. Note that each $A \in \Psi_m(E,F)$ has a principal symbol σ_A in $S^m(E,F)/S^{m-1}(E,F)$.

A **can be represented by its symbol,** (amplitude, being more precise). To present it we detour to Riemannian structures on the manifold and Riemannian covariant derivatives. Recall that a *Riemannian manifold* is a real smooth manifold X in which each tangent space is equipped with an inner product $g_{ij} =< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >$ (where x_is are local coordinate on X,) which varies smoothly from point to point. The metric allows us to define various notions such as angles, lengths of curves, curvature. For real vector bundle E over X, we define the *Riemannian vector bundle*, i.e. there is an inner product in each fiber and smoothly depends on the point on the manifold. For complex vector bundle with continuous inner product in each fiber, the name is *Hermition vector bundle*. Every real (complex) vector bundle admits a Riemannian (Hermition) structure. Let (E, p) be a vector bundle over X. A *connection* on E is a linear map

$$\nabla : C^\infty(E) \to C^\infty(T^*X \otimes E), \text{ such that } \nabla(fe) = df \otimes e + f\nabla e, \text{ for all } f \in C^\infty(X), e \in C^\infty(E).$$

The connection can be defined on bases with a matrix of 1-form. Let $e_1, \cdots, e_m$ be basis of frames in E. Then $\nabla e_i = \sum_{i=1}^n \Gamma^i_j \otimes e_j$, where Γ^i_j are 1-forms over base space X. Given a vector field V on X, i.e. a section of the tangent bundle, we define a map from $C^\infty(E)$ to itself by $\nabla_V(e) =< \nabla e, V >$, where $<,>$ means pairing of V with 1-form. ∇_V is called the *covariant derivative* with respect to V, which generalizes the directional derivative. The covariant derivative is *Riemannian* if $V < e, f >=< \nabla_V e, f > + < e, \nabla_V f >$ and it is used to define differentiation of frames in vector bundle

[5] As a convention for simplicity we use the word symbol to mean principal symbol. Denote σ_A the principal symbol of A.

E. A section s in vector bundle E is *parallel along a vector field* V if $(\nabla_{V(x)}e)(x) = 0$ for all x in X where vector field is defined.

The *amplitude* of A is defined similarly as (II.6) and can be represented by its symbol $\sigma_A(x,\xi)$:

$$a(x,y,\xi) = \alpha(x,y)\sigma_A(q(y,(x,\xi_x))), \tag{II.10}$$

where $\alpha \in C^\infty(X \times X)$ has support contained in a small neighborhood of the diagonal such that $\alpha(x,x) = 1, \alpha(x,y) \geq 0$ for all $x, y \in X$ and $q : X \times T^*X \to T^*X : (y,(x,\xi_x)) \mapsto (y,\xi_y)$ where ξ_y is parallel transport of ξ_x from x to y.

Remark II.1.10. The symbol of A is $\sigma_A(x,\xi) = a(x,x,\xi)$.

Since $a(x,y,\xi)$ vanishes outside a small neighborhood U of the diagonal of $X \times X$, there is a homeomorphism

$$v : U \to \text{a neighborhood of zero section in } T^*X : (x,y) \mapsto (x, \exp_x^{-1} y),$$

where $\exp_x : T_xX \to X$ is the exponential map. We define pseudo-differential operator $A : C_c(X,E) \to C_c(X,F)$ by

$$Au(x) = \int_{X \times T_x^*X} e^{i<v(x,y),\xi>} a(x,y,\xi)u(y)\mathrm{d}y\mathrm{d}\xi_x = \int_{X \times T_x^*X} e^{i\Phi(x,y,\xi)} a(x,y,\xi)u(y)\mathrm{d}y\mathrm{d}\xi_x, \tag{II.11}$$

where $\Phi(x,y,\xi) = < \exp_x^{-1}(y), \xi_x >$ is the *phase function.*

Definition II.1.11. An operator $A \in \Psi^m(E,F)$ is *elliptic* if for any compact subset K in X, there exists a constant $c_K > 0$ such that the matrix inverse of $\sigma_A(x,\xi)$ exists and satisfies

$$|\sigma_A(x,\xi)^{-1}| \leq c_K(1+|\xi|)^{-m}$$

for any $(x,\xi) \in T^*X|_K$.

A can be modified and "replaced" without changing its index. The following **assumptions** on A are **standard** in the sense that, both A and σ_A represent some equivalent classes in the KK-group (Appendix E) and we can change A or σ_A without changing the equivalent classes. The reductions/assumptions are explained as follows. These assumptions are used in the thesis without loss the generality.

1. Assume A to be properly supported.

   Under this circumstance, $A : C_c(X,E) \to C_c(X,F)$ extends to a densely defined operator $A : L^2(X,E) \to L^2(X,F)$, where $L^2(X,E)$ is the completion of $C_c(X,E)$ under the inner product $(f,g) = \int_X < f(x), g(x) > \mathrm{d}x$.

   Definition II.1.12. *A is properly supported* if the projection from its amplitude's $X \times X$ support to each X is proper map, equivalently, The projection from the support of the distribution kernel of A to both factors of $X \times X$ is proper map. This definition is equivalent to (II.8).

2. Assume A to be odd and (essentially) self-adjoint acting on $\mathbb{Z}/2$-graded space.

   Replace $A : L^2(X,E) \to L^2(X,F)$ by $\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$ acting on $\mathbb{Z}/2$-graded Hilbert space $L^2(X, E \oplus F)$. Then we can apply functional calculus of A.

3. It is sufficient to consider 0-order pseudo differential operators (Ψ^0).

   In fact, if a self-ajoint, properly supported differential operator $A : L^2(X,E) \to L^2(X,E)$ has order $m > 0$, it can be normalized into a 0-order pseudo-differential operator via functional calculus: $B = \frac{A}{\sqrt{1+A^2}}$. The fact that $\sqrt{1+A^2}$ is invertible and positive implies that A, B have the same index. The reason to consider a 0-order operator is because such an operator is bounded on $L^2(X,E)$ in the case we are interested (Refer to the section II-3). Of course, another way to get a bounded operator is to use Riemannian metric to define Sobolev space $H^s(X,E)$ analogous to the line (II.5). Then $A : H^s(X,E) \to H^{s+m}(X,F)$ is bounded, if $A \in \Psi^m(X;E,F)$ is properly supported.

4. When $\|\xi\|$ is large, the invertibility of σ_A can be modified into σ_A being unitary.[6]

   Hence the ellipticity for a 0-order self-adjoint properly supported odd operator A acting on a $\mathbb{Z}/2$ graded Hilbert space $L^2(X,E)$ is carried out as follows.

[6]Replace σ_A by $(\sigma_A \sigma_A^*)^{-\frac{1}{2}} \sigma_A$. Note that $(\sigma_A \sigma_A^*)^{-\frac{1}{2}}$ is not defined when $\|\xi\|$ is small.

Definition II.1.13. $A \in \Psi^0(E)$ with symbol σ_A is *elliptic* if for any compact subset $K \subset X$,

$$\|\sigma_A(x,\xi)^2 - 1\| \to 0 \text{ uniformly for } x \in K, \text{ and for } \xi \to \infty \text{ in } T_x^*X. \tag{II.12}$$

The reason to study ellipticity is motivated by the following Theorem II.1.14. Being invertible up to smoothing operators is a key feature of elliptic operators. There are several interesting corollaries. An immediate observation is that the solutions for an elliptic operator are smooth. A not very obvious corollary is that elliptic operators have indices. But the elliptic operators acting on a compact X are Fredholm (Appendix A) and have a Fredholm index, which is well-known in Operator Theory and which motivates the study of higher indices for elliptic operators.

Theorem II.1.14. *For elliptic operator P of order m there is a pseudo-differential operator Q (called paramatrix of P) of order $-m$, uniquely defined up to a difference of smoothing operator, such that $PQ - Id$ and $QP - Id$ are smoothing operators.*

II.2 Dirac and Dolbeault operators

Dirac operator is an important and manageable example of an elliptic differential operator. Dolbeault operator is a special case of a Dirac operator. The canonical Dirac operator captures the geometry of the manifold it acts on. The knowledge on Dirac operators is summarized from [20] Chapter 2 and that on "the Dolbeault operators" is summarized from [20], [8].

Clifford algebra

Let V be a vector space over a field k ($\mathbb{C}$ or $\mathbb{R}$) and let q be a bilinear quadratic form on V(i.e. homogenous polynomial of degree two in several variables). Let

$$\mathscr{T}(V) = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

be the free tensor algebra of V. *The Clifford algebra* $\mathrm{Cl}(V,q)$ is an associate algebra with unit defined by:

$$\mathscr{T}(V)/\text{Ideal generated by } \{v \otimes v + q(v,v)1, \text{ for all } v \in V\}.$$

Remark II.2.1. Let $q(v) = q(v,v)$. Then $q(u,v) = \frac{1}{2}(q(u+v) - q(u) - q(v))$.

Example II.2.2.
- Let e_1, e_2 be basis of $V = \mathbb{R}^2$ and let $q(x_1e_1 + x_2e_2) = x_1^2 - x_2^2$. Then $\mathrm{Cl}(\mathbb{R}^2, q)$ and is isomorphic to $\mathbb{C}$ as algebras.
- $\mathrm{Cl}(V, 0)$ is isomorphic to the exterior algebra Λ^*V. Clifford algebra is an enhancement of Λ^*V and sometimes we use elements of Λ^*V to represent elements of $\mathrm{Cl}(V, q)$.

Clifford algebra is "universal" in the following sense and it is equivalent to the definition of Clifford algebra.

Proposition II.2.3. *Let $f : V \to A$ be a linear map from V to any associative k-algebra with unit, such that $f(v) \cdot f(v) = -q(v)1$ for all $v \in V$. Then f extends uniquely to a k-algebra homomorphism $\tilde{f} : \mathrm{Cl}(V, q) \to A$. Moreover, up to isomorphism $\mathrm{Cl}(V, q)$ is the unique associative k-algebra with this property.*

Remark II.2.4. Clifford algebra is $\mathbb{Z}_2$-graded. In fact, $\alpha : V \to V : v \mapsto -v$ extends to an automorphism of $\mathrm{Cl}(V, q)$ using Proposition II.2.3. Let

$$\mathrm{Cl}^i(V, q) = \{\phi \in \mathrm{Cl}(V, q) : \alpha(\phi) = (-1)^i\phi\}, i = 0, 1.$$

Then we have $\mathrm{Cl} = \mathrm{Cl}^0 \oplus \mathrm{Cl}^1$, and $\mathrm{Cl}^i \cdot \mathrm{Cl}^j \subseteq \mathrm{Cl}^{i+j}$, where the indices are taken module 2.

Example II.2.5. There are some special Clifford algebras used in this thesis: $\mathrm{Cl}_{r,s} \doteq \mathrm{Cl}(V, q)$ where $V = \mathbb{R}^{r+s}$ and $q(x) = x_1^2 + \cdots x_r^2 - x_{r+1}^2 - \cdots x_{r+s}^2$. In particular, $\mathrm{Cl}_n \doteq \mathrm{Cl}_{n,0}$ and $\mathrm{Cl}_n^* \doteq \mathrm{Cl}_{0,n}$. If $V = \mathbb{C}^{r+s}$ we denote the corresponding algebras by $\mathbb{C}l_{r,s}, \mathbb{C}\mathrm{l}_n, \mathbb{C}\mathrm{l}_n^*$. $\mathrm{Cl}_{r,s}$ is generated by the q-orthonormal basis in $\mathbb{R}^{r+s} : e_1, \cdots, e_{r+s}$ with the relation:

$$e_ie_j + e_je_i = -2\delta_{ij} \text{ when } i \leq r \text{ and } e_ie_j + e_je_i = 2\delta_{ij} \text{ when } i > r.$$

The are classification of Cl_n and $\mathbb{C}\mathrm{l}_n$ can be found at [20] theorem I.4.3. We need $\mathbb{C}\mathrm{l}_{2n} \simeq M_{2^n}(\mathbb{C})$ in this thesis.

Clifford module

Let W be a vector space over field k and let $\mathrm{End}_k(W, W)$ be the linear transformation of W. W is a $\mathrm{Cl}(V, q)$-*module* over k if there is a k-representation of $\mathrm{Cl}(V, q)$:

$$\rho : \mathrm{Cl}(V, q) \to \mathrm{End}_k(W, W).$$

If V is a k-manifold and $\Lambda^*(T^*V)$ is the exterior bundle over cotangent bundle T^*V, then $\Lambda^*(T^*V)$ is a $\mathrm{Cl}(T^*V)$-module as is shown in the following example.

Example II.2.6.

- Let V be an n-dimensional real manifold and $\Lambda^*(T^*V)$ be the exterior algebra over T^*V. Define $q(v,w) = (v,w), v,w \in V$, where $(\cdot,\cdot)$ is the Riemannian metric on T^*V and where T^*V and TV are identified using the Riemannian metric. Define

$$\mathfrak{c} : T^*V \to \mathrm{End}(\Lambda^*(T^*V)) : v \mapsto \varepsilon(v) - \iota(v),$$

where $\varepsilon(v)w = v \wedge w$ and $\iota(v)$ is the *contraction* with the co-vector $q(v,\cdot) \in V^*$:

$$\iota(v)v_1 \wedge \cdots \wedge v_p = \sum_{i=1}^{n} (-1)^{i+1} q(v_i, v) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p.$$

Since

$$\varepsilon(v)\iota(w) + \iota(w)\varepsilon(v) = q(v,w), v,w \in V, \tag{II.13}$$

then $\mathfrak{c}(v)^2 = -q(v)^2 = -(v,v) \cdot 1 = -\|v\|^2 \cdot 1$ for all $v \in V$ and then we can extend $\mathfrak{c}$ to the Clifford algebra $\mathrm{Cl}(T^*V) \simeq \mathrm{Cl}_n$. We call $\mathfrak{c}(x)$ *Clifford multiplication* on the exterior algebra by $x \in \mathrm{Cl}(T^*V)$.

- Let V be a n dimensional $\mathbb{C}$-linear space with a inner product $(\cdot,\cdot)$ ($\mathbb{C}$-conjugate linear in the first variable, $\mathbb{C}$-linear in the second variable) and a anti-linear involution $x \to x^*$. Define the quadratic form $q(x) = (x^*, x)$. Define the Clifford multiplication by

$$\mathfrak{c} : V \to \mathrm{End}_{\mathbb{C}}(\Lambda^*V) : x \mapsto \varepsilon(x^*) - \iota(x).$$

Using (II.13) we have $\mathfrak{c}(x)^2 = -q(x,x^*) \cdot 1 = -\|x\|^2 \cdot 1$. Then $\mathfrak{c}$ extends to $\mathbb{C}\mathrm{l}(V) \simeq \mathbb{C}\mathrm{l}_n$.

In the example II.2.6 the $\mathrm{Cl}, \mathbb{C}\mathrm{l}$ module is not ireducible. The following is an irreducible $\mathbb{C}\mathrm{l}_{2n}$-module. There is a classification of irreducible modules of Cl_n and $\mathbb{C}\mathrm{l}_n$ in [20] theorem I.5.8.

Example II.2.7. Let V be a $2n$-dimensional real space underlying $\mathbb{C}^n$, where $x_1, \ldots, x_n, y_1, \ldots, y_n$ are orthonormal basis. Choose the Hermitian inner product on V, $\mathbb{C}$-linear in the second coordinate, and define $q(x) = (x^*, x)$ The map

$$\mathfrak{c} : V \to \mathrm{End}_{\mathbb{C}}(\Lambda^{0,*}V) : x_i \mapsto \frac{1}{\sqrt{2}}(\varepsilon(\bar{z_i}) - \iota(z_i); y_j \mapsto \frac{1}{\sqrt{2}}(i\varepsilon(\bar{z_j}) + i\iota(z_j))$$

extends to a homomorphism on $\mathrm{Cl}(V) \otimes \mathbb{C}$ and $\mathrm{Cl}(V) \otimes \mathbb{C} \simeq \mathrm{End}(\Lambda^{0,*}V)$. Then $S = \Lambda^{0,*}V$ is a irreducible $\mathbb{C}\mathrm{l}_{2n}$-module.

A Clifford module can be constructed using a principal bundle. Recall that a *principal G-bundle* is a fiber bundle $\pi : P \to X$ with fiber G (for every point $x \in X$ there is an open set U containing x such that $\pi^{-1}(U)$ is homeomorphic to $U \times G$) together with a continuous left action $G \times P \to P$ by a topological group G such that G preserves each fiber of P and acts freely and transitively on it.

All vector bundles can be obtained from a principal bundle using the associate bundle construction. Let $\pi : P \to X$ be a principal G-bundle over X and let $\mathrm{Homeo}\, F$ (with compact-open topology) be the group of homeomorphisms of another space F. For a continuous homomorphism $\rho : G \to \mathrm{Homeo}\, F$, we construct a vector bundle over X of fiber F by taking the quotient of $P \times F$ by the orbit of the free group action:

$$g(p, f) = (pg^{-1}, \rho(g)f), g \in G, p \in P.$$

The quotient is the bundle associate to P by ρ and denoted by $P \times_\rho F$.

Example II.2.8. Let X be a Riemannian manifold and $P = P_O X$ be the principal $O_n(\mathbb{R})$-bundle of the tangent frames and let $\rho : O_n(\mathbb{R}) \to O(\mathbb{R}^n)$ denote the identity representation and $\rho^* : O_n(\mathbb{R}) \to O(\mathbb{R}^n)$ be defined by $\rho^*(g) = \rho(g^{-1})^T$ (T means transpose). Then $TX = P_O(X) \times_\rho \mathbb{R}^n$ and $T^*X = P_O(X) \times_{\rho^*} (\mathbb{R}^n)^*$. Also, $\Lambda^*(T^*X) = P_O \times_\rho \Lambda^*(\mathbb{R}^n)^*$.

Let M be a Cl_n (or a $\mathbb{C}\mathrm{l}_n$) module and let X be a Riemannian manifold. A *Clifford module* is the associated bundle $P_O \times_{\mathfrak{c}} M$ where $\mathfrak{c} : \mathbb{R}^n \to \mathrm{End}\, M$ is the Clifford representation on M. As a special case, a *Clifford bundle* is defined by $\mathrm{Cl}(T^*M) = P_O \times_{\mathfrak{c}} \Lambda^*(T^*X)$, where $\mathfrak{c}$ is the map in the first item of Example II.2.6

The Canonical Dirac operator is defined on a “irreducible” Clifford module S over an oriented manifold X (*spin structure* is further required). Recall that a *spin group* Spin_n s a two fold cover of SO_n [7] and X has a *spin structure* if the principal bundle P_{SO_n} over X, decided by the structure group SO_n, can be lifted to a principle Spin_n-bundle. Spin group is a group in the Clifford algebra: $\mathrm{Spin}_n \subset \mathrm{Cl}_n$[20].

Example II.2.9.

- When $n \geq 3$, a spin structure on E is a principal Spin_n bundle $P_{\mathrm{Spin}}(E)$ with a two sheet covering $\eta : P_{\mathrm{Spin}}(E) \to P_{SO}(E)$ such that $\eta(pg) = \eta(p)\eta_0(g), \eta_0 : \mathrm{Spin}_n \to SO_n, \forall p \in P_{\mathrm{Spin}}(E), g \in \mathrm{Spin}_n$.
- When $n = 2$, a spin structure of E is defined analogously with $Spin_n$ replaced by SO_2 and $\eta_0 : SO_2 \to SO_2$ being the two fold covering.

Let X be an oriented Riemannian manifold with a spin structure $\eta : P_{\mathrm{Spin}}(E) \to P_{SO}(X)$, a *(real) spinor bundle* X is a bundle of the form

$$S(TX) = P_{\mathrm{Spin}}(TX) \times_{\mu} M,$$

where M is a left irreducible Cl_n-module and $\mu : \mathrm{Spin}_n \to SO(M)$ is the representation of elements of $\mathrm{Spin}_n \subset \mathrm{Cl}_n$ on M. $S(T^*X)$ is a Clifford module over $\mathrm{Cl}(T^*X)$. Sometimes we use elements from the exterior algebra to represent the elements in $\mathrm{Cl}(T^*X)$ because of the vector space isometry $\Lambda^*(T^*X) \simeq \mathrm{Cl}(T^*X)$.

Dirac operators

Let X be an n-dimensional Riemannian manifold with Clifford bundle $\mathrm{Cl}(TX)$. Let V be a left $\mathrm{Cl}(TX)$-module. To define Dirac operator we need to define a connection on V. Let ∇ be a Levi-Civita connection on TX. ∇ extends to $\mathrm{Cl}(T^*X)$, the bundle of Clifford algebra constructed from TM. Hence ∇ extends to $\mathrm{Cl}(TM)$.

A connection $\nabla^V : C^\infty(M,V) \to C^\infty(M, T^*M \otimes V)$ on V is a *Clifford connection* if ∇^V is compatible with ∇,

$$\text{i.e. } [\nabla^V_X, \mathfrak{c}(a)] = \mathfrak{c}(\nabla_X a), X \in C^\infty(M,TM), a \in C^\infty(M,TM),$$

where $\mathfrak{c}(a)$ is Clifford multiplication of a on V. A Clifford connection always exists for Clifford module ([8] Corollary 3.41).

[7] When $n \geq 3$, Spin_n is the universal cover.

Definition II.2.10. Let ∇^V be a Clifford connection on X. The the first-order differential operator

$$D : L^2(X,V) \to L^2(X,V), Du = \sum_{j=1}^{n} \mathfrak{c}(e^j) \cdot \nabla^V_{e_j} u$$

is called *the Dirac operator*, where $e_1, \ldots, e_n$ are orthonormal basis for TX, $\varepsilon^1, \ldots, e^n$ are dual bases in T^*X and "$\mathfrak{c}(\cdot)$" denotes Clifford multiplication of $e^j \in \mathrm{Cl}(T^*X)$ on V via module structure. We say that D^2 is a *generalized Laplacian*. A spin structure on X is needed if $V = S$ is a irreducible $\mathrm{Cl}(T^*X)$-module.

Remark II.2.11. If $X = \mathbb{R}^n, e_i$s are constants satisfying $\mathfrak{c}(e_j) \cdot \mathfrak{c}(e_j) + \mathfrak{c}(e_j) \cdot \mathfrak{c}(e_i) = -2\delta_{ij}$, then D^2 is reduced to

$$-(\frac{\partial^2}{\partial e_1^2} + \cdots + \frac{\partial^2}{\partial e_n^2}).$$

Proposition II.2.12 ([20]). *Let D be a Dirac operator of bundle S defined above, the (principal) symbol*

$$\sigma(D)(x,\xi) = i\mathfrak{c}(\xi)$$

and $\sigma(D^2)(x,\xi) = \|\xi\|^2$, where $\mathfrak{c}(\cdot)$ means Clifford multiplication by ξ on the fiber E_x. Dirac operator is elliptic.

Example II.2.13. Let $n = 1, X = \mathbb{R}, \mathrm{Cl}_1 = V = \mathbb{C}$, $e_1 = i$, $\nabla_{e_1} = \dfrac{\partial}{\partial x_1}$, so $D = i\dfrac{\partial}{\partial x_1}$.

Dirac operators are commonly used in index theory of elliptic operators. Next we introduce a class of examples of Dirac operators that are used in the formulation of the index formula, namely, Dolbeault operators. They are first order differential operators acting on differential forms on an almost complex manifold.

Let V be a real vector space and let $J : V \to V$ be a $\mathbb{R}$-linear isomorphism. Then J is *a complex structure* on V if

$$J^2 = -\mathrm{Id}.$$

We may equip real vector space V with a structure of complex vector space if V admits a complex structure J. Let $v \in V$ and then a scaler multiplication of a complex number $a + ib(a, b \in \mathbb{R})$ is defined by $(a+ib)v = av + bJ(v)$.

Example II.2.14. The complex space $\mathbb{C}^n$ has a complex structure. In fact, the element $(z_1, \cdots, z_n), z_j =$

$x_j + iy_j, x_j, y_j \in \mathbb{R}$ viewed as in a $\mathbb{R}$-space is expressed as $(x_1, y_1, \cdots, x_n, y_n)$and the complex structure J is defined to be

$$J(x_1, y_1, \cdots, x_n, y_n) = (-y_1, x_1, \cdots, -y_n, x_n).$$

It corresponds the multiplication by i in $\mathbb{C}^n$.

Similar to the exterior algebra $\Lambda^* V$ of real vector space V we build the complex exterior algebra for V with a complex structure J: Consider $V \otimes_{\mathbb{R}} \mathbb{C}$ and extend J to a $\mathbb{C}$-linear map on $V \otimes_{\mathbb{R}} \mathbb{C}$ by $J(v \times a) = J(v) \otimes a$ for $v \in V, a \in \mathbb{C}$ and still $J^2 = -Id$. Let $V^{1,0}$ and $V^{0,1}$ be the eigenspace correpond the eigenvalues $i, -i$ of J respectively. Then there is an isomorphism $V \otimes_{\mathbb{R}} \mathbb{C} \simeq V^{1,0} \oplus V^{0,1} : ix_j \mapsto x_j - iy_j, y_j \mapsto x_j + iy_j$ where $V^{1,0}$ and $V^{0,1}$ are $\mathbb{R}$-isomorphic and related by conjugation: $\overline{v \times a} = v \times \overline{a}$ for $v \in V, a \in \mathbb{C}$. Let $\Lambda^{p,q}$ be the subspace of $\Lambda^* V \otimes_{\mathbb{R}} \mathbb{C}$ of form $u \wedge v$ where $u \in \Lambda^p V^{1,0}, v \in \Lambda^q V^{0,1}$ and we have the direct sum $V \otimes_{\mathbb{R}} \mathbb{C} = \sum_{r=0}^{2n} \sum_{p+q=r} \Lambda^{p,q} V$.

To some manifold J can be similarly defined. Let X be a manifold. A vector bundle isomorphism

$$J : TX \to TX$$

is called an *almost complex structure* for X if the isomorphism $J_x : T_x X \to T_x X$ of fiber over each point $x \in X$ is a complex structure for $T_x X$. A $2n$-dimensional differentiable manifold has an almost complex structure.

Example II.2.15. Let X be a manifold and T^*X be its cotangent bundle and (x, ξ) be its local coordinate. There is a almost complex structure on

$$T^*X : J(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \xi_n}) = (\frac{\partial}{\partial \xi_1}, -\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial \xi_n}, -\frac{\partial}{\partial x_n}).$$

(Horizontal part x imaginary and vertical part ξ real.)

Define $\Lambda^{p,q} T^*X$, the complex exterior algebra bundle of type (p,q) on X with almost complex structure, by working on each fiber of T^*X. The sections of the vector bundle ΛT^*X over X is the $\mathbb{C}$-valued differential forms of type (p,q) (also denoted by $\Lambda^{p,q}(X)$).

Let $\{w_1, \cdots, w_n\}$ be local frame for $T^*X^{1,0}$ restrict to some open set $U \subset X$, then $\{\overline{w_1}, \cdots, \overline{w_n}\}$ is a local frame for $T^*X^{0,1}$. If we let d be the exterior derivative from $\Lambda^{p,q}(T^*X)$ to $\sum_{r+s=p+q-1} \Lambda^{r,s}(T^*X)$.

Then Set $\partial = \pi_{p+1,q} \circ d, \overline{\partial} = \pi_{p,q+1} \circ d$ where $\pi_{p,q}$ is the projection to the subspace space of (p,q)-forms.

Example II.2.16. For a complex manifold [8] with local holomorphic coordinate $z_1, \cdots, z_n$, then

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j, \overline{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z_j}} d\overline{z_j}.$$

Definition II.2.17. *Dolbeault operator* is defined as

$$\bar{\partial} + \bar{\partial}^* : \Lambda^{0,*} \to \Lambda^{0,*}$$

for manifold X with an almost complex structure. Alternatively, *Dolbeault operator* is an operator on $L^2(X, \Lambda^{0,*}T^*X)$ with symbol

$$\sigma(x, \mathrm{d}f) = i(\varepsilon(\overline{\partial} f) - \iota(\partial f))$$

where $\iota(\mathrm{d}z)$ is defined by contraction using $\mathbb{C}$ linear quadratic form $q(\mathrm{d}\bar{z}, \mathrm{d}z) = (\mathrm{d}z, \mathrm{d}z)$ where $(\cdot,\cdot)$ is a Hermitian metric $\mathbb{C}$-linear in the second coordinate. [9]

Remark II.2.18. Since $\sigma(x, \mathrm{d}x_j)^2 = \frac{i}{2}\varepsilon(\mathrm{d}\bar{z}_j) - \frac{i}{2}\iota(\mathrm{d}z_j)$, and $\sigma(x, \mathrm{d}y_j) = -\frac{1}{2}\varepsilon(\mathrm{d}\bar{z}_j) - \frac{1}{2}\iota(\mathrm{d}z_j)$, then $\sigma(x, \mathrm{d}x_j)^2 = \frac{1}{2}\|x_j\|^2, \sigma(x, \mathrm{d}y_j)^2 = \frac{1}{2}\|y_j\|^2$. So Dolbeaut operators are special kinds of Dirac operators. [10]

Example II.2.19. Consider TX with an almost complex structure in the previous example, and let (x, ξ) be the local coordinate and η, ζ be the coordinate of the cotangent vector in $T^*(TX)$ in the x and ξ direction respectively. Similar to the case of complex manifold $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial x})$, the Dolbeault operator is $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ acting on the exterior algebra bundle $\Lambda^{0,*}(T^{0,1}(TX))^*$. Recall that $\mathrm{d}z = \mathrm{d}\xi + \mathrm{d}x = \zeta + i\eta$ and $\mathrm{d}\bar{z} = \mathrm{d}\xi - i\mathrm{d}x = \zeta - i\eta$ The symbol of the Dolbeault operator is

$$\sigma((x,\xi),\zeta) = \frac{1}{2}(\varepsilon(\eta + i\zeta) - \iota(-\eta + i\zeta)), \sigma((x,\xi),\eta) = \frac{1}{2}(\varepsilon(-\zeta + i\eta) + \iota(-\zeta - i\eta)),$$

which are elements in $\pi^*(\Lambda^*(T^*X))$ and where $\pi : T(T^*X) \to T^*X$ is vector bundle projection.

[8] The manifold is locally homeomorphic to $\mathbb{C}^n$ and the transition functions are holomorphic.

[9] The Dolbeault operator defined using symbol represent a larger class of operators, which differ by a 0-order terms.

[10] Being more precise, there is a multiple of $\sqrt{2}$ on the operator or symbol, so that $\sigma(x,v)^2 = \|v\|^2$.

Remark II.2.20. Dolbeault operator on an almost complex manifold can be represented in terms of Definition II.2.10 using a *superconnection.* If X is Kähler, which means that the Levi-Civita connection ∇ on $\Lambda^* T^* X$ preserves $\Lambda^{0,*}(T^{0,1})^*$, then it can be shown that the restriction of ∇ on $\Lambda^{0,*}(T^{0,1})^*$ is a Clifford connection. The Dirac operator constructed using this connection coincides with $\sqrt{2}(\bar{\partial}+\bar{\partial}^*)$, which coincide with Definition II.2.17 of the Dolbeault operator. In fact, when X is Kähler,

$$\bar{\partial} = \sum \varepsilon(\bar{z}^i)\nabla_{\bar{z}^i}, \bar{\partial}^* = -\iota(\bar{z}^i)\nabla_{z_i}(\text{[8] Proposition 3.67}).$$

In general we can choose the Levi-Civita connection ∇ [11] of the bundle $\Lambda^{0,*}(T^{0,1})^*$ and define the Dirac operator by

$$D = \sqrt{2}\sum(\varepsilon(\bar{z}^i)\nabla_{\bar{z}_i} - \iota(z^i)\nabla_{z_i}). \tag{II.14}$$

Recall that $\mathfrak{c}(\bar{z^j}) = \sqrt{2}\varepsilon(\bar{z^j})$ and $\mathfrak{c}(z^j) = -\sqrt{2}\iota(z^j)$ by Example II.2.7, then (II.14) coincides with Definition II.2.10. The Dirac operator D in (II.14) and the Dolbeault operator $\sqrt{2}(\bar{\partial}+\bar{\partial}^*)$ are Dirac operators of the same Clifford module and they have the same symbol.

