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American Options: Symmetry Properties

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American Options: Symmetry Properties*

Jérôme Detemple†

Résumé / Abstract

Une propriété utile des options européennes et américaines, dans le cadre du modèle standard de Black-Scholes, est la « symétrie ». Celle-ci énonce que la valeur d'une option d'achat au prix d'exercice K et à date d'échéance T est identique à la valeur d'une option de vente au prix d'exercice S, date d'échéance T dans un marché financier auxiliaire où le taux d'intérêt est δ et où le titre support paye des dividendes au taux r et est valorisé à K. Cet article fait une synthèse des généralisations récentes de cette propriété et établit certains résultats complémentaires. La validité de la propriété de symétrie est établie pour une classe générale de modèles des marchés financiers qui comprend des spécifications nonmarkoviennes à coefficients stochastiques du sous-jacent. En effet, la symétrie se généralise de manière naturelle aux actifs contingents nonstandards de style américain, tels que (i) les options à échéance aléatoires (options à barrières et options plafonnées), (ii) les produits dérivés sur titres supports multiples, (iii) les produits dérivés sur temps d'occupation et (iv) les titres dont les valeurs d'échéance sont homogènes de degré $v \neq 1$. La méthode de changement de numéraire, qui est essentielle pour la démonstration de ces résultats, est également passée en revue.

A useful feature of European and American options in the standard financial market model with constant coefficients is the property of put-call symmetry. This property states that the value of a put option with strike price K *and maturity date* T *is the same as the value of a call option with strike price* S, *maturity date* T *in an auxiliary financial market with interest rate* δ *and in which the underlying asset price pays dividends at the rate* r *and has initial value* K. *In this paper we review recent generalizations of this property and provide complementary results. We show taht put-call symmetry is a general property which holds in a large class of financial market models including nonmarkovian models with stochastic coefficients. The property extends naturally to nonstandard American claims such as (i) options with random maturity which include barrier options and capped options, (ii) multiasset derivatives, (iii) occupation time derivatives and (iv) claims whose payoffs are homogeneous of degree* $\nu \neq 1$. *Changes of numeraire which are instrumental in establishing symmetry properties are also reviewed and discussed.*

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Mots Clés : Évaluation, options américaines, politique optimale d'exercice, propriété de symétrie, changement de mesure, échéance aléatoire, options plafonnées, options à barrières, titres supports multiples, temps d'occupation, homogénéité, changement de numéraire, formules de représentation des prix

Keywords: Option pricing, American options, early exercise policy, symmetry properties, change of measure, random maturity, barrier options, capped options, multiasset options, occupation time derivatives, homogeneity, changes of numeraire, replicating portfolio, reprensentation of prices

1 Introduction.

Put-call symmetry (PCS) holds when the price of a put option can be deduced from the price of a call option by relabelling its arguments. For instance, in the context of the standard financial market model with constant coefficients the value of an American put equals the value of an American call with strike price S, maturity date T, in a financial market with interest rate δ and in which the underlying asset price pays dividends at the rate r. This result was originally demonstrated by McDonald and Schroder (1990, 1998) using a binomial approximation of the lognormal model and by Bjerksund and Stensland (1993) in the continuous time model using PDE methods; it is a version of the international put-call equivalence (Grabbe (1983)).

Put-call symmetry is a useful property of options since it reduces the computational burden in implementations of the model. Indeed, a consequence of the property is that the same numerical algorithm can be used to price put and call options and to determine their associated optimal exercise policy. Another benefit is that it reduces the dimensionality of the pricing problem for some payoff functions. Examples include exchange options or quanto options. PCS also provides useful insights about the economic relationship between contracts. Puts and calls, forward prices and discount bonds, exchange options and standard options are simple examples of derivatives that are closely connected by symmetry relations.

Some intuition for PCS is based on the properties of the normal distribution. Indeed, in the model with constant coefficients the distribution of the terminal stock price is lognormal. Symmetry of the put and call option payoff function combined with the symmetry of the normal distribution then suggest that the put and call values can be deduced from each other by interchanging the arguments of the pricing functions. This can be verified directly from the valuation formulas for standard European and American options. As demonstrated by Gao, Huang and Subrahmanyam (2000) it is also true for European and American barrier options, such as down and out call and up and out put options, in the model with constant coefficients.

Since option values depend only on the volatility of the underlying asset price it seems reasonable to conjecture that PCS will hold in diffusion models in which the drift is an arbitrary function of the asset price but the volatility is a symmetric function of the price. This intuition is exploited by Carr and Chesney (1994) who show that PCS indeed extends to such a setting. Since alternative assumptions about the behavior of the underlying asset price destroy the symmetry of the terminal price distribution it would appear that the property cannot hold in more general contexts. Somewhat surprisingly, Schroder (1999), relying on a change of numeraire introduced by Geman, El Karoui and Rochet (1995), is able to show that the result holds in very general environments including models with stochastic coefficients and discontinuous underlying asset price processes.[1]

This paper surveys the latest results in the field and provides further extensions. Our basic market structure is one in which the underlying asset price follows an Ito process with progressively measurable coefficients (including the dividend rate) and the interest rate is an adapted stochastic process. We show that a version of PCS holds under these general market conditions. One feature behind the property is the homogeneity of degree one of the put and call payoff functions with respect to the stock price and the exercise price. For such payoffs the standard symmetry property of prices follows from a simple change of measure which amounts to taking the asset price as numeraire.

The identification of the change of numeraire as a central feature underlying the standard PCS property permits the extension of the result to more complex contracts which involve liquidation provisions. A *random maturity* option is an option (put or call) which is automatically liquidated at a prespecified random time and, in such an event, pays a prespecified random cash flow. A typical example is a down and out put option with barrier L. This option expires automatically if the underlying asset price hits the level L (null liquidation payoff), but pays off $(K - S)^+$ if exercised prior to expiration. Put-call symmetry for random maturity options states that the value of an

[1]Symmetry results in general market environments are also reported in Kholodnyi and Price (1998). Their proofs are based on no-arbitrage arguments and use operator theory and group theory notions.

American put with strike price K, maturity date T, automatic liquidation time τ_l and liquidation payoff H_{τ_l} equals the value of an American call with strike S, maturity date T, automatic liquidation time τ_l^* and liquidation payoff $H_{\tau_l}^*$ in an auxiliary financial market with interest rate δ and in which the underlying asset price pays dividends at the rate r and has initial value K. The liquidation characteristics τ_l^* and $H_{\tau_l}^*$ of the equivalent call can be expressed in terms of the put specifications K, τ_l and H_{τ_l} and the initial value of the underlying asset S. For a down and out put option with barrier L which has characteristics

$$\tau_L = \inf\{t \in [0,T] : S_t = L\} \text{ and } H_{\tau_L} = 0$$

the equivalent up and out call has characteristics

$$\tau_L^* = \tau_{L^*} = \inf\{t \in [0,T] : S_t^* = L^* \equiv \frac{KS}{L}\} \text{ and } H_{\tau_L}^* = 0.$$

where S^* denotes the price of the underlying asset in the auxiliary financial market.

Contingent claims which are written on multiple assets also exhibit symmetry properties when their payoff is homogeneous of degree one. In fact the same change of measure argument as in the one asset case identifies classes of contracts which are related by symmetry and therefore can be priced off each other. In particular, for contracts on two underlying assets, we show that American call max-options are symmetric to American options to exchange the maximum of an asset and cash against another asset, that American exchange options are symmetric to standard call or put options (on a single underlying asset) and that American capped exchange options with proportional cap are symmetric to both capped call options with constant caps and capped put options with proportional caps. In all of these relationships the symmetric contract is valued in an auxiliary financial market with suitably adjusted interest rate and underlying asset prices.

We then discuss extensions of the property to a class of contracts analyzed recently in the literature, namely occupation time derivatives. These contracts, typically, depend on the amount of time spent by the underlying asset price in certain presepecified regions of the state space. Examples of such path-dependent contracts are Parisian and Cumulative barrier options (Chesney, Jeanblanc-Picque and Yor (1997)), Step options (Linetsky (1999)) and Quantile options (Miura (1992)). More general payoffs based on the occupation time of a constant set, above or below a barrier, are discussed in Hugonnier (1998). While the literature has focused exclusively on European-style contracts in the context of models with geometric Brownian motion price processes, we consider American-style occupation time derivatives in models with Ito price processes. We also allow for occupation times of random sets. We show that occupation time derivatives with homogeneous payoff functions satisfy a symmetry property in which the symmetric contract depends on the occupation time of a suitably adjusted random set. Extensions to multiasset occupation time derivatives are also presented.

Symmetry-like properties also hold when the contract under consideration is homogeneous of degree $\nu \neq 1$. In this instance the interest rate in the auxiliary economy depends on the coefficient ν, the interest rate in the original economy and the dividend rate and volatility coefficients of the numeraire asset in the original economy. The dividend rates of other assets in the new numeraire are also suitably adjusted.

Since symmetry properties reflect the passage to a new numeraire asset it is of interest to examine the replicability of attainable payoffs under changes of numeraire. For the case of nondividend paying assets Geman, El Karoui and Rochet (1995) have established that contingent claims that are attainable in one numeraire are also attainable in any other numeraire and that the replicating portfolios are the same. We show that these results extend to the case of dividend-paying assets. This demonstrates that any symmetric contract can indeed be attained in the appropriate auxiliary economy with new numeraire and that its price satisfies the usual representation formula involving the pricing measure and the interest rate that characterize the auxiliary economy.

The second section reviews the property in the context of the standard model with constant coefficients. In section 3 PCS is extended to a financial market model with Brownian filtration and stochastic opportunity set. The markovian model with diffusion price process (and general volatility structure) is examined as a subcase of the general model. Extensions to random maturity options, multiasset contingent claims, occupation time derivatives and payoffs that are homogeneous of degree ν are carried out in sections 4-7. Questions pertaining to changes of numeraire, replicating portfolios and representation of asset prices are examined in section 8. Concluding remarks are formulated last.

2 Put-Call Symmetry in the Standard Model.

We consider the standard financial market model with constant coefficients (constant opportunity set). The underlying asset price, S, follows a geometric Brownian motion process

$$dS_t = S_t[(r-\delta)dt + \sigma d\tilde{z}_t], t \in [0,T]; S_0 \text{ given} \tag{1}$$

where the coefficients (r, δ, σ) are constant. Here r represents the interest rate, δ the dividend rate and σ the volatility of the asset price. The asset price process (1) is represented under the equivalent martingale measure Q: the process $\tilde{z}$ is a Q-Brownian motion.

In this complete financial market it is well known that the price of any contingent claim can be obtained by a no-arbitrage argument. In particular the value of a European call option with strike price K and maturity date T is given by the Black and Scholes (1973) formula

$$c(S_t, K, r, \delta, t) = S_t e^{-\delta(T-t)} N(d(S_t, K, r, \delta, T-t)) - K e^{-r(T-t)} N(d(S_t, K, r, \delta, T-t) - \sigma\sqrt{T-t}) \tag{2}$$

where

$$d(S, K, r, \delta, T-t) = \frac{\log(\frac{S}{K}) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{3}$$

Similarly the value of a European put with the same characteristics (K, T) is

$$p(S_t, K, r, \delta, t) = K e^{-r(T-t)} N(-d(S_t, K, r, \delta, T-t) + \sigma\sqrt{T-t}) - S_t e^{-\delta(T-t)} N(-d(S_t, K, r, \delta, T-t)). \tag{4}$$

Comparison of these two formulas leads to the following symmetry property

Theorem 1 *(European PCS). Consider European put and call options with identical characteristics K and T written on an asset with price S given by (1). Let $p(S, K, r, \delta, t)$ and $c(S, K, r, \delta, t)$ denote the respective price functions. Then*

$$p(S, K, r, \delta, t) = c(K, S, \delta, r, t) \tag{5}$$

Proof of Theorem 1: Substituting (K, S, δ, r) for (S, K, r, δ) in (2) and using

$$\begin{aligned} d(K, S, \delta, r, T-t) &= \frac{\log(\frac{K}{S}) + (\delta - r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= -\frac{\log(\frac{S}{K}) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} + \sigma\sqrt{T-t} \\ &= -d(S, K, r, \delta, T-t) + \sigma\sqrt{T-t} \end{aligned} \tag{6}$$

gives the desired result. ■

This results shows that the put value in the financial market under consideration is the same as the value of a call option with strike price S and maturity date T in an economy with interest rate δ and in which the underlying asset price follows a geometric Brownian motion process with dividend rate r, volatility σ and initial value K, under the risk neutral measure.

