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Short and Long Memory in Equilibrium Interest Rate Dynamics*

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Résumé / Abstract

Dans cet article, nous analysons une classe de processus pour le taux d'intérêt à court terme, qui sont dérivés dans un cadre d'équilibre en temps discret. La dynamique des taux d'intérêts et des rendements est commandée par la dynamique de la volatilité conditionnelle de la variable d'état. Sous des restrictions de paramètres appropriées, les taux d'intérêt dérivés dans ce cadre sont non-négatifs. Nous étudions les processus Markovien de taux d'intérêt, de même que des procédés Markoviens plus généraux, qui affichent une mémoire « courte » et « longue ». Ces processus affichent aussi des schémas d'hétéroscédasticité qui sont plus généraux que ceux des modèles d'équilibre existants. Nous trouvons que les déviations à la structure Markovienne améliorent de façon significative la performance empirique du modèle et que les données soutiennent la présence de mémoire longue. Nous trouvons également que les données soutiennent des schémas d'hétéroscédasticité qui diffèrent de ceux présents dans les modèles d'équilibre existants.

This paper analyzes a large class of processes for the short-term interest rate that are derived in a discrete-time equilibrium framework. The dynamics of interest rates and yields are driven by the dynamics of the conditional volatility of the state variable. Under appropriate parameter restrictions, interest rates derived in this framework are nonnegative. We study Markovian interest rate processes as well as more general non-Markovian processes that display "short" and "long" memory. These processes also display heteroskedasticity patterns that are more general than those of existing equilibrium models. We find that deviations from the Markovian structure significantly improve the empirical performance of the model and that the data support the presence of long memory. We also find that the data support heteroskedasticity patterns that are different from the ones present in existing equilibrium models.

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1 Introduction

The study of term structure models is a topic of considerable interest in asset pricing, and the existing literature contains several classes of term structure models that use different theoretical constructs and differ in aim and scope. An important class of models studies the term structure by analyzing a standard representative agent economy. The empirical performance of these models is typically studied by analyzing the short interest rate (the time series implications of the model) as well as the implied shape of the term structure (the cross sectional implications of the model). Even though we currently have a large number of these equilibrium models at our disposal, their empirical performance has not been entirely satisfactory. The main motivation for this paper is to investigate the empirical performance of a class of equilibrium term structure models that are substantially more general compared to existing equilibrium models. The empirical results in this paper focus on the dynamics of the short-term rate, and do not address the cross-sectional implications of the class of equilibrium models under study.

Duffie and Kan (1996) demonstrate that a large number of existing term structure models, such as Vasicek (1978) and Cox, Ingersoll and Ross (1985) can be nested within a unifying framework, which is known as the affine class of term structure models. This class of models has a number of advantages, the most important one being the existence of analytical solutions for bond prices and yields. However, it has become clear that the empirical performance of this class of models is unsatisfactory. Most notably, parameterization of these models that perform well in the time series dimension often perform poorly in the cross sectional dimension.

In response to these empirical failures, there have been two major developments.[1] One approach focuses on the problem that the affine models do not perform well in the cross-sectional dimension. This has serious consequences because to price term structure derivatives, it has proven necessary to match the initial (cross-sectional) term structure exactly. To accommodate this empirical problem, a number of models were constructed that essentially extend exponential affine models to allow for time-varying drifts, a feature that allows exact matching of the initial term structure. This is a large and expanding literature, including contributions such as Ho and Lee (1986), Hull and White (1990,1993), Black, Derman and Toy (1990), Black and Karasinski (1991) and Heath, Jarrow and Morton (1992). For our purpose, we note that the Ho and Lee and Hull and White models are both Markovian models, just like the original exponential affine models. The Heath, Jarrow and Morton framework is substantially more general and allows for non-Markovian structure. However, most of the models implemented in the Heath-Jarrow-Morton framework are for practical reasons restricted to special forms that are low dimensional Markovian systems. A second empirical innovation has been the extension of the initial one-factor exponential affine models such as Vasicek (1978) and Cox, Ingersoll and Ross (1985) to multiple factors (e.g. see Litterman and Scheinkman (1991), Chen and Scott (1993) and Pearson and Sun (1994)). Observable factors used in this context include the long yield (Brennan and Schwartz (1979)), the volatility of the short rate (Longstaff and Schwartz (1992), Balduzzi, Das, Foresi and Sundaram (1996)), and the long-run mean of the short rate (Jegadeesh and Pennacchi (1996), Balduzzi, Das, Foresi and Sundaram (1996)).[2] It

[1] There is some controversy regarding the relative merits of these developments (e.g. see Backus, Foresi and Zin (1998)).