II.3 Equivariant elliptic operators

This section introduces our target–the elliptic operator for proper cocompact action and proves an elliptic property (Theorem II.1.14) in this case. Given a complete Riemannian manifold X, a vector bundle (E,π) and an properly supported elliptic operator $A : C_c(X,E) \to C_c(X,E)$, we study the case when X admits a group action and when A commutes with this group G.

We impose the following assumptions on the group G :

1. G is ***locally compact***, i.e. the group is also a locally compact Hausdorff topological space, which means on G there is a unique, up to scaler multiple, left-invariant Haar measure. Recall that a *left-invariant Haar measure* μ on G is a nonzero left invariant Radon measure, i.e. $\mu(sE) = \mu(E)$ for any Borel set $E \subset G$ and $s \in G$.

2. G is ***uni-modular***, i.e. there exists a bivariant Haar measure on G. The name uni-modular comes from the "modular function". In fact, if G is not unimodular, μ-is not right invariant,

[11] This connection needs to be projected to the sub-bundle because the original operator may not preserve the subspace.

instead, there will be a continuous homomorphism $\Delta : G \to \mathbb{R}$ called *modular function*, so that $\mu(Es) = \Delta(s)\mu(E)$. When G is unimodular, $\Delta(g) = 1, \forall g \in G$. A uni-modular locally compact group contains two most interesting cases,

- G is compact with $\mu(G) = 1$;
- G is discrete, with $\mu(e) = 1$.

For simplicity, write $\mathrm{d}g \doteq \mathrm{d}\mu(g)$. From the above assumption we have $\mathrm{d}(tg) = \mathrm{d}g, \mathrm{d}(gt) = \mathrm{d}g$ and $\mathrm{d}(g^{-1}) = \mathrm{d}g$ for any $g, t \in G$.

The action G on X is assumed to be

1. ***Proper***, i.e. the pre-image of a compact set via the continuous G-action $G \times X \to X \times X : (g, x) \mapsto (g \cdot x, x)$ is compact.

   Remark II.3.1. Isotropy group $G_x = \{g \in G \mid gx = x\}$ is compact.

2. ***Cocompact***, i.e. the quotient X/G is compact.

3. ***Isometric***, i.e. $< x, y >=< gx, gy >, g \in G, x, y \in TX$. [12]

Remark II.3.2. The reason to assume isometric action is to make sure the sphere bundle $SX \in TX$ is a G-manifold. This is a technical requirement. One reason to choose properly cocompact action is the existence of the cutoff function for any X with properly cocompact action.

Definition II.3.3. A function $c \in C_c^\infty(X)$ is said to be a *cutoff function* if it is a non-negative, compact supported function $c \in C_c^\infty(X)$ so that $\int_G c(g^{-1}x)\mathrm{d}g = 1$ for all $x \in X$.

Proposition II.3.4. *A properly cocompact space has a cutoff function $c \in C_c^\infty(X)$ given by*

$$c(x) = \frac{h(x)}{\int_G h(g^{-1}x)\mathrm{d}g},$$

where $h(x) \in C_c^\infty(X)$ is nonnegative and has non empty intersection with each orbit.

Remark II.3.5. Another reason for the assumptions is the structure theorem of proper space and the existence of volume element on X.

[12] Properness and cocompactness imply the isometry condition.

Example II.3.6. Let G be a Lie group and H be a compact subgroup, and let $X = G/H$, a homogeneous space consisting of all the left cosets of H in G. It has a proper G-action. Let E be a representation space of H. The induced representation space $Y = G \times_H E$, the orbits of $G \times E$ by the action of H: $h(g,e) = (gh, h^{-1}e), g \in G, e \in E, h \in H$, which forms a G-vector bundle over X, is a proper G-space. Every proper space turns out to have such a local structure by the slice theorem appearing below.

Definition II.3.7. [27] Let X be a G-space and K (referred as the *slicing subgroup*) a close subgroup of G. A K-invariant subset $S \subset X$ is a *K-slice* in X if

1. The union $G(S)$(called tubular set) of all orbits intersecting S is open;

2. There is a G-equivariant map $f : G(S) \to G/K$ called the *slicing map,* such that $S = f^{-1}(eK)$.

Theorem II.3.8 (Slice theorem). *Let G be a locally compact group and X be a proper G-space and $x \in X$. Then for any neighborhood O of x in X, there exists a compact subgroup K of G with $G_x \subset K$ and a K-slice S such that $x \in S \subset O$*

An introduction of the slice theorem may be found at [2] section 2. According to [10] Ch. II, Theorem 4.2, the tubular set $G(S) \subset X$ with a compact slicing subgroup K is a twisted product, i.e. $G(S)$ is G-homeomorphic to $G \times_K S$. Since X is covered by G-invariant neighborhood and X/G is compact, then X has a finite sub-cover:

$$X = \cup_{i=1}^N G \times_{K_i} S_i = \cup_{i=1}^N G(S_i). \tag{II.15}$$

The local structure (II.15) of X defines a G-invariant measure on X, denoted by the volume element dx. In fact, The measure of a set in $G(S_i)$ can be calculated from the measure on G and on S_i divided by the measure of K_i. To get the measure of a set in X, intersect it with each slice and times the measure of the portion in the slice by the G-invariant partition of unity function and figure out the weighted sum.

With uni-modular group G acting on X, let (E, π) be a *G-vector bundle* over X, i.e.

1. There is a smooth G action on E such that $\pi(gv) = g\pi(v)$ for $v \in E$;

2. The maps of the fibers $g : E_x \to E_{gx}$ are linear.

We move on to the properly supported pseudo-differential operator $A : C_c^\infty(X,E) \to C_c^\infty(X,E)$. There is a natural G action on $C_c(X,E)$, the set of continuous sections of the G-bundle E vanishing at infinity, by

$$(g \cdot f)(x) = g(f(g^{-1}x)), g \in G, f \in C_c^\infty(X,E).$$

Choose a G-invariant Hermitian structure on E and let $L^2(X,E)$ be the completion of $C_c(X,E)$ under inner product, which is invariant under the left action of G:

$$< f,g >_{L^2} = \int_X < f(x), g(x) >_{E_x} \mathrm{d}x, x \in X.$$

Let $E = E_0 \oplus E_1$ be a $\mathbb{Z}/2$-graded G-vector bundle and let A be a self-adjoint operator with odd grading i.e. $A = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix}$. Recall that A is of order m if its symbol $\sigma_A(x,\xi)$ satisfies

$$|\frac{\partial^a}{\partial x^{|a|}} \frac{\partial^b}{\partial \xi^{|b|}} \sigma_A(x,\xi)| \leq C_{a,b,K}(1+\|\xi\|)^{m-|b|} \tag{II.16}$$

for any compact set $K \subset X$, where $C_{a,b,K}$ is a constant depending on a,b,K. And A can be constructed from its amplitude via line (II.11). Conversely, the principle symbol $\sigma_A(x,\xi)$ of A can be computed locally by the term with highest degree in ξ in $e^{-ix\cdot\xi}Ae^{ix\cdot\xi}$ for properly supported A. Another concept associating A and σ_A is the integral kernel of A.

Definition II.3.9. $K_A(x,y) \in \mathrm{End}(E_y,F_x)$ is said to be the *Schwartz kernel* of A if

$$Au(x) = \int_X K_A(x,y)u(y)\mathrm{d}y \text{ for all } u(x) \in C_c^\infty(X,E).$$

$K_A(x,y)$ is expressed in the distributional sense

$$K_A(x,y)(w) = \int_{X\times T^*X} e^{i\Phi(x,y,\xi)} a(x,y,\xi) w(x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi, w \in C_c^\infty(X\times X). \tag{II.17}$$

Definition II.3.10. $A \in \Psi^0(X;E,E)$ is a *G-invariant* operator, if

$$A(gf) = gA(f), f \in L^2(X,E), g \in G.$$

The action of G on A is then defined by $g(A) = gAg^{-1}$.

Remark II.3.11. The Schwartz kernel of a G-invariant operator A satisfies

$$K_A(x,y) = K_A(gx,gy) \text{ for all } x,y \in X, g \in G. \tag{II.18}$$

The G action on vector bundle E,F gives rise to a G-bundle $\mathrm{End}(\pi^*E,\pi^*F)$, and the symbol of a G-invariant operator A is a G-invariant section in this vector bundle. Conversely, if $\sigma_A(x,\xi)$ a G-invariant symbol, then there is a G-invariant operator with symbol $\sigma_A(x,\xi)$:

Denote by $\Psi^n_G(\Psi^n_{G,p})$ the subset of G-invariant (properly supported) operators in Ψ^n and by Ψ^n_c the subset of operators with compact support. There is an *averaging process* defined by [12]:

$$\mathrm{Av}_G : \Psi^*_c \to \Psi^*_{G,p} : P \mapsto \int_G gPg^{-1}\mathrm{d}g. \tag{II.19}$$

Clearly, Av_G is surjective, and $Av_G(cA)$, where $c(x)$ is a cutoff function, is a G-invariant operator with the symbol $a(x,\xi)$. Note that if $A \in \Psi^*_{G,p}$ then $A = \mathrm{Av}_G(cA)$.

Next we focus on elliptic pseudo-differential operators. A pseudo-differntial operator $A \in \Psi^m$ is *elliptic* if there exists $B \in \Psi^{-m}$ so that

$$\|\sigma_A\sigma_B - I\| \to 0 \text{ and } \|\sigma_B\sigma_A - I\| \to 0 \tag{II.20}$$

uniformly in $x \in K$, where K is any compact subset in X, as $\xi \to \infty$ in T^*_xX.

Proposition II.3.12. *1. If $A \in \Psi^n_c$, then* $\mathrm{Av}(A) \in \Psi^n_{G,p}$.

2. *If $A \in \Psi^n_{G,p}(X)$ is elliptic, then there exists a parametrix $Q \in \Psi^{-n}_{G,p}(X)$ such that*

$$1 - AQ = S_1 \in \Psi^{-\infty}_{G,p}(X), 1 - QA = S_2 \in \Psi^{-\infty}_{G,p}(X), \tag{II.21}$$

where $\Psi^{-\infty}_{G,p}(X) = \cap_{n\in\mathbb{R}}\Psi^n_{G,p}(X)$ is the set of smoothing operators.

3. *If $S \in \Psi^{-\infty}_{G,p}(X)$, then $K_S(x,y)$ is smooth and properly supported.*

Proof. 1. Clearly, $\mathrm{Av}_G(A) \in \Psi^*_{G,p}(X)$. Let $a(x,y,\xi) \in S^m(X \times T^*X)$ be the amplitude and by definition $K = \{(x,y) \in X \times X | a(x,y,\xi) \neq 0\}$ is compact. Using the fact that the Riemannian metric on T^*X is G-invariant and the measure on X is G-invariant, we calculated the amplitude for $\mathrm{Av}_G(A)$ as

$$\int_G a(g^{-1}x, g^{-1}y, \xi_{g^{-1}x})\mathrm{d}g$$

which is in S^n because the integral is taken over a compact set $\{g \in G | (g^{-1}x, g^{-1}y) \in K\}$.

2. Let $A \in \Psi^n_{G,p}(X)$ be elliptic and $c \in C_c^\infty(X)$ be the cutoff function of X. Cover X by bounded open balls and finitely many open balls $\{U_i\}_{i=1}^N$ such that supp(c) : supp$(c) \subset \cup_{i=1}^N U_i$. Let $\{a_i\}_{i=1}^N$ be a partition of unity subordinate to the finite cover. Since A is elliptic $|\sigma_A| \geq C(1+|\xi|)^n$ for all $|\xi| \geq C_K$ uniformly for compact $K \subset X$, then there exists $Q_i \in \Psi_c^{-n}(U_i), 1 \leq i \leq N$ so that

$$AQ_i - a_i = R_{1,i}, Q_iA - a_i = R_{2,i}$$

are elements in $\Psi_c^{-\infty}(U_i)$. Extend elements in $\Psi_c^*(U_i)$ to $\Psi_c^*(X)$ and then

$$c\sum_{i=1}^N Q_iA - c = c\sum_{i=1}^N R_{2,i}.$$

Since $\sum_{i=1}^N Q_i \in \Psi_c^{-n}(X)$, $\sum_{i=1}^N R_{2,i} \in \Psi_c^{-\infty}(X)$, then set

$$Q = \int_G g(c\sum_{i=1}^N Q_i)\mathrm{d}g \in \Psi^{-n}_{G,p}(X) \text{ and } S = \int_G g(c\sum_{i=1}^N R_{2,i})\mathrm{d}g \in \Psi^{-\infty}_{G,p}(X).$$

Then $QA = \int_G g(c\sum_{i=1}^N Q_i)A\mathrm{d}g = \int_G g(c)g(\sum_{i=1}^N Q_iA)\mathrm{d}g = \int_G g(c)\mathrm{d}g + \int_G g(c)g(\sum_{i=1}^n R_{2,i})\mathrm{d}g = I + S$.

Similarly, there is a $Q' = \int_G g(\sum_{i=1}^N Q_ic)\mathrm{d}g \in \Psi^{-n}_{G,p}(X)$ and $S' \in \Psi^{-\infty}_{G,p}(X)$ so that $AQ' - I = S'$. Since $Q' + SQ' - Q = (1+S)Q' - Q = Q(AQ' - 1) = QS'$, then $Q' - Q \in \Psi^{-\infty}_{G,p}(X)$. Hence there are $S_1, S_2 = S \in \Psi^{-\infty}_{G,p}(X)$ such that $AQ = 1 + S_1, QA = 1 + S_2$.

3. If $A \in \Psi^{-\infty}_{G,p}(X)$, then $cA \in \Psi_c^{-\infty}(X)$.

We know that $cA \in \Psi_c^{-\infty}(X)$ is equivalent to the fact that $K_{cA}(x,y)$ is smooth and compactly supported in $X\times X$. Therefore the statement follows from the fact that

$$K_A(x,y)=K_{\mathrm{Av}_G(cA)}(x,y)=\int_G K_{cA}(g^{-1}x,g^{-1}y)\mathrm{d}g$$

and the fact that the integral vanish outside a compact set in G.

□

II.4 Von Neumann trace and L^2-index

Let X be a complete Riemannian manifold on which a locally compact unimodular group G acts properly and cocompactly. Let $c\in C_c^\infty(X)$ be a cutoff function of X. Let E,F be G-bundles over X. Let $A:L^2(X,E)\to L^2(X,F)$ ($A\in\Psi^*_{G,p}(X,E,F)$) be a G-invariant, properly supported elliptic operator with the distribution kernel $K_A:X\times X\to\mathrm{End}(E,F)$. When X is compact and when G is trivial, $\dim\mathrm{Ker}A,\dim\mathrm{Ker}A^*$ are finite and their difference defines the index of A. In the equivariant case we measure the size of $\mathrm{Ker}A,\mathrm{Ker}A^*$ by a real number (von Neumann dimension). An L^2-index of A, analogous to the Fredholm index is defined at the end of the section. Refer to Appendix B for a brief introduction for von Neumann algebra and trace.

We start with a few examples of defining von Neumann trace of the projections.

Example II.4.1 (Atiyah's L^2-index theorem for free cocompact action [4].). Let X be a compact manifold, where $\tilde{X}$ is its universal cover and $G=\pi_1(X)\curvearrowright\tilde{X}$ freely with fundamental domain U. Let $D:L^2(X)\to L^2(X)$ be an elliptic differential operator and $\tilde{D}:L^2(\tilde{X})\to L^2(\tilde{X})$, a G-invariant operator, be the lift of D to $\tilde{X}$. To find the correct notion in measuring $\mathrm{Ker}\tilde{D}$ and $\mathrm{Ker}\tilde{D}^*$, the fact that $\dim\mathrm{Ker}D=\mathrm{tr}P_{\mathrm{Ker}D}$ inspires to apply appropriate trace to $P_{\mathrm{Ker}\tilde{D}}$, the projection onto $\mathrm{Ker}\tilde{D}$. Now $P_{\mathrm{Ker}\tilde{D}}$ is a G-invariant operator acting on the Hilbert space:

$$L^2(\tilde{X})=L^2(G)\otimes L^2(U). \tag{II.22}$$

In (II.22), G acts on $L^2(G)$ by the left regular representation $g\cdot k(h)=k(g^{-1}h),g,h\in G,k\in L^2(G)$ and on $L^2(U)$ trivially. Then $P_{\mathrm{Ker}\tilde{D}}$ is an element in

$$\mathscr{R}(L^2(G))\otimes\mathscr{B}(L^2(U)), \tag{II.23}$$

where $\mathscr{R}(L^2(G))$ is the commutant of the left regular representation of G, i.e. the weak closure of the right regular representation of G. There is a natural von Neumann trace defined on (II.23):

$$\mathrm{tr}_G(S \otimes T) = \tau(S) \cdot \mathrm{tr}(T), \tag{II.24}$$

where

$$\tau(\sum_{g \in G} c_g U_g) = c_0, c_g \in \mathbb{C}, U_g(k)(h) = k(hg)$$

and $\mathrm{tr}(T) = \sum_i < Te_i, e_i >$, with $\{e_i\}$ the orthonormal basis for $L^2(U)$.

Hence the L^2-index of $\tilde{D}$ is defined as

$$\mathrm{ind}\tilde{D} = \mathrm{tr}_G P_{\ker \tilde{D}} - \mathrm{tr}_G P_{\ker \tilde{D}^*}. \tag{II.25}$$

Example II.4.2 (L^2-index theorem of homogeneous space for Lie group [12].). Let G be a unimodular Lie group and H be a compact subgroup. Consider the homogenous space $M = G/H$ of left cosets of H in G and a G-bundle E over $M = G/H$, then the fiber of E at eH, denoted by $V = E|_{eH}$, is a H-space,

$$\begin{array}{ccc} V = E|_{eH} & \xrightarrow{h \in H} & V = E|_{eH} \\ \downarrow & & \downarrow \\ eH & \xrightarrow{h \in H} & eH \end{array}$$

It is easy to see that E is a representation of G induced by the representation of H on V:

$$E = G \times_H V, E = \mathrm{induce}_H^G V.$$

Let $A : C_c^\infty(M, E_1) \to C_c^\infty(M, E_2)$ be a G-invariant and properly supported pseudo-differential operators of order n. Then $C_c^\infty(M, E)$ is identified with $(C_c^\infty(G) \otimes V)^H$, the elements in $C_c^\infty(G) \otimes V$ that are invariant under the action of H, where H acts on $C_c(G) \otimes V$ by

$$h(f(g), e) = (f(gh^{-1}), h(e)), \forall h \in H, g \in G, e \in V.$$

The actions of G and that of H on $L^2(G) \otimes V$ commutes, because G acts on $L^2(M, \mathscr{E}) = (L^2(G) \otimes$

$E)^H$ by

$$g(f(h) \otimes e) = f(g^{-1}h) \otimes e, \forall g, h \in G, f \in C_c^\infty(G), e \in E.$$

Since Ker$A$ is a G-invariant subspace of $L^2(M,E) = (L^2(G) \otimes V)^H$, the projection $P_{\mathrm{Ker}A}$ belongs to $\mathscr{R}(L^2(G)) \otimes \mathrm{End}(V)$, on which there is a well defined trace. $\mathscr{R}(L^2(G))$ has a dense subset S such that $\forall x \in S$, there exist $f \in L^2(G)$ such that $x = R(f) = \int_G f(g) U_g \mathrm{d}g$, where $U_g(k(h)) = k(hg), k \in L^2(G), h, g \in G$. There is a canonical faithful, finite and normal trace defined by the property

$$\tau(R(f)^* R(f)) = \int_G |f(g)|^2 \mathrm{d}g \text{ for } f \in L^2(G) \text{ with } R(f) \in \mathscr{R}(L^2(G)). \tag{II.26}$$

Define $\mathrm{tr}_G(P_{\mathrm{Ker}A})$ by multiplying the trace τ on $\mathscr{R}(L^2(G))$ by the matrix trace Tr on End$E$. Then we have the L^2-index

$$\mathrm{ind} A = \mathrm{tr}_G P_{\mathrm{ker}A} - \mathrm{tr}_G P_{\mathrm{ker}A^*} \in \mathbb{R}.$$

In order to find a finite "L^2-index" for the G-invariant operators on properly cocompact manifold X, we make the following definition for the G-trace. At the first glance, the definition looks different from the one discussed in the previous examples. But the two traces coincide on the projections onto the kernel of an elliptic operator.

Definition II.4.3. A bounded operator $S : L^2(X,E) \to L^2(X,E)$, which commutes with the action of G, is of G-trace class if S is G-invariant and if $\phi S \psi$ is of trace class for all $\phi, \psi \in C_c^\infty(X)$. Recall that a bounded operator T on Hilbert space H is of trace class if $\sum_i | < Te_i, e_i > | < \infty$ where e_is are orthonormal basis of the Hilbert space and its trace is defined as $\mathrm{tr}(T) = \sum\limits_i < Te_i, e_i >$. If S is a G-trace class operator, we calculate the G-trace by the formula

$$\mathrm{tr}_G(S) = \mathrm{tr}(c_1 S c_2), \tag{II.27}$$

where $c_1, c_2 \in C_c^\infty(X)$ being positive and satisfying $c_1 c_2 = c$ for any cutoff function c on X.

Remark II.4.4. This definition is essentially the definition of the G-trace class operator appeared in [4] where G is assumed to be discrete. Similar to Lemma 4.9 of [4], we prove in the following proposition that tr_G is well defined, i.e. tr_G is independent of the choice of c_1, c_2 and c.

Proposition II.4.5. *Let S (bounded, G-invariant and positive) be a G-trace class operator and*

$c_1, c_2, d_1, d_2 \in C_0^\infty(X)$ *be positive functions satisfying* $\int_G c_1(g^{-1}x)c_2(g^{-1}x)\mathrm{d}g = 1$ *and* $\int_G d_1(g^{-1}x)d_2(g^{-1}x)\mathrm{d}g = 1$*, i.e.,* $c = c_1c_2, d = d_1d_2$ *are cutoff functions on* X*. Then* $\mathrm{tr}(c_1Sc_2) = \mathrm{tr}(d_1Sd_2)$.

Proof. Let $K = \{g \in G | \operatorname{supp}(g \cdot (d_1d_2)) \cap \operatorname{supp} c \neq \emptyset\}$ and then K is compact by the properness of the group action. Hence,

$$\begin{aligned}
\mathrm{tr}(c_1Sc_2) &= \mathrm{tr}(\int_G [g \cdot (d_1d_2)]c_1Sc_2\mathrm{d}g) = \mathrm{tr}(\int_K [g \cdot (d_1d_2)]c_1Sc_2\mathrm{d}g) \\
&= \int_K \mathrm{tr}([g \cdot d_1][g \cdot d_2]c_1Sc_2)\mathrm{d}g = \int_K \mathrm{tr}(c_1[g \cdot d_1]D[g \cdot d_2]c_2)\mathrm{d}g \\
&= \int_K \mathrm{tr}([g^{-1} \cdot c_1]d_1Sd_2[g^{-1} \cdot c_2])\mathrm{d}g = \mathrm{tr}([\int_G g(c_1c_2)\mathrm{d}g]d_1Sd_2) = \mathrm{tr}(d_1Dd_2).
\end{aligned}$$

□

Using the fact that tr is a well-defined trace on the compactly supported operators on X, it is easy to see that tr_G is linear, faithful, normal and semi-finite. The tracial property of tr_G is proved in the following proposition together with some other properties of tr_G.

Proposition II.4.6.

1. *A properly supported smoothing operator* $A \in \Psi_{G,p}^{-\infty}$ *is of G-trace class. If* $K_A : X \times X \to \operatorname{End} E$ *is the kernel of the operator then its G-trace is calculated by*

$$\mathrm{tr}_G(A) = \int_X c(x) \operatorname{Tr} K_A(x,x)\mathrm{d}x, \tag{II.28}$$

   where c is a cutoff function and Tr *is the matrix trace of* $\operatorname{End} E$*. In fact, this formula holds for all G-invariant operators having smooth integral kernel.*

2. *If* $A \in \Psi_G^*$ *is of G-trace class, so is* A^*.

3. *If* $A \in \Psi_G^*$ *is of G-trace class and* $B \in \Psi_G^*$ *is bounded, then AB and BA is of G-trace class.*

4. *If AB and BA are of G-trace class, then* $\mathrm{tr}_G(AB) = \mathrm{tr}_G(BA)$

Proof. Let $\phi, \psi \in C_c(X)$ and let $\{\alpha_i^2\}_{i=1}^N$ be the G-invariant partition of unity in the last proposition.

1. Proposition II.3.12 item 3 shows that $A \in \Psi_{G,c}^{-\infty}$ has smooth kernel $K_A(x,y)$. Then $K_{\phi A\psi}(x,y) = \phi(x)K_A(x,y)\psi(y)$, is smooth and compactly supported, which means $\phi A\psi$ is of trace class.

The integral formula for smoothing operator is classical. A proof may be found at [32] Section 2.21.

2. Because $\bar{\psi}A\bar{\phi}$ has finite trace by definition, then $\phi A^*\psi = (\bar{\psi}A\bar{\phi})^*$ is of trace class.

3. Assume G-trace class operator $A \in \Psi^*_{G,c}$. Since $\operatorname{supp}\psi$ is compact then by proper supportness of A, there is a compact set K so that $\operatorname{supp} A\psi \subset K$. Choose $\eta, \zeta \in C_c^\infty(X)$ with $K \subset \operatorname{supp}\eta$ and $\eta\zeta = \eta$. Then $\eta A\psi = A\psi$ and for bounded $B \in \Psi^*_G$, $\phi BA\psi = \phi B\zeta\eta A\psi = (\phi B\zeta)(\eta A\psi)$. Since $\phi B\zeta$ is bounded operator with compact support and $\eta A\psi$ is trace class operator then their product is also a trace-class operator. So BA is of G-trace class. AB is of G-trace class because B^*A^* is of G-trace class.

   If $A \in \Psi^*_G$, then we have $A = A_1 + A_2$ so that $A_1 \in \Psi^*_{G,c}$ and A_2 has smooth kernel (which follows from a classical statement saying that the Schwartz kernel is smooth off the diagonal). Then the statement follows from the fact that $\phi A_1\psi$ has smooth, compactly supported Schwartz kernel.

4. We first prove a special case when AB and BA have smooth integral kernels. Use the slice theorem (II.15) and let $\{G \times_{K_i} S_i = G(S_i)\}_{i=1}^N$ be G-invariant tubular open sets covering X, then there exist G-invariant maps $\alpha_i : X \to [0,1]$ with $\operatorname{supp}\alpha_i \subset G(S_i)$ such that $\sum_{i=1}^N \alpha_i^2 = 1$. In fact, let $\tilde{\alpha}_i{}^2$ be the partition of unity of X/G subordinate to the open sets $G(S_i)/G$. Lift $\tilde{\alpha}_i$ to α_i on X, then $\{\alpha_i^2\}$ is the G-invariant partition of unity of X. Then:

$$
\begin{aligned}
&\operatorname{tr}_G(AB) = \int_X \int_X c(x)\operatorname{Tr}(K_A(x,y)K_B(y,x))\mathrm{d}y\mathrm{d}x \\
=&\sum_{i,j} \int_{G\times_{K_i}S_i}\int_{G\times_{K_j}S_j} \alpha_i^2(x)\alpha_j^2(y)c(x)\operatorname{Tr}(K_A(x,y)K_B(y,x))\mathrm{d}y\mathrm{d}x \\
=&\sum_{i,j} \frac{1}{\mu(K_i)\mu(K_j)}\int_{S_i}\int_{S_j}\alpha_i^2(\bar{s})\alpha_j^2(\bar{t})\int_G\int_G c(\overline{ht})\operatorname{Tr}(K_A(\overline{gs},\overline{ht})K_B(\overline{ht},\overline{gs}))\mathrm{d}g\mathrm{d}s\mathrm{d}h\mathrm{d}t \\
=&\sum_{i,j} \frac{1}{\mu(K_i)\mu(K_j)}\int_{S_i}\int_{S_j}\alpha_i^2(\bar{s})\alpha_j^2(\bar{t})\int_G \operatorname{Tr}(K_A(\bar{s},\overline{ht})K_B(\overline{ht},\bar{s}))\mathrm{d}g\mathrm{d}s\mathrm{d}h\mathrm{d}t = \operatorname{tr}_G(BA).
\end{aligned}
$$

   Note that in the third equality, $\bar{gs} \doteq (g,s)K_i = x \in G\times_{K_i} S_i$ and $\bar{ht} \doteq (h,t)K_j = y \in G \times_{K_j} S_j$ and by definition $\alpha_i(\bar{s}) = \alpha_i(\bar{gs}), \alpha_j(\bar{t}) = \alpha_j(\bar{ht})$. Also, we have used (II.18), $\mathrm{d}h^{-1} = \mathrm{d}h, \mathrm{d}(h^{-1}g) = \mathrm{d}g$, and change of variable in the fourth equality.

If either A or B are properly supported, (say A), then $\mathrm{tr}_G(AB) = \mathrm{tr}(c_1ABc_2) = \mathrm{tr}(\int_G c_1Ag \cdot (c_1c_2)Bc_2)$. So the $g \in G$ making $c_1Ag \cdot c_1$ form a compact set K, which allow us to interchange tr and $\int_K$, use tracial property of tr and G-invariance of A and B to prove $\mathrm{tr}_G(AB) = \mathrm{tr}_G(BA)$

In general let $A = A_1 + A_2$ and $B = B_1 + B_2$ where A_1, B_1 are properly supported and A_2, B_2 are bounded and have smooth kernel. Then $\mathrm{tr}_G(AB) = \mathrm{tr}_G(BA)$ using the special cases discussed above.

□

Remark II.4.7. Let S be a bounded G-invariant operator with smooth integral kernel and let $S_i \doteq \alpha_i S \alpha_i \in \Psi_c^{-\infty}(X; E, E)$. Then $\alpha_i^2 S$ is of G-trace class by Proposition II.4.6 item 3. We may calculate $\mathrm{tr}_G(S)$ as follows,

$$\mathrm{tr}_G(S) = \mathrm{tr}_G(\sum_{i=1}^N \alpha_i^2 S) = \sum_{i=1}^N \int_{G\times_{K_i} S_i} \alpha_i(x)c(x)\,\mathrm{Tr}\,K_S(x,x)\alpha_i(x)\mathrm{d}x$$
$$= \sum_{i=1}^N \int_{G\times_{K_i} S_i} c(x)\,\mathrm{Tr}\,K_{S_i}(x,x)\mathrm{d}x = \sum_{i=1}^N \mu(K_i)^{-1} \int_{G\times S_i} c((g,s))\,\mathrm{Tr}\,K_{S_i}((g,s),(g,s))\mathrm{d}g\mathrm{d}s$$
$$= \sum_{i=1}^N \mu(K_i)^{-1} \int_{G\times S_i} c((g,s))\,\mathrm{Tr}\,K_{S_i}((e,s),(e,s))\mathrm{d}g\mathrm{d}s = \sum_{i=1}^N \mu(K_i)^{-1} \int_{S_i} \mathrm{Tr}\,K_{S_i}(s,s)\mathrm{d}s.$$

The above trace formula coincides with the trace formulas in the special cases.