This symmetry property between the value of puts and calls is even more striking when we consider American options. For these contracts (Kim (1990), Jacka (1991) and Carr, Jarrow and Myneni (1992)) have shown that the value of a call has the early exercise premium representation (EEP)

$$C(S_t, K, r, \delta, t, B^c(\cdot)) = c(S_t, K, r, \delta, t) + \pi(S_t, K, r, \delta, t, B^c(\cdot)) \tag{7}$$

where $C(S, K, r, \delta, t, B^c(\cdot))$ is the value of the American call, $c(S, K, r, \delta, t)$ represents the value of the European call in (2) and $\pi(S, K, r, \delta, t, B^c(\cdot))$ is the early exercise premium

$$\pi(S_t, K, r, \delta, t, B^c(\cdot)) = \int_t^T \phi(S_t, K, r, \delta, v-t, B_v^c)dv \tag{8}$$

with

$$\phi(S_t, K, r, \delta, v-t, B_v^c) = \delta S_t e^{-\delta(v-t)} N(d(S_t, B_v^c, r, \delta, v-t)) - rKe^{-r(v-t)} N(d(S_t, B_v^c, r, \delta, v-t) - \sigma\sqrt{v-t}). \tag{9}$$

The exercise boundary $B^c(\cdot)$ of the call option solves the recursive integral equation

$$B_t^c - K = C(B_t^c, K, r, \delta, t, B^c(\cdot)) \tag{10}$$

subject to the boundary condition $B_T^c = \max(K, \frac{r}{\delta}K)$. Let $B^c(K, r, \delta, t)$ denote the solution.

The EEP representation for the American put can be obtained by following the same approach as for the call. Alternatively the put value can be deduced from the call formula by appealing to the following result (McDonald and Schroder (1998)).

Theorem 2 *(American PCS). Consider American put and call options with identical characteristics K and T written on an asset with price S given by (1). Let $P(S, K, r, \delta, t, B^p(\cdot))$ and $C(S, K, r, \delta, t, B^c(\cdot))$ denote the respective price functions and $B^p(K, r, \delta, \cdot)$ and $B^c(S, r, \delta, \cdot)$ the corresponding immediate exercise boundaries. Then*

$$P(S, K, r, \delta, t, B^p(K, r, \delta, \cdot)) = C(K, S, \delta, r, t, B^c(S, \delta, r, \cdot)) \tag{11}$$

and for all $t \in [0, T]$

$$B^p(K, r, \delta, t) = \frac{SK}{B^c(S, \delta, r, t)} \tag{12}$$

This result can again be demonstrated by substitution along the lines of the proof of theorem 1. A more elegant approach relies on a change of measure detailed in the next section.

Hence, even for American options the value of a put is the same as the value of a call with strike S, maturity date T, in an economy with interest rate δ and in which the underlying asset price, under the risk neutral measure, follows a geometric Brownian motion process with dividend rate r, volatility σ and initial value K. Furthermore the exercise boundary for the American put equals the inverse of the exercise boundary for the American call with characteristics (S, δ, r) multiplied by the product SK.

Some intuition for this result rests on the properties of normal distributions. In models with constant coefficients (r, δ, σ) the value of put and call options can be expressed in terms of the cumulative normal distribution. Combining the symmetry of the normal distribution with the symmetry of the put and call payoffs leads to the relationship between the option values and the exercise boundaries.

A priori this intuition may suggest that the property does not extend beyond the financial market model with constant coefficients. As we show next this conjecture turns out to be incorrect.

3 Put-Call Symmetry with Ito Price Processes.

In this section we demonstrate that a version of PCS holds under fairly general financial market conditions. The key to the approach is the adoption of the stock as a new numeraire. Changes of numeraire have been discussed thoroughly in the literature, in particular in Geman, El Karoui and Rochet (1995). The extension of options' symmetry properties to general uncertainty structures based on this change of numeraire is due to Schroder (1999). This section considers a special case of Schroder, namely a market with Brownian filtration.

Suppose an economy with finite time period $[0,T]$, a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_{(\cdot)}$. A Brownian motion process z is defined on $(\Omega, \mathcal{F})$ and takes values in $\mathbb{R}$. The filtration is the natural filtration generated by z and $\mathcal{F}_T = \mathcal{F}$.

The financial market has a stochastic opportunity set and nonmarkovian price dynamics. The underlying asset price follows the Ito process,

$$dS_t = S_t[(r_t - \delta_t)dt + \sigma_t d\widetilde{z}_t], t \in [0,T]; S_0 \text{ given} \tag{13}$$

under the Q-measure. The interest rate r, the dividend rate δ and the volatility coefficient σ are progressively measurable and bounded processes of the Brownian filtration $\mathcal{F}_{(\cdot)}$ generated by the underlying Brownian motion process z. The process $\widetilde{z}$ is a Q-Brownian motion.

At various stages of the analysis we will also be led to consider an alternative financial market with interest rate δ, in which the underlying asset price S^* satisfies

$$dS_t^* = S_t^*[(\delta_t - r_t)dt + \sigma_t dz_t^*], t \in [0,T]; S_0^* \text{ given} \tag{14}$$

under some risk neutral measure Q^*. In this market the asset has dividend rate r and volatility coefficient σ. The process z^* is a Brownian motion under the pricing measure Q^*. Both z^* and Q^* will be specified further as we proceed.

We first state a relationship between the values of European puts and calls in the general financial market model under consideration.

Theorem 3 *(generalized European PCS). Consider a European put option with characteristics K and T written on an asset with price S given by (13) in the market with stochastic interest rate r. Let $p(S, K, r, \delta; \mathcal{F}_t)$ denote the put price process. Then*

$$p(S_t, K, r, \delta; \mathcal{F}_t) = c(S_t^*, S, \delta, r; \mathcal{F}_t) \tag{15}$$

where $c(S_t^*, S, \delta, r; \mathcal{F}_t)$ is value of a call with strike price $S = S_t$ and maturity date T in a financial market with interest rate δ and in which the underlying asset price follows the Ito process (14) for $v \in [t,T]$ with initial value $S_t^* = K$ and with z^* defined by

$$dz_v^* = -d\widetilde{z}_v + \sigma_v dv \tag{16}$$

for $v \in [0,T]$, with $z_0^* = 0$.

This result extends the PCS property of the previous section to nonmarkovian economies with Ito price processes and progressively measurable interest rates. The key behind this general equivalence is a change of measure, detailed in the proof, which converts a put option in the original economy into a call option with symmetric characteristics in the auxiliary economy. Note that the equivalence is obtained by switching (S, K, r, δ) to (S^*, S, δ, r), but keeping the trajectories of the Brownian motion the same, i.e. the filtration which is used to compute the value of the call in the auxiliary financial market is the one generated by the original Brownian motion z. Thus information is preserved across economies. In effect the change of measure creates a new asset whose price is the inverse of the original asset price adjusted by a multiplicative factor which depends only on the initial conditions.

As we shall see below in the context of diffusion models the change of measure is instrumental in proving the symmetry property without placing restrictions on the volatility coefficient.

Proof of Theorem 3: In the original financial market the value $p_t \equiv p(S_t, K, r, \delta; \mathcal{F}_t)$ of the put option with characteristics (K, T) has the (present value) representation

$$p_t = \widetilde{E}\left[\exp(-\int_t^T r_v dv)(K - S_t \exp(\int_t^T \alpha_v dv + \int_t^T \sigma_v d\widetilde{z}_v))^+ \mid \mathcal{F}_t\right]$$

where $\alpha \equiv r - \delta - \frac{1}{2}\sigma^2$ and the expectation is taken relative to the equivalent martingale measure Q. Simple manipulations show that the right hand side of this equation equals

$$\widetilde{E}\left[\exp(-\int_t^T (\delta_v + \frac{1}{2}\sigma_v^2)dv + \int_t^T \sigma_v d\widetilde{z}_v)(K\exp(-\int_t^T \alpha_v dv - \int_t^T \sigma_v d\widetilde{z}_v) - S_t)^+ \mid \mathcal{F}_t\right]$$

Consider the new measure

$$dQ^* = \exp(-\frac{1}{2}\int_0^T \sigma_v^2 dv + \int_0^T \sigma_v d\widetilde{z}_v)dQ \tag{17}$$

which is equivalent to Q. Girsanov's Theorem (1960) implies that the process

$$dz_v^* = -d\widetilde{z}_v + \sigma_v dv \tag{18}$$

is a Q^*-Brownian motion. Substituting (18) in the put pricing formula and passing to the Q^*-measure yields

$$p_t = E^*\left[\exp(-\int_t^T \delta_v dv)(K\exp(\int_t^T (\delta_v - r_v - \frac{1}{2}\sigma_v^2)dv + \int_t^T \sigma_v dz_v^*) - S_t)^+ \mid \mathcal{F}_t\right]. \tag{19}$$

But the right hand side is the value of a call option with strike $S = S_t$, maturity date T in an economy with interest rate δ, asset price with dividend rate r and initial value $S_t^* = K$, and pricing measure Q^*. ■

An even stronger version of the preceding result is obtained if the coefficients of the model are adapted to the subfiltration generated by the process z^*. Let $\mathcal{F}_{(\cdot)}^*$ denote the filtration generated by this Q^*-Brownian motion process.

Corollary 4 *Suppose that the coefficients (r, δ, σ) are adapted to the filtration $\mathcal{F}_{(\cdot)}^*$. Then*

$$p(S_t, K, r, \delta; \mathcal{F}_t) = c(S_t^*, S, \delta, r; \mathcal{F}_t^*)$$

where $c(S_t^*, S, \delta, r; \mathcal{F}_t^*)$ is value of a call with strike price $S = S_t$ and maturity date T in a financial market with information filtration $\mathcal{F}_{(\cdot)}^*$ generated by the Q^*-Brownian motion process (16), interest rate δ and in which the underlying asset price follows the Ito process (14) with initial value $S_t^* = K$.

In the context of this Corollary part of the information embedded in the original information filtration generated by the Brownian motion z may be irrelevant for pricing the put option. Since all the coefficients are adapted to the subfiltration generated by z^* this is the only information which matters in computing the expectation under Q^* in (19).

Remark 1 *Note that the standard European PCS in the model with constant coefficients is a special case of this Corollary. Indeed in this setting direct integration over z^* leads to the call value in the auxiliary economy and the put value in the original economy.*

Let us now consider the case of American options. For these contracts early exercise, prior to the maturity date T, is under the control of the holder. At any time prior to the optimal exercise time the put value $P_t \equiv P(S_t, K, r, \delta; \mathcal{F}_t)$ in the original economy is (see Bensoussan (1984) and Karatzas (1988))

$$P_t = \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}\left[\exp(-\int_t^\tau r_v dv)(K - S_t \exp(\int_t^\tau (r_v - \delta_v - \frac{1}{2}\sigma_v^2)dv + \int_t^\tau \sigma_v d\widetilde{z}_v))^+ \mid \mathcal{F}_t\right]$$

where $\mathcal{S}_{t,T}$ denotes the set of stopping times of the filtration $\mathcal{F}_{(\cdot)}$ with values in $[t, T]$. Using the same arguments as in the proof of Theorem (3) we can write

$$P_t = \sup_{\tau \in \mathcal{S}_{t,T}} E^*\left[\exp(-\int_t^\tau \delta_v dv)(K \exp(\int_t^\tau (\delta_v - r_v - \frac{1}{2}\sigma_v^2)dv + \int_t^\tau \sigma_v dz_v^*) - S_t)^+ \mid \mathcal{F}_t\right]$$

where the expectation is relative to the equivalent measure Q^* and conditional on the information $\mathcal{F}_t$. Since the change of measure performed does not affect the set of stopping times over which the holder optimizes the following result holds.