[2] There is also an extensive literature that investigates continuous-time and discrete-time models of the short rate

must be noted that these multifactor approaches also maintain the Markovian structure of the early exponential affine term structure models. This is somewhat remarkable given the fact that departures from the Markovian structure are popular in empirical modeling across many disciplines. Moreover, even though the multifactor extensions of the exponential affine models have enjoyed some empirical success, much work remains to be done to improve empirical performance. Recent empirical work has for example indicated that the specification of multifactor models is far from obvious (see Dai and Singleton (1999)). It seems therefore interesting to explore other modeling approaches, such as deviations from the Markovian structure. Backus and Zin (1995) explore such deviations using a discrete-time framework with Gaussian innovations (see also Campbell, Lo and MacKinlay (1997)). Their work indicates that departures from the Markovian structure can substantially improve model performance, but a disadvantage of their modeling strategy is that interest rates can become negative.[3] Nevertheless, it seems interesting to further explore to what extent non-Markovian models can capture some of the stylized facts of term structure data.

This paper uses a discrete-time representative-agent setup to analyze a large class of term structure models that can be nested in a single equilibrium framework. In general the framework is non-Markovian, but restrictions can be imposed to create a Markovian model and these restrictions can be tested. The main difference with existing equilibrium interest rate models is that the interest rate dynamic is driven by the time-varying volatility of consumption, as opposed to the time-varying mean of consumption. In principle, this modeling strategy can be implemented by modeling the volatility of aggregate consumption using a class of nonnegative processes. In this paper we use a chi-square dynamic for consumption volatility, borrowing from the extensive work done in the area of stock market volatility using the GARCH framework (see Engle (1982) and Bollerslev (1986)). The resulting dynamics for the short-term interest rate are extremely tractable for empirical purposes. These dynamics inherit the structure of consumption volatility, and are also governed by innovations of chi-square distributions, in contrast with most existing interest rate models which are governed by normally distributed innovations. The advantage of the chi-square innovations is that interest rates can stay positive under suitable parameter restrictions, whereas it is not possible to ensure positive interest rates with normally distributed innovations.

Our interest rate dynamics are very general and include a range of dynamics resembling those of models based on the ARMA specification. Building on the evidence of long memory in interest rates uncovered by Backus and Zin (1993), we also investigate a long-memory specification of our equilibrium model. In general, because we use a setup where interest rate dynamics are determined by the conditional variance of consumption instead of the conditional mean, we obtain much greater flexibility in our modeling approach. We illustrate this flexibility by modeling fairly

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outside the affine class of models. These specifications do not lead to closed form solutions for bond prices and yields (e.g. see Chan, Karolyi, Longstaff and Sanders (1992), Brenner, Harjes and Kroner (1996), Andersen and Lund (1997, 1999), Koedijk, Nissen, Schotman and Wolff (1997) and Pfann, Schotman and Tschernig (1996) on the incorporation of more general heteroskedasticity patterns. See Hamilton (1988) and Gray (1996) on the importance of regime switching for interest rates).

[3] Interest rates can become negative in the Backus-Zin setup because of the assumption of Gaussian shocks within a discrete time framework. Even though the continuous-time exponential affine models also use Gaussian shocks, it is possible to avoid negative interest rates because shocks are of infinitesimal nature (see Cox, Ingersoll and Ross (1985)).

general heteroskedasticity patterns in the interest rate dynamic. It is well known that allowing for heteroskedasticity improves the empirical performance of the short-term interest rate and term structure models. The empirical results of Chan, et al. (1992) indicate that the use of the constant elasticity of variance (CEV) specification to reflect heteroskedasticity can improve the fit of the model to the interest rate data. We show in this paper that the CEV type of heteroskedasticity can be easily incorporated into our equilibrium interest rate model. Heteroskedasticity is incorporated into interest rates by directly modeling the time variation in the conditional heteroskedasticity of the consumption process. The typical term structure models such as the exponential affine models of Vasicek (1978), Cox, Ingersoll and Ross (1985) and Duffie and Kan (1996) do not explore this dimension, and therefore cannot model such heteroskedasticity patterns.[4]