1. Let X be compact with G being trivial and $H_0 = L^2(X, E_0), H_1 = L^2(X, E_1)$ be separable Hilbert spaces, with orthonormal basis $\{e_i\}, \{f_i\}$. There is a faithful, normal and semi-finite trace Tr on $\mathscr{B}(H_0, H_1)$ so that $\mathrm{tr}(A) = \sum (Ae_i, f_i)$ for $A \geq 0$. The set of trace class operators is defined as the linear span of positive operators with finite trace.

2. When the action is free and cocompact, we have $X = G \times U$, and for bounded positive self adjoint operator S with smooth kernel, we have that $\mathrm{tr}_G(S) = \displaystyle\int_U \mathrm{Tr}\,K_S(x,x)\mathrm{d}x$.

3. For the homogeneous space of Lie group $X = G/H$, and for $S \in \Psi_{G,p}^{-\infty}(X)$, we have $\mathrm{tr}_G(S) = K_S(e,e)$, where $e \in G$ is the identity.

Proposition II.4.8. *If $P \in \Psi_{G,p}^m$ is an elliptic operator, then $\mathrm{tr}_G(P_{\mathrm{Ker}\,P_0}), \mathrm{tr}_G(P_{\mathrm{Ker}\,P_0^*}) < \infty$. Moreover, $P_{\mathrm{Ker}\,P_0}, P_{\mathrm{Ker}\,P_0^*} \in \Psi_G^{-\infty}$.*

Proof. There is a $Q \in \Psi^{-m}_{G,p}$ so that $1 - QP_0 = S \in \Psi^{-\infty}_{G,p}$. Denote $(1-S^*)(1-S) = 1-T$ and T^*T is of G-trace class because it is a smoothing operator (smoothing operators are closed under addition and product). It is sufficient to prove $\dim_G(\operatorname{Ker}(1-T)) < \infty$ because

$$\operatorname{Ker} P_0 \subset \operatorname{Ker}(1-S) = \operatorname{Ker}(1-T).$$

Observe that $\operatorname{Ker}(1-T)$ is the eigenspace of T^*T at 1, then $P_{\operatorname{Ker}(1-T)} = \delta_1(T^*T) \leq T^*T$. So the fact that T^*T is of G-trace class implies that $\operatorname{tr}_G(P_{\operatorname{Ker}(1-T)}) < \infty$.

For the second statement, $P_{\operatorname{Ker} P_0}$ is a smoothing operator because it maps L^2-sections to elements in $\operatorname{Ker} P_0$, which are smooth by Proposition II.3.12. To prove to, we apply $1 - QP_0 = S \in \Psi^{-\infty}_{G,p}$ to $P_{\operatorname{Ker} P_0}$ and get $P_{\operatorname{Ker} P_0} = SP_{\operatorname{Ker} P_0} \in \Psi^{-\infty}_G$. $\square$

Remark II.4.9. Using the G-trace property, there is another way to define the G-trace of $P_{\operatorname{Ker} A}$. Let $\{\alpha_i^2\}$ be the G-invariant partition of unity in Proposition II.4.6 item 4. Then by the same property,

$$\operatorname{tr}_G P_{\operatorname{Ker} A} = \sum_i \operatorname{tr}_G(\alpha_i P_{\operatorname{Ker} A} \alpha_i)$$

where every summand $\alpha_i P_{\operatorname{Ker} A} \alpha_i$ is G-invariant and restricts to a slice $G \times_{K_i} S_i \subset X$. Similar to the example of homogeneous space for unimodular Lie groups (Example II.4.2), The action of G on E is induced by the action of K_i on $V = E|_{S_i}$ [13]

$$\begin{array}{ccc} V = E|_{S_i} & \xrightarrow{k \in K_i} & V = E|_{S_i} \\ \downarrow & & \downarrow \\ \{e\} \times S_i & \xrightarrow{k \in K_i} & \{e\} \times S_i \end{array}$$

there is decomposition $L^2(G \times_{K_i} S_i, E) = (L^2(G) \otimes L^2(S_i, V))^{K_i}$ be the elements of $L^2(G) \otimes L^2(S, V)$ invariant under the action of K, where $k \in K$ acts by

$$k(f(g), h(s)) = (f(gk^{-1}), k \cdot h(s)), g \in G, s \in S, f \in L^2(G), h \in L^2(S, V).$$

[13] In $E|_{S_i}$, being more precise, $S_i = \{(e,s)K_i \subset X | s \in S_i\}$.

The G-invariance of $\ker A$ implies that $\alpha_i P_{\ker A} \alpha_i$ is an element of

$$\mathscr{R}(L^2(G)) \otimes \mathscr{B}(L^2(S_i, V)). \tag{II.29}$$

and this element commutes with the action of the group K_i on $\mathscr{R}(L^2(G)) \otimes \mathscr{B}(L^2(S_i, V))$. Here $\mathscr{R}(L^2(G))$ is the weak closure of the right regular representation of G ($L^1(G)$ more precisely) represented on $L^2(G)$. On this set there is a natural von Neumann trace determined by

$$\tau(R(f)^* R(f)) = \int_G |f(g)|^2 \mathrm{d}g,$$

where $f \in L^2(G) \cap L^1(G)$ and $R(f) = \int_G f(g) R(g) \mathrm{d}g$. Here $R(g)$ is the right regular representation of $g \in G$ on $L^2(G)$. Also $\mathscr{B}(L^2(S_i, V))$ also has a subset where an operator trace tr can be defined. There is a natural normal, semi-finite and faithful trace defined on $\mathscr{R}(L^2(G)) \otimes \mathscr{B}(L^2(S_i, V))$ given by $\tau \otimes \mathrm{tr}$ on algebraic tensors. Refer to [28] Section 2 for a detailed description.

This trace coincides with the G-trace in Definition II.4.3 on the set of bounded operator with smooth kernel. In fact, let $S = A \otimes B \in \mathscr{R}(L^2(G)) \otimes \mathscr{B}(L^2(S_i, V))$, which commutes with the action of K_i, and where A and B have smooth kernel. In [12], it has been shown that $\tau(A) = K_A(e, e)$. Let $d \in C_c^\infty(G)$ be any cutoff function for G. Then $\tau(A) = \int_G d(g) K_A(g, g) \mathrm{d}g$. Then

$$\begin{aligned}
\tau(A)\,\mathrm{tr}(B) &= \int_G d(g) K_A(g, g) \mathrm{d}g \int_{S_i} K_B(s, s) \mathrm{d}s \\
&= \int_{G \times S_i} \frac{1}{\mu(K_i)} c((g, s)) K_S((g, s), (g, s)) \mathrm{d}g \mathrm{d}s \\
&= \int_{G \times_{K_i} S_i} c(x) K_S(x, x) \mathrm{d}x.
\end{aligned}$$

Therefore we have proved the following proposition.

Proposition II.4.10. *Over $\Psi_{G,p}^{-\infty}(X; E, E)$ the G-trace equals the natural von Neumann trace on the von Neumann algebra $\mathscr{R}(L^2(X, E))$, the weak closure of all the natural bounded operators on $L^2(X, E)$ which commute with the action of G. The L^2-index is the difference of the von Neumann trace of $P_{\mathrm{Ker}\, P_0}$ and $P_{\mathrm{Ker}\, P_0^*}$.*

Example II.4.11. When G is a discrete group acting on itself by the left translation, choose $c(g) =$

$\begin{cases} 1 & g=e \\ 0 & g\neq e \end{cases}$ then $\operatorname{tr} cT = \sum_{g\in G} < T\delta_g, \delta_g > = < \sum_{g\in G} g^{-1}(cT)g\delta_e, \delta_e > = < \mathrm{Av}(cT)\delta_e, \delta_e > = \operatorname{tr}_G \mathrm{Av}(cT)$.

In general, Av(c·) : $\mathscr{B}(L^2(G)) \to \mathscr{R}(L^2(G))$ extends to the map $\Psi_c \to \Psi_{G,p} : cT \to \mathrm{Av}(cT)$ which preserve the corresponding trace. When T is G-invariant, $T = \mathrm{Av}(cT)$ and the following provides the motivation for the tr_G formula: $\operatorname{tr}_G T = \operatorname{tr}_G \mathrm{Av}(cT) = \operatorname{tr} cT$.

When $A : L^2(X,E) \to L^2(X,E)$ in Ψ_G^* is an elliptic operator over X with properly cocompact G action. According to proposition II.4.8, define a real valued *G-dimension* of K, a closed G-invariant subspace of $L^2(X,E)$, by

$$\dim_G K = \operatorname{tr}_G P_K$$

where P_K is the projection from $L^2(X,E)$ onto K, which is a G-invariant operator.

Definition II.4.12. The L^2*-index* of A is

$$\operatorname{ind} A = \dim_G \operatorname{Ker} A - \dim_G \operatorname{Ker} A^*. \tag{II.30}$$

An immediate computation of the L^2-index is the following proposition.

Proposition II.4.13. *Let $A \in \Psi^m_{G,p}$ be elliptic and P be the operator so that $1-PA = S_1, 1-AP = S_2$ are of G-trace class, then*

$$\operatorname{ind} A = \operatorname{tr}_G S_1 - \operatorname{tr}_G S_2.$$

Proof. The proof is based on [4]. We have

$$S_1 P_{\operatorname{Ker} A} = P_{\operatorname{Ker} A} \text{ and } P_{\operatorname{Ker} A^*} S_2 = P_{\operatorname{Ker} A^*}$$

by composing $PA = 1 - S_1$ with $P_{\operatorname{Ker} A}$ and by composing $P_{\operatorname{Ker} A^*}$ with $1 - S_2 = AP$ respectively. Also, $A(PA) = (AP)A$ implies $AS_1 = S_2 A$. Set $Q = \delta_0(A^*A)A^*$ where $\delta_0(0) = 1, \delta_0(x) = 0$ for $x \neq 0$, then

$$QA = 1 - P_{\operatorname{Ker} A}, AQ = 1 - P_{\operatorname{Ker} A^*}.$$

On one hand $\operatorname{tr}_G S_1 - \operatorname{tr}_G P_{\operatorname{Ker} A} = \operatorname{tr}_G S_1(1 - P_{\operatorname{Ker} A}) = \operatorname{tr}_G(S_1 QA)$, on the other hand $\operatorname{tr}_G S_2 - \operatorname{tr}_G P_{\operatorname{Ker} A^*} = \operatorname{tr}_G S_2(1 - P_{\operatorname{Ker} A^*}) = \operatorname{tr}_G(S_2 AQ) = \operatorname{tr}_G(AS_1 Q)$. Therefore $\operatorname{tr}_G S_1 - \operatorname{tr}_G S_2 = \operatorname{tr}_G P_{\operatorname{Ker} A} - \operatorname{tr}_G P_{\operatorname{Ker} A^*}$ by

Proposition II.4.6. $\square$

The purpose of proposition III.3 is to derive the following *McKean-Singer formula*, which is used later.

Corollary II.4.14. *If $D \in \Psi^1_G(X, E^0, E^1)$ is a first order elliptic differential operator, $t > 0$, then*

$$\operatorname{ind} D = \operatorname{tr}_G(e^{-tD^*D}) - \operatorname{tr}_G(e^{-tDD^*}), \tag{II.31}$$

which in particular means that $\operatorname{ind} D$ *is independent of $t > 0$.*

Proof. To prove (II.31) we need the following lemma.

Lemma II.4.15. *Let D be as above, then e^{-tDD^*} and e^{-tD^*D} are of G-trace class.*

Let $P = \int_0^t e^{-sD^*D} D^* \mathrm{d}s$, which is the parametrix of D because

$$1 - PD = e^{-tD^*D}, I - DP = e^{-tDD^*}$$

and they are of G-trace class by the lemma. The statement follows from Proposition III.3. $\square$

Proof of lemma II.4.15. It is sufficient to prove the case when $t = 1$. (The independence of t can be carried out by a modification of the second proof of [8] Theorem 3.50.) The proof is based on the ideas in [17] [12].

If $\lambda \in \mathbb{C} - [0, \infty)$, then $\lambda I - D^*D$ is invertible. Let $L = \{\lambda \in \mathbb{C} | d(\lambda, \mathbb{R}_+) = 1\}$ be a curves in $\mathbb{C}$ and be clock-wise oriented. Then

$$e^{-D^*D} = \frac{1}{2\pi i} \int_L \frac{e^{-\lambda}}{\lambda I - D^*D} \mathrm{d}\lambda.$$

Let $\phi, \psi \in C_c(X)$ be supported in a compact set $K \subset X$ and let $\{\alpha_k\}_{i=1}^N$ be a partition of unity subordinated to a open cover of K of local coordinate. We "approximate" $\phi e^{-D^*D} \psi$ by an operator in $\Psi_c^{-\infty}$ (with smooth and compactly supported Schwartz kernel) by inverting $\lambda I - D^*D$ "locally".

Let p_k be the symbol (not the principal symbol) of $\alpha_i \phi (\lambda I - D^*D)^{-1} \psi$. Then p_k has the asymptotic sum

$$p_k \sim \sum_{j=2}^{\infty} a_{-j} \text{ on a local coordinate (defined in (II.4)), i.e. } \mathrm{Op}(p - \sum_{j=2}^{m} a_{-j}) \in \Psi_c^{-m-1} \forall m > 1, \tag{II.32}$$

where Op means the operator corresponding to the local symbol.

For any $l > 0$ and $n > 0$, choose a large enough M and set the operator approximating $\alpha_i \phi(\lambda I - D^* D)^{-1} \psi$ to be

$$P_k(\lambda) = \mathrm{Op}(\sum_{j=2}^{M} a_{-j}), \tag{II.33}$$

in the sense that $P_k(\lambda)$ is analytic in λ and for any fixed $u \in L^2(X, E)$,

$$\|(P_k(\lambda) - \alpha_i \phi(\lambda I - D^* D)^{-1} \psi) u\|_l \le C(1 + |\lambda|)^{-n}, \tag{II.34}$$

where the norm is the Soblev l-norm $\| \cdot \|_l$. The estimate (II.34) is made possible by the asymptotic sum (II.32). In fact, let $r(x, \xi)$ be the symbol of $R \doteq P_k(\lambda) - \alpha_i \phi(\lambda I - D^* D)^{-1} \psi$ which is in S^{-M-1} and then the left hand side of (II.34) is $\|Ru\|_l = \int (1 + |\xi|^2)^l |\widehat{Ru}(\xi)| \mathrm{d}\xi$, where $Ru(x) = \int e^{<x-y,\xi>} r(x, \xi) u(y) \mathrm{d}y \mathrm{d}\xi$ can be controlled by the right hand side of (II.34) when $M >> 2l + 2n$ because by definition of $r(x, \xi)$ there is a constant C so that $|r(x, \xi)| < C(1 + |\xi|)^{-M-1}$.

Set

$$E(\lambda) = \sum_{k=1}^{N} E_k(\lambda) = \sum_{k=1}^{N} \frac{1}{2\pi i} \int_L e^{-\lambda} P_k(\lambda) \mathrm{d}\lambda, \tag{II.35}$$

Then the following two observations prove that $\phi e^{-D^* D} \psi$ is of trace class.

1. The operator $E(\lambda)$ is a compactly supported operator with smooth Schwartz kernel.

   Proof of claim. We need to show that the Schwartz kernel of $E_k(\lambda)$ is smooth. In view of (II.33) and (II.35), it is sufficient to show that $\mathrm{Op}(a_j), j \le -2$ has smooth kernel and $\int_L e^{-\lambda} \partial^\beta (\mathrm{Op}(a_j) u) \mathrm{d}\lambda$ is integrable for all β. This claims can be proved by the symbolic calculus ([17]). The crucial part in the argument. is that by Proposition II.1.1, all $a_j, j \le -2$ contains the factor $e^{-\sigma_2(D^* D)}$ and the fact that $e^{-t\sigma_2(D^* D)}$ is rapid decreasing in ξ contributes to the convergence of the integrals. □

2. the function $(E(\lambda) - \phi e^{-D^* D} \psi) u \in H^l$ for any fixed $u \in L^2$.

Proof of claim. Using (II.34), and fix a $u \in L^2(X,E)$

$$\begin{aligned}\|(E(\lambda) - \phi e^{-D^*D}\psi)u\|_l &\leq \frac{1}{2\pi}\sum_{k=1}^{N}\int_L e^{-\lambda}\|(P_k(\lambda) - \alpha_k\phi(D^*D - \lambda I)^{-1}\psi)u\|_l \mathrm{d}\lambda \\ &\leq C\int_L e^{-\lambda}(1+|\lambda|)^{-n}\mathrm{d}\lambda \to 0 \text{ as } n \to \infty.\end{aligned}$$

□

Note that $E(\lambda)$ depends on the number M, which is chosen based on l,n, and it it always has compactly supported smooth kernel by the first claim and hence $E(\lambda)u \in C_c^\infty \subset H^l$. The second claim shows that $\phi e^{-D^*D}\psi$ is in H^l. [14]

Let $l \to \infty$, then by the Sobolev Embedding Theorem $(\phi e^{-D^*D}\psi)u$ is smooth for all $u \in L^2$. Therefore $\phi e^{-D^*D}\psi$ has compactly supported smooth kernel and is a trace-class operator. □

[14] When $n \to \infty$, there is a sequence of $E(\lambda) \in H^l$ approaching $\phi e^{-D^*D}\psi$ in $\|\cdot\|_l$ norm.

CHAPTER III

THE HIGHER INDEX AND THE L^2-INDEX

In this chapter, we discuss Kasparov's K-theoretic index formula and construct the $\mathbb{R}$-valued index by taking "trace" of the "higher index" of elliptic operators in the context of KK-theory. This $\mathbb{R}$-valued index is analytical in the same spirit as the Fredholm index, and equals to the L^2-index in the last chapter. We introduce the motivation and the formulation of the higher index in section 1 and a topological formula, the K-theoretic index formula, in section 2. In the third section we discuss the $\mathbb{R}$-valued index and prove that the L^2-index is stable and depends only on the topological information of the manifold and of the vector bundle.

III.1 Higher index

Let A be an elliptic operator on a compact manifold. The index is an integer calculated by the difference of the dimensions of the null spaces of A and A^*. In this section we investigate the spirit of taking index and apply the inspiration to define the index for elliptic operators that may not be Fredholm. Recall that for a Fredholm operator A, two important observations are that A is invertible module a compact operator and that A has an integer index. The link between the two is obvious when the index is viewed as an element in K-theory. The first part of the section is summarized from any standard K-theory textbook such as [9] and the last part of the section is based on [24].

K-theory for a locally compact Hausdorff space X is a cohomology theory $K^*(X)$. The elements in the group represent classes of complex vector bundles under some equivalent relation. The group is a homotopy invariant and measures how far the vector bundle is from being trivial.

- When X is compact,

  1. $K^0(X)$ is the Grothendieck group of semigroup of the stable isomorphism class of complex vector bundles over X. Recall that the direct sum of two complex vector bundles $(E,p),(F,q)$ over X is defined as $E\oplus F=\{(x,(e,f)) : p(e)=q(f), x\in X, e\in E, f\in F\}$. A general element in $K^0(X)$ is the formal difference of the two isomorphism class of vector bundles $[E]-[F]$ and the class of trivial bundles represents the 0 element of

the group. For example, $K_0(\mathrm{pt}) = \mathbb{Z}$ is calculated by the difference of dimension of two $\mathbb{C}$ vector spaces. The morphism is defined by the pull back bundle.[1]

2. $K^1(X)$ is the abelian group of the homotopy classes of the set $\{f : X \to GL_\infty(\mathbb{C})$ is continuous$\}$, where $GL_\infty(\mathbb{C}) = \lim_n GL_n(\mathbb{C})$ under inclusion.

- The K group of locally compact space X is defined by $K^*(X) = \mathrm{Ker}\{i^* : K^*(X^+) \to K^*(\mathrm{pt})\}$, where X^+ is the one point compactification of X, and $i : \mathrm{pt} \to X^+$ is the inclusion map.

Remark III.1.1. A proper continuous map $f : X \to Y$ between locally compact Hausdorff spaces defines an algebra homomorphism $F : C_0(Y) \to C_0(X) : F(g) = g \circ f$. Then a complex vector bundle E over X^+(the one point compactification of X), corresponds a *projective module*, which is a direct summand of the free module $\oplus_{i=1}^n C_0(X)$. E is determined by the projection from $\oplus_{i=1}^n C_0(X)$ to the summand. [2]

K-theory can be defined on C^*-algebras. Recall that a C^*-*algebra* $\mathscr{A}$ is a Banach algebra[3] with a $*$-operation on $\mathscr{A}$ with: $(aA+bB)^* = \overline{a}A^* + \overline{b}B^*$, for any $a,b \in \mathbb{C}, A,B \in \mathscr{A}$; $(A^*)^* = A$; $(AB)^* = B^*A^*$; $\|A^*A\| = \|A\|^2$. A C^*-algebra having a unit is *unital.* The unitalization of a C^*-algebra[4] is analogous to the one point compactification of a space. If $M_n(A)$ is the set of all $n \times n$ A-valued matrices, then $M(A)$, the inductive limit of $M_n(A)$ by identify element in $R \in M_n(A)$ to $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+m}(B)$, is still a C^*-algebra.

- $K_0(A)$ group of a unital C^* algebra A is the Grothendieck group of the semigroup of projection classes of $M(A)$ under $\oplus$, where $P \oplus Q = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$ $(P,Q \in M_n(A)$ for some large $n)$ and $P \sim Q$ if there exists a continuous projection path $S_t \in C([0,1], M(A))$ such that $S_0 = P, S_1 = Q$. The element in this group is denoted by $([P],[Q]) or [P]-[Q]$. For enough large n and for $P,Q,B,S \in M_n(A)$, we say that $[P]-[Q] = [R]-[S]$ if their exists an idempo-

[1] If $f : X \to Y$ is continuous, then $f^* : K^0(Y) \to K^0(X) : E \mapsto f^*E$.

[2] There exists a bundle F so that $E \oplus F = T$ is an n dimensional trivial bundle, there is a direct sum of sections $C_0(E) \oplus C_0(F) = C_0(T) = C_0(X^+) \oplus \cdots \oplus C_0(X^+)$.

[3] A is a *Banach algebra* if it is a complex algebra and complete under its *norm* $\|\cdot\|$, i.e. $\|\cdot\| : A \to [0,\infty) : x \mapsto \|x\|$ such that (1) $\|c \cdot x\| = |c|\|x\|$ for any $x \in A, c \in \mathbb{C}$; (2)$\|x+y\| \leq \|x\| + \|y\|$ for any $x,y \in A$; (3)$\|x\| = 0$ if and only if $x = 0$ where $x \in A$.

[4] $B^+ = \{(a,c) : a \in B, c \in \mathbb{C}\}$ where the multiplication is $(a,c)(b,d) = (ab+bc+ad, cd)$ and unit is $(0,1)$, with norm $\|a+c\| = \|a\| + \|c\|$.

tent $T \in M_n(A)$ such that $P+S+T \sim Q+R+T$.) [5] The morphism is that if $f: A \to B$ is a $*$-homomorphism preserving identity, then $f_*: K_0(A) \to K_0(B): [P] \mapsto [f \circ P]$.

K_0 is a homotopy invariant.

- $K_1(A)$ is defined as the abelian group $GL(A)$ over the equivalent relation $\sim$ under direct sum or matrix multiplication, where $GL_n(A)$ is the set of all $n \times n$ invertible A-valued-matrices and $GL(A)$ is the inductive limit of $GL_n(A)$ by identifying $B \in GL_n(A)$ to $\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \in GL_{n+m}(A)$., and equivalent relation is by continuous path of unitaries[6] in $GL(A)$.

- When A is not unital, $K_*(A) = \operatorname{Ker} p_*$ where $p: A^+ \to \mathbb{C}$ is the projection map.

Proposition III.1.2. *1. the group K_0 and K_1 are related by the isomorphism $K_1(A) \simeq K_0(A(0,1))$, where $A(0,1)$, the* suspension *of A, is the set of all continuous functions from $[0,1]$ to A vanishing at 0 and 1.*

2. K-theory preserves morita equivalence: $K_*(A) \simeq K_*(A \otimes \mathscr{K})$, where $\mathscr{K}$ is the set of all compact operators on some Hilbert space.

3. K-theory has the six term exact sequence: If $0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} A/I \longrightarrow 0$ is short exact, then there is a six term exact sequence

$$\begin{tikzcd} K_1(I) \arrow[r, "i_*"] & K_1(A) \arrow[r, "\pi_*"] & K_1(A/I) \arrow[d, "\partial"] \\ K_0(A/I) \arrow[u, "\delta"] & K_0(A) \arrow[l, "\pi_*"] & K_0(I) \arrow[l, "i_*"] \end{tikzcd}.$$

The boundary map ∂ is called index map and δ is called exponential map.

4. K-theory has the Bott periodicity*,*

$$K_0(A) = K_0(A(0,1)^2).$$

[5] P is a projection if $P^2 = P, P^* = P$.
[6] U is a unitary if $U^*U = UU^* = I$.

Example III.1.3. The six-term exact sequence allows us to compute more K-groups. For example, when $I = \mathscr{K}$ and $A = \mathscr{B}$, we have that

$$\begin{tikzcd} K_1(\mathscr{K}(H)) \arrow[r, "i_*"] & K_1(\mathscr{B}(H)) \arrow[r, "\pi_*"] & K_1(\mathscr{B}/\mathscr{K}) \arrow[d, "\partial"] \\ K_0(\mathscr{B}/\mathscr{K}) \arrow[u, "\delta"] & K_0(\mathscr{B}(H)) \arrow[l, "\pi_*"] & K_0(\mathscr{K}(H)) \arrow[l, "i_*"] \end{tikzcd}$$

is exact. Since $K_*(\mathscr{B}(H)) \simeq 0$ and $K_*(\mathscr{K}(H)) \simeq K_*(\mathbb{C})$, then$K_0(\mathscr{B}/\mathscr{K}) \simeq K_1(\mathscr{K}) \simeq 0$ and $K_1(\mathscr{B}/\mathscr{K}) \simeq K_0(\mathscr{K}) \simeq \mathbb{Z}$.

Remark III.1.4. As a corolllary of Example III.1.3 we see the link between Fredholm operator F and its index. In fact, F being a Fredholm module means that $[F] \in K_1(\mathscr{B}/\mathscr{K})$. Up to a compact perturbation of F, its poler decomposition has form $F = PV$, where P is positive invertible and V is a partial isometry and F and V have the same Fredholm index. Lift $V \in \mathscr{U}(\mathscr{B}/\mathscr{K})$ (where $\mathscr{U}$ means unitary) to an invertible element

$$w = \begin{pmatrix} V & 1 - VV^* \\ 1 - V^*V & V^* \end{pmatrix} \in M_2(\mathscr{B}(H)),$$

with $w^{-1} = \begin{pmatrix} V^* & 1 - V^*V \\ 1 - VV^* & V \end{pmatrix}$. A direct calculation of $\partial[V] \doteq [wp_0w^{-1}] - [p_0]$ shows that

$$\partial[F] = \partial[V] = [1 - VV^*] - [1 - V^*V] = [P_{\mathrm{Ker}\,V}] - [P_{\mathrm{Ker}\,V^*}] \in K_0(\mathscr{K}(H)) = \mathbb{Z}.$$

Since $K_0(\mathscr{K}(H)) \to \mathbb{Z}$ is given by the difference of the dimension of two projections in $K_0(\mathscr{K}(H))$, $\partial[F]$ is the Fredholm index of $[F]$. The index of Fredholm operator is an element of $K_0(\mathscr{K}(H))$.

Analogous to the Fredholm operators, the elliptic operators with proper cocompact group action are invertible up to some "small algebras" (the "Compact operators" over $C^*(G)$. Refer to the definition of the group C^*-algebra $C^*(G)$ in Appendix C). Using the boundary map of a six-term exact sequence, a finer index in $K_0(C^*(G))$ is obtained. Being precise, extend the elliptic differential operator $A : C_c(X,E) \to C_c(X,E)$, where $A \in \Psi^0_{G,p}(X,E,E)$ by completing $C_c(X,E)$ into a Hilbert-$C^*(G)$ module (The definition of Hilbert B-module can be found in Appendix E).

The construction was done in [23] and sketched as follows. Embed $C_c(X,E)$ into $C_c(G,L^2(X,E))$ by

$$f \mapsto (g \mapsto c^{\frac{1}{2}} g \cdot f), f \in C_c(X,E), g \in G, \text{ where } c \text{ is cutoff function of } X.$$

And there are pre-Hilbert $C_c(G)$-module structures on $C_c(X,E)$ defined by

$$< f,h >_{C_c(G)} (g) = \int_X < f(x),(g \cdot h)(x) >_{E_x} \mathrm{d}x$$

with the module multiplication

$$f \cdot b(x) = \int_G g \cdot f(x) \cdot b(g^{-1}) \mathrm{d}g, f,h \in C_c(X,E), b \in C_c(G),$$

and on $C_c(G,L^2(X,E))$ defined by

$$< f,h >_{C_c(G)} (g) = \int_G < f(s),h(sg) >_{L^2} \mathrm{d}s$$

with module multiplication

$$f \cdot b(g) = \int_G (h^{-1} \cdot f)(gh^{-1}) b(h) \mathrm{d}h, f,h \in C_c(G,L^2(X,E)), b \in C_c(G).$$

The norm of the Hilbert module is $\|f\|_{C^*(G)} = \| < f,f >_{C^*(G)} \|^{\frac{1}{2}}$ The embedding

$$i : C_c(X,E) \hookrightarrow C_c(G,L^2(X,E))$$

preserves the $C^*(G)$-valued inner product. Further, completing both side under $\| \cdot \|_{C^*(G)}$, we get a submodule $\mathscr{E}$. (It is the completion of $C_c(X,E)$ in the Hilbert $C^*(G)$-module $C^*(G,L^2(X,E))$.) $\mathscr{E}$ is a direct summand of $C^*(G,L^2(X,E))$. In fact, i has an adjoint

$$p : C_c(G,L^2(X,E)) \to C_c(X,E) : p(f) = \int_G g(c)^{\frac{1}{2}} \cdot g(f(g^{-1})) \mathrm{d}g.$$

Since it is easy to check that $p \circ i = id$ then $i \circ p$ is a projection. $\mathscr{E}$ is the convolution of the projection $i \circ p$ with the algebra $C^*(G,L^2(X,E))$.

The operator $A : C_c(X,E) \to C_c(X,E)$ in $\Psi^0_G(X,E,E)$ extends to two bounded maps $A : L^2(X,E) \to L^2(X,E)$ and $A : \mathscr{E} \to \mathscr{E}$ with $\|A\|_{\mathscr{E}} \leq \|A\|_{L^2(E)}$.

Denote by $\mathscr{L}(\mathscr{E})$ the C^*-algebra of all bounded operators on $\mathscr{E}$ having adjoint. Then $A : \mathscr{E} \to \mathscr{E}$ defines an element in $\mathscr{L}(\mathscr{E})$ according to the following lemma.