Theorem 5 *(generalized American PCS). Consider an American put option with characteristics K and T written on an asset with price S given by (13) in the market with stochastic interest rate r. Let $P(S, K, r, \delta; \mathcal{F}_t)$ denote the American put price process and $\tau^p(K, r, \delta)$ the optimal exercise time. Then, prior to exercise, the put price is*

$$P(S_t, K, r, \delta; \mathcal{F}_t) = C(S_t^*, S, \delta, r; \mathcal{F}_t) \tag{20}$$

where $C(S_t^*, S, \delta, r; \mathcal{F}_t)$ is the value of an American call with strike price $S = S_t$ and maturity date T in a financial market with interest rate δ and in which the underlying asset price follows the Ito process (14) with initial value $S_t^* = K$ and with z^* defined by (16)*. The optimal exercise time for the put option is*

$$\tau^p(S, K, r, \delta) = \tau^c(K, S, \delta, r) \tag{21}$$

where $\tau^c(S, \delta, r)$ denotes the optimal exercise time for the call option.

Remark 2 *Consider the model with constant coefficients (r, δ, σ). In this setting the optimal exercise time for the call option in the auxiliary financial market is*

$$\tau^c(K, S, \delta, r) = \inf\{t \in [0, T] : K \exp((\delta - r - \frac{1}{2}\sigma^2)t + \sigma z_t^*) = B^c(S, \delta, r, t)\}$$

On the other hand the optimal exercise time for the put option in the original financial market is

$$\tau^p(S, K, r, \delta) = \inf\{t \in [0, T] : S \exp((r - \delta - \frac{1}{2}\sigma^2)t + \sigma \widetilde{z}_t) = B^p(K, r, \delta, t)\}$$

where $B^p(K, r, \delta, t)$ is the put exercise boundary. Using the definition of z^* in (16) we conclude immediately that

$$B^p(K, r, \delta, t) = \frac{SK}{B^c(S, \delta, r, t)}.$$

3.1 Diffusion Financial Market Models.

Suppose that the stock price satisfies the stochastic differential equation

$$dS_t = S_t[(r(S_t,t) - \delta(S_t,t))dt + \sigma(S_t,t)d\tilde{z}_t], t \in [0,T]; S_0 \text{ given} \tag{22}$$

under the Q-measure. In this market the interest rate r may depend on the stock price and along with the other coefficients of (22) satisfies appropriate Lipschitz and Growth conditions for the existence of a unique strong solution (see Karatzas and Shreve (1988)). We assume that the solution is continuous relative to the initial conditions.

Since this markovian financial market is a special case of the general model of the previous section PCS holds. However, in the model under consideration the exercise regions of options have a simple structure which leads to a clear comparison between the put and the call exercise policies.

Define the discount factor

$$R_{t,s} = \exp(-\int_t^s r(S_v,v)dv)$$

for $t,s \in [0,T]$ and the Q-martingale

$$M_{t,s} \equiv \exp(-\frac{1}{2}\int_t^s \sigma(S_v,v)^2 dv + \int_t^s \sigma(S_v,v)d\tilde{z}_v)$$

for $t,s \in [0,T], s \geq t$.

Consider an American call option and let $\mathcal{E}$ denote the exercise set. Continuity of the strong solution of (22) relative to the initial conditions implies that the option price is continuous and that the exercise region is a closed set. Thus we can meaningfully define its boundary B^c.[2] Let $\mathcal{E}(t)$ denote the t-section of the exercise region. The EEP representation for a call option with strike K and maturity date T is

$$C(S_t,K,r,\delta,t,B^c(\cdot)) = c(S_t,K,r,\delta,t) + \pi(S_t,K,r,\delta,t,B^c(\cdot)) \tag{23}$$

where $C(S,K,r,\delta,t,B^c(\cdot))$ is the value of the American call, $c(S,K,r,\delta,t)$ represents the value of the European call

$$c(S_t,K,r,\delta,t) = \tilde{E}[(S_t \exp(-\int_t^T \delta(S_v,v)dv)M_{t,T} - KR_{t,T})^+ \mid S_t] \tag{24}$$

and $\pi_t \equiv \pi(S_t,K,r,\delta,t,B^c(\cdot))$ is the early exercise premium

$$\pi_t = \tilde{E}\left[\int_t^T (\delta(S_v,v)S_t \exp(-\int_t^s \delta(S_v,v)dv)M_{t,s} - r(S_s,s)KR_{t,s})1_{\{S_s \in \mathcal{E}(s)\}}ds \mid S_t\right]. \tag{25}$$

In these expressions dependence on r and δ is meant to represent dependence on the functional form of $r(\cdot)$ and $\delta(\cdot)$. The boundary $B^c(\cdot)$ of the exercise set for the call option solves the recursive integral equation

$$B^c_t - K = C(B^c_t,K,r,\delta,t,B^c(\cdot)) \tag{26}$$

subject to the boundary condition $B^c_T = \max(K, \frac{r(B^c_T,T)}{\delta(B^c_T,T)}K)$. Let $B^c(K,r,\delta,t)$ denote the solution. The optimal exercise policy for the call is to exercise at the stopping time

$$\tau^c(S,K,r,\delta) = \inf\{t \in [0,T] : SR^{-1}_{0,t}\exp(-\int_0^t \delta(S_v,v)dv)M_{0,t} = B^c(K,r,\delta,t)\}. \tag{27}$$

In this context put-call symmetry leads to

[2] If the exercise region is up-connected the exercise boundary is unique. Failure of this property may imply the existence of multiple boundaries.

Proposition 6 *Consider an American put option with characteristics K and T written on an asset with price S given by (22) in the market with interest rate $r(S,t)$. Let $P(S,K,r,\delta,t)$ denote the American put price process and $\tau^p(S,K,r,\delta)$ the optimal exercise time. Then, prior to exercise, the put price is*

$$P(S_t,K,r,\delta,t) = C(S_t^*,S,\delta,r,t) \tag{28}$$

where $C(S_t^*,S,\delta,r;t)$ is value of an American call with strike price $S=S_t$ and maturity date T in a financial market with stochastic interest rate δ and in which the underlying asset price S^* satisfies the stochastic differential equation

$$dS_v^* = S_v^*\left[\left(\delta(\frac{1}{S_v^*},v) - r(\frac{1}{S_v^*},v)\right)dv + \sigma(\frac{1}{S_v^*},v)dz_v^*\right],\ \textit{for } v\in[t,T] \tag{29}$$

with initial value $S_t^*=K$ and with z^* defined by (16). *The optimal exercise time for the put option is $\tau^p(S,K,r,\delta)=\tau^c(K,S,\delta,r)$ and the exercise boundaries are related by*

$$B^p(K,r,\delta,t) = \frac{SK}{B^c(S,\delta,r,t)} \tag{30}$$

In the financial market setting of (22) all the information relevant for future payoffs is embedded in the current stock price. Any strictly monotone transformation of the price is also a sufficient statistic. Thus the passage from the original economy to the auxiliary economy with stock price (29) preserves the information required to price derivatives with future payoffs. No information beyond the current price S_t^* is required to assess the correct evolution of the coefficients of the underlying asset price process. This stands in contrast with the general model with Ito price processes in which the path of the Brownian motion needs to be recorded in the auxiliary economy for proper evaluation of future distributions.

Note also that the change of measure converts the original underlying asset into a symmetric asset with inverse price up to a multiplicative factor depending only on the initial conditions. Since the change of measure can be performed independently of the structure of the coefficients the results are valid even in the absence of symmetry-like restrictions on the volatility coefficient.

Proof of Proposition 6: The first part of the proposition follows from Theorem 5. To prove the relationship between the exercise boundaries note that the call boundary at maturity equals

$$B^c = \max(K,b^c)$$

where b^c solves the nonlinear equation

$$r(\frac{SK}{b^c},T)b^c - \delta(\frac{SK}{b^c},T)S = 0.$$

In this expression we used the relation $S_T = \frac{SK}{S_T^*}$. Now with the change of variables $b^p = \frac{SK}{b^c}$ it is clear that b^p solves

$$r(b^p,T)K - \delta(b^p,T)b^p = 0$$

and that the put boundary at the maturity date satisfies (30). To establish the relation prior to the maturity date it suffices to use the recursive integral equation for the call boundary, pass to the Q^*-measure and perform the change of variables indicated. The resulting expression is the recursive integral equation for the put boundary. ■

The results in this section can be easily extended to multivariate diffusion models (S,Y) where Y is a vector of state variables impacting the coefficients of the underlying asset price process. Passage to the measure Q^*, in this case, introduces a risk premium correction in the state variables processes. Multivariate models in that class are discussed extensively in Schroder (1999).

4 Options with Random Expiration Dates.

We now consider a class of American derivatives which mature automatically if certain prespecified conditions are satisfied. Let τ_l denote a stopping time of the filtration and let $H = \{H_t : t \in [0,T]\}$ denote a progressively measurable process. A call option with maturity date T, strike K, automatic liquidation time τ_l and liquidation payoff H pays $(S-K)^+$ if exercised by the holder at date $t < \tau_l$. If τ_l materializes prior to T the option automatically matures and pays off H_{τ_l}. A random maturity put option with characteristics (K, T, τ_l, H) has similar provisions but pays $(K-S)^+$ if exercised prior to the automatic liquidation time τ_l. Options with such characteristics are referred to as *random maturity options.*

Popular examples of such contracts are barrier options such as down and out put options and up and out call options. Both of these contracts become worthless when the underlying asset price reaches a prespecified level L (i.e. the liquidation payoff is a constant $H = 0$).

Another example is an American capped call option with automatic exercise at the cap L. This option is automatically liquidated at the random time

$$\tau_l = \tau_L \equiv \inf\{t \in [0,T] : S_t = L\}$$

or $\tau_L = \infty$ if no such time materializes in $[0,T]$ and pays off the constant $H = L - K$ in that event. If $\tau_L > T$ the option payoff is $(S-K)^+$.[3] Capped options with growing caps and automatic exercise at the cap are examples in which the automatic liquidation payoff is time dependent

Consider again the general financial market model with underlying asset price given by (13). Recall the definitions of the discount factor

$$R_{t,s} = \exp(-\int_t^s r_v dv)$$

for $t, s \in [0,T]$ and the Q-martingale

$$M_{t,s} \equiv \exp(-\frac{1}{2}\int_t^s \sigma_v^2 dv + \int_t^s \sigma_v d\tilde{z}_v)$$

for $t, s \in [0,T], s \geq t$.

Let $P_t = P(S, K, T, \tau_l, H, r, \delta; \mathcal{F}_t)$ denote the value of an American random maturity put with characteristics (K, T, τ_l, H). In this financial market the put value is given by

$$P_t = \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}\left[R_{t,\tau}\left(K - S_t R_{t,\tau}^{-1}\exp(-\int_t^\tau \delta_v dv) M_{t,\tau}\right)^+ 1_{\{\tau < \tau_l\}} + R_{t,\tau_l} H_{\tau_l} 1_{\{\tau \geq \tau_l\}} |\mathcal{F}_t\right]$$

Performing the same change of measure as in the previous section enables us to rewrite the put value P_t as

$$\begin{aligned}
&\sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}\left[\exp(-\int_t^\tau \delta_v dv) M_{t,\tau}\left[K R_{t,\tau}\exp(\int_t^\tau \delta_v dv) M_{t,\tau}^{-1} - S_t)^+ 1_{\{\tau < \tau_l\}} + \frac{S_t H_{\tau_l}}{S_{\tau_l}} 1_{\{\tau \geq \tau_l\}}\right] |\mathcal{F}_t\right] \\
&= \sup_{\tau \in \mathcal{S}_{t,T}} E^*\left[\exp(-\int_t^\tau \delta_v dv)\left[(K R_{t,\tau}\exp(\int_t^\tau \delta_v dv) M_{t,\tau}^{-1} - S_t)^+ 1_{\{\tau < \tau_l\}} + H^*_{\tau_l} 1_{\{\tau \geq \tau_l\}}\right] |\mathcal{F}_t]\right]
\end{aligned}$$

where we define the stochastic process H^* as

[3] Note that, in the case of constant cap, an American capped call option without an automatic exercise clause when the cap is reached is indistinguishable from an American capped call option with an automatic exercise provision at the cap but otherwise identical features. It is indeed easy to show that the optimal exercise time for such an option is the minimum of the hitting time of the cap and the optimal exercise time for an uncapped call option with identical features (see Broadie and Detemple (1995) for a derivation of this result in a market with constant coefficients).

$$H_v^* = \frac{S_t H_v}{S_v}$$

for $v \in [t, T]$.

With these transformations it is apparent that the following result holds.

Theorem 7 *(random maturity options PCS). Let τ_l denote a stopping time of the filtration and let $H = \{H_t : t \in [0,T]\}$ be a progressively measurable process. Consider an American random maturity put option with maturity date T, strike K, automatic liquidation time τ_l and liquidation payoff H, written on an asset with price S given by (13) in the market with stochastic interest rate r. Denote the put price by $P(S, K, T, \tau_l, H, r, \delta; \mathcal{F}_t)$ and the optimal exercise time by $\tau^p(S, K, T, \tau_l, H, r, \delta)$. Then, prior to exercise, the put price equals*

$$P(S_t, K, T, \tau_l, H, r, \delta; \mathcal{F}_t) = C(S_t^*, S, T, \tau_l^*, H^*, \delta, r; \mathcal{F}_t) \tag{31}$$

where $C(S_t^*, S, T, \tau_l^*, H^*, \delta, r; \mathcal{F}_t)$ is the value of an American random maturity call with strike price $S = S_t$, maturity date T , automatic liquidation time τ_l^* and liquidation payoff H^* in a financial market with interest rate δ and in which the underlying asset price follows the Ito process (14) with initial value $S_t^* = K$ and with z^* defined by (16). The liquidation payoff is given by

$$H_t^* = \frac{S H_t}{S_t} = \frac{S_t^* H_t}{K}$$

and the liquidation time is $\tau_l^* = \tau_l$. The optimal exercise time for the put option is

$$\tau^p(S, K, \tau_l, H, r, \delta) = \tau^c(K, S, \tau_l^*, H^*, \delta, r) \tag{32}$$

where $\tau^c(K, S, \tau_l^*, H^*, \delta, r)$ denotes the optimal exercise time for the random maturity call option.

Remark 3 *Suppose that the automatic liquidation provision of the random maturity put is defined as*

$$\tau_l = \inf\{t \in [0,T] : S_t \in A\}$$

where A is a closed set in $\mathbb{R}^+$, i.e. τ_l is the hitting time of the set A. Then the liquidation time of the corresponding random maturity call can be expressed in terms of the underlying asset price in the auxiliary market as

$$\tau_l^* = \inf\{t \in [0,T] : S_t^* \in A^*\}$$

where $A^* = \{x \in \mathbb{R}^+ : x = \frac{KS}{y}$ and $y \in A\}$. Given the definition of the process for S^* and the fact that the information filtration is the same in the auxiliary market it is immediate to verify that $\tau_l^* = \tau_l$.

As an immediate corollary of Theorem 7 we get the symmetry property for down and out put options and up and out call options. This generalizes results of Gao, Huang and Subrahmanyam (2000) who consider barrier options when the underlying asset price follows a geometric Brownian motion process.

Corollary 8 *(barrier options PCS). Let $\tau_L = \inf\{t \in [0,T] : S_t = L\}$. Consider an American down and out put option with maturity date T, strike price K and automatic liquidation time τ_L (and liquidation payoff $H = 0$), written on an asset with price S given by (13) in the market with stochastic interest rate r. Prior to exercise or liquidation, the put price equals*

$$P(S_t, K, T, \tau_L, 0, r, \delta; \mathcal{F}_t) = C(S_t^*, S, T, \tau_{L^*}, 0, \delta, r; \mathcal{F}_t) \tag{33}$$

where $C(S_t^*, S, T, \tau_L^*, 0, \delta, r; \mathcal{F}_t)$ is the value of an American up and out call with strike price $S = S_t$, maturity date T and automatic liquidation time τ_{L^*} (and liquidation payoff $H^* = 0$) in a financial market with interest rate δ and in which the underlying asset price follows the Ito process (14) with initial value $S_t^* = K$ and with z^* defined by (16). The liquidation time is

$$\tau_{L^*} = \inf\{t \in [0,T] : S_t^* = L^* \equiv \frac{KS}{L}\}$$

The optimal exercise time for the put option is

$$\tau^p(S, K, \tau_L, 0, r, \delta) = \tau^c(K, S, \tau_{L^*}, 0, \delta, r) \tag{34}$$

where $\tau^c(K, S, \tau_{L^*}, 0, \delta, r)$ denotes the optimal exercise time for the up and out call option.

Another corollary covers the case of American capped put and call options.

Corollary 9 *(capped options PCS). Let $\tau_L = \inf\{t \in [0,T] : S_t = L\}$. Consider an American capped put option with maturity date T, strike price K, cap $L < K$ and automatic liquidation time τ_L (and liquidation payoff $H = K - L$), written on an asset with price S given by (13) in the market with stochastic interest rate r. Prior to exercise, the put price equals*

$$P(S_t, K, T, \tau_L, 0, r, \delta; \mathcal{F}_t) = C(S_t^*, S, T, \tau_{L^*}, 0, \delta, r; \mathcal{F}_t) \tag{35}$$

where $C(S_t^*, S, T, \tau_L^*, 0, \delta, r; \mathcal{F}_t)$ is the value of an American capped call with strike price $S = S_t$, maturity date T, cap $L^* = \frac{KS}{L}$ and automatic liquidation time τ_{L^*} (and liquidation payoff $H^* = L^* - S$) in a financial market with interest rate δ and in which the underlying asset price follows the Ito process (14) with initial value $S_t^* = K$ and with z^* defined by (16). The liquidation time is

$$\tau_{L^*} = \inf\{t \in [0,T] : S_t^* = L^* \equiv \frac{KS}{L}\}$$

The optimal exercise time for the capped put option is

$$\tau^p(S, K, \tau_L, 0, r, \delta) = \tau^c(K, S, \tau_{L^*}, 0, \delta, r) \tag{36}$$

where $\tau^c(K, S, \tau_{L^*}, 0, \delta, r)$ denotes the optimal exercise time for the capped call option.

5 Multiasset Derivatives.

In this section we consider American-style derivatives whose payoffs depend on the values of n underlying asset prices.

The setting is as follows. The underlying filtration is generated by an n-dimensional Brownian motion process z. The price S^j of asset j follows the Ito process

$$dS_t^j = S_t^j[(r_t - \delta_t^j)dt + \sigma_t^j d\tilde{z}_t] \tag{37}$$

where r, δ^j and σ^j are progressively measurable and bounded processes, $j = 1, ..., n$. The financial market is complete, i.e. the volatility matrix σ of the vector of prices is invertible. Let $S = (S^1, ..., S^n)$ denote the vector of prices.

The derivatives under consideration have payoff function $f(S, K)$ with parameter K. In some applications the parameter K can be interpreted as a strike price; in others it represents a cap. We assume that the function f is continuous and homogeneous of degree one in the $n+1$-dimensional vector (S, K). Examples of such contracts are call and put options on the maximum or the minimum of n assets, spread options, exchange options, capped exchange options and options on a weighted average of assets. Capped multiasset options such as capped options on the maximum or minimum of multiple assets are also obtained if K is a vector.

For a constant λ define $\lambda \circ_j S$ as

$$\lambda \circ_j S = (S^1, ..., S^{j-1}, \lambda S^j, S^{j+1}, ..., S^n)$$

i.e. $\lambda \circ_j S$ represents the vector of prices whose j^{th} component has been rescaled by the factor λ. Also for a given f-claim with parameter K and for any j we define the associated f^j-claim obtained by permutation of the j^{th} argument and the parameter

$$f^j(S, K) = f(\lambda^j \circ_j S, S^j)$$

with $\lambda^j = \frac{K}{S^j}$, $j = 1, ..., n$.

For the contracts under consideration the approach of the previous sections applies and leads to the following symmetry results.

Theorem 10 *Consider an American f-claim with maturity date T and a continuous and homogeneous of degree one payoff function $f(S, K)$. Let $V(S, K, r, \delta; \mathcal{F}_t)$ denote the value of the claim in the financial market with filtration $\mathcal{F}_{(\cdot)}$, asset prices S_t satisfying (37) and progressively measurable interest rate r. Pick some arbitrary index j and define*

$$\lambda^j \equiv \frac{K}{S^j} \text{ and } \lambda^j(\delta) \equiv \frac{r}{\delta^j}.$$

Prior to exercise the value of the claim is

$$V(S_t, K, r, \delta; \mathcal{F}_t) = V^j(S_t^*, S^j, \delta^j, \lambda^j(\delta) \circ_j \delta; \mathcal{F}_t)$$

where $V^j(S_t^*, S^j, \delta^j, \lambda^j(\delta) \circ_j \delta; \mathcal{F}_t)$ is the value of the f^j-claim with parameter S^j and maturity date T in an auxiliary financial market with interest rate δ^j and in which the underlying asset prices follow the Ito processes

$$\begin{cases} dS_v^{i*} = S_v^{i*}[(\delta_v^j - \delta_v^i)dv + (\sigma_v^j - \sigma_v^i)dz_v^{j*}]; \text{ for } i \neq j \text{ and } v \in [t, T] \\ dS_v^{j*} = S_v^{j*}[(\delta_v^j - r_v)dv + \sigma_v^j dz_v^{j*}]; \text{ for } i = j \text{ and } v \in [t, T] \end{cases}$$

with respective initial conditions $S_t^{i*} = S^i$ for $i \neq j$ and $S_t^{j*} = K$ for $i = j$. The process z^{j*} is defined by

$$dz_v^{j*} = -d\widetilde{z}_v + \sigma_v^{j'} dv, \text{ for } v \in [0, T]; z_0^{j*} = 0.$$

The optimal exercise time for the f-claim is the same as the optimal exercise time for the f^j-claim in the auxiliary financial market.

Theorem 10 is a natural generalization of the one asset case. It establishes a symmetry property between a claim with homogeneous of degree one payoff in the original financial market and related claims whose payoffs are obtained by permutation of the original one in auxiliary financial markets $j = 1, ..., n$. In the j^{th} auxiliary market the interest rate is the dividend rate of asset j in the original economy, the dividend rate of asset i is δ^i for $i \neq j$ and r for asset j, and the volatility coefficients of asset prices are $\sigma^j - \sigma^i$ for $i \neq j$ and σ^j for asset j. The initial (date t) value of asset j is the payoff parameter K of the f-claim under consideration. Clearly the results of the previous sections are recovered when we specialize the payoff function to the earlier cases considered.

Proof of Theorem 10: Define $S^j = S_t^j$. Proceeding as in section 2 we can write the value of the contract

$$V(S_t, K, r, \delta; \mathcal{F}_t) \quad = \quad \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}\left[\exp(-\int_t^\tau r_v dv) f(S_\tau, K) \,|\mathcal{F}_t\right]$$

$$
\begin{aligned}
&= \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}\left[\exp(-\int_t^\tau r_v dv)\frac{S_\tau^j}{S^j} f(S_\tau \frac{S^j}{S_\tau^j}, K\frac{S^j}{S_\tau^j}) \,|\mathcal{F}_t\right] \\
&= \sup_{\tau \in \mathcal{S}_{t,T}} E^{j*}\left[\exp(-\int_t^\tau \delta_v^j dv) f(S_\tau \frac{S^j}{S_\tau^j}, S_\tau^{j*}) \,|\mathcal{F}_t\right] \\
&= \sup_{\tau \in \mathcal{S}_{t,T}} E^{j*}\left[\exp(-\int_t^\tau \delta_v^j dv) f^j(S_\tau^*, S^j) \,|\mathcal{F}_t\right] \\
&= V^j(S_t^*, S^j, \delta^j, \lambda^j(\delta) \circ_j \delta; \mathcal{F}_t)
\end{aligned}
$$

The second equality above uses the homogeneity property of the payoff function, the third is based on the definition $S_\tau^{j*} = K\frac{S^j}{S_\tau^j}$ and the passage to the measure Q^{j*} and the fourth relies on the definition of the permuted payoff f^j. The final equality uses the definition of the value function V^j.

To complete the proof of the theorem it suffices to use Ito's lemma to identify the dynamics of the asset prices in the auxiliary economy. This leads to the processes stated in the theorem. ■

The interest of the Theorem becomes apparent when we specialize the payoff function to familiar ones. The following results are valid.