Lemma III.1.5. *[24] Let X be a complete Riemannian manifold and G be a locally compact group which acts on X properly and isometrically with compact quotient X/G. Let $A : C_c(X,E) \to C_c(X,E)$ be a properly supported G-invariant pseudo differential operator of order 0 on a vector bundle E on X. Then A extends to a bounded operators on $\mathscr{E}$, which is the completion of $C_c(X,E)$ in the norm $\| < e,e > \|^{\frac{1}{2}}_{C^*(G)}$.*

On the Hilbert $C^*(G)$ module $\mathscr{E}$, the rank one operator is defined by

$$\theta_{e_1,e_2}(e)(x) = e_1(e_2,e)(x) = \int_X (\int_G \theta_{g(e_1)(x),g(e_2)(y)} dg) e(y) \mathrm{d}y, \forall x \in X.$$

The closure of the the linear combinations of the rank one operators under the norm of $\mathscr{L}(\mathscr{E})$ is the set of "compact operators" denoted by $\mathscr{K}(\mathscr{E})$. The elements of $\mathscr{K}(\mathscr{E})$ can be identified to be the integral operator with G-invariant continuous kernel and with proper support. Using the following feature of the compact operators of the Hilbert module and the definition of ellipticity, the index in K-theory group can be obtained.

Proposition III.1.6. *[24] If the symbol of the G-invariant properly supported operator A of order 0 is bounded in the cotangent direction by a constant. Then the norm of A in $\mathscr{B}(\mathscr{E})/\mathscr{K}(\mathscr{E})$ does not exceed that constant. The operator A is compact ($A \in \mathscr{K}(\mathscr{E})$) if the symbol of A is 0 at infinity (in cotangent direction).*

Recall that A is elliptic if

$$\|\sigma_A(x,\xi)^2 - 1\| \to 0 \text{ as } \xi \to 0, x \in K$$

uniformly for any compact set $K \subset X$. Then using Proposition III.1.6 we have that $A^2 - Id \in \mathscr{K}(\mathscr{E})$. If A acts on graded space with $A = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix}$, then $[A_0] \in K_1(\mathscr{B}(\mathscr{E})/\mathscr{K}(\mathscr{E}))$. The image of the class in the K-theory of the quotient algebra under the boundary map in the six term exact sequence

$$\partial : K_*^G(\mathscr{B}(\mathscr{E})/\mathscr{K}(\mathscr{E})) \to K_{*+1}^G(\mathscr{K}(\mathscr{E}))$$

assigns A an element in $K_0^G(\mathscr{K}(\mathscr{E}))$. This is the *$K$-theoretic index* of A, denoted by $\mathrm{Ind}A$. Note that when $\mathscr{E}$ is a Hilbert space without group action, $\mathrm{Ind}A$ coincides with that discussed in Remark III.1.4.

The compact $C^*(G)$-module $\mathscr{K}(\mathscr{E})$ and $C^*(G)$ are equivalent in the following sense.

Definition III.1.7. Let E be a Hilbert B-module, a E-B equivalence bi-module X is an E-B-module equipped with E and B valued inner products with respect to X is a right Hilbert B module and a left Hilbert E-module, satisfying $< x,y >_E z = x < y,z >_B, \forall x,y,z \in X$ and $< X,X >_B$ spans a dense subset of B and $< X,X >_E$ spans a dense subset of E. We say that E and B are *strongly Morita equivalent* if there is a E-B equivalent bi-module.

Example III.1.8. Let $\mathscr{E}$ be a left $\mathscr{K}(\mathscr{E})$ module defined by the inner product $\mathscr{E} \times \mathscr{E} \to \mathscr{K}(\mathscr{E}) : (x,y) \mapsto \theta_{x,y}$ and let $\mathscr{E}$ be a right Hilbert $C^*(G)$-module. It is easy to verify the conditions of the above definition and conclude that $\mathscr{K}(\mathscr{E})$ and $C^*(G)$ are strongly Morita equivalent. In particular, A is strongly Morita equivalent with $A \otimes \mathscr{K}$ for any C^*-algebra A.

Using the classical fact that K-theory preserves Morita equivalence we have that

$$K_*(\mathscr{K}(\mathscr{E})) \simeq K_*(C^*(G))$$

and

$$\mathrm{Ind}A \in K_0(C^*(G)).$$

The K-theoretic index $\mathrm{Ind}A$ can also be formulated topologically. Let $f \in C_0(X)$ be the operator on $L^2(X,E)$ defined by point-wise multiplication and let $A \in \Psi_p^0(X;E,E)$ be elliptic. Using Rellish lemma one may check that $A_0 : L^2(X,E_0) \to L^2(X,E_1)$ satisfies the following conditions: $(A_0A_0^* - I)f \in \mathscr{K}(L^2(X,E_1))$, $(A_0^*A_0 - I)f \in \mathscr{K}(L^2(X,E_0))$, $Af - fA \in \mathscr{K}(L^2(X,E))$ and $A_0 - g \cdot A_0 \in \mathscr{K}(L^2(X,E_1),L^2(X,E_2))$ for all $g \in G$. This shows that $A \in K_G^0(C_0(X))$, a group dual to K-theory called *K-homology*. Recall K-homology in the Appendix D. Both K-homology and K-theory are special cases of KK-theory (Appendix E), a more general group, $KK^G(B,C)$, a group depending on two C^*-algebras. In fact we have

$$KK^G(\mathbb{C},C) = K_0^G(C), KK^G(B,\mathbb{C}) = K_G^0(B)$$

and the good thing about this group is that it has an associative intersection product

$$KK^G(A,B) \times KK^G(B,C) \to KK^G(A,C).$$

Proposition III.1.9. *The elliptic operator $A : L^2(X,E) \to L^2(X,E)$ in $\Psi^0_{G,p}$ gives a class of cycle*

$$[(L^2(X,E),A)] \in KK^G(C_0(X),\mathbb{C})$$

where $C_0(X)$ acts on $L^2(X,E)$ by pointwise multiplication. Its K-theoretic index is in fact defined by the following assembly map:

$$KK^G(C_0(X),\mathbb{C}) \xrightarrow{j^G} KK(C^*(G,C_0(X)),C^*(G)) \xrightarrow{[p]\otimes_{C^*(G,C_0(X))}} KK(\mathbb{C},C^*(G)) \ ,$$

where $[p]$ is defined by some projection associated to the cut-off function c.

Proof. Let $p \doteq c \cdot g(c) \in C_c(X, C_c(X))$ and it is easy to check that P is an idempotent. The direct summand $\mathscr{E} \subset C^*(G,L^2(X,E))$ is obtained by convoluting p on the left to $C^*(G,L^2(X,E))$. We denote $[p]$ as the class for the cycle

$$[(pC^*(G,C_0(X)),0)] \in KK(\mathbb{C},C^*(G,C_0(X))).$$

Applying the descent map j^G to $[(L^2(X,E),A)] \in KK^G(C_0(X),\mathbb{C})$, we get

$$[(C^*(G,L^2(X,E)),\tilde{A})] \in KK(C^*(G,C_0(X)),C^*(G)),$$

where $\tilde{A}f(g) = A(f(g)), g \in G, f \in C^*(G,L^2(X,E))$ and the representation of $C^*(G,C_0(X))$ on the Hilbert module $C^*(G,L^2(X,E))$ can be made from the $C_0(X)$ action and G action on $L^2(X,E)$. Finally by compressing this cycle with $[p]$, we get a cycle $[(\mathscr{E}, p\tilde{A}i)]$, which is an $[(\mathscr{E},A)]$ module a "compact operator" in $\mathscr{K}(\mathscr{E})$. Therefore the image under the assembly map is

$$[(\mathscr{E},A)] \in KK^G(\mathbb{C},C^*(G)) = KK_0^G(C^*(G)).$$

$\square$

III.2 K-theoretic index formula

This section talks about a topological formula of the K-theoretic index defined in the last section. The important fact to be used in later calculations is the following K-homological formula by Kasparov. And the K-theoretic index formula is a corollary of the K-homological formula.

Theorem III.2.1. *[24] Let X be a complete Riemannian manifold and let G be a locally compact group acting on X properly and isometrically. Let A be a G-invariant elliptic operator on X of order* 0. *Then*

$$[A] = [\sigma_A] \otimes_{C_0(T^*X)} [D] \in K_G^*(C_0(X)), \tag{III.1}$$

where $[D]$ *is the class defined by the dolbeault operator* $\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial x}$ *on* T^*X.

Theorem III.2.1 means that the class of an elliptic operator A depends only on its symbol and the manifold on which it acts. When X is compact and when G is trivial, The formula (III.1) explains the map structure in the duality of K-theory and K-homology:

$$K_0(X) \to K^0(TX) : [A] \to [\sigma_A].$$

The theorem says that $[A]$ is given by the index pairing of the symbol with some fundamental operator (Dolbeult) on TX. This is the essence of the Atiyah-Singer index theorem. To understand Theorem III.2.1, the symbol σ_A and the paring in (III.1) is explained.

III.2.1 The symbol class $[\sigma_A]$

The symbol of a 0-order elliptic operator A is easily observed to be an element in $KK^G(C_0(X),C_0(TX))$ using ellipticity

$$\|\sigma_A^2(x,\xi) - I\| \to 0, \xi \to \infty, \text{ in } T_x^*X, x \in K \subset X. \tag{III.2}$$

In fact, consider $C_0(TX, \pi^*E)$, where $\pi : T^*X \to X$, as a Hilbert module over $C_0(TX)$ using the Hermitian structure on E, and the set of "compact operators" is $C_0(TX, \mathrm{End}(\pi^*E))$. Also $C_0(X)$ acts on $C_0(\pi^*E_0 \oplus \pi^*E_1)$ by pointwise multiplication. Hence for all $f \in C_0(X)$, $(\sigma_A^2 - I)f$ is compact by (III.2) and $[\sigma_A, f]$ is compact by the fact that $[\sigma_A, f]$ is of order -1. Therefore the symbol $\sigma_A : \pi^*(E_0) \to \pi^*(E_1)$ defines the following element in KK-theory:

$$[(C_0(TX, \pi^*E_0 \oplus \pi^*E_1), \begin{pmatrix} 0 & \sigma_{A_0}^* \\ \sigma_{A_0} & 0 \end{pmatrix})] \in KK^G(C_0(X), C_0(T^*X)).$$

The symbol σ_A also defines a G-bundle ([5] section 7). Let $B(X) \subset TX$ be the ball bundle with sphere bundle $S(X) \subset TX$ as its boundary. A new manifold ΣX is obtained by gluing two copies of $B(X)$ along their boundaries:

$$\Sigma X = B(X) \cup_{S(X)} B(X). \tag{III.3}$$

The action of G on TX extends naturally to ΣX because G acts on X isometrically. And a G-vector bundle over ΣX is built out of σ_A as follows: The ellipticity of A implies the invertibility of the symbol restricted to $S(X)$: $\sigma_A|_{S(X)}$. Define a G-vector bundle over $\Sigma(X)$ by gluing the σ_A on the boundary:

$$V(\sigma_A) = \pi^*E_0|_{B(X)} \cup_{\sigma_A|_{S(X)}} \pi^*E_1|_{B(X)}, \tag{III.4}$$

Note that $V(\sigma_A)$ is a class in the representable K-theory $RKK_G^0(C_0(X), C_0(\Sigma X))$.

Proposition III.2.2. *The homomorphism*

$$KK^G(C_0(X), C_0(TX)) \to KK^G(C_0(X), C_0(\Sigma X)) : [(C_0(TX, \pi^*E), \sigma_A)] \mapsto [(C_0(\Sigma X, V(\sigma_A)), 0)]$$

naturally follows from the inclusion map $i : C_0(TX) \to C_0(\Sigma X)$.

Proof. The pre-image of $[(V(\sigma_A), 0)]$ of i_* in $KK^G(C_0(X), C_0(TX))$ is $[(V(\sigma_A)|_{TX}, 0)]$. We construct a projection out of $M_1 \doteq \begin{pmatrix} 0 & \sigma_A^* \\ \sigma_A & 0 \end{pmatrix}$ as follows,

$$P = [\frac{1}{1+\sigma_A\sigma_A^*}\begin{pmatrix} \sigma_A\sigma_A^* & \sigma_A^* \\ \sigma_A & 1 \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}].$$

Then

$$\begin{aligned}
[(C_0(TX,\pi^*E_0 \oplus \pi^*E_1),M_1)] =& [(C_0(TX,\pi^*E_0 \oplus \pi^*E_1),P)] \\
=& [(PC_0(TX,\pi^*E_0 \oplus \pi^*E_1),1)] + [((1-P)C_0(TX,\pi^*E_0 \oplus \pi^*E_1),0)] \\
=& [(C_0(TX,V(\sigma_A)|_{TX}),0)]
\end{aligned}$$

□

Remark III.2.3. When X is compact and when G is trivial, the inclusion $\mathbb{C} \to C(X)$ further reduces σ_A to an element of $KK(\mathbb{C},C_0(TX))$ by "forgetting" the action of $C(X)$ on the Hilbert-$C_0(TX)$ module. Therefore,

$$KK(\mathbb{C},C_0(TX))) \simeq K_0(C_0(TX))$$

is identified using the Fredholm picture of KK-theory, i.e. σ_A is invertible module compact operators of some Hilbert-$C_0(TX)$ module, which is strong Morita equivalent to $C_0(TX)$. σ_A provides an alternative definition to $K_0(C_0(TX))$ instead of using a vector bundle. The vector bundle that is trivial at infinity in TX and that corresponds σ_A is constructed by gluing $\pi^*E|_{B(X)}$ and $\pi^*E|_{TX\ B(X)^\circ}$ along the boundary using the fact that $\sigma_A|_{S(X)}$ is invertible. This bundle is the restriction of $V(\sigma_A)$ to TX.

III.2.2 Index pairing

Formula (III.1) is some form of pairing. When X is compact with trivial group action, apply the map

$$c^* : K_0(C(X)) \to K_0(\mathbb{C})$$

induced by constant map c to both sides of (III.1). The left hand side of (III.1) is then the Fredholm index of A and the right hand side is the intersection product of $[\sigma_A] \in K_0(TX)$ with $[D] \in K^0(TX)$. When $[\sigma_A]$ is viewed as a class of vector bundle V, the intersection product is well known to be the Fredholm index of the Dirac operator D with coefficients in V. Then the formula (III.1) is reduced to

$$[A] = [\sigma_A] \otimes [D] = [D_V].$$

Definition III.2.4. Let $[(H, \phi, F)] \in K^0(C_0(X))$ be an even Fredholm module and let $[P] \in K_0(C(X))$ be an equivalent class of projections. [7] Extend $\phi(P) : M_n(C_0(X)) \to \mathscr{B}(\oplus_{i=1}^n H)$ by applying ϕ to each entry. Since ϕ has degree 0, so does $\phi(P)$, which maps $\oplus_{j=1}^n H_i$ to itself ($i = 1, 2$). Define an operator

$$\tilde{T} = \phi(P)\mathrm{diag}(T, \cdots, T)\phi(P) : \phi(P)(\oplus_{i=1}^n H_0) \to \phi(P)(\oplus_{i=1}^n H_1)$$

which is Fredholm and the index of which is independent of the choice of the representatives in K-theory or K-homology. The *index pairing* is defined as

$$K^0(A) \times K_0(A) \to \mathbb{Z} : ([H, \phi, F], [P]) \mapsto \mathrm{Ind}(\tilde{T}). \tag{III.5}$$

Example III.2.5. The index pairing of $[(L^2(S^1), M, \frac{-i\frac{d}{d\theta}}{\sqrt{1+(i\frac{d}{d\theta})^2}})] \in K^1(C(S^1))$ and $[e^{i\theta}] \in K_1(C(S^1))$ under (III.5) is -1 in $K_0(\mathbb{C}) \simeq \mathbb{Z}$.

In (III.1) the Dolbeault operator D is a first order differential operator $D = \sqrt{2}(\partial + \partial^*)$ acting on smooth sections of $\Lambda^*(T^*X)$ on T^*X (ExampleII.2.19). Let H be Hilbert space of L^2-forms of bi-degree $(0, *)$ on T^*X graded by the odd and even degree forms. Then D is an essentially self adjoint operator on H of degree 1. The C^*-algebra $C_0(T^*X)$ acts on H by point-wise multiplication. *The Dolbeault element* is

$$[(H, \frac{D}{\sqrt{1+D^2}})] \in K_G^0(C_0(T^*X)) = KK^G(C_0(T^*X), \mathbb{C}).$$

Proof of Theorem III.2.1. (Kasparov [24]) Let $A \in \Psi^0_{G,p}(X; E, E)$ be an odd, self-adjoint operator. The intersection product $\sigma_A \sharp D$ defines an operator on T^*X. Let $D_1 = d_\xi + d_\xi^* + \varepsilon(\xi) + \iota(\xi)$ be

[7] $H = H_0 \oplus H_1, F = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$, and $P^2 = P \in M_n(C_0(X))$.

the family of the Bott-Dirac operators on each fibers of T^*X. Intersect A with D_1 and we get a pseudo-differential operator $A\sharp D_1$ on T^*X. The K-homology class defined by D_1 represents the multiplicative identity in the ring $KK^G(C_0(X), C_0(X))$ because D_1 having Fredholm index 1 in the fiber. Then $A\sharp D_1$ represents the same element with A in $KK^G(C_0(TX), \mathbb{C})$. The statement follows from the following observations.

- The symbols of $A\sharp D_1$ and $\sigma_A\sharp D$ are of order 0.
- The symbol of $A\sharp D_1$ and $\sigma_A\sharp D$ are computed as follows by the intersection product. Let $\nu : \mathbb{R} \to [0,1]$ is a smooth nondecreasing function such that $\nu(t) = 0$ when t is in some small open set containing 0 and $\nu(t) = 1$ for t in some small open set of 1.

  $$\sigma_{A\sharp D_1} = M_1(\sigma_A \hat{\otimes} 1) + N_1(1 \hat{\otimes} \sigma_{D_1})$$

  where $M_1^2 = \nu(\frac{\|\eta\|^2+1}{\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2+1})$ and $N_1^2 = 1 - M_1^2$.

  $$\sigma_{\sigma_A\sharp D} = M(\sigma_A \hat{\otimes} 1) + N(1 \hat{\otimes} \sigma_D)$$

  where $M^2 = \nu(\frac{\|\xi\|^2+1}{\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2+1})$ and $N^2 = 1 - M^2$. Recall that $\sigma_D = \varepsilon(-\eta + i\zeta) + \iota(-\eta + i\zeta)$.
- The two symbols represent the same element in $KK^G(C_0(TX), \mathbb{C})$. In fact the two differ by a rotation in (ξ, η) and therefore they are homotopic and define the same element in $K^*(C_0(X))$.

$\square$

The Dolbeault operator on T^*X extends to the proper cocompact G-manifold ΣX which has an almost complex structure. We just glue two Dolbeault operators on $B(X) \subset TX$ along the boundary (The normal directions of $S(X)$ in $B(X)$ need to switch signs on different pieces). The new Dolbeault operator $[\bar{D}]$ is clearly G-invariant and defines an element in $KK^G(C_0(\Sigma X), \mathbb{C})$. [8]

[8] In Chapter 4 we simply denote $[\bar{D}]$ by $[D]$.

Obviously $i : C_0(TX) \to C_0(\Sigma X)$ induces the natural map

$$i^* : KK^G(C_0(\Sigma X), \mathbb{C}) \to KK^G(C_0(TX), \mathbb{C}) : [\bar{D}] \mapsto [D]. \tag{III.6}$$

Corollary III.2.6. *Suppose we have the same conditions in the K-homological formula Theorem III.2.1. Then*

1. *The elliptic operator $[A]$ coincides with the intersection product in $[V(\sigma_A)] \otimes [\bar{D}]$ in*

$$KK^G(C_0(X), C_0(\Sigma X)) \times KK^G(C_0(\Sigma X), \mathbb{C}) \to KK^G(C_0(X), C_0(\Sigma X)).$$

2. *The elliptic operator A relates to an Dirac operator in the following sense:*

$$[A] = j^*[\bar{D}_{V(\sigma_A)}] \tag{III.7}$$

where $j^* : KK^G(C_0(\Sigma X), \mathbb{C}) \to KK^G(C_0(X), \mathbb{C})$ is induced by the inclusion $j : C_0(X) \to C_0(\Sigma X)$.

Proof. The first statement is a result of functorality of the intersection product

$$[A] = [\sigma_A] \otimes_{C_0(TX)} [D] = [\sigma_A] \otimes_{C_0(TX)} i^*[\bar{D}] = i_*[\sigma_A] \otimes_{C_0(\Sigma X)} [\bar{D}] = [V(\sigma_A)] \otimes_{C_0(\Sigma X)} [\bar{D}].$$

To prove the second statement, we calculate

$$[V(\sigma_A)] \otimes_{C_0(\Sigma X)} [\bar{D}] = [(C_0(\Sigma X, V(\sigma_A)), \phi_1, 0)] \otimes_{C_0(\Sigma X)} [(L^2(\Sigma X, \Lambda^*(\Sigma X)), \phi_2, F)] = [(H, \eta, G)]$$

where $H = C_0(\Sigma X, V(\sigma_A)) \otimes_{C_0(\Sigma X)} L^2(\Sigma X, \Lambda^*(\Sigma X))$ and $F = \frac{\bar{D}}{\sqrt{1+\bar{D}^2}}$. By Definition E.0.43, G needs to satisfy the following two conditions.

1. G is an F-connection;
2. G has the property $\eta(a)[0 \otimes 1, G]\eta(a) \geq 0$ module $\mathscr{K}(H)$.

By the stabilization theorem E.0.21, there is a $C_0(\Sigma X)$-valued projection P so that $C_0(\Sigma X, V(\sigma_A)) =$

$P(\oplus C_0(\Sigma X))$. Therefore,

$$H = P(\oplus C_0(\Sigma X)) \otimes_{C_0(\Sigma X)} L^2(\Sigma X, \Lambda^*(\Sigma X)) = \phi_2(P)(\oplus L^2(\Sigma X, \Lambda^*(\Sigma X))).$$

We claim that

$$G = \phi_2(P)(\oplus F)\phi_2(P) \tag{III.8}$$

First note that the statement is proved if the claim is true. In fact, one needs only to observe that $\phi_2(P)(\oplus L^2(\Sigma X, \Lambda^*(\Sigma X))) = L^2(\Sigma X, \Lambda^*(\Sigma X) \otimes V(\sigma_A))$ and $\phi_2(P)(\oplus \bar{D})\phi_2(P) = \bar{D}_{V(\sigma_A)}$ on H. To prove the claim (III.8), it is sufficient to show the following observations.

- $(G^2 - 1)\eta(f) \in \mathscr{K}(H), f \in C_0(X)$;
- $[G, \eta(f)] \in \mathscr{K}(H), f \in C_0(X)$;
- $[\tilde{T}_\xi, F \oplus G] \in \mathscr{K}(L^2(\Sigma X, \Lambda^*(\Sigma X)) \oplus H), \forall \xi \in C_0(\Sigma X, V(\sigma_A))$, where $\tilde{T}_\xi = \begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix} \in \mathscr{B}(L^2(\Sigma X, \Lambda^*(\Sigma X)) \oplus H), T_\xi \in \mathscr{B}(L^2(\Sigma X, \Lambda^*(\Sigma X)), H)$ is defined by $T_\xi(\eta) = \xi \hat{\otimes} \eta \in H$. (Definition E.0.40)

□

An important corollary to Theorem III.2.1 is the K-theoretic index formula. The right hand side represents topological information from the operator and the manifold.

Theorem III.2.7. *[22] Let X be a complete Riemannian manifold, on which a locally compact group G acts properly and isometrically with compact quotient. Let A be a properly supported G-invariant elliptic operator on X of order 0. Then*

$$\mathrm{Ind} A = [c] \otimes_{C^*(G, C_0(X))} j^G([A]) = [c] \otimes_{C^*(G, C_0(X))} j^G([\sigma_A]) \otimes_{C^*(G, C_0(T^*X))} j^G([D]) \in K_*(C^*(G)).$$

Where $[c]$ is the projection in $C^*(G, L^2(X, E))$ defined by $[c] = (c \cdot g(c))^{\frac{1}{2}}$ and $[D]$ is the Dolbeault element.

III.3 Link to the L^2-index

In the K-theoretic index theorem, there is no general algorithm for $K_0(C^*(G))$. The "trace" of $K_0(C^*(G))$, which is a real number, is easier to compute. We prove in this section that this numerical index coincides with the L^2-index.

Recall that the class under the isomorphism $KK(\mathbb{C}, C^*(G)) \cong K_0(C^*(G))$ of element $[(\mathscr{E}, A)] \in KK(\mathbb{C}, C^*(G))$ is *the K-theoretic index* of A. This index is represented by a formal difference of some $C^*(G)$-valued matrices due to the following remark.

Remark III.3.1. First of all, by adding a degenerate cycle $[(H_{C^*(G)}, 1)] \in KK(\mathbb{C}, C^*(G))$ to $[(\mathscr{E}, A)]$ and by the Kasparov stabilization theorem $\mathscr{E} \oplus H_{C^*(G)} \cong H_{C^*(G)}$, we may replace $\mathscr{E}$ by $H_{C^*(G)}$. By the axioms of KK-cycle, A is an odd self adjoint operator $\begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix}$ on graded Hilbert module $H_{C^*(G)}$, then $A^2 - 1 \in \mathscr{K}(\mathscr{E})$ implies that

$$A_0 A_0^* - I \in \mathscr{K}(H_{C^*(G)}) \text{ and } A_0^* A_0 - I \in \mathscr{K}(H_{C^*(G)})$$

and so A_0 is a generalized Fredholm operator. Now the generalized Atkinson theorem ([34] section 17.1) claims that the set of the generalized Fredholm operators in $\mathscr{L}(H_{C^*(G)})$ coincide with the following set $\{F \in \mathscr{L}(H_{C^*(G)}) | \exists K \in \mathscr{K}(H_{C^*(G)}), s.t. \operatorname{Ker}(F+K)$ and $\operatorname{Ker}(F+K)^*$ are finitely generated, $\operatorname{im}(F+K)$is closed.$\}$ Hence up to a compact perturbation we may assume that A_0 has closed range and that $\operatorname{Ker}(A_0), \operatorname{Ker}(A_0^*)$ are *finitely generated* $C^*(G)$-submodule. Now we have the boundary map (This is the Fredholm picture of KK-theory): $KK(\mathbb{C}, C^*(G)) \to K_0(\mathscr{K}(H_{C^*(G)})) \cong K_0(C^*(G))$:

$$(\mathscr{E}, A) \mapsto [\begin{pmatrix} A_0 A_0^* & A_0\sqrt{1 - A_0^* A_0} \\ \sqrt{1 - A_0^* A_0} A_0^* & 1 - A_0^* A_0 \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]. \tag{III.9}$$

If the elliptic operator $A_0 \in \mathscr{L}(H_{C^*(G)})$ has a closed range, then there exists polar decomposition $A_0 = U\sqrt{A_0^* A_0}$ with $\operatorname{Ker} A_0 = \operatorname{Ker} U$, where U is the partial isometry from $\operatorname{im} A_0^*$ to $\operatorname{im} A_0$ ([34] Theorem 15.3.8). Also, $1 - A_0^* A_0 \in \mathscr{K}(\mathscr{E})$ implies that $U - A_0 \in \mathscr{K}(\mathscr{E})$ and so we may replace A_0 by U in the boundary map (III.9). Since $1 - U^*U = P_{\ker A_0}, 1 - UU^* = P_{\ker A_0^*}$, then the image of the boundary map is reduced to

$$[1-U^*U]-[1-UU^*]=[P_{\ker A_0}]-[P_{\ker A_0^*}]\in K_0(C^*(G)),$$

which is the K-theoretic index.

Recall that topologically, the *K-theoretic index* of $[A]\in K^0_G(C_0(X))$, according to [22] is defined to be $\mathrm{Ind}_t A \doteq [p]\otimes_{C^*(G,C_0(X))} j^G([A])\in K_0(C^*(G))$, which is the image of $[A]$ under the descent map $j^G: KK^G(\mathbb{C},C_0(X))\to KK(C^*(G),C_0(G,C_0(X)))$ composed with the intersection product with a projection $[p]\in KK(\mathbb{C},C^*(G,C_0(X)))$, $[p]\otimes_{C^*(G,C_0(X))}: KK(C^*(G,C_0(X)),C^*(G))\to KK(\mathbb{C},C^*(G))$. Here $[p]\doteq(c\cdot g(c))^{\frac{1}{2}}$ is the projection represented as an element in $K_0(C^*(G,C_0(X)))$. Analytically, the K-theoretic index of A is constructed explicitly in the first section of the chapter. As a generalization of Atiyah-Singer index theorem, Kasparov proved that Ind_a and Ind_t coincide [22], [24]. We will simply use Ind to denote the K-theoretic index. In summary, the K-theoretic index under the homomorphism $\mathrm{Ind}: K^0_G(C_0(X))\to KK(\mathbb{C},C^*(G))\simeq K_0(\mathscr{K}(\mathscr{E}))$ is calculated by

$$[(L^2(X,E),A)]\mapsto[(\mathscr{E},\bar{A})]\mapsto[\begin{pmatrix}\bar{A}_0\bar{A}_0^* & \bar{A}_0\sqrt{1-\bar{A}_0^*\bar{A}_0}\\ \sqrt{1-\bar{A}_0^*\bar{A}_0}\bar{A}_0^* & 1-\bar{A}_0^*\bar{A}_0\end{pmatrix}]-[\begin{pmatrix}1&0\\0&0\end{pmatrix}], \qquad \text{(III.10)}$$

where $\bar{A}=\mathrm{Av}(cA)$. Note that the second arrow is the Fredholm picture of $KK(\mathbb{C},C^*(G))$ via boundary map.

Given the K-theoretic index $\mathrm{Ind}A\in K_0(\mathscr{K}(\mathscr{E}))$, we will define the a homomorphism $K_0(\mathscr{K}(\mathscr{E}))\to\mathbb{R}$. To do this we find a dense subalgebra $\mathscr{S}(\mathscr{E})$ of $\mathscr{K}(\mathscr{E})$ on which a "trace" can be defined and which is closed under holomorphic functional calculus. Since $\mathscr{K}(\mathscr{E})$ is generated by G-invariant operators with continuous and properly supported kernel, we define $\mathscr{S}(\mathscr{E})$ to be a subset of the bounded G-invariant operators with smooth kernels. Let $S: C_c^\infty(X,E)\to C_c^\infty(X,E)$ be a G-invariant smoothing operators. Extend S to an operator $\bar{S}\in\mathscr{B}(\mathscr{E})$ and then $\bar{S}\in\mathscr{S}(\mathscr{E})$. Define the trace on $\bar{S}\in\mathscr{S}(\mathscr{E})$ by $\mathrm{tr}_G(S)$ and still denote by tr_G The trace is well defined for all the elements of $\mathscr{S}(\mathscr{E})$. In fact, $\mathscr{S}(\mathscr{E})\subset\mathscr{S}(\mathscr{E}\otimes_{C^*(G)}\mathscr{R}(G))\subset\mathscr{R}(L^2(X,E))$ and $\mathscr{S}(\mathscr{E}\otimes_{C^*(G)}\mathscr{R}(G))$ is a subset of all G-trace class operators. Recall that tr_G is defined on a dense subset of the G-invariant operators on

$L^2(X,E)$,which can be represented as elements of

$$\mathscr{R}(L^2(G)) \otimes (\oplus_{i,j}\mathscr{B}(L^2(U_i,E),L^2(U_j,E)),$$

and an element of the set can be expressed in terms of a $\mathscr{R}(L^2(G))$-valued matrix.

Proposition III.3.2. • *We have the isomorphism* $K_0(\mathscr{K}(\mathscr{E})) \simeq K_0(\mathscr{S}(\mathscr{E}))$.

- *The G-trace* tr_G *on* $\mathscr{S}$ *defines a group homomorphism*

$$\mathrm{tr}_{G*} : K_0(\mathscr{K}(\mathscr{E})) \to \mathbb{R}.$$

Proof. Proposition II.4.6 part 4 suggests that $\mathscr{S}(\mathscr{E})$ is an ideal of $\mathscr{B}(\mathscr{E})$. Let $S \in \mathscr{S}(\mathscr{E})$ and for any holomorphic function f defined on the spectrum of S with $f(0) = 0$, there is another holomorphic function g so that $f(z) = zg(z)$. We require $f(0) = 0$ because $\mathscr{K}(\mathscr{E})$ does not contain identity. Since the spectrum of S is bounded and g is continuous which implies that g is a bounded function, i.e. $g(S) \in \mathscr{B}(\mathscr{E})$, $f(S) = Sg(S)$, then $f(S) \in \mathscr{S}(\mathscr{E})$. Hence $\mathscr{S}(\mathscr{E})$ is a dense subalgebra of $\mathscr{K}(\mathscr{E})$ closed under holomorphic functional calculus. Therefore $K_*(\mathscr{K}(\mathscr{E})) = K_*(\mathscr{S}(\mathscr{E}))$.

An element of $K_0(\mathscr{S}(\mathscr{E}))$ is represented by projection matrix with entry in $\mathscr{S}(\mathscr{E})$, on which there is a natural trace consists of the combination of the matrix trace with τ on $\mathscr{S}(\mathscr{E})$. Note that if the element was represented by difference of two classes of matrices with entries in $\mathscr{S}(\mathscr{E})^+$, the algebra by adding a unit, then we define the trace of this extra unit to be 0. Hence we obtain a homomorphism $\mathrm{tr}_{G*} : K_*(\mathscr{S}(\mathscr{E})) \to \mathbb{R}$ by the properties of the trace τ. □

Composing with the K-theoretic index, A has a numerical index in the image of the map

$$K^0_G(C_0(X)) \xrightarrow{\text{K-theoretic index}} K_0(\mathscr{S}) \xrightarrow{\mathrm{tr}_{G*}} \mathbb{R}$$

and this number depends only on the symbol class and the manifold according to Kasparov's K-theoretic index formula. We show that this number is in fact the L^2-index.

Proposition III.3.3. *Let* $P \in \Psi^0_{G,p}(X;E,E)$ *be elliptic, then its* L^2*-index coincides with the trace of its K-theoretic index, i.e.* $\operatorname{ind} P = \mathrm{tr}_{G*}(\mathrm{Ind}[P])$.

Proof. Let $P = A$ and then $P = \bar{A} = \mathrm{Av}(cA)$ in III.10. Then

$$\mathrm{Ind}P = [\begin{pmatrix} P_0P_0^* & P_0\sqrt{1-P_0^*P_0} \\ \sqrt{1-P_0^*P_0}P_0^* & 1-P_0^*P_0 \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}].$$

We shall alter the matrix representatives without changing the equivalent class, so that we may apply tr_G to the 2×2-matrices.

Given $P_0 \in \Psi^0_{G,p}(X; E_0, E_1)$ and using Proposition II.3.12, there is a $Q \in \Psi^0_{G,p}$ so that $1 - QP = S_0, 1 - PQ = S_1$. According to the boundary map construction in [13] section 2, we lift $\begin{pmatrix} 0 & -Q \\ P & 0 \end{pmatrix}$ which is invertible in $M_2(\mathscr{B}(\mathscr{E})/\mathscr{S}(\mathscr{E}))$ to an invertible element $u = \begin{pmatrix} S_0 & -(1+S_0)Q \\ P & S_1 \end{pmatrix}$ in $M_2(\mathscr{B}(\mathscr{E}))$ and then

$$\mathrm{Ind}P \doteq [u\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}u^{-1}] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}] = [\begin{pmatrix} S_0^2 & S_0(1+S_0)Q \\ P_0S_1 & 1-S_1^2 \end{pmatrix}] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}].$$

Therefore, $\mathrm{tr}_{G*}(\mathrm{Ind}P) = \mathrm{tr}_G(S_0^2) + \mathrm{tr}_G(1-S_1^2) - \tau(1) = \mathrm{tr}_G(S_0^2) - \mathrm{tr}_G(S_1^2)$. Choose another $Q' \doteq 2Q - QPQ$, then $1 - Q'P_0 = S_0^2, 1 - PQ' = S_1^2$ with S_0^2, S_1^2 being smoothing operators. Then using Proposition , we conclude that $\mathrm{tr}_G(S_0^2) - \mathrm{tr}_G(S_1^2) = \mathrm{ind}\, P$. Hence $\mathrm{tr}_{G*}(\mathrm{Ind}P) = \mathrm{ind}\, P$. □

Remark III.3.4. Let $X = G/H$ be a homogeneous space of a unimodular Lie group G (where H is a compact subgroup). In [12] section 3, it was shown directly that the L^2-index depends only on the symbol class $[\sigma_P]$ of P in $K_0^G(C_0(TX))$. Plus, there exists a homomorphism $i: K_0^G(C_0(TX)) \to \mathbb{R}$ so that $i[\sigma_P] = \mathrm{ind}\, P$.

Note that the Poincaré duality between K-homology and K-theory implies that $K_0^G(C_0(TX)) \simeq K_G^0(C_0(X))$. So L^2-index essentially gives a homomorphism:

$$\mathrm{ind}: K_G^0(C_0(X)) \to \mathbb{R}. \tag{III.11}$$

But the L^2-index is not well-defined for all the representing cycles of the K-homology.

Remark III.3.5. In this section we work on the cycles in $K_G^0(C_0(X))$ as elliptic pseudo-differential operators on X. Let Y be another properly cocompact manifold and E be a G-bundle where $L^2(X,E)$ admits $C_0(X)$ representation. And if we have $[(L^2(Y,E),Q)] \in K_G^0(C_0(X))$ with $Q \in \Psi^0_{G,p}(Y;E,E)$, we may carry out similar statements of this section easily.

To compute $\operatorname{ind} A$, the L^2-index of A, we use the fact that $\operatorname{ind} A$ factors through the K-theoretic index homomorphism together with $[A] = j^*[\bar{D}_{V(\sigma_A)}]$ in (III.7). As indicated by the following commuting diagram,

$$\begin{tikzcd} KK^G(C_0(X),\mathbb{C}) \arrow[r, "\mu"] & KK(\mathbb{C},C^*(G)) \arrow[r, "\tau"] & \mathbb{R} \\ KK^G(C_0(\Sigma X),\mathbb{C}) \arrow[u, "j^*"] \arrow[ur, "\mu"'] & & \end{tikzcd} \tag{III.12}$$

the L^2-index of A can be computed by the L^2-index of a Dirac type operator constructed out of the symbol of A. The statement is summarized in the following proposition.

Proposition III.3.6. *Let A be a properly supported G-invariant elliptic operator of order 0, D be the Dolbealt operator on ΣX and $V(\sigma_A)$ is the G-vector bundle over ΣX built using σ_A. Then*

$$\operatorname{ind} A = \operatorname{ind} D_{V(\sigma_A)} \tag{III.13}$$

CHAPTER IV

L^2-INDEX OF DIRAC TYPE OPERATORS

To find a cohomological formula for the L^2-index of A, it is sufficient to figure out a formula for the Dirac type operator according to Proposition III.3.6. Let M be an even-dimensional ($\dim M = n$) properly cocompact G-manifold with a G-Clifford bundle V, which is a $\mathbb{C}l(TM)$-module via Clifford multiplication. Let $\mathscr{D}$ be a Dirac type operator acting on sections in V. We compute $\operatorname{ind}\mathscr{D}$ using the heat kernel method in section IV.1 IV.2 and apply it to the case when

$$M = \Sigma X, \mathscr{D} = D_{V(\sigma_A)}$$

in section IV.3. In section IV.4 the L^2-index formula for free cocompact action [4] and for homogeneous space for Lie group [12] are obtained as corollaries. In the following we assume $V = V_0 \oplus V_1$ be $\mathbb{Z}/2$ graded and $\mathscr{D}$ be odd and essentially self-adjoint, i.e. $\mathscr{D} = \begin{pmatrix} 0 & \mathscr{D}_0^* \\ \mathscr{D}_0 & 0 \end{pmatrix}$.

IV.1 Heat kernel method

The Schwartz kernel $k_t(x,y)$ of the solution operator $e^{-t\mathscr{D}^2}$ of the heat equation $u_t + \mathscr{D}^2 u = 0$ on M is the heat kernel. The L^2-index of $\mathscr{D}$ is expressed using $k_t(x,x)$ by the Mckean-Singer formula (II.31) and is independence of t. The heat kernel method is to calculate $\operatorname{ind}\mathscr{D}$ by finding $\lim_{t\to 0+} k_t(x,x)$. The method was invented by Patodi for Dirac operators on compact manifold and further studied by Atiyah, Bott, Gilkey and Getzler, etc. Now the heat kernel method is a commonly-used analytical method in proving various versions of index formula. The outline of the method in the context of proper cocompact group action is explained in this section, and the proof of some technical theorems (Theorem IV.1.8, Lemma IV.1.11) are done in the next section.

We start by recalling *the Mckean-Singer formula* in the line (II.31):

Proposition IV.1.1. *The index of the odd self adjoint G-invariant Dirac operator $\mathscr{D} : L^2(M,V) \to L^2(M,V)$ is calculated by*

$$\operatorname{ind}\mathscr{D} = \operatorname{str}_G(e^{-t\mathscr{D}^2}), \tag{IV.1}$$

where $\mathrm{str}_G(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \mathrm{tr}_G(a) - \mathrm{tr}_G(d)$ *and* $\mathscr{D}^2 = \begin{pmatrix} \mathscr{D}_0^*\mathscr{D}_0 & 0 \\ 0 & \mathscr{D}_0\mathscr{D}_0^* \end{pmatrix}$.

The Dirac type operators $\mathscr{D}$ are constructed as follows. On TM there is a G-invariant Levi-Civita connection ∇. In fact, because of the G-invariant partition of unity on M, it is sufficient to find a G-invariant metric when $M = G \times_H S$. But such a metric can be defined using an H-invariant metric on S. The Levi-Civita connection ∇ extends to $\mathrm{Cl}(TM)$. Let the $\mathbb{Z}_2$ graded vector bundle V be a Clifford module over $\mathbb{C}\mathrm{l}(TM) = \mathrm{Cl}(TM) \otimes \mathbb{C}$. Let ∇^V be the G-invariant Clifford connection on V. A Dirac operator $\mathscr{D} : C^\infty(M,V) \to C^\infty(M,V)$ is defined under the composition of the connection ∇^V and the Clifford multiplication $\mathfrak{c} : C^\infty(M, T^*M \times V) \to C^\infty(M,V)$ by

$$\mathscr{D} = \sum_i \mathfrak{c}(e^i)\nabla^V_{e_i},$$

where e_is are orthonormal basis of the bundle TM and e^is are the dual basis of T^*M.

Let $R^V = {\nabla^V}^2$ be the curvature tensor of connection ∇^V, then

$$\begin{aligned}
\mathscr{D}^2 &= \sum_{ij} \mathfrak{c}(e^i)\nabla^V_{e_i}\mathfrak{c}(e^j)\nabla^V_{e_j} = \sum_{ij} \mathfrak{c}(e^i)[\nabla^V_{e_i}, \mathfrak{c}(e^j)\nabla^V_{e_j}] \\
&= \sum_{ij} \mathfrak{c}(e^i)[\nabla^V_{e_i}, \mathfrak{c}(e^j)]\nabla^V_{e_j} + \sum_{ij} \mathfrak{c}(e_i)\mathfrak{c}(e^j)[\nabla^V_{e_i}, \nabla^V_{e_j}] \\
&= \sum_{ij} \mathfrak{c}(e^i)\mathfrak{c}(\nabla_{e_i}e_j)\nabla^V_{e_j} - \sum_{i=1}^n (\nabla^V_{e_i})^2 + \sum_{i<j} \mathfrak{c}(e_i)\mathfrak{c}(e_j)[\nabla^V_{e_i}, \nabla^V_{e_j}] \\
&= -\sum_i (\nabla^V_{e_i})^2 + \sum_i \nabla^V_{\nabla_{e_i}e_i} + \sum_{i<j} \mathfrak{c}(e^i)\mathfrak{c}(e^j)R^V(e_i,e_j) \doteq \Delta^V + \sum_{i<j} \mathfrak{c}(e^i)\mathfrak{c}(e^j)R^V(e_i,e_j)
\end{aligned}$$

is a generalized Laplacian. $R^V \in \Lambda^2(M, \mathrm{End}\, V)$ can be further decomposed, which is explained as follows.

Let S be the spinor (irreducible) representation of $\mathrm{Cl}(T_xM)$. It is a standard fact that

$$\mathrm{End}\, S = S \otimes S^* = \mathbb{C}\mathrm{l}(T_xM).$$

The fiber of the Clifford module V at x has the decomposition $V_x = S \otimes W$. Therefore on the endomorphism level we have

$$\mathrm{End}\, V_x = \mathbb{C}\mathrm{l}(T_xM) \otimes \mathrm{End}\, W. \tag{IV.2}$$

Here W is the set of vectors in V_x that commute with the action of $\mathbb{C}\mathrm{l}(T_xM)$ and $\mathrm{End}\,W$ is the set of transformations of V_x that commute with $\mathbb{C}\mathrm{l}(T_xM)$. Denote $\mathrm{End}_{\mathbb{C}\mathrm{l}(T_xM)}(V_x) \doteq \mathrm{End}\,W$.

Proposition IV.1.2 ([8] Proposition 3.43)**.** *The curvature R^V of a Clifford connection ∇^V on V decomposes under the isomorphism (IV.2) as*

$$R^V = R^S + F^{V/S}$$

where $R^S(e_i,e_j) = \frac{1}{4}\sum_{kl}(R(e_i,e_j)e_k,e_l)c^kc^l$ is the action of the Riemannian curvature $R = \nabla^2$ of M on the bundle V and $F^{V/S} \in \Lambda^2(M,\mathrm{End}_{\mathbb{C}\mathrm{l}}V)$ is called the twisting curvature *of V.*

Proposition IV.1.3 (Lichnerowicz Formula, [8] Proposition 3.52)**.** *Using the previous notation, the generalized Laplacian is calculated by:*

$$\mathscr{D}^2 = -\sum_{i=1}^{n}(\nabla^V_{e_i})^2 + \sum_i \nabla^V_{\nabla_{e_i}e_i} + \frac{1}{4}r_M + \sum_{i<j} F^{V/S}(e_i,e_j)\mathfrak{c}(e_i)\mathfrak{c}(e_j),$$

where e_i is a local orthonormal frame of $TM \simeq T^*M$ and $F^{V/S}(e_i,e_j) \in \mathrm{End}_{\mathbb{C}\mathrm{l}}V$ are the coefficient of the twisting curvature of the Clifford connection ∇^V.

Having $\mathscr{D}^2$ we can consider the heat equation

$$\frac{\partial}{\partial t}u(t,x) + \mathscr{D}^2u(t,x) = 0, u(x,0) = f(x)$$

and then $(e^{t\mathscr{D}^2}f)(x)$ is the solution. Denote by $k_t(x,y)$ the Schwartz kernel of $e^{-t\mathscr{D}^2}$ and k_t is a smooth map $M\times M \to \mathrm{Hom}(V,V)$ and satisfies $e^{-t\mathscr{D}^2}f(x) = \int_M k_t(x,y)f(y)\mathrm{d}y$. We call $k_t(x,y)$ *the heat kernel*.

Remark IV.1.4. Fix y then $k_t(x,y)$ is the fundamental solution of $u_t + \mathscr{D}_x^2u = 0$ with initial condition $\delta(x-y)$. We study heat kernel to compute the L^2-index of $\mathscr{D}$.

The following lemma is some property of the heat kernel to be used later.

Lemma IV.1.5. *1. For $f(x) \in L^2(M)$, $e^{-t\mathscr{D}^2}f$ is a smooth section;*

2. The kernel $k_t(x,y)$ of $e^{-t\mathscr{D}^2}$ tends to δ function weakly, i.e. $e^{-t\mathscr{D}^2}s(x) = \int_M k_t(x,x_0)s(x_0)\mathrm{d}x_0 \to s(x)$ uniformly on compact set in x as $t \to 0$.

Proof. We have proved that the Schwartz kernel of $ce^{-t\mathscr{D}^2}$ is smooth in Lemma II.4.15. So

$$(e^{-t\mathscr{D}^2}f)(x) = \int_{G\times M} c(g^{-1}x)k_t(x,y)f(y)\mathrm{d}y\mathrm{d}g \doteq \int_G h_g(x)\mathrm{d}g,$$

where $h_g(x) = \int_M c(g^{-1}x)k_t(x,y)f(y)\mathrm{d}y$ is smooth in $x \in M$ for fixed $g \in G$. Using the fact that $e^{-t\mathscr{D}^2}$ is a bounded operator and that $c(x)$ is smooth and compactly supported, we conclude that $h_g(x)$ depends smoothly on $g \in G$. Let K be any compact neighborhood of x, then by the properness of the group action, the set

$$Z \doteq \{g \in G | c(g^{-1}x) \neq 0, x \in K, g \in G\}$$

is compact and then $(e^{-t\mathscr{D}^2}f)(x) = \int_Z h_g(x)\mathrm{d}g$ is smooth for $x \in K$. Therefore the first statemante is proved.

To prove the second one, let u be a smooth function with norm 1. Then $< e^{-t\mathscr{D}^2}u, u >= \int_{\lambda\in \mathrm{sp}(\mathscr{D})} e^{-t\lambda^2}\mathrm{d}P_{u,u}$ (sp means spectrum). Since the set of integrals for $0 < t \leq 1$ is bounded by 1, then by the dominant convergent theorem,

$$< e^{-t\mathscr{D}^2}u, u > \to \int_{\lambda\in \mathrm{sp}(\mathscr{D})} 1\mathrm{d}P_{u,u} =< u, u > \text{ as } t \to 0.$$

$\square$

Example IV.1.6. Given the heat equation

$$u_t - \sum_{i=1}^{n} \frac{\partial^2}{\partial^2 x_i} = 0$$

on $\mathbb{R}^n$, the heat kernel is

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t} \tag{IV.3}$$

from the theory of partial differential equation.

The heat kernel for curved manifold M is more complicated. The formula (IV.3) suggests a first approximation for the heat kernel on M. The small time behavior of heat kernel $k_t(x,y)$ for x near y depends on the local geometry of x near y. This is made precise by the *asymptotic expansion* for

$k_t(x,y)$.

Definition IV.1.7. Let B be a Banach space with norm $\|\cdot\|$ and $f:\mathbb{R}^+\to B: t\mapsto f(t)$ be a function. A formal series $\sum_{k=0}^{\infty} a_k(t)$ with $a_k(t)\in E$ is called *an asymptotic expansion* for f, denoted by $f(t)\sim \sum_{i=0}^{\infty} a_k(t)$, if for any $m>0$, there are M_m and $\varepsilon_m>0$. So that for all $l\geq M_m, t\in(0,\varepsilon_m]$, we have

$$\|f(t)-\sum_{k=0}^{l} a_k(t)\|\leq Ct^m.$$

When M is compact and when $B=C^0(M,\mathrm{End}(V,V))$ has C^0-norm $\|f\|=\sup_{x\in M}|f(x)|$, it is the standard fact that the heat kernel $k_t(x,x)$ of $e^{-t\mathscr{D}^2}$ has an asymptotic expansion

$$k_t(x,x)\sim\frac{1}{(4\pi t)^{n/2}}\sum_{j=0}^{\infty} t^j a_j(x)$$

where $a_j(x)\in\mathrm{Hom}(V_x,X_x), x\in M$ are smooth sections. (Refer to [29] Theorem 7.15). In the case of non-compact M having group action, this theorem can be modified as follows. The theorem is proved in the next section.

Theorem IV.1.8. *Let M be a proper cocompact Riemannian manifold and $\mathscr{D}$ be a Dirac type operator acting on the sections of Clifford bundle V, and k_t be the heat kernel of $e^{-t\mathscr{D}^2}$. There is an asymptotic expansion for $c(x)k_t(x,x)$ under the C^0-norm $\|f\|=\sup_{x\in M}|f(x)|$:*[1]

$$c(x)k_t(x,x)\sim c(x)\frac{1}{(4\pi t)^{n/2}}\sum_{j=0}^{\infty} t^j a_j(x) \tag{IV.4}$$

where $a_j\in C^\infty(M,\mathrm{End}V)$ and $a_j(x)$ depends only on the the geometry at x (involving metrics, connection coefficients and their derivatives). In particular $a_0(x)=1$.

Remark IV.1.9. From (IV.4) $\lim_{t\to 0+} c(x)\,\mathrm{str}\,k_t(x,x)=\lim_{t\to 0+} c(x)\frac{1}{(4\pi t)^{n/2}}\sum_{j=0}^{l} t^j\,\mathrm{str}\,a_j(x)$ for sufficient large l. Then to calculate the left hand side it is sufficient to investigate a_js on the right hand side.

In the Remark (IV.1.9) $\mathrm{str}\,a_i, a_i(x)\in\mathrm{End}V_x$, needs to be figured out. If $a\in\mathrm{End}V_x$ then a has decomposition

$$a=b\otimes c, b\in\mathbb{C}\mathrm{l}(T_xM), c\in\mathrm{End}W \text{ under } (IV.2).$$

[1] The asymptotic expansion works for any C^l-norm for $l\geq 0$. But we only need $l=0$.

The super-trace $\mathrm{str}\, a$ is then calculated by

$$\mathrm{str}(b \otimes c) = \tau(b) \cdot \mathrm{str}^{V/S}(c) \tag{IV.5}$$

where $\mathrm{str}^{V/S}$ in (IV.5) is the super-trace of $\mathbb{C}$-linear endomorphism of W under the identification $\mathrm{End}_{\mathbb{C}l(TM_x)}(V_x) = \mathrm{End}_{\mathbb{C}}(W)$ and τ_s is the the super-trace on $\mathrm{End}\, S = S \otimes S^* = \mathbb{C}\mathrm{l}(T_x M)$. Note that by definition of the super-trace, $\tau_s(t) = \tau(vt)$, where v is the grading operator of $\mathbb{C}\mathrm{l}(T_x M)$ [2].

The super-trace τ_s on $\mathbb{C}\mathrm{l}(T_x M)$ is explicitly calculated by the following lemma:

Lemma IV.1.10 ([8] Proposition 3.21)**.** *Let* $\dim M = n = 2m$ *and let* $c = \sum c_{i_1 i_2 \cdots i_k} e_{i_1} e_{i_2} \cdots e_{i_k}$ *be an element in* $\mathbb{C}\mathrm{l}(\mathbb{R}^{2m}) \simeq \mathbb{C}\mathrm{l}(T_x M) = \mathrm{End}(S)$*, where* e_i*s are orthonormal frames of* $T_x M$ *and* $c_{i_1 i_2 \cdots i_k}, 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$ *be the coefficient of the base* $e_{i_1} e_{i_2} \cdots e_{i_k}$ *in* $\mathbb{C}\mathrm{l}(T_x M)$*. Then*

$$\tau_s(c) = (-2i)^{\frac{n}{2}} c_{12\cdots n}.$$

Proof. Since the grading operator of $\mathbb{C}\mathrm{l}(T_x M)$ is $i^m e_1 \cdots e_{2m}$, then by definition $\tau_s(c) = \tau(i^m e_1 \cdots e_{2m} c)$. It is sufficient to check when c is in the basis of $\mathbb{C}\mathrm{l}(T_x M)$:

- First of all we have $\tau(e_1 \cdots e_{2m}) = (-2i)^m$. It follows from the fact that $(e_1 \cdots e_{2m})^2 = (-1)^m$ and that the action of 1 on S has trace equals to 2^m, the dimension of S.
- If $a = e_{i_1} \cdots e_{i_k}$ does not contain e_j, then $\tau_s(a) = 0$. Because a is written as graded commutator $a = \frac{1}{2}[e_j a, e_j]$, $\tau_s(a) = \tau_s(\frac{1}{2}[e_j a, e_j]) = 0$.

□

We say that A is a filtered algebra if $A = \cup_0^\infty A_i$ and $A_i \subset A_{i+1}, A_i \cdot A_j \subset A_{i+j}$. The Clifford algebra is a filtered algebra and we have

$$\mathbb{C}\mathrm{l}(T_x M) = \mathbb{C}\mathrm{l}(\mathbb{R}^n) = \cup_{i=0}^n \mathbb{C}\mathrm{l}_i$$

[2]This is a general definition of the supertrace. For $M_2(A)$ with standard even grading, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the grading operator and then $\mathrm{str}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \mathrm{tr}\, a - \mathrm{tr}\, d$

where $\mathbb{C}\mathrm{l}_i$ is the linear combination of $e_{j_1}\cdots e_{j_k}, k \le i$. In proving Theorem IV.1.8 in the next section, the following lemma is obtained (Corollary IV.2.4)

Lemma IV.1.11. *Let $a_i(x)$ be the ith term in the asymptotic expansion. Then*

$$a_i(x) \in \mathbb{C}\mathrm{l}_{2i} \otimes \mathrm{End}_{\mathbb{C}l(TM_x)}(S_x). \tag{IV.6}$$

Remark IV.1.12. An easy corollary of Lemma IV.1.10 and the line (IV.6) is that $\mathrm{str}\, a_i(x) = 0$ for $i \le \frac{n}{2}$. Therefore

$$\mathrm{ind}\,\mathscr{D} = \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{i \ge \frac{n}{2}} t^i \int_M c(x)\,\mathrm{str}(a_i(x))\mathrm{d}x. \tag{IV.7}$$

Furthermore, since the index is independent of t and n is even, we have the following theorem.

Theorem IV.1.13. *The index of the graded Dirac operator $\mathscr{D}$ is equal to*

$$\mathrm{ind}\,\mathscr{D} = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M c(x)\,\mathrm{str}(a_{n/2}(x))\mathrm{d}x. \tag{IV.8}$$

Example IV.1.14. ([29] Proposition 7.19) For 2-dim manifold, consider the de Rham operator $D = d + d^*$ on differential forms, (IV.8) reduces to the Gauss-Bonnet theorem $\mathrm{ind}(D) = \dfrac{1}{4\pi}\displaystyle\int_M k dx$ where k is twice the Gaussian curvature.

The $\mathrm{ind}\,\mathscr{D}$ in (IV.8) can be calculated analytically in terms of differential forms on M by the following is the **main theorem of this section**.

Theorem IV.1.15. *Let R be the curvature 2-form with respect to the Levi-Civita connection on the manifold (on TM). Then $\mathrm{str}(a_{\frac{n}{2}}(x))$ is the n form part of $(-2i)^{n/2}\det^{\frac{1}{2}}(\frac{R/2}{\sinh R/2})\,\mathrm{tr}^{V/S}(e^{-F})$. In particular,*

$$\mathrm{ind}\,\mathscr{D} = (2\pi i)^{-\frac{n}{2}} \int_M c(x)\hat{A}(M)\cdot \mathrm{ch}(V/S). \tag{IV.9}$$

where

$$\hat{A}(M) = \det{}^{\frac{1}{2}}\left(\frac{R/2}{\sinh R/2}\right)$$

is the $\hat{A}$-class of TM and

$$\mathrm{ch}(V/S) = \mathrm{tr}^{V/S}(e^{-F^{V/S}})$$

is the relative Chern character, i.e. Chern character of the twisted curvature $F^{V/S}$ of bundle S.

Remark IV.1.16. The characteristic class is using Chern-Weil's definition and to be compatible with the topological definition, we sometimes need to replace the curvature R by $\dfrac{R}{2\pi i}$. In the proof of the theorem, we use R to define $\hat{A}$. But starting from the next section, we use $\dfrac{R}{2\pi i}$ to define $\hat{A}$. Then the L^2-index formula is

$$\operatorname{ind}\mathscr{D} = \int_M c(x)\hat{A}(M)\cdot \operatorname{ch}(V/S).$$

Because of Theorem IV.1.13, the proof of Theorem IV.1.15 is to calculate $\operatorname{str}(a_{n/2}(x)) \in \operatorname{End}(V_x)$. The idea of doing that is to localize the operator $\mathscr{D}$ and the heat kernel $k_t(x,y)$ at a point x. Because the local calculation is irrelevant to M being compact or not, **we use the classical calculation of** $\operatorname{str} a_{\frac{n}{2}}$ **on a compact manifold without modification.** In the rest of the section, I summarize the idea of the calculation without further verification. For details, please refer to [8] Chapter 4.

Getzler symbol

The first step is to "change" the operators on M to TM, without changing the initial condition of the heat kernel. Getzler [14] gives a systematic way of investigating the top order part of an operator by introducing a generalized version of symbol. The formulation needs the *(Getzler) symbol* for $a_{\frac{n}{2}}(x) \in \operatorname{End}(V_x)$ and for differential operators acting on $L^2(M,V)$, such as $\mathscr{D}$.

Definition IV.1.17. Let $A = \cup_0^\infty A_i$ be a filtered algebra. Denote by $G(A)$ the new graded algebra

$$G(A) = \oplus_0^\infty G(A)^i = \oplus_0^\infty A_i/A_{i-1},$$

then the projection $\sigma : \oplus_0^\infty A_i \to G(A)$ is called *the symbol map.* Its component of degree i is the projection restricted to A_i,

$$\sigma_i : A_i \to G(A).$$

Remark IV.1.18.

1. If $a \in A_{m-1}$ then $\sigma_m(a) = 0$;
2. If $a \in A_m$ and $a' \in A_{m'}$, then $\sigma_m(a)\sigma_{m'}(a') = \sigma_{m+m'}(aa')$.

Example IV.1.19.

- Let A_i be the set of degree i pseudo-differential operators, then $G(A)$ is the set of (principal) symbols.

- Let T be a vector space and $A = \mathbb{C}l(T)$ be the complex Clifford algebra of T. Then $A_i = \mathbb{C}l_i$, the linear span of elements of form $v_1 \cdots v_k, v_j \in V, k \leq i$, and $G(A) = \Lambda^* T$ and

$$\sigma : \mathbb{C}l(T) \to \Lambda^* T : a \mapsto \mathfrak{c}(a) \cdot 1,$$

where $1 \in \Lambda^* T$, $\Lambda^* T$ is the representation space of $\mathbb{C}l(T)$.

- The filtration used in the calculation of the L^2-index is the *Clifford filtration*. $\mathrm{End}(V_x)$, under the decomposition $\mathrm{End}(V_x) = \mathbb{C}l(T_x M) \otimes \mathrm{End}(W)$, is a filtered algebra using the standard filtration on $\mathbb{C}l(T_x M)$.

$$\sigma : \mathrm{End}(V_x) = \mathbb{C}l(T_x M) \otimes \mathrm{End}(W) \to \Lambda^* T_x M \otimes \mathrm{End}(W) : a \otimes b \mapsto c(a) \cdot 1 \otimes b. \tag{IV.10}$$

Recall from Lemma IV.1.11, the degree of a_i is $2i$ under Clifford filtration on small neighborhood of x_0 in $T_x M$. i.e. $a_i(x) \in \mathbb{C}l_{2i}(T_{x_0} M) \otimes \mathrm{End}_{\mathbb{C}l(T_{x_0} M)}(V_{x_0})$. Since the supertrace of an element c of the Clifford algebra $\mathbb{C}l(\mathbb{R}^{2m})$ is equal to the top degree part of the of c: $\mathrm{str}(c) = (-2i)^m c_{12\cdots 2m}$, we have the following relationship between $a_{\frac{n}{2}}(x)$ and its symbol.