1. Call max-option on two assets $(f(S^1, S^2, K) = (\max(S^1, S^2) - K)^+)$: One symmetric contract is an option to exchange the maximum of an asset and cash against another asset (or, equivalently, an exchange option with put floor) whose payoff is

$$f^2(S^{1*}, S^{2*}, K') = (\max(S^{1*}, K') - S^{2*})^+ = (S^{1*} - S^{2*})^+ \vee (K' - S^{2*})^+$$

where $K' = S^2$ in the auxiliary financial market obtained by taking $j = 2$ as reference. A similar contract emerges if $j = 1$ is taken as reference. The Theorem implies that the valuation of any one of these contracts is obtained by a simple reparametrization of the values of the symmetric contracts.

2. Exchange option on two assets $(f(S^1, S^2) = (S^1 - S^2)^+)$: A symmetric contract is a standard call option with payoff

$$f^2(S^{1*}, K') = (S^{1*} - K')^+$$

and $K' = S^2$ in auxiliary market $j = 2$ in which S^{1*} satisfies

$$dS_t^{1*} = S_t^{1*}[(\delta_t^2 - \delta_t^1)dt + (\sigma_t^2 - \sigma_t^1)dz_t^{2*}] = S_t^{1*}[(\delta_t^2 - \delta_t^1)dt + (\sigma_{1t}^2 - \sigma_{1t}^1)dz_{1t}^{2*} + (\sigma_{2t}^2 - \sigma_{2t}^1)dz_{2t}^{2*}].$$

In the second equality we used $\sigma^i = (\sigma_1^i, \sigma_2^i)$, for $i = 1, 2$. Bjerksund and Stensland (1993) prove this result for financial markets with constant coefficients using PDE methods (see also Rubinstein (1991) for a proof in a binomial setting and Broadie and Detemple (1997) for a proof based on the EEP representation). The case of European options is treated in Margrabe (1978). Our Theorem establishes the validity of this symmetry in a much broader setting. The second symmetric contract is a standard put option with strike price $K' = S^1$ in auxiliary market $j = 1$.

3. Capped exchange option with proportional cap $(f(S^1, S^2) = LS^2 \wedge (S^1 - S^2)^+)$: In this instance one symmetric contract (in auxiliary financial market $j = 2$) is a capped call option with constant cap whose payoff is

$$f^2(S^{1*}, K') = LK' \wedge (S^{1*} - K')^+$$

where $K' = S^2$. The Theorem thus provides a simple and immediate proof of this result derived in Broadie and Detemple (1997) for models with constant coefficients. Alternatively we can also consider the symmetric contract in the auxiliary market $j = 1$. We find the payoff

$$f^1(K', S^{2*}) = LS^{2*} \wedge (K' - S^{2*})^+,$$

with $K' = S^1$. In other words the capped exchange option with proportional cap is symmetric to a put option with proportional cap in the market in which asset 1 is chosen as the numeraire.

4. Capped exchange option with constant cap ($f(S^1, S^2, K) = (S^1 \wedge K - S^2)^+$): The symmetric contract in any auxiliary market $j = 2$ is a call option on the minimum of two assets with payoff

$$f^2(S^{1*}, S^{2*}, K') = (S^{1*} \wedge S^{2*} - K')^+$$

where $K' = S^2$. An analysis of min-options in the context of the model with constant coefficients is carried out in Detemple and Tian (1998).

5. The symmetry relations of theorem 10 also apply to multiasset derivatives whose payoffs are homogeneous of degree one relative to a subset of variables. An interesting example is provided by quantos. These are derivatives written on foreign asset prices or indices but whose payoff is denominated in domestic currency. For instance a quanto call option on the Nikkei pays off $(S - K)^+$ dollars at the exercise time where S is the value of the Nikkei quoted in Yen. The payoff in foreign currency is $e(S - K)^+$ where e is the Y/\$ exchange rate. From the foreign perspective the contract is homogeneous of degree $\nu = 2$ in the triplet (e, S, K). However, for interpretation purposes it is more advantageous to treat it as a contract homogeneous of degree $\nu = 1$ in the exchange rate e. If r_f denotes the foreign interest rate and the dividend rate on the index is δ the American quanto call is valued at

$$C_t^Q = \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}^f \left[\exp(-\int_t^\tau r_v^f dv) e_\tau (S_\tau - K)^+ \,|\mathcal{F}_t \right]$$

in Yen where the expectation is taken relative to the foreign risk neutral measure and

$$\begin{cases} dS_t = S_t[(r_t^f - \delta_t)dt + \sigma_t d\widetilde{z}_t^f] \\ de_t = e_t[(r_t^f - r_t)dt + \sigma_t^e d\widetilde{z}_t^f]. \end{cases}$$

Here r is the domestic interest rate and σ, σ^e are the volatility coefficients of the foreign index and the exchange rate. The process $\widetilde{z}^f$ is a 2-dimensional Brownian motion relative to the foreign risk neutral measure. Using the exchange rate as new numeraire yields

$$C_t^Q = \sup_{\tau \in \mathcal{S}_{t,T}} E^{f*} \left[\exp(-\int_t^\tau r_v dv)(S_\tau - K)^+ \,|\mathcal{F}_t \right]$$

where

$$dS_t = S_t \left[(r_t^f - \delta_t + \sigma_t \sigma_t^{e\prime})dt - \sigma_t dz_t^{f*} \right].$$

Hence, from the foreign perspective the quanto call option is symmetric to a standard call option on an asset paying dividends at the rate $\delta - \sigma\sigma^{e\prime}$ in an auxiliary financial market with interest rate r. Similarly a quanto forward contract is symmetric to a standard forward contract in the same auxiliary financial market. The forward price is

$$F_t = \frac{E^{j*}\left[\exp(-\int_t^\tau r_v dv) S_\tau \,|\mathcal{F}_t\right]}{E^{j*}\left[\exp(-\int_t^\tau r_v dv) \,|\mathcal{F}_t\right]}.$$

For the case of constant coefficients $F_t = S_t \exp((r^f - \delta + \sigma\sigma^{e\prime})(T - t))$. Alternative representations for these prices can be derived by using the homogeneity of degree 2 relative to (e, S, K); they are discussed in section 7.

6. Lookback options: The exercise payoff depends on an underlying asset value and its sample path maximum or minimum. A lookback put pays off $f(S_v, M_v) = (M_v - S_v)^+$ where $M_v =$

$\sup_{s\in[0,v]} S_s$; the lookback call payoff is $f(S_v, m_v) = (S_v - m_v)^+$ where $m_v = \inf_{s\in[0,v]} S_s$. Even though there is only one underlying asset the contract depends on 2 state variables, namely the underlying asset price and one of its sample path statistics. Since renormalizations do not affect the order of a sample path statistic it is easily verified that the lookback call is symmetric to a put option on the minimum of the price expressed in a new numeraire $(S - m_v^*)$ where $m_v^* = (S/S_v) \inf_{s\in[0,v]} S_s = \inf_{s\in[0,v]}(SS_s/S_v)$. Likewise, a lookback put is related to a call option on the maximum of the price expressed in a new numeraire. European Lookback option pricing is discussed in Goldman, Sosin and Gatto (1979) and Garman (1989) in the context of the model with constant coefficients. Similar symmetry relations can be established for average options (Asian options).

6 Occupation Time Derivatives.

An occupation time derivative is a derivative whose payoff has been modified to reflect the time spent by the underlying asset price in certain regions of space. Various special cases have been considered in the recent literature such as Parisian and Cumulative Barrier options (Chesney, Jeanblanc-Picque and Yor (1997)), Step options (Linetsky (1999)) and Quantile options (Miura (1992), Akahori (1995), Dassios (1995)). The general class of occupation time claims is introduced by Hugonnier (1998) who discusses their valuation and hedging properties. So far the literature has focused exclusively on European-style derivatives when the underlying asset follows a geometric Brownian motion process. In this section we provide symmetry results applying to both European and American-style contracts and when the underlying asset follows an Ito process. Extensions to multiasset occupation time derivatives are also discussed.

We consider an American occupation time f-claim with exercise payoff

$$f(S, K, O^{S,A})$$

at time t, where S satisfies the Ito process (1), K is a constant representing a strike price or a cap and $O^{S,A}$ is an occupation time process defined by

$$O_t^{S,A} = \int_0^t 1_{\{S_v \in A_v\}} dv, \quad t \in [0, T].$$

for some random, progressively measurable, closed set $A(\cdot,\cdot) : [0,T] \times \Omega \to \mathcal{B}(\mathbb{R}^+)$. Thus $O_t^{S,A}$ represents the amount of time spent by S in the set A during the time interval $[0,t]$. Examples treated in the literature involve occupation times of constant sets of the form $A = \{x \in \mathbb{R}^+ : x \geq L\}$ or $A = \{x \in \mathbb{R}^+ : x \leq L\}$ with L constant, which represent time spent above or below a constant barrier L. Simple generalizations of these are when the barrier L is a function of time or a progressively measurable stochastic process.

The value of this American claim is

$$V(S_t, K, O^{S,A}, r, \delta; \mathcal{F}_t) = \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}\left[\exp(-\int_t^\tau r_v dv) f(S_\tau, K, O_\tau^{S,A}) \mid \mathcal{F}_t\right].$$

Assume that the claim is homogeneous of degree one in (S, K). Then we can perform the usual change of measure and obtain

Theorem 11 *Consider an American occupation time f-claim with maturity date T and a payoff function $f(S, K, O)$ which is homogeneous of degree one with respect to (S, K). Let $V(S, K, O^{S,A}, r, \delta; \mathcal{F}_t)$ denote the value of the claim in the financial market with filtration $\mathcal{F}_{(\cdot)}$, asset price S satisfying (1) and progressively measurable interest rate r. Prior to exercise the value of the claim is*

$$V(S_t, K, O^{S,A}, r, \delta; \mathcal{F}_t) = V^1(S_t^*, S, O^{S^*,A^*}, \delta, r; \mathcal{F}_t)$$

where $A^* = \{A^*(v,\omega), v \in [t,T]\}$ *with* $A^*(v,\omega) = \{x \in \mathbb{R}^+ : x = \frac{KS}{y}$ *and* $y \in A(v,\omega)\}$ *and* $O_t^{S^*,A^*} \equiv O_t^{S,A}$. *Also* $V^1(S_t^*, S, O^{S^*,A^*}, \delta, r; \mathcal{F}_t)$ *is the value of the permuted claim* $f^1(S_t^*, S, O_t^{S^*,A^*}) = f(S, K\frac{S}{S_t}, O_t^{S^*,A^*})$ *with parameter* $S = S_t$, *occupation time* $O_t^{S^*,A^*}$, *and maturity date* T *in an auxiliary financial market with interest rate* δ *and in which the underlying asset price follows the Ito process*

$$dS_v^* = S_v^*[(\delta_v - r_v)dt + \sigma_v dz_v^*], \quad \text{for } v \in [t,T]$$

with initial condition $S_t^* = K$. *The process* z^* *is defined by* $dz_v^* = -d\tilde{z}_v + \sigma_v dv, v \in [0,T], z_0^* = 0$. *The optimal exercise time for the* f*-claim is the same as the optimal exercise time for the* f^1*-claim in the auxiliary financial market.*

Proof of Theorem 11: Fix $t \in [0,T]$ and set $O_t^{S,A} = O_t^{S^*,A^*}$. For any stopping time $\tau \in \mathcal{S}_{t,T}$ the occupation time can be written

$$O_\tau^{S,A} = O_t^{S,A} + \int_t^\tau 1_{\{S_v \in A_v\}} dv = O_t^{S^*,A^*} + \int_t^\tau 1_{\{S_v^* \in A_v^*\}} dv = O_\tau^{S^*,A^*}$$

where $S_v^* = KS/S_v, v \in [t,T]$ and $O_\tau^{S^*,A^*}$ denotes the occupation time of the random set A^* by the process S^*. Performing the change of measure leads to the results. ■

Special cases of interest are as follows.