Proposition IV.1.20. *The asymptotic coefficient $a_{\frac{n}{2}}(x)$ and its symbol are related as follows:*

$$\mathrm{str}(a_{\frac{n}{2}}(x)) = (-2i)^{n/2} \, \mathrm{str}^{V/S}(\sigma_n(a_{\frac{n}{2}}(x))). \tag{IV.11}$$

This proposition together with Lemma IV.1.11 and Theorem IV.1.13,

$$\mathrm{ind}\, \mathscr{D} = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_M c(x) \, \mathrm{str}^{V/S}(\sigma_n(a_{\frac{n}{2}}(x))) = \frac{(4\pi)^{\frac{n}{2}}}{(2\pi i)^{\frac{n}{2}}} \int_M c(x) \, \mathrm{str}^{V/S}(\sigma(k_t(x,x))|_{t=1}, \tag{IV.12}$$

where $k_t(x,x) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{i=0}^{\infty} t^i a_i(x)$ and $\sigma(k_t(x,x)) = (4\pi)^{-\frac{n}{2}} \sum_{i=0}^{\frac{n}{2}} t^{-\frac{n}{2}+i} \sigma_{2i}(a_i(x)))$.

The geometric meaning of the symbol $\sigma : \mathrm{End}(V_{x_0}) \to \Lambda^* T_{x_0} M \otimes \mathrm{End}_{\mathbb{C}l}(V_{x_0})$ in (IV.10) is to localize $a_i(x)$ and the Schwartz kernel $k_t(x, x_0)$ to $T_{x_0} M$.

Restrict the Dirac operator $\mathscr{D}$ on a coordinate neighborhood O of x_0 in M (then $V|_O$ is trivial). Let U be the neighborhood of x_0 in $T_{x_0} M$ homeomorphic to O. The heat kernel $k_t(x, x_0), x \in O$

localized to O can be transformed to a function on $\mathbb{R}_+ \times U$ taking values in $\operatorname{End} V_{x_0}$:

$$\sigma(k_t(\exp \mathbf{x}, x_0)) = k(t, \mathbf{x}) = \tau(x_0, x) k_t(x, x_0), x \in O, \mathbf{x} \in U, t \in \mathbb{R}_+,$$[3]

where $\tau(x_0, x) : V_x \to V_{x_0}$ is the identification of the fiber V_x and V_{x_0}. We are interested in $k(t, \mathbf{0}) = k_t(x_0, x_0)$. Note that $k(t,\mathbf{x})_{[k]} = \begin{cases} t^{\frac{k-n}{2}} \sigma_k(a_{\frac{k}{2}}(x_0)) + o(t^{\frac{k-n}{2}}) & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$ by Lemma IV.1.11

Also, consider $\mathscr{D}$ acting on a neighborhood of the tangent space $C^\infty(U, \operatorname{End} V_{x_0})$ instead of $C^\infty(M, \operatorname{End} V)$. Then $\frac{\partial}{\partial t} + \mathscr{D}^2$ induces a new operator $\frac{\partial}{\partial t} + L$ acting on $C^\infty(\mathbb{R}_+ \times U, \Lambda^* TM \otimes \operatorname{End}_{\mathbb{Cl}}(V))$ after localization.

Proposition IV.1.21 ([8] Lemma 4.16)**.** *L is represented by the following fomula.*

$$L = -\sum_{i=1}^n (\nabla^V_{e_i})^2 + \sum_{i=1}^n \nabla^V_{\nabla_{e_i} e_i} + \frac{1}{4} r_M + \sum_{i<j} F^{V/S}(e_i, e_j) \mathfrak{c}(\mathrm{d}\mathbf{x}_i) \mathfrak{c}(\mathrm{d}\mathbf{x}_j),$$

where e_is are the local orthonormal frame obtained by parallel transport from the orthonormal basis of $T_{x_0}M$ and where

$$\nabla^V_{e_i} = \frac{\partial}{\partial x_i} + \frac{1}{4} \sum_{j;k<l} R_{klij} \mathbf{x}^{\mathbf{j}} c(\mathrm{d}\mathbf{x}_k) c(\mathrm{d}\mathbf{x}_l) + \sum_{k<l} f_{ikl}(\mathbf{x}) \mathfrak{c}(\mathrm{d}\mathbf{x}_k) \mathfrak{c}(\mathrm{d}\mathbf{x}_l) + g_i(\mathbf{x}).$$

In the last line, $R_{klij} = (R(e_i, e_j)e_l, e_k)_{x_0}$ is the Riemannian curvature at x_0 and $f_{ikl}(\mathbf{x}) = O(|\mathbf{x}|^2)$, $g_i(\mathbf{x}) = O(|\mathbf{x}|)$ are error terms.

As is discussed in [8], $k(t, \mathbf{x})$ is the solution to the equation

$$(\frac{\partial}{\partial t} + L) u(t, \mathbf{x}) = 0, \tag{IV.13}$$

Rescaling process

To calculate $\sigma(k_t(x,x))$ in (IV.12), a rescaling process on the space of functions $C^\infty(\mathbb{R}_+ \times U, \Lambda^* TM \otimes \operatorname{End}_{\mathbb{Cl}}(V))$ is used. For all $a \in C^\infty(\mathbb{R}_+ \times U, \Lambda^* TM \otimes \operatorname{End}_{\mathbb{Cl}}(V))$, define the rescaling $(0 < \lambda \leq 1)$ as

$$\delta_\lambda a(t, \mathbf{x}) = \sum_{k=0}^n \lambda^{-\frac{k}{2}} a(\lambda t, \lambda^{\frac{1}{2}} \mathbf{x})_{[k]}$$

[3]We identify $\operatorname{End} V_{x_0}$ with $\Lambda^* T^*_{x_0} M \otimes \operatorname{End} W$.

where $[k]$ means the filtered degree in Λ^*TM.

Define the rescaled heat kernel by

$$r(\lambda,t,\mathbf{x}) = \lambda^{\frac{n}{2}}(\delta_\lambda k)(t,\mathbf{x})$$

and the motivation is the fact that

$$\lim_{\lambda\to 0} r(\lambda,t=1,\mathbf{x}=0) = \lim_{\lambda\to 0}\sum_{i=0}^{n}\lambda^{\frac{n-i}{2}}k(\lambda t,\mathbf{x})_{[i]}|_{(t,\mathbf{x})=(1,0)} = \sum_{0\le i\le n, i \text{ is even}} \sigma_i(a_{\frac{i}{2}}(x_0)) = (4\pi)^{\frac{n}{2}}\sigma(k_t(x_0,x_0))|_{t=1}. \tag{IV.14}$$

Definition IV.1.22. A differential operator A acting on $C^\infty(\mathbb{R}\times U, \Lambda^*TM\otimes \mathrm{End}_{\mathbb{C}l}(V))$ has Getzler order m if the following limit exists

$$\lim_{\lambda\to 0}\lambda^{\frac{m}{2}}\delta_\lambda A\delta_\lambda^{-1}.$$

Example IV.1.23.

1. A polynomial $p(x)$ has order $-\deg p$; A polynomial $p(t)$ has order $-2\deg p$;
2. $\frac{\partial}{\partial x_i}$ has order 1; $\frac{\partial}{\partial t}$ has order 2 ($\delta_\lambda\partial_t\delta_\lambda^{-1} = \lambda^{-1}\partial_t$; $\delta_\lambda\partial_{x_i}\delta_\lambda^{-1} = \lambda^{-\frac{1}{2}}\partial_{x_i}$);
3. Exterior multiplication by a convector α has order 1 and interior -1

   $\delta_\lambda \mathrm{ext}(\alpha)\delta_\lambda^{-1} = \lambda^{-\frac{1}{2}}\mathrm{ext}(\alpha)$; $\delta_\lambda \mathrm{int}(\alpha)\delta_\lambda^{-1} = \lambda^{\frac{1}{2}}\mathrm{int}(\alpha)$.

Remark IV.1.24. Using (IV.13) and the Example IV.1.23-2, it is trivial to check that $(\partial_t + \lambda\delta_\lambda L\delta_\lambda^{-1})r(\lambda,t,\mathbf{x}) = 0$.

Using the asymptotic expansion of $k_t(x,x_0)$ and parallel transport, $k(t,\mathbf{x})$ has an asymptotic expansion and then the rescaled heat kernel

$$r(\lambda,t,\mathbf{x}) = \lambda^{\frac{n}{2}}(\delta_\lambda k)(t,\mathbf{x})$$

has an asymptotic expansion in λ. It is proved in [8] Proposition 4.17-4.20 that $\lim_{\lambda\to 0} r(\lambda,t,\mathbf{x})$ exists and is the solution to the equation

$$\partial_t + \sigma_2(\mathscr{D}^2) \doteq \partial_t + \lim_{\lambda\to 0}\lambda\delta_\lambda L\delta_\lambda^{-1} = 0.^4 \tag{IV.15}$$

[4] $\sigma_2(\mathscr{D}^2)$ is called the *Getzler symbol* of $\mathscr{D}^2$.

Hence by (IV.14) the heat kernel of the heat equation (IV.15) evaluated at $t = 1, \mathbf{x} = 0$ is exactly

$$\sigma_0(a_0(x)) + \cdots + \sigma_n(a_{\frac{n}{2}}(x)) = (4\pi)^{\frac{n}{2}}\sigma(k_t(x,x))|_{t=1}.$$

Harmonic Oscillator

The limiting operator exists and is of the form of some harmonic oscillator, expressed by the following proposition.

Proposition IV.1.25. $\lim_{\lambda\to 0}\delta_\lambda L\delta_\lambda^{-1}$ *exists, i.e. L has Getzler order* 2 *and* $\sigma_2(\mathscr{D}^2) \doteq \lim_{\lambda\to 0}\delta_\lambda L\delta_\lambda^{-1}$, *the Getzler symbol of* $\mathscr{D}^2$*, relative to an orthonormal basis of* $T_{x_0}M$ *is*

$$-\sum_i\left(\frac{\partial}{\partial x^i} - \frac{1}{4}\sum_j R_{ij}x^j\right)^2 + F \tag{IV.16}$$

where R_{ij} *is the Riemann curvature of the manifold at p(skew symmetric matrix of two forms) and F is the twisting* 2*-form at p (2 forms with values in* $\mathrm{End}_{\mathbb{C}\mathrm{l}}(V)$*, then entries of R commutes with that of F)*

The heat kernel of (IV.16) can be solved explicitly.

- It is sufficient to show solve the equation when $F = 0$.[5]
- The curvature matrix R is skew symmetric and we may assume it is a diagonal of 2×2 matrix $\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$. Then (IV.16) reduces to $\frac{\partial}{\partial t} - (\frac{\partial}{\partial x} - \frac{1}{4}\theta y)^2 - (\frac{\partial}{\partial y} + \frac{1}{4}\theta x)^2 = 0$
- We look for a solution of kernel invariant under the rotation of $\mathbb{R}^2$ about $(0,0)$. The it is sufficient to solve

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \frac{1}{16}\theta^2 x^2 w = 0 \tag{IV.17}$$

whose solution is given by the *Mehler's formula*

$$\frac{1}{(4\pi t)^{1/2}}\left(\frac{it\theta/2}{\sinh it\theta/2}\right)^{\frac{1}{2}} e^{-\frac{1}{8}i\theta x^2\coth(it\theta/2)}. \tag{IV.18}$$

[5]The solution of the non-linear equation has an extra multiple of e^{-tF} to the solution of the linear one.

Let $R_1 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}$ and $\mathbf{x} = (x,y)^T$. Since $\det \frac{R_1}{\sinh R_1} = \det \frac{R_2}{\sinh R_2} = (\frac{i\theta}{\sinh i\theta})^2$, then the heat kernel of $\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \frac{1}{16}\theta^2 x^2 w + \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial y^2} - \frac{1}{16}\theta^2 y^2 w = 0$ evaluated at $t = 1, \mathbf{x} = 0$ is $\frac{1}{(4\pi)^{\frac{n}{2}}} \det^{\frac{1}{2}} (\frac{R_1/2}{\sinh R_1/2})$. Then Theorem IV.1.15 is easily verified.

IV.2 Proofs of the assertions

In this section we study the heat kernel $k_t(x,y)$ of $e^{-t\mathscr{D}^2}$. We construct an "approximate heat kernel" of $k_t(x,y)$ and obtain Theorem IV.1.8 and Lemma IV.1.11 as corollaries. The proof is a modification of the case of operators on "compact" manifold, which can be found in [29] Theorem 7.15 or in [8] Chapter 2.

Since $K_t(x,y)$ satisfies the heat equation:

$$\frac{\partial}{\partial t} k_t(x,y) + \mathscr{D}^2 k_t(x,y) = 0, k_0(x,y) = \delta_y(x) \tag{IV.19}$$

where $\mathscr{D}$ operates on x-coordinate only, we fix y and denote it by x_0 and solve this equation locally on a coordinate neighborhood $x \in O_{x_0}$ of x_0. we approximate the heat kernel $k_t(x,x_0), x \in O_{x_0}$ locally by looking for a formal solution

$$p_t(x,x_0) \sum_{i=0}^{\infty} t^i \alpha_i(x) \tag{IV.20}$$

to the equation (IV.19), where

$$p_t(x,x_0) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{r^2}{4t}} (r = |\mathbf{x}| = d(x,x_0))$$

is the heat kernel to the standard Dirac operator on Euclidean space (IV.3). Denote by $s_t(x,x_0) = \sum_{i=0}^{\infty} t^i \alpha_i(x)$ in (IV.20) and the heat kernel is written as

$$k_t(x,x_0) = p_t(x,x_0) s_t(x,x_0). \tag{IV.21}$$

$\mathscr{D}^2$, given by the Lichnerowicz Formula (Proposition IV.1.3) on O_{x_0}, has the Lemma IV.2.1

when operating on (IV.21).

Lemma IV.2.1 ([29] equation 7.16)**.** *Let* $r = |\mathbf{x}|$ *and* $g = \det(g_{ij})$ *where* (g_{ij}) *is the Riemannian metric on* M. *Then*

$$[\frac{\partial}{\partial t} + \mathscr{D}^2](p_t s_t) = p_t[\frac{\partial}{\partial t} + \mathscr{D}^2 + \frac{r}{4gt}\frac{\partial g}{\partial r} + \frac{1}{t}\nabla_{r\frac{\partial}{\partial r}}]s_t. \tag{IV.22}$$

To find the formal solution (IV.20), set the right hand side of (IV.22) to be 0 and then the vanishing of the coefficients for t^is enables us to find α_i inductively via

$$\nabla_{\frac{\partial}{\partial r}}(r^i g^{\frac{1}{4}}\alpha_j(x)) = \begin{cases} 0 & i = 0 \\ -r^{i-1}g^{\frac{1}{4}}\mathscr{D}^2\alpha_{i-1}(x) & i > 0 \end{cases} \quad [6] \tag{IV.23}$$

- (Solve $\alpha_0(x)$) It is trivial to see that $p_t(x, x_0) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{r^2}{4t}} \to \delta_{x_0}(x)$ uniformly as $t \to 0+$. From Lemma IV.1.5, $k_t(x, x_0) \to \delta_{x_0}(x)$ uniformly as $t \to 0+$ for all $x \in K$, where $K \subset X$ is any compact subset. Therefore $\alpha_0(x_0) = 1$ is the necessary assumption. The first line in (IV.23) indicates that $g^{\frac{1}{4}}\alpha_0(x) = g(x_0)^{\frac{1}{4}}\alpha_0(x_0) = 1$, and then $\alpha_0(x) = g^{-\frac{1}{4}}(x)$ is determined by $\alpha_0(x_0)$.

- (Solve $\alpha_i(x), i > 0$) Inductively the smoothness of α_i implies the uniqueness of the smooth solution α_{i+1}. In fact, when solving the equation in (IV.23), the constant term has to be 0 otherwise α_{i+1} is not smooth at $r = 0$. Then α_{i+1} is smooth except that it may blow up at 0. But by setting $r = 0$ in the second line in (IV.23) we have $\alpha_{i+1}(x_0) = -\frac{1}{j}(\mathscr{D}^2\alpha_i)(x_0)$ which makes sense if α_i is smooth. Therefore, there exists a sequence of smooth sections $\{\alpha_i(x)\}$ in $\mathrm{End}(V_{x_0}, V_x)$ uniquely determined by $\alpha_0(x_0) = 1$.

- From solving the equations we see that α_is depends on the local geometry at x_0. For example, $\alpha_1(x) = \frac{1}{6}k(x) - K(x)$, where k is scaler curvature and K satisfies $\mathscr{D}^2 = \Delta + K$.

Note that α_is are defined on a coordinate neighborhood O_{x_0} and depend smoothly on the local geometry around x_0. Denote $\alpha_i(x)$ by $\alpha_i(x, x_0), x \in O_{x_0}$. Now for any $x_0 \doteq y \in M$, we obtain the formal solution $\alpha_i(x, y)$ which smoothly depends on both x and y for $x \in O_y$ (O_y is a coordinate

[6] Refer to [29] equation 7.17.

neighborhood of $y \in M$). Choose $O' \subset M \times M$ such that $\{(x,x)|x \in M\} \subset O' \subset \cup_{y\in M} O_y$ and choose

$$\phi(x,y) \in C^\infty(M \times M) \text{ so that } \phi(x,y) = \begin{cases} 1 & (x,y) \in O' \\ 0 & (x,y) \notin \cup_{y\in M} O_y \end{cases}.$$ [7]

Definition IV.2.2. Let (IV.21) be the true heat kernel. Define

$$h_t^n(x,y) = p_t(x,y) \sum_{i=0}^{n} t^i a_i(x,y) \tag{IV.24}$$

where $a_i(x,y) = \phi(x,y)\alpha_i(x,y) \in C^\infty(M \times M)$ and supported in a neighborhood of the diagonal.

Proposition IV.2.3. *Let $k_t(x,y)$ be the heat kernel and $h_t^n(x,y)$ be the one in (IV.24). Let $c(x) \in C^\infty(M)$ be the cutoff function of the properly cocompact G-manifold M. Choose $\bar{c} \in C_c^\infty(M)$ satisfying $c(x)\bar{c}(x) = c(x), x \in M$. For all $m > 0$, there is a N_m, so that for all $l > N_m$ and $t \in (0,1]$,*

$$\|c(x)h_t^l(x,y)\bar{c}(y) - c(x)k_t(x,y)\bar{c}(y)\| < Ct^m \tag{IV.25}$$

where $\|f\| = \sup_{x,y\in M} |f(x,y)|$.

Corollary IV.2.4. *Theorem IV.1.8 and Lemma IV.1.11 are true assuming the proposition.*

Proof.

- Let $x = y$ in Proposition IV.2.3, then (IV.25) reduces to $\|c(x)h_t^l(x,x) - c(x)k_t(x,x)\| < Ct^m$ and therefore Theorem IV.1.8 is proved.

- We define $a_i(x) = a_i(x,x)$ to be $\alpha(x,x)$. To see Lemma IV.1.11, we need to show that $\alpha_i(y,y) \in \mathbb{C}\mathrm{l}_{2i} \otimes \mathrm{End}_{\mathbb{C}\mathrm{l}}(V_y)$. Set $x = y$ in (IV.23), then

$$\alpha_0(y,y) = 1 \text{ and } \alpha_j(y,y) = -\frac{1}{j}(\mathscr{D}^2\alpha_{j-1})(y,y).$$

  with $\alpha_0(y,y) = 1 \in \mathbb{C}\mathrm{l}_0 \otimes \mathrm{End}_{\mathbb{C}\mathrm{l}}(V_y)$. Inductively, the fact that $\mathscr{D}^2$ contains the factor $\mathfrak{c}(e_i)\mathfrak{c}(e_j)$, makes sure that the degree of $\alpha_i(x)$ does not increase more than 2 compared to $\alpha_{i-1}(x)$.

□

[7] This definition is based on a cutoff function appeared in defining the approximate heat kernel in [8] Definition 2.28.

Proof of Proposition IV.2.3. For all m, let $N_m > \max\{n+1, m+\frac{n}{2}\}$, where $n = \dim M$. By definition $h_t^{N_m}(x,y)$ approximately satisfies the heat equation in the sense that

$$(\frac{\partial}{\partial t} + \mathscr{D}^2)h^{N_m} = t^{N_m} p_t(x,y)\mathscr{D}^2 a_{N_m}(x,y) + O(t^\infty) \doteq r_t(x,y), \tag{IV.26}$$

where the first term in (IV.26) comes from the calculation of the formal solution [8] and what remains $O(t^\infty)$ is of order t^∞, because this term contains the derivatives of ϕ, which are of 0-value for x near y, and $p_t(x,y), (x \neq y)$, which decreases faster than any positive power t^k as $t \to 0+$. $r_t(x,y)$ has the following properties:

- The remainder $r_t(x,y)$ is smooth for any fixed $t > 0$. Because $p_t(x,y) = \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{d(x,y)^2}{4t}}$ and $a_i(x,y)$s (in Definition IV.2.2) are smooth functions, for all $t > 0$.

- Denote the kth Sobolev norm on $C^m(M \times M)$ by $\|\cdot\|_k$. Then for all fixed $t > 0$ and for all k: $\|c(x)r_t(x,y)\bar{c}(y)\|_k$ exists. Because $c(x)r_t(x,y)\bar{c}(y)$ is smooth and compactly supported on $M \times M$.

- $$\|c(x)r_t(x,y)\bar{c}(y)\|_{\frac{n}{2}+1} < Ct^m \text{ uniformly for all } t \in (0,1]. \tag{IV.27}$$

  Proof. In the first term

  $$c(x)t^{N_m}p_t(x,y)(\mathscr{D}^2 a_{N_m}(x,y))\bar{c}(y)$$

  of $c(x)r_t(x,y)\bar{c}(y)$, only $t^{N_m}p_t(x,y)$ has t, it is sufficient to know the order of t in the kth derivative (in x or y) of $t^{N_m}p_t(x,y)$, where $k \leq \frac{n}{2}+1$ and the order is: $t^{N_m}t^{-\frac{n}{2}}t^{-k} = t^{N_m-\frac{n}{2}-k}$. So

  $$\|c(x)t^{N_m}p_t(x,y)(\mathscr{D}^2 a_{N_m}(x,y))\bar{c}(y)\|_{\frac{n}{2}+1} \leq \sum_{k=0}^{\frac{n}{2}+1} c_k t^{N_m-\frac{n}{2}-k}.$$

  Since $N_m > n+1$, there are no terms of non-positive order in t on the right hand side. In addition, since $N_m > \frac{n}{2}+m$, then for all $t \in (0,1]$, there is a constant C so that

  $$\|c(x)t^{N_m}p_t(x,y)(\mathscr{D}^2 a_{N_m}(x,y))\bar{c}(y)\|_{\frac{n}{2}+1} \leq C_1 t^{N_m-\frac{n}{2}} \leq C_1 t^m.$$

[8] In fact, using (IV.22), (IV.23), we have $(\frac{\partial}{\partial t} + \mathscr{D}^2)[p_t(x,y)\sum_{j=0}^{N_m} t^j \alpha_j(x,y)] = t^{N_m} p_t(x,y)\mathscr{D}^2 \alpha_{N_m}(x,y)$.

The derivatives of $c(x)O(t^\infty)\bar{c}(y)$ does not has any terms containing negative power of t so $\|c(x)O(t^\infty)\bar{c}(y)\|_{\frac{n}{2}+1} < C_2 t^m$ for all $t \in (0,1]$. $\square$

Next, we want to use the $r_t(x,y)$ to relate $k_t(x,y)$ and $h_t^{N_m}(x,y)$ using the following claim:

Claim: There is a unique smooth solution for the following equation:

$$\begin{cases} (\frac{\partial}{\partial t} + \mathscr{D}^2)u_t(x,y) = r_t(x,y) & ^{9} \\ u_0(x,y) = 0 \end{cases} \tag{IV.28}$$

Proof. It is trivial to check that $u_1 = \int_0^t e^{-(t-\tau)\mathscr{D}^2} r_\tau(x,x_0)\mathrm{d}\tau$ is smooth and satisfies the equation. If u_2 is another smooth solution, then $u = u_1 - u_2$ satisfies $(\frac{\partial}{\partial t} + \mathscr{D}^2)u = 0, u_0 = u(t=0) = 0$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t} < u,u >= - < u, \mathscr{D}^2 u > - < \mathscr{D}^2 u, u >= -2\|\mathscr{D}u\|_{L^2}^2$$

implies that $\|u\|^2$ is non-decreasing in t, and so $\|u(t=0)\| = 0$ forces $u = u_1 - u_2 = 0$. $\square$

Since $h_t^{N_m}(x,y) - k_t(x,y)$ is also a solution to the equation (IV.28), by the uniqueness of solution we have

$$h_t^{N_M}(x,y) - k_t(x,y) = \int_0^t e^{-(t-\tau)\mathscr{D}^2} r_\tau(x,y)\mathrm{d}\tau,$$

then, for all $t \in (0,1]$,

$$\|c(x)k_t(x,y)\bar{c}(y) - c(x)h_t^{N_m}(x,y)\bar{c}(y)\|_{\frac{n}{2}+1} \leq t\sup\{\|c(x)r_\tau(x,y)\bar{c}(y)\|_{\frac{n}{2}+1} | 0 \leq \tau \leq t\} \leq Ct^m,$$

where second inequality is because of (IV.27).

By the Sobolev embedding theorem, for all $p > \frac{n}{2}$, $\|u\| \leq C_0\|u\|_p$ for $u \in H^p$, where $\|\cdot\|$ is the C^0 sup norm and $\|\cdot\|_p$ is the Sobolev p-norm. Therefore,

$$\|c(x)k_t(x,y)\bar{c}(y) - c(x)h_t^{N_m}(x,y)\bar{c}(y)\| \leq C'\|c(x)k_t(x,y)\bar{c}(y) - c(x)h_t^{N_m}(x,y)\bar{c}(y)\|_{\frac{n}{2}+1} \leq C'Ct^m.^{10}$$

$\square$

[9] $u_t(x,y)$ is regard as a function of t and x.

[10] Since $c(x)$ and $\bar{c}(x_0)$ are compactly supported, the function in the norm is supported in a compact set in $M \times M$, where the theorem can be applied.

In this section we prove the L^2-index formula for a G-invariant (G is unimodular) elliptic operator operator A acting on properly cocompact space X. Now we have Theorem III.3.6, and so to calculate $\mathrm{ind}A$ it is enough to figure out $\mathrm{ind}D_{V(\sigma_A)}$ where D is the Dolbeault operator on ΣX, and where $V(\sigma_A)$ is a bundle over ΣX. $D_{V(\sigma_A)}$ is a generalized Dirac operator and we use the heat kernel method to calculate the case when $\mathscr{D} = D_{V(\sigma_A)}$, $M = \Sigma X$ in the last two sections. The following proposition, as a corollary to Theorem IV.1.15, is essential in obtaining the L^2-index formula of A.

Proposition IV.3.1. *Let G be a locally compact unimodular group and let M be properly cocompact G-manifold of dimension n having an almost complex structure, curvature R, a cutoff function $c \in C_c^\infty(M)$ and a G-bundle E with curvature F. Let D be the Dolbeault operator on M. Then the L^2-index of the generalized Dirac operator D_E is,*

$$\mathrm{ind}D_E = \int_M c\mathrm{Td}(M)\mathrm{ch}(E),$$

where $\mathrm{Td}(M) = \det(\frac{R}{1-e^R})$ *and* $\mathrm{ch}(E) = \mathrm{tr}_s(e^{-F})$.

Both $\mathrm{Td}(M)$ and $\mathrm{ch}(E)$ are G-invariant forms. So the integral does not depend on the choice of the cutoff function. If $M = \Sigma X$, then the cutoff function on M can be obtained from the cutoff function on X by setting the values of the elements in the same fiber to be the same. The following index formula is immediate assuming the proposition.

Theorem IV.3.2. *Let X be a complete Riemannian manifold where a locally compact unimodular group G acts properly, cocompactly and isometrically. If A is a zero order properly supported elliptic pseudo-differential operator, then the L^2 index of A is given by the formula*

$$\mathrm{ind}(A) = \int_{TX} c(x)(\hat{A}(X))^2\mathrm{ch}(\sigma_A). \tag{IV.29}$$

Proof. Set $M = \Sigma X, V = V_{\sigma_A}$. By Proposition IV.3.1,

$$\mathrm{ind}A = \int_{\Sigma X} c(x)\mathrm{Td}(\Sigma M)\mathrm{ch}(V_{\sigma_A}) = \int_{TX} c(x)\mathrm{Td}(TX \otimes \mathbb{C})\mathrm{ch}(\sigma_A).$$

Observe that $\mathrm{Td}(TX \otimes \mathbb{C}) = \hat{A}(X)^2$, the statement follows. $\square$

Proof of Proposition IV.3.1. The prove is essentially a summary of a part of section 4.1 in [8] Let X be a m dimensional proper cocompact G-manifold, then $M = \Sigma X$ has an almost complex structure J. Say $x_i, y_i, 1 \leq i \leq m$ are a local frame of TM and $J(x_i) = y_i, J(y_i) = -x_i$. J extend $\mathbb{C}$-linearly to $TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$ where

$$TM^{1,0} = \{v - iJv | v \in TM\},$$

the set of holomorphic tangent vectors of the form $z_i \doteq x_j - iy_j$, is the i-eigenspace of J and

$$TM^{0,1} = \{v + iJv, v \in TM\},$$

the set of anti-holomorphic tangent vectors of form $\bar{z_j} \doteq x_j + iy_j$, is the $-i$-eigenspace of J. We have real isomorphisms

$$\pi^{1,0} : TM \to TM^{1,0}, v \mapsto v^{1,0} = \frac{1}{2}(v - iJv) \text{ and } \pi^{0,1} : TM \to TM^{0,1}, v \mapsto v^{0,1} = \frac{1}{2}(v + iJv).$$

Therefore $(TM, J) \simeq TM^{1,0} \simeq \overline{TM^{0,1}}$ as an almost complex bundle. Similarly, the complexified cotangent bundle decomposes as $T^*M \otimes \mathbb{C} = T^*M^{1,0} \oplus T^*M^{1,0}$ where

$$T^*M^{1,0} = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = i\eta(v)\},$$

covectors of form $z^j \doteq x^j + iy^j$, [11] is the $\mathbb{C}$-dual of $TM^{1,0}$ and

$$T^*M^{0,1} = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = -i\eta(v)\},$$

covectors of form $\bar{z^j} \doteq x^j - iy^j$, is the $\mathbb{C}$-dual of $TM^{0,1}$. We also have real isomorphism

$$T^*M \to T^*M^{1,0} : \eta \mapsto \eta^{1,0} = \frac{1}{2}(\eta - i(\eta \circ J)) \text{ and } T^*M \to T^*M^{0,1} : \eta \mapsto \eta^{0,1} = \frac{1}{2}(\eta + i(\eta \circ J)).$$

If g is the G-invariant Riemannian metric on M, then the G-invariant Hermitian metric is defined

[11] $x^j(x_i) = \delta_{ij}, y^j(y_i) = \delta_{ij}$

by $h(X,Y) = g(X,Y) + \sqrt{-1}g(X,J(Y))$ for vector fields $X,Y \in TM$. Let $\Lambda^* M$ be the bundle of exterior algebra of M and $\Omega^* M$, the set of smooth sections, splits into type $(p,q), p+q = *$ with

$$\Lambda^{p,q}T^*M = (\Lambda^p T^*M^{1,0}) \otimes (\Lambda^q T^*M^{0,1}) = \oplus_{p+q=k}\Lambda^{p,q}T^*M.$$

If $\alpha \in \Omega^{p,q}(M)$, then the differential decomposes into $\mathrm{d}\alpha = \sum_{i=0}^{p+q+1}(\mathrm{d}\alpha)^{1,p+q+1-i}$ and set $\partial\alpha = (\mathrm{d}\alpha)^{p+1,q}, \bar{\partial}\alpha = (\mathrm{d}\alpha)^{p,q+1}$. The Dolbeault operator $\bar{\partial} : \Omega^{0,q} \to \Omega^{0,q+1}$ is the order 1 differential operator given in the local coordinate by $\bar{\partial} = \frac{\partial}{\partial \xi} + i\frac{\partial}{\partial x}$ in the local coordinate $(x,\xi) \in \Sigma X = M$.[12]

Dolbeault operator "is" the canonical Dirac operator on M. In fact, the bundle $S = \Lambda^{0,*}T^*M$ has an action of cotangent vectors via Clifford multiplication:

$$c(\eta)s = \sqrt{2}(\varepsilon(\eta^{0,1})(s) - \iota(\eta^{1,0})s), \eta \in T^*M, s \in \Lambda^{0,*}T^*M,^{13}$$

where ε is the exterior multiplication and ι is the $\mathbb{C}$-linear compression by a vector.

There is a G-invariant Hermitian metric $(\cdot,\cdot)$ and Levi-Civita connection ∇^L on S. The Dolbeault operator is defined to be $D = \sum c(e^i)\nabla^L_{e_i}$ where e_is are local orthonormal basis of TM. Now if there is an auxiliary complex G-vector bundle $E \to M$, with a G-invariant Hermitian metric and G-invariant connection ∇^E the Dolbeault operator D_E acting on $V = S \otimes E$ with coefficient in E can be represented by (up to an lower order term):

$$D_E = \sum c(e_i)\nabla^V_{e_i}, \text{ where } \nabla^V = \nabla^L \otimes 1 + 1 \otimes \nabla^E.$$

Let ∇ be the Levi-Civita connection of M and $R = \nabla^2 \in \Lambda^2(M, so(TM))$ be Riemannian curvature, the matrix with two forms coefficient representing the curvature of M,

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, X,Y \in C^\infty(M,TM).$$

[12]If use grading, the Dolbeault is $\bar{\partial} + \bar{\partial}^*$ on $\Omega^{0,*}M$.

[13]$c(x^i) = \frac{1}{\sqrt{2}}\varepsilon(\bar{z}) - \iota(z), c(x^i)c(x^j) + c(x^j)c(x^i) = -2\delta_{ij}$.

In the orthonormal frame e_i of TM,

$$R(e_i,e_j) = -\sum_{k<l}(R(e_i,e_j)e_l,e_k)e^k\wedge e^l,$$

where we identify $so(TM)$ with two forms on M. Now we have a Clifford module $S=\Lambda(T^{0,1}M)^*$, $Cl(TM)\otimes\mathbb{C}=\mathrm{End}(S)$, on which T^*M acts by Clifford multiplication. On S there is a Clifford connection ∇^S so that the Clifford multiplication by unit vectors preserve the metric and ∇^S is compatible with the connection on M. [14] Let $R^S=(\nabla^S)^2$ be the curvature associated to ∇^S. It is well know that the Lie algebra isomorphism $\mathrm{spin}_n\simeq so_n$ given by $\frac{1}{4}[v,w]\mapsto v\wedge w$ implies that

$$R^S(e_i,e_j) = \frac{1}{2}\sum_{k<l}(R(e_i,e_j)e_k,e_l)c(e_k)c(e_l) = \frac{1}{4}\sum_{kl}(R(e_i,e_j)e_k,e_l)c(e_k)c(e_l).$$

On S, there is also a Levi-Civita connection, denoted by ∇^L. The associated curvature $R^L=(\nabla^V)^2\in\mathrm{End}S$ is written as

$$R^L = R^S+F$$

where $R^S(\cdot,\cdot)=\frac{1}{4}\sum_{kl}(R(\cdot,\cdot)\bar{z_k},z_l)c(\bar{z_k})c(z_l)+\frac{1}{4}\sum_{kl}(R(\cdot,\cdot)z_k,\bar{z_l})c(z_k)c(\bar{z_l})\in Cl(TM)$ and $F\in\mathrm{End}_{Cl}V$ is the twisting curvature.

Recall that the curvature of the Levi-Civita connection on ΛV^* is the derivation of the algebra ΛV^* which coincide with $R(e_i,e_j)$ on V and is given by the formula

$$\sum_{kl} < e^k, R(e_i,e_j)e_l > \varepsilon(e_k)\iota(e^l) = \sum_{kl}(R(e_i,e_j)e_l,e_k)\varepsilon(e^k)\iota(e^l).$$

Let R^- be the curvature of the Levi-Civita connection on $T^{0,1}M$. Note that $R=R^-$. Then the curvature of the ∇^V on $S=\Lambda(T^{0,1})^*$ is given by

$$R^L(\cdot,\cdot) = \frac{1}{4}\sum_{i,j}(R^-(\cdot,\cdot)z_i,\bar{z_j})\varepsilon(\bar{z_j})\iota(z_i) = -\frac{1}{8}\sum_{i,j}(R^-(\cdot,\cdot)z_i,\bar{z_j})c(\bar{z_j})c(z_i).$$

[14] $\nabla(\phi)\cdot\sigma=(\nabla\phi)\cdot\sigma+\phi\cdot(\nabla\sigma), \phi\in\Omega^1, \sigma\in\Omega^{0,*}$.

Using the fact that $c(z_i)^2 = 0, c(\bar{z_i})^2 = 0, c(z_i)c(\bar{z_j}) + c(\bar{z_j})c(z_i) = -4\delta ij$[15] we have

$$F^{V/S} = R^V - R^S = \frac{1}{2}\sum_k (Rz_k, \bar{z_k}) = \frac{1}{2}\operatorname{Tr} R + F^E$$

and a direct calculation shows that

$$\hat{A}(M)e^{F^{V/S}} = \det\frac{R/2}{\sinh R/2}e^{\frac{1}{2}\operatorname{Tr} R}(e^{F^E}) = \det\frac{R}{e^R - 1}(e^{F^E}) = \operatorname{Td}(M)\operatorname{Tr}(e^{-F^E}).$$

□

IV.4 Spectial cases

IV.4.1 Atiyah's L^2-index theorem

Corollary IV.4.1. *Let D be an elliptic operator on a compact manifold X and $\tilde{D}$ be the $\pi_1(X)$-invariant operator defined on the universal cover space $\tilde{X}$, then* $\operatorname{ind}\tilde{D} = \operatorname{ind} D$.

Proof. The Atiyah-Singer index theorem for compact manifold states that $\operatorname{ind} D = \int_{TX} \hat{A}(X)^2 \operatorname{ch}\sigma_D$. The L^2-index theorem states that $\operatorname{ind}\tilde{D} = \int_{T\tilde{X}} c(x)\hat{A}(\tilde{X})^2 \operatorname{ch}\sigma_{\tilde{D}}$. The statement follows because $c(x)$ adds to 1 on each orbit and $\hat{A}(\tilde{X})^2$ and $\operatorname{ch}\sigma_{\tilde{D}}$, being $\pi_1(X)$ invariant is the lift of $\hat{A}(X)^2$ and $\operatorname{ch}\sigma_D$ respectively. □

IV.4.2 L^2-index theorem for homogeneous space for Lie group G.

Let G be a unimodular Lie group and H be a compact subgroup. Let $M = G/H$, the space of the left coset of H, be the homogeneous G space and D be a G-invariant elliptic operator on the bundle $\mathscr{E}$ over M. Recall that for all G-bundle $\mathscr{E}$ there is a H-space $E = \mathscr{E}|_{eH}$ so that $\mathscr{E} = G \times_H E$. In particular, any tangent vector is determined by a tangent vector in $V = T_{eH}M$, because

$$TM = G \times_H V.$$

[15] $c(z) = c(x) + ic(y), c(\bar{z}) = c(x) - ic(y)$.

Let $\Omega \in \Lambda^2(TM)^* \otimes gl(TM)$ be the curvature of M, associated to the G-invariant Levi-Civita connection on TM. Then we have the G-invariant $\hat{A}$-class

$$\hat{A}(M) = \det{}^{\frac{1}{2}} \frac{\Omega/2}{\sinh \Omega/2}.$$

Let ΣM be the G-manifold obtained by gluing two copies of $\pi^* E|_{BM} \to BM$ along the boundary and $\Omega^E \in \Lambda^2(\Sigma M)^* \otimes gl(V(\sigma_A))$ be a curvature form associated to some G-invariant connection on $V(\sigma_D)$ over ΣM. Then

$$\mathrm{ch}(\sigma_D) = \mathrm{Tr}\, e^{\Omega^E}|_{TM}$$

is the Chern character of $V(\sigma_A)$ restricted to TM. Let Ω_V be the curvature tensor Ω restricted to $V = T_{eH}M$ and Ω_V^E be the curvature tensor Ω^E restricted to V. Then we define the corresponding $\hat{A}$-class and Chern character as

$$\hat{A}(M)_V \doteq \det{}^{\frac{1}{2}} \frac{\Omega_V/2}{\sinh \Omega_V/2} \text{ and } \mathrm{ch}(\sigma_D)_V \doteq \mathrm{Tr}\, e^{\Omega_V^E}$$

We have a corollary of the L^2-index theorem for homogeneous space.

Corollary IV.4.2. *The L^2-index of G-invariant elliptic operator $D : L^2(M, \mathscr{E}) \to L^2(M, \mathscr{E})$ is*

$$\mathrm{ind}\, D = \int_V \hat{A}^2(M)_V \mathrm{ch}(\sigma_A)_V. \tag{IV.30}$$

Proof. The L^2-index theorem of D says that

$$\mathrm{ind}\, D = \int_{TM} c\hat{A}^2(M)\mathrm{ch}(\sigma_A).$$

Since $TM = G \times_H V$, the integration of form $c\hat{A}^2(M)\mathrm{ch}(\sigma_A)$ on TM can be lifted to an H-invariant form on $G \times V$ and then integrated over the group part and then the tangent space at eH. Since $\hat{A}^2(M)\mathrm{ch}(\sigma_A)$ is G-invariant, then at any $g \in G$, the form will be the same as its value at the unit e of G: $\hat{A}^2(M)_V \mathrm{ch}(\sigma_A)_V$. Hence,

$$\int_{TM} c\hat{A}^2(M)\mathrm{ch}(\sigma_A) = \int_V \hat{A}^2(M)_V \mathrm{ch}(\sigma_A)_V \int_G c(g^{-1}v)\mathrm{vol} = \int_V \hat{A}^2(M)_V \mathrm{ch}(\sigma_A)_V,$$

where vol is the volume form on G. $\square$

Remark IV.4.3. This formula IV.30 coincides with the L^2-index formula in [12]. The components of the formula in IV.30 are sketched as follows. On the Lie algebra g of G there is an H-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an H-invariant complement. $V = T_{eH}(G/H)$ is a candidate for $\mathfrak{m}$. There is a curvature form on m defined by

$$\Theta(X,Y) = -\frac{1}{2}\theta([X,Y]), X, Y \in \mathfrak{m} \tag{IV.31}$$

where θ is the connection form given by the projection $\theta : \mathfrak{g} \to \mathfrak{h}$. Θ composed with $r : h \to gl(E)$, the differential of a unitary representation of H on some vector space E, is an H-invariant curvature form

$$\Theta_r(X,Y) = r(\Theta(X,Y)), X, Y \in \mathfrak{m}.$$

Then

$$\mathrm{ch} : R(H) \to H^*(g,H) : r \mapsto \mathrm{Tr}\, e^{\Theta_r}$$

is a well-defined Chern character ([12] page 309). Also, compose the curvature form (IV.31), with $\mathfrak{h} \to gl(V)$, the differential of the H-module structure of V. And a curvature form on V $\Theta_V \in \Lambda^2\mathfrak{m}^* \otimes gl(V)$ is constructed and the $\hat{A}$-class is defined as

$$\hat{A}(\mathfrak{g},H) = \det{}^{\frac{1}{2}} \frac{\Theta_V/2}{\sinh \Theta_V/2}.$$

The L^2-index formula of D is

$$\mathrm{ind}\, D = \int_V \mathrm{ch}(a)\hat{A}(\mathfrak{g},H), \tag{IV.32}$$

where a is an element of the representation ring $R(H)$ so a is the pre-image of $V(\sigma_D)|_{V^+}$[16] under the Thom isomorphism $R(H) \to K_H(V)$.[17]

To see that IV.30 and IV.32 are the same one, we prove the following assertions.

1. The restriction of $\hat{A}(M)$ to V is the same as $\hat{A}(\mathfrak{g},H)$ in the cohomology group. $(\hat{A}(M)_V =$

[16] V^+ is space V adding one point at infinity. It is the ball fiber in ΣM at eH.

[17] This Thom isomorphism exists only for the case when the action of H on V $H \to SO(V)$ lifts to $Spin(V)$. The general case was done by introducing a double covering of H and by reducing the problem to this situation. Please refer to the construction in [12] on page 307.

$\hat{A}(\mathfrak{g}, H)$).

Proof. Since $TM = G \times_H V$ is a principal G-bundle over V/H and V is a principal H-bundle over V/H, then by [25] II Prop. 6.4, the connection form on TM restricted to V is also a connection form. Also, on the homogeneous space G/H, the restriction of any G-invariant tensor on TM to V is an H-invariant tensor on V. Therefore Ω_V is an H-invariant curvature form on V and the restriction $\hat{A}(M)_V$ is the $\hat{A}$-class defined by curvature Ω_V. By definition $\hat{A}(\mathfrak{g}, H)$ is the $\hat{A}$-class of the curvature Θ_V on V, $\hat{A}$-class of another connection on the same V. Then the statement is proved because $\hat{A}$ is a topological invariant and is independent of the choice of connection on V. □

2. The restriction of $\mathrm{ch}\sigma_D$ to V is the same as in the cohomology group ($\mathrm{ch}(\sigma_D)_V = \mathrm{ch}(a)$).

Proof. Similar to the last proof, Ω_V^E is an H-invariant curvature form of $V(\sigma_D)|_{V^+}$ restricted to V. Recall that $V(\sigma_D)$ is glued by G-invariant symbol σ_D and therefore it is determined by its restriction at the ball fiber, V^+. By definition $V(\sigma_D)|_{V^+}$ is glued two copies of $BV \times E$ on the boundary by $\sigma_D|_{SV}$. Note that the evaluation of $\sigma_D|_{SV}$ at $\xi \in SV$ is

$$\sigma_D(eH, \xi) \in GL(E), \xi \in V, \|\xi\| = 1.$$

H-bundle $V(\sigma_D)|_V = V \times_H E$ where $r : H \to E$. Hence the curvature Ω_V^E is r composed with some curvature form on V. Hence the statement follows from the fact that $\mathrm{ch}(r)$ is independent of the connection and the choice of the H-invariant splitting of G. □

APPENDIX I

FREDHOLM OPERATORS

This section is a brief review of the Fredholm theory. Let H be a Hilbert space and $\mathscr{B}(H)$ is the set of the bounded linear operators, and $\mathscr{K}(H)$ is the set of all compact operators, i.e. the completion of the finite rank operators under operator norm. $T \in \mathscr{B}(H)$ is called a *Fredholm operator* if $T + \mathscr{K}(H)$ is invertible in $\mathscr{B}(H)/\mathscr{K}(H)$. Equivalently, $T \in \mathscr{B}(H)$ is Fredholm if and only if

1. The range of T is closed;

2. The dimension of the $\ker T$ and $\ker T^*$ are finite.

The *index* for the Fredholm operator T is defined as

$$\operatorname{ind} T = \dim \ker(T) - \dim \ker(T^*).$$

Example A.0.4. Let $H = \{f \in C(S^1) | \int f(\theta)e^{in\theta}d\theta = 0\}$, and let $T = PM_{e^{i\theta}}$ be the Toeplitz operator where $M_{e^{i\theta}}$ is the multiplication operator by $e^{i\theta}$ and P is the projection onto H. The Fredholm index $\operatorname{ind} T = -1$.

Fredholm index is a nice analytical invariant.

Proposition A.0.5. *Let S and T be Fredholm, K be compact, and let T_t be a path of Fredholm operators continuous in $t, 0 \le t \le 1$. Then*

1. $\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T$;

2. $\operatorname{ind} T = \operatorname{ind}(T + K)$;

3. $\operatorname{ind} T_0 = \operatorname{ind} T_1$.

Elliptic operator on compact manifold X is Fredholm. In fact, Rellich Lemma implies that for X, if $s < t$, then the inclusion map $H^s(X) \to H^t(X)$ is a compact map. Hence we have the following statement saying that a pseudo-differential operator acting on a compact manifold with negative order is compact. In particular, smoothing operator over compact manifold is compact.

Proposition A.0.6. *Let $a(x,\xi) \in S^m(X), m < 0$ with x-coordinate supported in K, a compact set, then $a(x,D) : H^s(X) \to H^s(X)$ is a compact operator for any s.*

Hence we have the following corollary to theorem II.1.14: An elliptic pseudo-differential operator P over a compact manifold is Fredholm. If X is non-compact and let M_f be a multiplication operator by $f(x) \in C_0(X)$, then

$$(PQ - Id)M_f, (QP - Id)M_f$$

are compact. The analytical index of the elliptic operator in the case of compact manifold is defined to be the Fredholm index. The elliptic operators on compact manifold are almost invertible and the index of elliptic operators measures how far the operators are from being invertible.

APPENDIX II

VON NEUMANN ALGEBRA AND TRACE

We introduce the "relative dimension" in this section and define the analytic index of equivariant elliptic operator in the next subsection.

Let M be a subset of $\mathscr{B}(H)$ and we define its commutant as $M' = \{A \in \mathscr{B}(H) : AB = BA, \forall B \in M\}$. A Von Neumann algebra is a $*$-subalgebra $M \subset \mathscr{B}(H)$ satisfying $M = M''$.

The weak operator topology on $\mathscr{B}(H)$ is defined by the following set of basic neighborhoods of any operator $A \in \mathscr{B}(H)$:

$$U(x_1,\cdots,x_N;y_1,\cdots,y_N;\varepsilon;A) = \{B \in \mathscr{B}(H) || ((B-A)x_i,y_i)| \leq \varepsilon, i = 1,\cdots,N\}.$$

Let $\{A_\gamma\}$ be a net of operators in $\mathscr{B}(H)$ we say $A \in \mathscr{B}(H)$ the weak limit of $\{A_\gamma\}$ if A_γ converge to$A$ in the weak operator topology. A subalgebra M of $\mathscr{B}(H)$ is said to be weakly closed if M is closed under weak operator topology.

Theorem B.0.7 (von Neumann Double commutant theorem)**.** *Let M be a $*$-subalgebra in $\mathscr{B}(H)$ containing 1, then the following conditions are equivalent:*
(1)$M = M''$; (2) M is weakly closed; (3) M is strongly closed.

Example B.0.8. $L^\infty(\mathbb{R})$. Every commutative von Neumann algebra is isomorphic to $L^\infty(X)$ for some measure space (X,μ) and for some σ-finite measure space X. The theory of von Neumann algebras has been called noncommutative measure theory, while the theory of C^*-algebras is sometimes called noncommutative topology.

Let M be a von Neumann algebra. We say $A \in M$ is positive if there exist $B \in M$ such that $A = BB^*$ and denote $A \geq 0$ if A positive. Let $M^+ = \{A \in M : A \geq 0\}$ and we define trace of M on M^+ as follows:

Definition B.0.9. A trace on M is a linear map $\tau : M^+ \to [0,\infty]$ satisfying the following conditions:
(1)$\tau(AA^*) = \tau(A^*A)$(tracial);
(2)$\tau(A) = 0$ implies $A = 0$ (faithful);

(3) If A_γ is an increasing net of elements converge to A, then $\tau(A_\gamma)$ increasing and converge to $\tau(A)$ (normal);

(4) For every $A \in M^+$, $\tau(A) = sup\{\tau(B) : B \in M^+, B \leq A, \tau(B) < \infty)\}$(semifinite).

A von Neumann algebra M whose center consists of $\mathbb{C} \cdot 1$ (1 is the identity operator) is called a factor. von Neumann showed that every von Neumann algebra on a separable Hilbert space is isomorphic to a direct integral of factors. This decomposition is essentially unique. Thus, the problem of classifying isomorphism classes of von Neumann algebras on separable Hilbert spaces can be reduced to that of classifying isomorphism classes of factors.

Let $P \in M$ is be a projection, i.e., $P = P^2 = P^*$. Hence $P \in M^+$. There is a partial order $<$ on the set of projections ($P < Q$ if $Im(P) \subset Im(Q)$) and an equivalent relation(Two projections are said to be equivalent if there is a partial isometry $u \in M$ such that $uu^* = P, u^*u = Q$). A projection P is said to be finite if $P \sim Q < P$ implies $P = Q$, and said to be infinite if it is not finite. It is a fact that any factor has a trace such that the trace of a non-zero projection is non-zero and the trace of an infinite projection is infinite. Such a trace is unique up to scaler multiple. The type of a factor can be defined from the possible values of this trace. [32]

Definition B.0.10. Let $D_M(H) = \{\mathrm{im} P \in H | P = P^* = P^2 \in M\}$ for any von Neumann algebra $M \in \mathscr{B}(H)$. Given a trace $\tau : M \to [0, \infty]$, define the corresponding dimension function

$$\dim_\tau : D_M(H) \to [0, \infty] : \mathrm{Im} P \mapsto \tau(P).$$

We denote it by $\dim_M$ if τ is fixed before. The following are some important properties of $\dim_M$.

Proposition B.0.11. *[32] (1)* $\dim_M(L) = 0$ *if and only if* $L = 0$;

(2)If $L_1, L_2 \in D_M(H), L_1 \subset L_2$*, then* $\dim_M(L_1) \leq \dim_M(L_2)$;

(3)If $\{L_i, i \in I\}$ *are closed subspace of* H *and orthogonal to each other,* $L_i \in D_M(H), \forall i$ *and* $L = \sup_i L_i$ *is the smallest closed subspace in* H *including all* $L_i, i \in I$*, then* $L \in D_M(H)$ *and*

$$\dim_M(L) = \sum_{i \in I} \dim_M(L_i)$$

where the sum is the least upper bound of the finite sums;

(4) If $L \in D_M(H)$ *and* $U \in M$ *is unitary then* $U(L) \in D_M(H)$ *and* $\dim_M(U(L)) = \dim_M(L)$.

APPENDIX III

CROSSED PRODUCT

Let A be a C^*-algebra and G be a locally compact group, G acts on A by continuous homomorphism $\alpha : G \to Aut(A)$, where $Aut(A)$ is the group of $*$-isomorphism between A and itself. We call (A,G,α) a *covariant system.* We construct a space including both A and G such that the action of G on A is "inner" in the space.

Definition C.0.12. A *covariant representation* of covariant system (A,G,α) is a pair of representations (π,ρ) of A and G on the same Hilbert space such that

$$\rho(g)\pi(a)\rho(g)^* = \pi(\alpha_g(a)) \text{ for all } a \in A, a \in G,$$

where $\pi : A \to \mathscr{B}(H)$ is a $*$-homomorphism and $\rho : G \to \mathscr{U}(H)$ is a unitary representation of G.

Remark C.0.13. A covariant representation always exists for a covariant system. In fact, let π be a $*$-representation of C^*-algebra A on Hilbert space H. Consider the Hilbert space $L^2(G,H)$, the square integrable H-valued functions on G with norm $\|x\|^2 = \int_G \|x(t)\|^2 \mathrm{d}t$. G acts on $L^2(G,H)$ by left regular representation $\rho(g) \cdot x(t) = x(g^{-1}t)$ and $(\pi(a)x)(s) = \pi(\alpha_s^{-1}a)x(s), s \in G$. Then (π,ρ) is a covariant representation of (A,G,α).

Definition C.0.14. The convolution algebra $C_c(G,A)$ (A-valued function on G with compact support) has convolution as product $(a_1 \cdot a_2)(t) = \int_G a_1(s) \cdot \alpha_s((a_2(s^{-1}t)))\mathrm{d}s$ and involution $a^*(t) = \alpha_t((a(t^{-1}))^*) \cdot \Delta(t)^{-1}$ where $a, a_1, a_2 \in C_c(G,A)$.

Proposition C.0.15. *If (A,G,α) has a covariant representation π,ρ on a Hilbert space H, then there is a non-degenerate representation $(\pi\times\rho)$ of $C_c(G,A)$ on H such that $(\pi\times\rho)(y) = \int \pi(y(t))\rho_t \mathrm{d}t$ for any $y \in C_c(G,A)$*

For each representation $C_c(A,G) \to \mathscr{B}(H)$, an element x in the convolution algebra has a norm through representation. We consider all the representation of $C_c(A,G)$ and take the supremum of the norm of x, we get a new norm $\|\cdot\|$ and the completion of the convolution algebra under this norm is the crossed product of G with A: $A \rtimes_{\alpha} G$ or $C^*(G,A,\alpha)$. The completion of $C_c(G,A)$ under the norm defined by the representation in remark C.0.13 is called reduced crossed product of A by G: $A \rtimes_{\alpha r} G$ or $C_r^*(G,A)$.

Remark C.0.16. Let $A = \mathbb{C}$, on which G acts trivially, the cross product is the group C^*-algebra $C^*(G)$ and the reduced cross product by reduced group C^*-algebra $C_r^*(G)$. $C^*(G)$ and $C_r^*(G)$ are the same if and only if G is amenable.

Example C.0.17. $C(\mathbb{Z}_2) \rtimes \mathbb{Z}_2 \simeq M_2(\mathbb{C}); C_0(\mathbb{Z}) \rtimes \mathbb{Z} \simeq \mathscr{K}(l^2(\mathbb{Z})); C_0(\mathbb{R} \rtimes \mathbb{Z}) \simeq C(S^1) \otimes \mathscr{K}(l^2(\mathbb{Z}))$; If G acts on X properly and freely, then $C_0(X) \rtimes G \simeq C_0(X/G) \otimes \mathscr{K}$.

Example C.0.18.

1. $C_0(G/H) \rtimes G = C^*(H) \otimes \mathscr{K}$.
2. Let H be a compact subgroup of G and M is a compact smooth manifold with action of H smoothly and isometrically. H acts on $G \times M$ by $h(g,m) = (gh, h^{-1}m), \forall h \in H, g \in G, m \in M$, then $C_0((G \times M)/H) \rtimes G \simeq C_0(M) \rtimes H \otimes \mathscr{K}$.

APPENDIX IV

ANALYTIC K-HOMOLOGY

The section is a brief formulation of elliptic operators as an element in K-homology. Let E,F be complex vector bundles over X with a Hermitian metric and $A : C_c(X,E) \to C_c(X,F)$ be a 0-order elliptic pseudo-differential operator with proper support , and it extends to a bounded map $A : L^2(X,E) \to L^2(X,F)$. One of the properties characterizing A is that A is locally a Fredholm operator, i.e. $(AA^* - Id)M_f \in \mathscr{K}(L^2(X,F)), (A^*A - Id)M_f \in \mathscr{K}(L^2(X,E))$, for any $f \in C_0(X)$, where M_f is the operator of multiplication by f. A has another property called pseudo local:

$$[A,M_f] = AM_f - M_fA \in \mathscr{K}(L^2(E))$$

where $f(x) \in C_0(X)$.

If $E = F$ and A self-adjoint $A = A^*$, The properties reduced to

$$(A^2 - Id)M_f \in \mathscr{K}(L^2(E)), [A,M_f] \in \mathscr{K}(L^2(E)).$$

where A is replaced by an odd operator $\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$. The properties of the 0-order elliptic operators can be summarized as $A \in H$, graded or without grading:

- $(A^2 - Id)M_f \in \mathscr{K}(H)$,
- $[A,M_f] \in \mathscr{K}(H)$,
- $A^* = A$.

Let A be a separable C^*-algebra.

- An *odd Fredholm module* over A is a triple (H,ϕ,F) where
  - H is a Hilbert space;
  - $\phi : A \to \mathscr{B}(H)$ is a $*$-homomorphism;

- $F \in \mathscr{B}(H)$ such that $F = F^*$, $[F, \phi(a)] \in \mathscr{K}(H)$, $(F^2 - 1)\phi(a) \in \mathscr{K}(H)$ for any $a \in A$.

- An *even Fredholm module* is defined with additional assumptions that H is graded with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\phi(a)$ has degree 0 and F has degree 1. Recall that H is graded if there is a grading operator $\varepsilon : H \to H$ such that $\varepsilon^2 = Id$. So H splits into a direct sum of $H_0 = \{v \in H : Jv = v\}$ and $H_1 = \{v \in H : Jv = -v\}$. An operator $F : H \to H$ is said have degree 0 if $F\varepsilon = \varepsilon F$ and have degree 1 if $F\varepsilon = -\varepsilon F$.

The group of K-homology $K^0(A)/K^1(A)$, a cohomology theory for C^*-algebra A, is defined by the set of even/odd Fredholm A modules under the direct sum operation

$$(H_1, \phi_1, F_1) \oplus (H_2 \phi_2, F_2) = (H_1 \oplus H_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2)$$

module equivalent relations $\sim$, i.e. $x \sim y$ if there exists a degenerated Fredholm module z, w such that $x \oplus z$is homotopic to $y \oplus w$. Recall that (H, ϕ, F) is *degenerate* if $\phi(a)F = F\phi(a), (1 - F^2)\phi(a) = 0$ for all $a \in A$ and (H_1, ϕ_1, F_1) is *homotopic* to (H_2, ϕ_2, F_2) if there is a continuous path (H_1, ϕ_1, T_t) of Fredholm module under strong operator topology such that $T_0 = F_1$ and (H_1, ϕ_1, T_1) is isomorphic [1] to (H_2, ϕ_2, F_2). The morphism of the functor is given by the composition of the representation

$$\phi^* : K^i(B) \to K^i(A) : [(H, \psi, F)] \mapsto [(H, \psi \circ \phi, F)],$$

given a $*$-homomorphism $\phi : A \to B$. If A is further a G-algebra, the *equivariant K-homology* $K^i_G(A)$ can be defined out of homotopy classes of cycle (H, ϕ, F), where H has a unitary representation π of G and $\phi : A \to \mathscr{B}(H)$ is a G-invariant $*$-homomorphism [2] and $g \cdot F - F \in \mathscr{K}(H)$[3]. For example, the 0-order equivariant elliptic operator we are interested defines an element in $K^i_G(C_0(X))$.

[1] Two Fredholm A-module $(H_1, \phi_1, F_1), (H_2, \phi_2, F_2)$ are isomorphic if ϕ_1, ϕ_2 and F_1, F_2 are unitary equivalent(i.e. there exist a unitary operator $U : H_1 \to H_2$ such that $T_2 = UT_1U^*, \phi_2(a) = U\phi_1(a)U^*$).

[2] (H, ϕ, π) is a covariant representation of A.

[3] $g \cdot F = \pi(g)F\pi^{-1}(g)$.