1. Parisian options (Chesney, Jeanblanc-Picque and Yor (1997)): Let $g(L,t) = \sup\{s \le t : S_s = L\}$ denote the last time the process S has reached the barrier L (if no such time exists set $g(L,t) = t$) and consider the random time

$$O_t^{S,A^+(t,L)} = \int_{g(L,t)}^t 1_{\{S_v \ge L\}} dv = \int_0^t 1_{\{(v,S_v) \in A^+(t,L)\}} dv$$

where $A^+(t,L) = \{(v,S) : v \ge g(L,t), S \ge L\}$. Note that $O_t^{S,A^+(t,L)}$ measures the age of a current excursion above the level L. A Parisian up and out call with window D has null payoff as soon as an excursion of age D *above* L takes place. If no such event occurs prior to exercise the exercise payoff is $(S-K)^+$. A Parisian down and out call with window D looses all value if there is an excursion of length D *below* the prespecified level L. Parisian put options are similarly defined. Fix $t \in [0,T]$ and suppose that no excursion of age D has occurred before t. The symmetry relation for Parisian options can be stated as

$$C(S_t, K, O_t^{S,A^+(t,L)}, D, r, \delta; \mathcal{F}_t) = P(S_t^*, S, O_t^{S^*,A^-(t,KS/L)}, D, \delta, r; \mathcal{F}_t). \tag{38}$$

This follows from $g(L,t) = \sup\{s \le t : S_s = L\} = \sup\{s \le t : S_s^* = KS/L\} = g^*(KS/L)$ and

$$O_t^{S,A^+(t,L)} = \int_{g(L,t)}^t 1_{\{KS/L \ge KS/S_v\}} dv = \int_{g^*(KS/L,t)}^t 1_{\{KS/L \ge S_v^*\}} dv = O_t^{S^*,A^-(t,KS/L)},$$

with $A^-(t,KS/L) = \{(v,S^*) : v \ge g(KS/L,t), KS/L \ge S^*\}$, which ensures that the stopping times

$$H_t(L,D) = \inf\{v \in [t,T] : O_v^{S,A^+(v,L)} \ge D\}, \quad \text{and}$$
$$H_t^*(KS/L,D) = \inf\{v \in [t,T] : O_v^{S^*,A^-(v,KS/L)} \ge D\}$$

at which the call and put options loose all value coincide. In summary a Parisian up and out call with window D has the same value as a Parisian down and out put with window D, strike $S = S_t$, occupation time $O_t^{S^*,A^-(t,KS/L)}$, and maturity date T in an auxiliary financial market with interest rate δ and in which the underlying asset price follows the Ito process described in theorem 11. Chesney, Jeanblanc-Picque and Yor derive this symmetry property for European Parisian options in a financial market with constant coefficients. In this context they also provide valuation formulas for such contracts involving Laplace transforms.

2. Cumulative (Parisian) Barrier options (Chesney, Jeanblanc-Picque and Yor (1997)): The contract payoff is affected by the (cumulative) amount of time spent above or below a constant barrier L. For instance let $A^{\pm}(L) = \{x \in \mathbb{R}^+ : (x - L)^{\pm} \geq 0\}$ and consider a call option that pays off if the amount of time spent above L exceeds some prespecified level D (up and in call). The following symmetry result applies:

$$C(S_t, K, O_t^{S,A^+(L)}, D, r, \delta; \mathcal{F}_t) = P(S_t^*, S, O_t^{S^*,A^-(KS/L)}, D, \delta, r; \mathcal{F}_t). \tag{39}$$

Here the left hand side is the value of the cumulative barrier call with payoff $(S-K)^+ 1_{\{O^{S,A^+(L)} \geq D\}}$ in the original economy; the right hand side is the value of a cumulative barrier put option with payoff $(S - S^*)^+ 1_{\{O^{S^*,A^-(KS/L)} \geq D\}}$ in an auxiliary economy with interest rate δ, dividend r and asset price process S^*. Chesney, Jeanblanc-Picque and Yor (1997) and Hugonnier (1998) examine the valuation of European cumulative barrier options when the underlying asset price follows a Geometric Brownian motion process. European cumulative barrier digital calls and puts satisfy similar symmetry relations and are discussed by Hugonnier. An analysis of these contracts is relegated to the next section since their payoffs are homogeneous of degree zero.

3. Step options (Linetsky (1999)): A step option is discounted at a rate which depends on the occupation time of a set. For instance the step call option payoff is $(S - K)^+ \exp(-\rho O_t^{S,A^{\pm}(L)})$ for some $\rho > 0$ where $A^{\pm}(L)$ is defined above. Again the PCS relation (39) holds in this case. Put and call step options are special cases of the occupation time derivatives in which the payoff function involves exponential discounting. Closed form solutions are provided by Linetsky for Geometric Brownian motion price process.

Occupation time derivatives can be easily generalized to the multiasset case. For a progressively measurable stochastic closed set $A \in \mathbb{R}_+^n$ and a vector of asset prices $S \in \mathcal{B}(\mathbb{R}_+^n)$ a multiasset f-claim has payoff $f(S, K, O^{S,A})$ where

$$O_t^{S,A} = \int_0^t 1_{\{S_v \in A_v\}} dv, \quad t \in [0, T].$$

A natural generalization of theorem 10 is

Theorem 12 *Consider an American occupation time f-claim with maturity date T and a payoff function $f(S, K, O^{S,A})$ which is homogeneous of degree one in (S, K). Let $V(S, K, O^{S,A}, r, \delta; \mathcal{F}_t)$ denote the value of the claim in the financial market with filtration $\mathcal{F}_{(\cdot)}$, asset prices S satisfying (37) and progressively measurable interest rate r. Pick some arbitrary index j and define*

$$\lambda^j \equiv \frac{K}{S^j} \text{ and } \lambda^j(\delta) \equiv \frac{r}{\delta^j}.$$

Prior to exercise the value of the multiasset occupation time f-claim is

$$V(S_t, K, O^{S,A}, r, \delta; \mathcal{F}_t) = V^j(S_t^*, S^j, O^{S^*,A^*}, \delta^j, \lambda^j(\delta) \circ_j \delta; \mathcal{F}_t)$$

where $A^* = \{A^*(v, \omega), v \in [t, T]\}$ with $A^*(v, \omega) = \{x \in \mathbb{R}_+^n : x_i = \frac{y_i S}{y_j}$, for $i \neq j, x_j = \frac{KS}{y_j}$ and $y = (y_1, ..., y_n) \in A(v, \omega)\}$ and $O_t^{S^*,A^*} \equiv O_t^{S,A}$. Also $V^j(S_t^*, S^j, O_t^{S^*,A^*}, \delta^j, \lambda^j(\delta) \circ_j \delta; \mathcal{F}_t)$ is the value of the f^j-claim with parameter $S^j = S_t^j$, maturity date T and occupation time $O_t^{S^*,A^*}$ in an auxiliary financial market with interest rate δ^j and in which the underlying asset prices follow the Ito processes

$$\begin{cases} dS_v^{i*} = S_v^{i*}[(\delta_v^j - \delta_v^i)dv + (\sigma_v^j - \sigma_v^i)dz_v^{j*}]; \text{ for } i \neq j \text{ and } v \geq t \\ dS_v^{j*} = S_v^{j*}[(\delta_v^j - r_v)dv + \sigma_v^j dz_v^{j*}]; \text{ for } i = j \text{ and } v \geq t \end{cases}$$

with respective initial conditions S^i for $j \neq i$ and K for $j = i$. The process z^{j*} is defined by

$$dz_v^{j*} = -d\widetilde{z}_v + \sigma_v^{j'} dv$$

for all $v \in [0,T], z_0^{j*} = 0$. The optimal exercise time for the f-claim is the same as the optimal exercise time for the f^j-claim in the auxiliary financial market.

Some particular cases are the natural counterpart of standard multiasset options.

1. Cumulative barrier max- and min-options: When there are two underlying assets call options in this category have payoff functions of the form $(S_t^1 \vee S_t^2 - K)^+ 1_{\{O_t^{S,A} \geq b\}}$ (max-option) or $(S_t^1 \wedge S_t^2 - K)^+ 1_{\{O_t^{S,A} \geq b\}}$ (min-option), where $b \in [0,T]$. Similarly for put options. It is easily verified that a cumulative barrier call max-option is symmetric to a cumulative barrier option to exchange the maximum of an asset and cash against another asset for which the occupation time has been adjusted.

2. Cumulative barrier exchange options: The payoff function takes the form $(S^1 - S^2) 1_{\{O_t^{S,A} \geq b\}}$. This exchange option is symmetric to cumulative barrier call and put options with suitably adjusted occupation times.

3. Quantile options (Miura (1992), Akahori (1995), Dassios (1995)): An α-Quantile call option pays off $(M(\alpha,t) - K)$ upon exercise where $M(\alpha,t) = \inf\{x : \int_0^t 1_{\{S_v \leq x\}} dv > \alpha t\} = \inf\{x : O_t^{S,A^-(x)} > \alpha t\}$. Consider an α-Quantile strike put with payoff $(M(\alpha,t) - S_t)$. Note that

$$\begin{aligned} M(\alpha,t) &= \inf\{x : \int_0^t 1_{\{S_v \leq x\}} dv > \alpha t\} = \inf\{x : \int_0^t 1_{\{SS_v/S_t \leq Sx/S_t\}} dv > \alpha t\} \\ &= (S_t/S) \inf\{y : \int_0^t 1_{\{SS_v/S_t \leq y\}} dv > \alpha t\} \equiv (S_t/S) M^*(\alpha,t) \end{aligned}$$

   where $M^*(\alpha,t)$ is the α-quantile of the normalized price $S_{v,t}^* \equiv SS_v/S_t$ for $v \leq t$. Thus $M(\alpha,t) = (S_t/S)M^*(\alpha,t)$ and an α-Quantile strike put is seen to be symmetric to an α-Quantile call option with (fixed) strike price S and quantile based on the normalized asset price $S_{v,t}^*, v \leq t$.

Multiasset step options can be also be defined in a natural manner and satisfy symmetry properties akin to those of standard multiasset options.

7 Symmetry Property without Homogeneity of Degree One.

Several derivative securities have payoffs that are not homogeneous of degree one. Examples include digital options and quantile options (homogeneous of degree $\nu = 0$) or product options (homogeneous of degree $\nu \neq 0, 1$). Product options (options on a product of assets) include options on foreign indices with payoff in domestic currency such as quanto options. As we show below, even in these cases, symmetry-like properties link various types of contracts.

Consider an f-claim on n underlying assets whose payoff is homogeneous of degree ν, i.e.,

$$f(\lambda S, \lambda K) = \lambda^\nu f(S,K)$$

for some $\nu \geq 0$ and for all $\lambda > 0$. The following result is then valid.

Theorem 13 *Consider an American f-claim with maturity date T and a continuous and homogeneous of degree ν payoff function $f(S,K)$. Let $V(S,K,r,\delta;\mathcal{F}_t)$ denote the value of the claim in the financial market with filtration $\mathcal{F}_{(\cdot)}$, asset prices S_t satisfying (37) and progressively measurable interest rate r. For $j=1,...,n$, define*

$$r^{j*} = (1-\nu)r + \nu\delta^j + \frac{1}{2}\nu(1-\nu)\sigma^j\sigma^{j\prime}$$

$$\delta^{i*} = (1-\nu)r + \delta^i + (\nu-1)\delta^j + (1-\nu)(-1+\frac{1}{2}\nu)\sigma^j\sigma^{j\prime} + (1-\nu)\sigma^i\sigma^{j\prime}, \text{ for } i \neq j$$

$$\delta^{j*} = (2-\nu)r + (\nu-1)\delta^j + (1-\nu)(-1+\frac{1}{2}\nu))\sigma^j\sigma^{j\prime}.$$

Prior to exercise the value of the claim is, for any $j=1,...,n$,

$$V(S_t,K,r,\delta;\mathcal{F}_t) = V^j(S_t^*,S^j,r^{j*},\delta^*;\mathcal{F}_t)$$

where $V^j(S_t^*,S^j,r^{j*},\delta^*;\mathcal{F}_t)$ is the value of the f^j-claim with parameter S^j and maturity date T in an auxiliary financial market with interest rate r^{j*} and in which the underlying asset prices follow the Ito processes

$$\begin{cases} dS_v^{i*} = S_v^{i*}[(r_v^{j*}-\delta_v^{i*})dv + (\sigma_v^j - \sigma_v^i)dz_v^{j*}]; \text{ for } i = j \text{ and } v \in [t,T] \\ dS_v^{j*} = S_v^{j*}[(r_v^{j*}-\delta_v^{j*})dv + \sigma_v^j dz_v^{j*}]; \text{ for } i = j \text{ and } v \in [t,T] \end{cases}$$

with respective initial conditions $S_t^{*i} = S^i$ for $i \neq j$ and $S_t^{*j} = K$ for $i = j$. The process z^{j*} is defined by

$$dz_v^{j*} = -d\widetilde{z}_v + \nu\sigma_v^{j\prime}dv, \text{ for } v \in [0,T]; z_0^{j*} = 0.$$

The optimal exercise time for the f-claim is the same as the optimal exercise time for the f^j-claim in the auxiliary financial market.