APPENDIX V

KK-THEORY

This section is a summary on some well-know facts in KK-theory as preliminary knowledge to the thesis. A complete discussion can be found in [9][19]. Cycles in KK-group are represented by abstract elliptic operators acting on the following "Hilbert space with coefficient".

Definition E.0.19. A *pre-Hilbert B-module* is a complex vector space E as well as a right B-module with inner product $<\cdot,\cdot>: E\times E\to B$ which is linear in the second variable and satisfies the following relations: For all $b\in B, x,y\in E$,

1. $<x,yb> = <x,y>b$,
2. $<x,y>^* = <y,x>$
3. $<x,x>\geq 0$ where $<x,x>=0$ if and only if $x=0$.

There is a norm defined on each Hilbert B-module: $\|e\| = \|<e,e>\|^{\frac{1}{2}}, e\in E$. A *Hilbert B-module* is the completion of a pre-Hilbert module in this norm.

For example, C^*-algebra B is a Hilbert B-module $<x,y>=x^*y$; Every closed right ideal of B is a Hilbert B-module; The completion of $\oplus_1^\infty B$(sequences in B that are eventually 0) under the norm

$$<(a_1,\cdots),(b_1,\cdots)> = \sum_n a_n^* b_n$$

is a Hilbert B-module, denoted by H_B. Theorem E.0.21 implies that any separable Hilbert B module is a direct summand of H_B. Analogous to the Hilbert space theory we look at bounded linear operator on Hilbert module and the "compact" ones. Let E_1,E_2 be Hilbert B-modules, and the following be the set of *bounded linear operators,*

$$\mathscr{B}_B(E_1,E_2) = \{T: E_1\to E_2 : \exists T^*: E_2\to E_1, \forall x\in E_1, y\in E_2, <Tx,y> = <x,T^*y>\}.$$

Denote $\mathscr{B}(E)=\mathscr{B}_B(E)=\mathscr{B}_B(E,E)$. Note that $T\in\mathscr{B}_B(E_1,E_2)$ is bounded with norm $\|T\| =$

$\sup\{\|Tx\| : \|x\| \leq 1\}$, and $\mathscr{B}(E)$ is a C^*-algebra. For $x \in E_1, y \in E_2$, define *rank one operator* by

$$\theta_{x,y} : E_1 \to E_2 : z \mapsto x < y,z > .$$

It is easy to see that $\theta_{x,y}^* = \theta_{y,x}, T\theta_{x,y} = \theta_{Tx,y}, \theta_{x,y}T = \theta_{x,T^*y}$ for $T \in \mathscr{B}(E_1,E_2)$. The *compact Hilbert B-module* $\mathscr{K}_B(E_1,E_2)$ is the closure of the linear span of the rank one operators $\theta_{x,y}$. It is a closed ideal of $\mathscr{B}_B(E_1,E_2)$. Write $\mathscr{K}(E) = \mathscr{K}_B(E,E)$ and $\mathscr{K}_B = \mathscr{K}(H_B)$. For example, $\mathscr{K}_{\mathbb{C}}$ is the set of all compact operators in a Hilbert space. Note that elements in $\mathscr{K}_B(E_1,E_2)$ may not be compact.

Example E.0.20. 1. The map $\mathscr{K}(B) = \mathscr{K}_B(B) \to B : \theta_{x,y} \mapsto xy^*$ is a $*$-isomorphism.

2. For Hilbert module E $\mathscr{B}(E \oplus \cdots \oplus E) \simeq M_n(\mathbb{C}) \otimes \mathscr{B}(E)$, $\mathscr{K}(E \oplus \cdots \oplus E) \simeq M_n\mathbb{C} \otimes \mathscr{K}(E)$. In particular, $\mathscr{K}(B \oplus \cdots \oplus B) \simeq M_n(\mathbb{C}) \otimes B = M_n(B)$.

3. We have that $\mathscr{K}_B \doteq \mathscr{K}(H_B) \simeq B \otimes \mathscr{K}$.

In the theory of Hilbert space, the set of bounded linear operators can be viewed as the multiplier algebra of the set of all compact operators. We have similar result for Hilbert module,

$$\mathscr{B}(B) \simeq \mathscr{M}(B), \mathscr{B}(B^n) \simeq M_n(\mathscr{M}(B)), \mathscr{B}(H_B) = \mathscr{M}(\mathscr{K} \otimes B).$$

The following *Stabilization Theorem* by Kasparov is a generalization of the Serre-Swan theorem.

Theorem E.0.21. *If E is a countably generated Hilbert B-module,* $E \oplus H_B \simeq H_B$.

The following two definitions concerns the tensor product of two Hilbert modules.

Definition E.0.22. Let E_i be a Hilbert B_i-module, $i = 1,2$ and $\phi : B_1 \to \mathscr{B}(E_2)$ be a $*$-homomorphism, view E_2 as a left B_1-module via ϕ. Define a B_2-valued pre-inner product of the algebraic tensor product of E_1, E_2 by

$$< x_1 \otimes x_2, y_1 \otimes y_2 > = < x_2, \phi(< x_1, y_1 >_1) y_2 >_2 .$$

The completion of the algebraic tensor product with respect to this inner product is the *tensor product* of E_1 and E_2, denoted by $E_1 \otimes_\phi E_2$ (or $E_1 \otimes_B E_2$), which is a Hilbert B_2-module.

Example E.0.23. $\phi : B_1 \to B_2$ is a $*$-homomorphism, then $B_1 \otimes_\phi B_2$ is isomorphic to $\overline{\phi(B_1)B_2}$ through $x \otimes y \mapsto \phi(x)y$.

Remark E.0.24. There is a natural homomorphism $\mathscr{B}(E_1) \to \mathscr{B}(E_1 \otimes_\phi E_2) : F \mapsto F \otimes Id$

Definition E.0.25. The *external tensor product* $E_1 \otimes E_2$ is the completion of the algebraic tensor product with respect to the inner product

$$< x_1 \otimes x_2, y_1 \otimes y_2 > = < x_1, y_1 > \otimes < x_2, y_2 > .$$

To state the definition in a nice form, graded C^*-algebras and graded Hilbert modules are used.

Definition E.0.26. A *grading* on C^*-algebra A is a $*$-automorphism J of A, satisfying $J^2 = Id$. Let

$$A^{(0)} = \{a : Ja = a\}, A^{(1)} = \{a : Ja = -a\},$$

we have $A = A^{(0)} \oplus A^{(1)}, xy \in A^{(m+n)}$ if $x \in A^{(m)}, y \in A^{(n)}$. A *grading operator* on A is a unitary and self-adjoint operator in $\mathscr{B}(A)$ such that $A^{(n)} = \{a \in A : g^*ag = (-1)^n a\}$. We say that A has an *even grading* if there is a grading operator. A $*$-homomorphism $\phi : A \to B$ is a *graded homomorphism* if $\phi(A^{(n)}) \subset B^{(n)}$. *Graded commutator* is $[a,b] = ab - (-1)^{deg(a)deg(b)} ba$ on the homogeneous elements a, b.

Example E.0.27. Let A be a C^*-algebra.

- $M_2(A)$ has the*standard even grading* with grading operator $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- $A \oplus A$ has the *standard odd grading* $J : A \oplus A \to A \oplus A : (x,y) \mapsto (y,x)$.

Definition E.0.28. Let A, B be graded C^*-algebras. Then the graded tensor C^*-algebra $A \hat{\otimes} B$ is the completion of the algebraic tensor product $A \odot B$ with the operations

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{deg(b_1)deg(a_2)} a_1 a_2 \hat{\otimes} b_1 b_2, (a \hat{\otimes} b)^* = (-1)^{deg(a)deg(b)} a^* \hat{\otimes} b^*.$$

In general the completion is not unique but we will consider the case when it is unique: $A, B = C_0(X)$.

Example E.0.29. A motivation for graded tensor is the construction of the Clifford algebra: Define $\mathbb{C}l_1$ by $\mathbb{C}^2$ with standard odd grading and define $\mathbb{C}l_p\hat{\otimes}\mathbb{C}l_q \simeq \mathbb{C}l_{p+q}$ inductively. The following statements are easy corollary of proposition E.0.30:

- When $n = 2m$, $\mathbb{C}l_n = M_{2^m}(\mathbb{C})$ with standard even grading.
- When $n = 2m+1$, $\mathbb{C}l_n = M_{2^m} \oplus M_{2^m}$ with standard odd grading.

Proposition E.0.30.
- *If A is evenly graded and $M_2(\mathbb{C})$ has the standard even grading, then $A\hat{\otimes}M_2 \simeq M_2(A)$;*
- *If A is evenly graded then $A\hat{\otimes}\mathbb{C}l_1 \simeq A \oplus A$ with the standard odd grading;*
- *Let A be graded C^*-algebra with $\mathbb{Z}_2$ action α, then $A\hat{\otimes}\mathbb{C}l_1 \simeq A \rtimes_\alpha \mathbb{Z}_2$, in particular, $\mathbb{C}l_1\hat{\otimes}\mathbb{C}l_1 \simeq M_2(\mathbb{C})$.*

Definition E.0.31. Let B be a graded C^*-algebra, a *graded Hilbert B-module E* is a decomposition $E^{(0)} \oplus E^{(1)}$ with $E^{(m)}B^{(n)} \subset E^{(m+n)}, < E^{(m)}, E^{(n)} > \in B^{(m+n)}$. The grading on Hilbert module induces a grading on $\mathscr{B}(E), \mathscr{K}(E)$.

Definition E.0.32 (Graded tensor product of Hilbert modules)**.** Let E_1, E_2 be graded Hilbert modules over A, B respectively, and $\phi : A \to E_2$ be a graded $*$-homomorphism, then graded tensor product $E_1\hat{\otimes}_\phi E_2$ is the ordinary tensor product with grading $\deg(x\hat{\otimes}y) = \deg(x) + \deg(y)$.

With the above preparation, we are ready to introduce the KK-cycles.

Definition E.0.33. Let A, B be graded C^*-algebras with action of a locally compact group G, an *(A,B)-bimodule (Kasparov A,B-module)* is a triple (E, ϕ, F) where

- E is a graded Hilbert B-module;
- $\phi : A \to \mathscr{B}(E)$ is a graded $*$-homomorphism and A acts on E through ϕ;
- $F \in \mathscr{B}(E)$, with degree 1, such that $[\phi(a), F], \phi(a)(F^2 - 1), \phi(a)(F - F^*), \phi(a)(g \cdot F - F)$ are all in $\mathscr{K}(E)$ for all $a \in A$(Sometimes for simplicity we replace $\phi(a)$ with a).

We denote the set of all (A,B)-bimodules by $\mathscr{E}^G(A,B)$. Note that the triples (E_1,ϕ_1,F_1) and (E_2,ϕ_2,F_2) are not distinguished if there is a graded (A,B)-bimodule isomorphism $u: E_1 \to E_2$ satisfying $F_2 = uF_1u^{-1}$. An (A,B)-bimodule is *degenerate* if

$$[\phi(a),F] = \phi(a)(F^2-1) = \phi(a)(F-F^*) = \phi(a)(g\cdot F - F) = 0, \forall a \in A.$$

Two (A,B)-modules are homotopic if there is a norm continuous path of (A,B)-bimodule is (E,ϕ,F_t). The addition of two (A,B)-bimodule is the direct sum

$$(E_1,\phi_1,F_1)\oplus(E_2,\phi_2,F_2) = (E_1\oplus E_2,\phi_1\oplus\phi_2,F_1\oplus F_2) = (E_1\oplus E_2, \begin{pmatrix}\phi_1 & 0\\ 0 & \phi_2\end{pmatrix}, \begin{pmatrix}F_1 & 0\\ 0 & F_2\end{pmatrix}).$$

The group $KK(A,B)$ is defined by the quotient of $\mathscr{E}^G(A,B)$ by homotopy up to stabilization of degenerate modules.

Remark E.0.34. $KK^G(A,B)$ is an abelian group. In fact, let $-E$ be the Hilbert with the opposite grading of E and $u \in \mathscr{B}(E,-E)$ be the identity map the inverse of (E,ϕ,F) is given by $(-E,\phi',-uFu^{-1})$ here the action ϕ' of A on -E is defined by $\phi(a)u(x) = u(\varepsilon(a)x)$ here ε is the grading for A. $\begin{pmatrix}\cos t\cdot F & \sin t u^{-1}\\ \sin t\cdot u & -\cos t uFu^{-1}\end{pmatrix}$ joints $(E,\phi,F)\oplus-(E,\phi,F)$ and a degenerate element.

Proposition E.0.35 (Functorial property)**.** *$f: A_2 \to A_1$, a homomorphism of graded C^*-algebras, gives a homomorphism*

$$f^*: KK^G(A_1,B)\to KK^G(A_2,B): (E,\phi,F)\mapsto (H,\phi\circ f,F).$$

$g: B_1\to B_2$, a homomorphism of graded C^*-algebra, induces a homomorphism of

$$h_*: KK^G(A,B_1)\to KK^G(A,B_2): (E,\phi,T)\mapsto (E\hat{\otimes}_g B_2,\phi\hat{\otimes}1,T\hat{\otimes}1).$$

$\tau_D: KK^G(A,B)\to KK^G(A\hat{\otimes}D,B\hat{\otimes}D): (E,\phi,F)\mapsto(E\hat{\otimes}D,\phi\hat{\otimes}1,F\hat{\otimes}1)$ is a homomorphism.

Definition E.0.36. Denote $KK_0^G(A,B) = KK^G(A,B)$ and define $KK_1^G(A,B) = KK^G(A,B\hat{\otimes}\mathbb{C}l_1)$.

Proposition E.0.37. *1. $KK^G(A_1\oplus A_2,B)\simeq KK^G(A_1,B)\oplus KK^G(A_2,B)$.*

2. $KK^G(A,B) \simeq KK^G(A\hat{\otimes}\mathscr{K},B) \simeq KK^G(A,B\hat{\otimes}\mathscr{K}).\tau_{\mathscr{K}} : KK^G(A,B) \to KK^G(A\hat{\otimes}\mathscr{K},B\hat{\otimes}\mathscr{K})$ *is an isomorphism.*

3. $g_1,g_2 : D \to B$ *are homotopic* $\Rightarrow g_{0*} = g_{1*} : KK^G(A,D) \to KK^G(A,B)$;
   $f_0,f_1 : A \to D$ *are homotopic* $\Rightarrow f_0^* = f_1^* : KK^G(D,B) \to KK^G(A,B)$.

4. *(Bott Periodicity)* $\tau_{\mathbb{C}l_1} : KK^G(A,B) \to KK^G(A\hat{\otimes}\mathbb{C}l_1,B\hat{\otimes}\mathbb{C}l_1)$ *is an isomorphism.*

Example E.0.38. $KK(\mathbb{C},\mathbb{C}) = \mathbb{Z}$. In fact, in $[(H,\phi,F)] \in KK(\mathbb{C},\mathbb{C})$, $H = H_0 \oplus H_1$, H_0,H_1 are trivially graded and H_1 has opposite grading to H_0, ϕ is determined by its value at 1, so we let $\phi(1) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$, P,Q are projections and $F = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$. (H,ϕ,F) is reduced to $(H_0 \oplus H_1, \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix})$.

Proposition E.0.39. *$KK^G(\mathbb{C},A) = K_0^G(A)$ when G is compact.*

KK-theory is not simply a generalization of K-theory and K-homology. There is an associative product between KK-groups

$$KK^G(A,D) \times KK^G(D,B) \to KK^G(A,B).$$

Let $(E_1,\phi,F_1) \in \mathscr{E}^G(A,D),(E_2,\phi,F_2) \in \mathscr{E}^G(D,B)$ and construct a (A,B)-bimodule (E,ϕ,F) where

$$E = E_1\hat{\otimes}_{\phi_2}E_2, \phi = \phi_1\hat{\otimes}_{\phi_2}1, F \in \mathscr{B}(E)$$

is a suitable combination of F_1 and F_2, which in precise is constructed using connection.

Definition E.0.40. Let E_1 be a Hilbert D-module and E_2 be a (D,B)-bimodule, $E = E_1\hat{\otimes}_D E_2, F_2 \in \mathscr{B}(E_2)$. An element $F \in \mathscr{B}(E)$ is said to be an F_2-*connection* for E_1 if and only if

$$[\tilde{T}_\xi, F_2 \oplus F] \in \mathscr{K}(E_2 \oplus E), \forall \xi \in E_1,$$

where $\tilde{T}_\xi = \begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix} \in \mathscr{B}(E_2 \oplus E), T_\xi \in \mathscr{B}(E_2,E)$ is defined by $T_\xi(\eta) = \xi\hat{\otimes}\eta \in E$.

Proposition E.0.41. *If E_1 is countably generated and $[F_2,\phi(d)] \in \mathscr{K}(E), \forall d \in D$ then there exists a F_2 connection.*

Example E.0.42. If $[F_2,\phi(D)] = 0$, then $1\hat{\otimes}F_2$ makes sense in $\mathscr{B}(E)$ and is a F_2 connection on E.

Definition E.0.43. Let A,B,D be graded G-algebras and $(E_1,\phi,F_1),(E_2,\psi,F_2)$ are $(A,D),(D,B)$-bimodule respectively. Let $E = E_1\hat{\otimes}_\psi E_2$ be (A,B)-bimodule and $F \in \mathscr{B}(E)$, the triple (E,η,F) is an *intersection product (Kasparov product)* if it satisfy the following conditions:

1. The cycle $(E,\eta,F) \in \mathscr{E}^G(A,B)$, a (A,B)-bimodule.
2. The operator F is a F_2 connection on E for E_1.
3. F have the property: $\eta(a)[F_1\hat{\otimes}1,F]\eta(a)^* \geq 0, \forall a \in A$, module $\mathscr{K}(E)$.

The set of all F such that (E,η,F) is an intersection product is denoted by $F_1\sharp_D F_2$. Denote $z = x\otimes_D y$ if z is intersection product of x and y.

Theorem E.0.44. *If A is separable, the intersection product exists and unique up to homotopy. Moreover, it passes through the quotient and defines a bilinear pairing:*

$$KK^G(A,D)\times KK^G(D,B) \to KK^G(A,B).$$

We can further have $KK_i^G(A,B)\times KK_j^G(A,B) \to KK_{i+j}^G(A,B)$(i+j mod 2). For example:

$$KK_1^G(A,D)\times KK_1^G(D,B) \to KK_0^G(A,B) : (x,y) \mapsto x\hat{\otimes}_{D\hat{\otimes}\mathbb{C}l_1}\tau_{\mathbb{C}l_1}y.$$

Theorem E.0.45. *Let A,D_1 be separable, $x_1 \in KK^G(A,D_1), x_2 \in KK^G(D_1,D_2), x_3 \in KK^G(D_2,B)$, then $(x_1\otimes_{D_1}x_2)\otimes_{D_2}x_3 = x_1\otimes_{D_1}(x_2\otimes_{D_2}x_3)$. The intersection product is associative.*

Definition E.0.46. In general for A_1,A_2 separable, we could define the pairing

$$KK^G(A_1,B_1\hat{\otimes}D)\times KK^G(D\hat{\otimes}A_2,B_2) \to KK^G(A_1\hat{\otimes}A_2,B_1\hat{\otimes}B_2) : (x,y) \mapsto x_1\otimes_D x_2 = \tau_{A_2}(x)\otimes_{B_1\hat{\otimes}D\hat{\otimes}A_2}\tau_{B_1}(y).$$

Remark E.0.47. When $D = \mathbb{C}$, the product $KK^G(A_1,B_1)\times KK^G(A_2,B_2) \to KK^G(A_1\hat{\otimes}A_2,B_1\hat{\otimes}B_2)$ is exterior intersection product and for $x \in KK^G(A_1,B_1), y \in KK^G(A_2,B_2)$ we have $x\otimes_{\mathbb{C}}y = y\otimes_{\mathbb{C}}x$. (They differ by $(-1)^{ij}$ if $x \in KK^i, y \in KK^j$)

Definition E.0.48. For $G-C^*$-algebra A, we have a identity element $1_A \in KK^G(A,A)$ as $[(A,0,0)]$.

Proposition E.0.49. *1 is a unit of the product:* $1 \otimes_{\mathbb{C}} x = x \otimes_{\mathbb{C}} 1 = x, \forall x \in KK^G(A,B)$ *for separable A.*

Remark E.0.50. $KK^G(\mathbb{C},\mathbb{C})$ is a ring with multiplication defined by intersection product. When G is compact, $KK^G(\mathbb{C},\mathbb{C}) \cong R(G)$ as rings. $KK^G(\mathbb{C},\mathbb{C})$ is a "representation" of non-compact group G.

Remark E.0.51. Let G be compact, $A_1 = B_2 = A_2 = B_1 = \mathbb{C}$, consider the pairing the pairing $KK^G(\mathbb{C},D) \times KK^G(D,\mathbb{C}) \to KK^G(\mathbb{C},\mathbb{C})$. Given $(E,\phi_1,F_1) \in KK^G(\mathbb{C},D), (E_2,\phi_2,F_2) \in KK^G(D,\mathbb{C})$, then $F \in F_1 \sharp F_2$ is defined by

$$\phi_2(F_1 \hat{\otimes} 1) + \phi_2(\sqrt{1-F_1^2} \hat{\otimes} 1) \cdot (1 \hat{\otimes} F_2).$$

Remark E.0.52 (Idea of building up the product). Given $(E_1,\phi_1,F_1) \in \mathscr{E}^G(A,D), (E_2,\phi_2,F_2) \in \mathscr{E}^G(D,B)$, want to build $F \in \mathscr{B}(E_1 \hat{\otimes}_{\phi_2} E_2)$ such that $F \in \mathscr{E}^G(A,B)$. $F_1 \hat{\otimes} 1 \in \mathscr{B}(E_1 \hat{\otimes}_{\phi_2} E_2)$ but it is not a candidate for F because $a \cdot (1 \hat{\otimes} 1 - F_1^2 \hat{\otimes} 1) = a \cdot ((1-F_1^2) \hat{\otimes} 1) \notin \mathscr{K}(E_1 \hat{\otimes}_{\phi_2} E_2)$. Similarly for $1 \hat{\otimes} F_2$(Note sometimes we need to replace it by F_2 connection on $E_1 \hat{\otimes}_{\phi_2} E_2$). So instead we consider a "convex" combination of $F_1 \hat{\otimes} 1$ and $1 \hat{\otimes} F_2$: $F = M(F_1 \hat{\otimes} 1) + N(1 \hat{\otimes} F_2)$ where M,N are positive operators on $E_1 \hat{\otimes}_{\phi_2} E_2$ and $M^2 + N^2 = 1$. And $F \in \mathscr{E}^G(A,B)$ when imposing suitable conditions on M,N (the technical part), but at the same time the conditions on M,N are not too strong to prevent their existence. The following is an example of using intersection product to produce Dirac operators on exterior algebra bundle over $\mathbb{R}^n$ from Dirac operator on $\mathbb{R}$.

Example E.0.53. Dirac operator $D_j = i\frac{d}{dx_j} : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ has symbol $\sigma_{D_j}(x,\xi) = i\xi e_j$, where e_j is basis for $\mathbb{R}$. Let $f_j = ie_j$, (then $f_i f_j + f_j f_i = 2\delta_{ij}$) and let $F_j = \frac{\sigma(D_j)}{1+\sigma(D_j)\sigma(D^*)} \in KK(\mathbb{C}l, C_0(\mathbb{R}))$. The exterior intersection product $KK(\mathbb{C}l, C_0(\mathbb{R})) \times KK(\mathbb{C}l, C_0(\mathbb{R})) \to KK(M_2(\mathbb{C}), C_0(\mathbb{R}^2))$ is defined (using the fact that $\mathbb{C}l \hat{\otimes} \mathbb{C}l \simeq M_2(\mathbb{C})$ and $C_0(\mathbb{R}) \hat{\otimes} C_0(\mathbb{R}) \simeq C_0(\mathbb{R}^2)$). Let

$$M = \frac{1+\xi_1 f_1}{1+\|\xi_1\|^2+\|\xi_2\|} \text{and} N = \frac{1+\xi_2 f_2}{1+\|\xi_1\|^2+\|\xi_2\|}$$

and $F = MF_1 + NF_2$. F is the normalization of Dirac operator $\dfrac{\partial}{\partial x_1} f_1 + \dfrac{\partial}{\partial x_2} f_2$ acting exterior algebra bundle (acted on by Clifford multiplication) over $\mathbb{R}^2$.

In the end of this section we define the induction homomorphism (or descent map)

$$j^G : KK^G(A,B) \to KK(C^*(G,A), C^*(G,B)).$$

Recall the special case (Green-Julg theorem): when G is compact, $K_i^G(B) \simeq K_i(C^*(G,B))$. It has its dual in K-homology: when G is discrete: $K_G^i(A) \simeq K^i(C^*(G,A))$. The construction goes as follows,

1. Let B be a G-C^*-algebra with G locally compact. Construct convolution algebra $C_c(G,B)$ by

$$(b_1 \cdot b_2)(t) = \int_G b_1(s) \cdot s(b_2(s^{-1}t))ds, b^*(t) = t(b(t^{-1}))^* \Delta(t)^{-1}.$$

   Its closure under maximal C^* norm $C^*(G,B)$ is the crossed product of G and B.

2. For a G-algebra B and a Hilbert B-module E, define Hilbert $C_c(G,B)$-module $C_c(G,E)$ by

$$(e \cdot b)(t) = \int_G e(s) \cdot s(b(s^{-1}t))ds$$

   and the $C_c(G,B)$-valued inner product

$$(e_1, e_2)(t) = \int_G s^{-1}(e_1(s), e_2(st))_E ds, \forall e, e_1, e_2 \in C_c(G,E), b \in C_c(G,B).$$

   The completion of $C_c(G,E)$ under norm $\|e\| = \|(e,e)\|_{C^*(G,B)}^{\frac{1}{2}}$, denoted by $C^*(G,E)$, is a Hilbert $C^*(G,B)$-module.

3. If $(E, \phi, T) \in \mathscr{E}^G(A,B)$, we have $(C^*(G,E), (\tilde{\phi}), \tilde{T})$, where $\tilde{\phi}$ is the induced action of $C^*(G,A)$ on $C^*(G,E)$ in step 1 and $\tilde{T}$ acts on $C^*(G,E)$ by $(\tilde{T}e)(s) = T(e(s)), \forall e \in C_c(G,E), s \in G$.

4. For G-algebra B_1, B_2, and Hilbert B_i module E_i and a homomorphism $B_1 \to \mathscr{B}(E_2)$, we have tensor product $C^*(G,E_1) \otimes_{C^*(G,B)} C^*(G,E_2) \simeq C^*(G, E_1 \otimes_{B_1} E_2)$.

Theorem E.0.54. *Let group G be second countable, for G-algebras A and B there exist natural homomorphism: $j^G : KK^G(A,B) \to KK(C^*(G,A), C^*(G,B))$ with the properties:*
(1)If $x_1 \in KK^G(A,D), x_2 \in KK^G(D,B)$ and $x_1 \otimes_D x_2$ exists, then $j^G(x_1 \otimes_D x_2) = j^G(x_1) \otimes_{C^*(G,D)} j^G(x_2)$.
(2) When A=B, $j^G(1_A) = 1_{C^*(G,A)}$.

BIBLIOGRAPHY

[1] S. Alinhac and P. Gerard. *Pseudo-differential operators and the Nash-Moser theorem*, volume 82. Graduate Studies in Math, 2007.

[2] S. A. Antonyan. Proper actions on topological groups: Applications to quotient spaces. *Proceedings of the American Mathematical Society*, 138:3707–3716, 2010.

[3] M. Atiyah. Global theory of elliptic operators. *Proc. Internat. Conf. on Functional analysis and related topics*, pages 21–30, 1970.

[4] M. Atiyah. Elliptic operators, discrete groups and von neumann algebras. *Société Mathématique de France*, 1976.

[5] M. Atiyah, R. Bott, and V. K. Patodi. On the heat equation and the index theorem. *Inventiones Mathematicae*, (19):279–330, 1973.

[6] M. Atiyah and I. Singer. The index of elliptic operators III. *The Annals of Mathematics*, 87(3):546–604, 1968.

[7] P. Baum and R. G. Douglas. K homology and index theory. *Proc. Symp. Pure Math*, 38(1):117–173, 1982.

[8] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*. Springer, 2003.

[9] B. Blackadar. *K-Theory for Operator Algebras*. Cambridge University Press, second edition, 1998.

[10] G. Bredon. *Introduction to compact transformation groups*. Academic Press, 1972.

[11] M. Breuer. Fredholm theories in von neumann algebras. i. *Math. Ann.*, 178:243–254, 1968.

[12] A. Connes and H. Moscovici. The L^2-index theorem for the homogeneous spaces of Lie groups. *Ann. of Math.*, 115:291–330, 1982.

[13] A. Connes and H. Moscovici. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology*, 29(3):345–388, 1990.

[14] E. Getzler. Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem. *Communications in Mathematical Physics*, 92:163–178, 1983.

[15] E. Getzler. A short proof of the local Atiyah-Singer index theorem. *Topology*, pages 111–117, 1986.

[16] P. B. Gilkey. Curvature and eigenvalues of the Laplacian for elliptic complexes. *Advances in Mathematics*, pages 433–382, 1973.

[17] P. B. Gilkey. *The index theorem and the heat equation*. Publish or Perish, inc. (Boston), 1974.

[18] L. Hörmander. Fourier integral operators I. *Acta Mathematica*, 127(1):79–183, 1971.

[19] K. Jensen and K. Thomsen. *Elements of KK-theory*. Mathematics: Theory and Applications, 1991.

[20] H. B. L. Jr. and M.-L. Michelsohn. *Spin Geometry*. Princeton University Press, 1989.

[21] G. Kasparov. Topological invariants of elliptic operators I: K-homology. *Mathematics of the USSR - Isvestiya*, 9(4):751–792, 1975.

[22] G. Kasparov. The index of invariant elliptic operators, K-theory, and lie group representations. *Dokl. Akad. Nauk SSSR*, (268):533–537, 1983.

[23] G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Inventiones Mathematicae*, 91(1):147–201, 1988.

[24] G. Kasparov. K-theoretic index theorem for elliptic operators. *Preprint*, 2008.

[25] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*, volume I. Weily-Interscience, weily classics library edition edition, 1996.

[26] M. Marcolli and V. Mathai. Twisted index theory on good orbifolds, ii: Fractional quantum numbers. *Communications in Mathematical Physics*, 217:55–87, 2001.

[27] R. S. Palais. On the existence of slices for actions of non-compact lie groups. *Ann. of Math.*, 73(2):295–323, March 1961.

[28] J. J. Perez. The G-fredholm property of the $\bar{\partial}$-neumann problems. *J. Geom. Anal.*, 2008.

[29] J. Roe. *Elliptic Operators, topology and asymptotic methods*. Chapman and Hall, second edition, 1998.

[30] J. Rosenberg. C^*-algebras, positive scalar curvature, and the Novikov conjecture. *Publications Mathematiques de l'Institut Hautes Études Scientifiques*, 58(1):197–212, 1983.

[31] T. Schick. L^2-index theorems, KK-theoory, and connections. *New York Journal of Mathematics*, 11:387–443, 2005.

[32] M. Shubin. Von neumann algebra and L^2 techniques in geometry and topology.

[33] M. Stern. L^2-index theorems on locally symmetric spaces. *Inventiones Mathematicae*, 96:231–282, 1989.

[34] N. E. Wegge-Olsen. *K-Theory and C^*-Algebras (A Friendly Approach)*. Oxford University Press, 1993.