Proof of Theorem 13: Define $S^j = S_t^j$. Let

$$r_v^{j*} = (1-\nu)r_v + \nu\delta_v^j + \frac{1}{2}\nu(1-\nu)\sigma_v^j\sigma_v^{j\prime}$$

and note that

$$\exp(-\int_t^\tau r_v dv)\left(\frac{S_\tau^j}{S^j}\right)^\nu = \exp(-\int_t^T r_v^{j*}dv)\exp(-\frac{1}{2}\nu^2\int_t^T \sigma_v^j\sigma_v^{j\prime}dv + \nu\int_t^T \sigma_v^j d\widetilde{z}_v).$$

Defining the equivalent measure

$$dQ^{j*} = \exp\left(-\frac{1}{2}\nu^2\int_0^T \sigma_v^j\sigma_v^{j\prime}dv + \nu\int_0^T \sigma_v^j d\widetilde{z}_v\right)dQ$$

enables us to write

$$\begin{aligned} V(S_t,K,r,\delta;\mathcal{F}_t) &= \sup_{\tau\in\mathcal{S}_{t,T}} \widetilde{E}\left[\exp(-\int_t^\tau r_v dv)f(S_\tau,K)\,|\mathcal{F}_t\right] \\ &= \sup_{\tau\in\mathcal{S}_{t,T}} \widetilde{E}\left[\exp(-\int_t^\tau r_v dv)\left(\frac{S_\tau^j}{S^j}\right)^\nu f(S_\tau\frac{S^j}{S_\tau^j}, K\frac{S^j}{S_\tau^j})\,|\mathcal{F}_t\right] \\ &= \sup_{\tau\in\mathcal{S}_{t,T}} E^{j*}\left[\exp(-\int_t^\tau r_v^{j*}dv)f(S_\tau\frac{S^j}{S_\tau^j}, S_\tau^{j*})\,|\mathcal{F}_t\right] \\ &= \sup_{\tau\in\mathcal{S}_{t,T}} E^{j*}\left[\exp(-\int_t^\tau r_v^{j*}dv)f^j(S_\tau^*, S^j)\,|\mathcal{F}_t\right] \\ &= V^j(S_t^*,S^j,r^{*j},\delta^*;\mathcal{F}_t). \end{aligned}$$

Under Q^{j*} the process

$$dz_v^{j*} = -d\tilde{z}_v + \nu\sigma_v^{j\prime}dv$$

is a Brownian motion and S^{i*} satisfies, for $i \neq j$ and $v \in [t, T]$

$$\begin{aligned} dS_v^{i*} &= S_v^{i*}[(\delta_v^j - \delta_v^i + (\sigma_v^j - \sigma_v^i)\sigma_v^{j\prime})dv - (\sigma_v^j - \sigma_v^i)d\tilde{z}_v] \\ &= S_v^{i*}[(\delta_v^j - \delta_v^i + (\sigma_v^j - \sigma_v^i)\sigma_v^{j\prime})dv + (\sigma_v^j - \sigma_v^i)[dz_v^{j*} - \nu\sigma_v^{j\prime}dv]] \\ &= S_v^{i*}[(\delta_v^j - \delta_v^i + (1-\nu)(\sigma_v^j - \sigma_v^i)\sigma_v^{j\prime})dv + (\sigma_v^j - \sigma_v^i)dz_v^{j*}] \\ &= S_v^{i*}[(r_v^{j*} - \delta_v^{i*})dv + (\sigma_v^j - \sigma_v^i)dz_v^{j*}] \end{aligned}$$

where

$$\delta_v^{i*} = (1-\nu)r_v + \delta_v^i + (\nu - 1)\delta_v^j + (1-\nu)(-1 + \frac{1}{2}\nu)\sigma_v^j\sigma_v^{j\prime} + (1-\nu)\sigma_v^i\sigma_v^{j\prime}$$

and for $i = j$ and $v \in [t, T]$

$$\begin{aligned} dS_v^{j*} &= S_v^{j*}[(\delta_v^j - r_v + \sigma_v^j\sigma_v^{j\prime})dv - \sigma_v^j d\tilde{z}_v] \\ &= S_v^{j*}[(\delta_v^j - r_v + (1-\nu)\sigma_v^j\sigma_v^{j\prime})dv + \sigma_v^j dz_v^{j*}] \\ &= S_v^{j*}[(r_v^{j*} - \delta_v^{j*})dv + \sigma_v^j dz_v^{j*}] \end{aligned}$$

where

$$\delta_v^{j*} = (2-\nu)r_v + (\nu - 1)\delta_v^j + (1-\nu)(-1 + \frac{1}{2}\nu)\sigma_v^j\sigma_v^{j\prime}.$$

This completes the proof of the Theorem. ■

Remark 4 *When the claim is homogeneous of degree* 1 *the interest rate and the dividend rates in the economy with numeraire j become $r_v^{j*} = \delta_v^j, \delta_v^{i*} = \delta_v^i$, for $i \neq j$, and $\delta_v^{j*} = r_v$. Thus we recover the prior results of Theorem 10.*

Another special case of interest is when the payoff function is homogeneous of degree 0. The economy with numeraire j then has characteristics

$$r^{j*} = r$$

$$\delta^{i*} = r + \delta^i - \delta^j - (\sigma^j - \sigma^i)\sigma^{j\prime}, \text{ for } i \neq j$$

$$\delta^{j*} = 2r - \delta^j - \sigma^j\sigma^{j\prime}$$

and the underlying asset prices follow the Ito processes

$$\begin{cases} dS_v^{i*} = S_v^{i*}[(r_v^{j*} - \delta_v^{i*})dv + (\sigma_v^j - \sigma_v^i)dz_v^{j*}]; \text{ for } i \neq j \text{ and } v \in [t, T] \\ dS_v^{j*} = S_v^{j*}[(r_v^{j*} - \delta_v^{j*})dv + \sigma_v^j dz_v^{j*}]; \text{ for } i = j \text{ and } v \in [t, T] \end{cases}$$

with respective initial conditions $S_t^{*i} = S^i$ for $i \neq j$ and $S_t^{*j} = K$ for $i = j$. The process z^{j*} is defined by $dz_v^{j*} = -d\tilde{z}_v$, for $v \in [0, T]$. It is a Brownian motion under $Q^* = Q$.

Examples of contracts in this category are

1. Digital options: A digital call option ($f(S, K) = 1_{\{S \geq K\}}$) is symmetric to a digital put option with strike $S = S_t$, written on an asset with dividend rate $\delta^* = 2r - \delta - \sigma^2$, in an economy with interest rate $r^* = r$.

2. Digital multiasset options: A digital call max-option ($f(S^1, S^2, K) = 1_{\{S^1 \vee S^2 \geq K\}}$) is symmetric to a digital option to exchange the maximum of an asset and cash against another asset ($f^2(S^1, S^2, K') = 1_{\{S^{*1} \vee K' \geq S^{*2}\}}$, where $K' = S^2$) in the economy with asset $j = 2$ as numeraire (with characteristics $r^{2*} = r, \delta^{1*} = r + \delta^1 - \delta^2 - (\sigma^2 - \sigma^1)\sigma^{2\prime}$, and $\delta^{2*} = 2r - \delta^2 - \sigma^2 \sigma^{2\prime}$). A digital call min-option ($f(S^1, S^2, K) = 1_{\{S^1 \wedge S^2 \geq K\}}$) is symmetric to a digital option to exchange the minimum of an asset and cash against another asset ($f^2(S^1, S^2, K') = 1_{\{S^{*1} \wedge K' \geq S^{*2}\}}$, where $K' = S^2$) in the same auxiliary economy. Similar relations hold for digital multiasset put options.

3. Cumulative barrier digital options: Symmetry properties for occupation time derivatives with homogeneous of degree zero payoffs can be easily identified by drawing on the previous section. A cumulative barrier digital call option with barrier L (i.e. payoff $f(S, K, O^{S, A^+(L)}) = 1_{\{S \geq K\}} 1_{\{O_t^{S, A^+(L)} \geq b\}}$ where $A^+(L) = \{x \in \mathbb{R}^+ : (x - L)^+ \geq 0\}$) is symmetric to a cumulative barrier digital put option with barrier $L^* = KS/L$ (i.e. payoff $f^1(S^*, K', O^{S^*, A^-(L^*)}) = 1_{\{K' \geq S^*\}} 1_{\{O_t^{S^*, A^-(L^*)} \geq b\}}$ where $K' = S$ and $A^-(L^*) = \{x \in \mathbb{R}^+ : (x - L^*)^- \geq 0\}$). A similar symmetry relation can be established for Parisian digital call and put options.

4. Quanto options: Consider again the quanto call option with payoff $e(S - K)^+$ in foreign currency where e is the Y/\$ exchange rate. From the foreign perspective the contract is homogeneous of degree $\nu = 2$ in the triplet (e, S, K). The results of Theorem 13 imply that the quanto call is symmetric to an exchange option in an economy with interest rate

$$r^{f*} = -r^f + 2r - \sigma^e \sigma^{e\prime}$$

and which underlying assets have dividend rates

$$\delta^{1*} = -r^f + \delta + r - \sigma \sigma^{e\prime}$$

$$\delta^{2*} = r.$$

The call value can be written

$$C_t^Q = e_t \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E}^{f*} \left[\exp(-\int_t^\tau r_v^{f*} dv)(S_\tau^{1*} - S_\tau^{2*})^+ \,|\mathcal{F}_t \right]$$

where

$$\begin{cases} dS_v^{1*} = S_v^{1*}[(r_v^{f*} - \delta_v^{1*})dv + (\sigma_v^e - \sigma_v)dz_v^{f*}]; \text{ for } v \in [t, T] \\ dS_v^{2*} = S_v^{2*}[(r_v^{f*} - \delta_v^{2*})dv + \sigma_v^e dz_v^{f*}]; \text{ for } v \in [t, T]. \end{cases}$$

with the initial conditions $S_t^{1*} = S_t$ and $S_t^{2*} = K$. An alternative representation for the quanto call was provided in section 7.

Remark 5 *Representation formulas involving the change of measure introduced in earlier sections can also be obtained with payoffs that are homogeneous of degree ν. In this case the coefficients of the underlying asset price processes reflect the homogeneity degree of the payoff function. Indeed letting $S^j = S_t^j$ we can always write*

$$\begin{aligned} V(S_t, K, r, \delta; \mathcal{F}_t) &= \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E} \left[\exp(-\int_t^\tau r_v dv) f(S_\tau, K) \,|\mathcal{F}_t \right] \\ &= \sup_{\tau \in \mathcal{S}_{t,T}} \widetilde{E} \left[\exp(-\int_t^\tau r_v dv) \left(\frac{S_\tau^j}{S^j} \right) f(S_\tau (\frac{S^j}{S_\tau^j})^{1/\nu}, K(\frac{S^j}{S_\tau^j})^{1/\nu}) \,|\mathcal{F}_t \right] \\ &= \sup_{\tau \in \mathcal{S}_{t,T}} E^{j*} \left[\exp(-\int_t^\tau \delta_v^j dv) f(\widehat{S}_\tau, \widehat{S}_\tau^{n+1}) \,|\mathcal{F}_t \right] \end{aligned}$$

where $\widehat{S}^i_v = S^i_v(\frac{S^j}{S^j_v})^{1/\nu}$ *for* $i = 1, ..., n$ *and* $\widehat{S}^{n+1}_v = K(\frac{S^j}{S^j_v})^{1/\nu}$ *for* $v \in [t, T]$*. The auxiliary economy has interest rate* δ^j *and the equivalent measure* Q^{j*} *is*

$$dQ^{j*} = \exp\left(-\frac{1}{2}\int_0^T \sigma^j_v \sigma^{j\prime}_v dv + \int_0^T \sigma^j_v d\widetilde{z}_v\right) dQ.$$

The process $dz^{j*}_v = -d\widetilde{z}_v + \sigma^{j\prime}_v dv$*, for* $v \in [0, T]$ *is a* Q^{j*}*-Brownian motion process.*

8 Changes of Numeraire and Representation of Prices.

In the financial markets of the previous sections the price of a contingent claim is the expectation of its discounted payoff where discounting is at the riskfree rate and the expectation is taken under the risk neutral measure. This standard representation formula is implied by the ability to replicate the claim's payoff using a suitably constructed portfolio of the basic securities in the model. Since symmetry properties are obtained by passing to a new numeraire a natural question is whether contingent claims that are attainable in the basic financial markets are also attainable in the economy with new numeraire. This question is in fact essential for interpretation purposes since the symmetry properties above implicitly assume that the renormalized claims can be priced in the new numeraire economy and that their price corresponds to the one in the original economy.

For the case of nondividend paying assets Geman, El Karoui and Rochet (1995) prove that contingent claims that are attainable in one numeraire are also attainable in any other numeraire and that the replicating portfolios are the same. Our next Theorem provides an extension of this result to dividend-paying assets. The framework of section 2 with Brownian filtration is adopted for convenience only; the results are valid for more general filtrations.

Theorem 14 *Consider an economy with Brownian filtration and complete financial market with* n *risky assets and one riskless asset. Suppose that risky assets pay dividends and that their prices follow Ito processes (37), and that the riskless asset pays interest at the rate* r*. Assume that all the coefficients are progressively measurable and bounded processes. If a contingent claim's payoff is attainable in a given numeraire then it is also attainable in any other numeraire. The replicating portfolio is the same in all numeraires.*

Proof of Theorem 14: Let $i = 0$ denote the riskless asset. The gains from trade in the primary assets are

$$\begin{aligned} dG^i_t &\equiv dS^i_t + S^i_t \delta^i_t dt = S^i_t[r_t dt + \sigma^i_t d\widetilde{z}_t], \text{ for } i = 1, ..., n \\ dG^0_t &\equiv dB_t = B_t r_t dt, \text{ for } i = 0. \end{aligned}$$

For $i = 0, ..., n$, gains from trade expressed in numeraire j are

$$G^{i,j}_t = \frac{S^i_t}{S^j_t} + \int_0^t \frac{1}{S^j_v}\delta^i_v S^i_v dv \tag{40}$$

so that

$$dG^{i,j}_t = \frac{1}{S^j_t} dS^i_t + S^i_t d(\frac{1}{S^j_t}) + \frac{1}{S^j_t} S^i_t \delta^i_t dt + d\left[S^i, \frac{1}{S^j}\right]_t = \frac{1}{S^j_t} dG^i_t + S^i_t d(\frac{1}{S^j_t}) + d\left[S^i, \frac{1}{S^j}\right]_t.$$

Now let π^i represent the amount invested in asset i and consider a portfolio $(\pi^0, \pi) \in \mathbb{R}^{n+1}$ such that $\int_0^T \pi_v \sigma_v \sigma'_v \pi'_v dv < \infty, (P - a.s.)$. The wealth process X generated by N where $N^j = \pi^j / S^j, j = 0, ..., n$ represents the number of shares of each asset in the portfolio satisfies

$$dX_t = \sum_{i=0}^n N^i_t dG^i_t$$

and $X_t = \sum_{i=0}^n N_t^i S_t^i$ (this portfolio is self financing since all dividends are reinvested). Using Ito's lemma gives

$$\begin{aligned} d\left(\frac{X_t}{S_t^j}\right) &= \sum_{i=0}^n N_t^i \left(\frac{dG_t^i}{S_t^j}\right) + X_t d\left(\frac{1}{S_t^j}\right) + \sum_{i=0}^n N_t^i d\left[G^i, \frac{1}{S^j}\right]_t \\ &= \sum_{i=0}^n N_t^i \left(\frac{dG_t^i}{S_t^j} + S_t^i d\left(\frac{1}{S_t^j}\right) + d\left[S^i, \frac{1}{S^j}\right]_t\right) \\ &= \sum_{i=0}^n N_t^i dG_t^{i,j} \end{aligned}$$

i.e. the normalized wealth process can be synthesized in the new numeraire economy in which all asset prices have been deflated by the numeraire asset j. Furthermore the investment policy which achieves normalized wealth is the same as in the original economy. Consequently, any deflated payoff is attainable in the new numeraire economy when the (undeflated) payoff is attainable in the original economy. ■

Remark 6 *(i) The proper definition of gains from trade in the new numeraire is instrumental in the proof above. Since dividends are paid over time they must be deflated at a discount rate which reflects the timing of the cash flows. This explains the discount factor inside the integral of dividends in (40).*

(ii) Note that theorem 14 applies even if the numeraire chosen is a portfolio of assets or any other progressively measurable process instead of one of the primitive assets. It also applies when the portfolio is not self financing, for example when there are infusions or withdrawal of funds over time.

(iii) The results above apply for payoffs that are received at fixed time as well as stopping times of the filtration: if there exists a trading strategy that attains the random payoff X_τ where $\tau \in \mathcal{S}_{0,T}$ in the original financial market then the normalized payoff X_τ / S_τ^j is attainable in the economy with numeraire asset j.

Our next result now follows easily from the above.

Theorem 15 *Suppose that asset j serves as numeraire and that S^j satisfies (37). Define the probability measure Q^{j*} by*

$$dQ^{j*} = \frac{\exp(-\int_0^T (r_v - \delta_v)dv) S_T^j}{S_0^j} dQ = \exp\left(-\frac{1}{2}\int_0^T \sigma_v^j \sigma_v^{j\prime} dv + \int_0^T \sigma_v^j d\tilde{z}_v\right) dQ \tag{41}$$

and consider the discount rate δ^j. Then the discounted prices of primary securities expressed in numeraire j are Q^{j*}-supermartingales (discounted gains from trade in numeraire j are Q^{j*}-martingales) and the price of any attainable security in the original economy can be represented as the expected discounted value of its cash flows expressed in numeraire j where the discount rate is δ^j and the expectation is under the Q^{j*}-measure.

Proof of Theorem 15: Using definition (40) of gains from trade expressed in numeraire j and Ito's lemma gives

$$\begin{aligned} dG_t^{i,j} &= \frac{1}{S_t^j} dS_t^i + S_t^i d(\frac{1}{S_t^j}) + \frac{1}{S_t^j} S_t^i \delta_t^i dt + d\left[S^i, \frac{1}{S^j}\right]_t \\ &= \frac{1}{S_t^j} S_t^i [r_t dt + \sigma_t^i d\tilde{z}_t] + S_t^i \frac{1}{S_t^j}[(\delta_t^j - r_t + \sigma_t^j \sigma_t^{j\prime})dt - \sigma_t^j d\tilde{z}_t] - S_t^i \frac{1}{S_t^j} \sigma_t^i \sigma_t^{j\prime} dt \end{aligned}$$

$$= \frac{1}{S_t^j} S_t^i[(\delta_t^j + (\sigma_t^j - \sigma_t^i)\sigma_t^{j\prime})dt + (\sigma_t^i - \sigma_t^j)d\tilde{z}_t]$$
$$= \frac{1}{S_t^j} S_t^i[\delta_t^j dt + (\sigma_t^j - \sigma_t^i)dz_t^{j*}],$$

where $dz_t^{j*} = -d\tilde{z}_t + \sigma_t^{j\prime}dt$ is a Q^{j*}-Brownian motion process. Defining $S_t^{i*} = S_t^i/S_t^j$ we can then write

$$dS_t^{i*} = S_t^{i*}[(\delta_t^j - \delta_t^i)dt + (\sigma_t^j - \sigma_t^i)dz_t^{j*}]$$

i.e. the discounted price of asset i in numeraire j, $\exp(-\int_0^t \delta_v^j dv)S_t^{i*}$, is a Q^{j*}-supermartingale where discounting is at the rate δ^j. Alternatively the discounted gains from trade process

$$\exp(-\int_0^t \delta_v^j dv)S_t^{i*} + \int_0^t \exp(-\int_0^v \delta_u^j du)S_v^{i*}\delta_v^i dv$$

is a Q^{j*}-martingale. Thus, we can write the representation formula

$$S_t^{i*} = E_t^{j*}\left[\exp(-\int_t^T \delta_v^j dv)S_T^{i*} + \int_t^T \exp(-\int_t^v \delta_u^j du)S_v^{i*}\delta_v^i dv \,|\mathcal{F}_t\right].$$

The relations satisfied by primary asset prices also apply to portfolios of primary assets and therefore to any contingent claim that is attainable. This completes the proof of the Theorem. ■

Remark 7 *When a dividend-paying primary asset price is chosen as deflator the auxiliary economy has an interest rate equal to the dividend rate of the deflator. In this new numeraire cash is converted into an asset that pays a dividend rate equal to the interest rate in the original economy. If we choose the discounted price $\widehat{S}_t^j = \exp(-\int_0^t (r_v - \delta_v^j)dv)S_t^j$, which is a martingale, as numeraire the process $S_t^{i*} = S_t^i/\widehat{S}_t^j$ satisfies*

$$dS_t^{i*} = S_t^{i*}[(r_t - \delta_v^i)dt + (\sigma_t^j - \sigma_t^i)dz_t^{j*}]$$

and its discounted value at the riskfree rate is a Q^{j*}-supermartingale where Q^{j*} is defined in (41). With this choice of numeraire the interest rate remains unchanged in the auxiliary economy. Cash is converted into an asset that pays a dividend rate equal to the interest rate and thus has null drift (martingale).

Remark 8 *(i) Note that a payoff expressed in a new numeraire is not necessarily the same as the payoff evaluated at normalized underlying asset prices (i.e. prices expressed in the new numeraire). There is clearly equivalence when the payoff is homogeneous of degree one. With homogeneity of degree ν the payoff in the new numeraire is equivalent to the payoff function evaluated at underlying asset prices that are normalized by a power of the numeraire price. Normalized asset prices (in the payoff function) then differ from asset prices expressed in the new numeraire.*

(ii) A byproduct of Theorem 15 is a generalized "symmetry" property which applies to any payoff function. In this interpretation of the property the symmetric contract is simply the payoff expressed in the new numeraire.

Some extensions are worth mentioning.

Remark 9 *Note that the results on the replication of attainable contingent claims, their financing portfolios and their representation under new measures are valid even when markets are incomplete. Indeed if the claims under consideration can be replicated in a given incomplete market equilibrium (i.e. if the claims' payoffs live in the asset span) so can they under a change of numeraire. The results are also valid when the market is effectively complete (single agent economies). In this case even when claims payoffs cannot be duplicated they have a unique price which can be expressed in different forms corresponding to various choices of numeraire.*

9 Conclusion.

In this paper we have reviewed and extended recent results on PCS. Features of the models considered include (i) financial markets with progressively measurable coefficients, (ii) random maturity options, (iii) options on multiple underlying asset, (iv) occupation time derivatives and (v) payoff functions that are homogeneous of degree $\nu \neq 1$. One important element in the proofs is the ability to renormalize a vector of prices and parameters which determine the payoff of the contract. Homogeneity of degree ν is sufficient in that regard but it is not a necessary condition. Another important element in the proofs is the separation between the role of informational variables and the change of measure (numeraire). Indeed while the change of measure converts the underlying assets into normalized or symmetric assets in the auxiliary financial market the information sets in the two markets are kept the same. This separation enables us to derive symmetry properties even for financial markets in which prices do not follow Markov processes. In the context of diffusion models the change of measure is instrumental for obtaining symmetry properties of option prices without restricting volatility coefficients.

Some of the results in the paper can be readily extended. Symmetry-like properties hold for multiasset contracts even when the payoff functions are not homogeneous of some degree ν (for instance when homogeneity of different degrees holds relative to different subsets of the underlying asset prices). In this instance normalized prices in the auxiliary economy involve further adjustments to dividends and volatilities. Likewise the methodology reviewed in this paper also applies, in principle, to complete financial markets with general semimartingales or even to incomplete markets provided that the securities under consideration lie in the asset span.

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