The empirical analysis of the short-term interest rate in this paper highlights several interesting stylized facts and suggests that the class of models proposed in this paper may be able to match some empirical regularities of the term structure. We analyze two time series containing daily observations on the short-term interest rate – overnight Eurodollar rates and one-week Eurodollar rates. When we restrict our analysis to the Markovian subclass of models, these two time series seem to exhibit similar characteristics. However, the estimation and test results for non-Markovian models are quite different for these two time series. Although formal statistical tests support deviations from the Markovian structure, the deviation is much more significant (statistically) for the one-week rates series. The results for the two time series also differ in terms of heteroskedasticity patterns. The weekly interest rate series supports heteroskedasticity models that are inherently different from the typical equilibrium models, but this is not the case if overnight rates are used. The results for overnight rates suggest that the existing heteroskedastic equilibrium interest rate models seem adequate. Finally, both interest rate time series imply some moderate levels of long memory. Although the long memory parameter is moderate in magnitude, allowing for long memory substantially improves model fit compared to the Markovian interest rate models. A small-scale simulation experiment indicates that the impact of the long memory parameter for the term structure is substantial.

2 Equilibrium interest rates with short and long memory

We characterize the short-term interest rate dynamic using an equilibrium approach. Specifically, we assume a representative-agent economy in which the agent has time-separable constant-relative-risk-aversion (TS-CRRA) preferences. Let C_t denote aggregate consumption at time t and let Ω_t denote the information set up to time t. For any asset with a price X_t and a dividend payment D_t at time t, equilibrium requires the following Euler equation to hold:

$$X_t = E[e^{-\delta}(\frac{C_{t+1}}{C_t})^{-\gamma}\,(X_{t+1} + D_{t+1})\,|\Omega_t] \qquad (1)$$

[4]Note that one can study any conceivable term structure model, including arbitrary forms of heteroskedasticity, outside the affine framework by using numerical techniques. However, the class of models under investigation in this paper leads to analytical solutions for some important objects of interest. Our modeling approach does not yield analytical solutions for bond yields nor for bond option prices, but it delivers analytical expressions for the short-term interest rate under risk neutrality, which greatly facilitates the valuation of bonds and options in this framework.

where γ is the relative risk aversion coefficient and δ is the intertemporal discount rate.[5] Let r_t denote the one-period risk-free interest rate (continuously compounded) at time t. Since the time-t value of a one-dollar time-$(t+1)$ payoff is e^{-r_t}, we have the following result:

$$e^{-r_t} = E[e^{-\delta}(\frac{C_{t+1}}{C_t})^{-\gamma}|\Omega_t] \tag{2}$$

Given this setup, the dynamic for the interest rate process is determined by the dynamic for the aggregate consumption growth rate.

We model the logarithm of the aggregate consumption growth rate by a GARCH-in-mean process; that is,

$$\ln\frac{C_{t+1}}{C_t} = \mu + \kappa q_{t+1} + \sqrt{q_{t+1}}\varepsilon_{t+1} \tag{3}$$

$$\varepsilon_{t+1}|\Omega_t \sim N(0,1) \tag{4}$$

We investigate the interest rate dynamic using two different classes of models for the conditional volatility of the consumption dynamic. Because consumption growth is unobservable at high frequencies, we use the GARCH class of processes that have proven useful for describing stock price volatility. These processes are therefore also likely to be useful for describing the time variation of the volatility of the state variable that drives financial markets. The first class of models assumes the GARCH(p,q) specification of Bollerslev (1986).[6]

$$q_{t+1} = \beta_0 + \sum_{j=1}^{p}\beta_j L^{j-1}q_t + \sum_{i=1}^{q}\alpha_i L^{i-1}q_t\varepsilon_t^2 \tag{5}$$

The volatility dynamic in (5) can accommodate fairly rich autocorrelation patterns.[7] However, it has recently been shown that a wide variety of phenomena display long memory, and that these patterns cannot be adequately captured by (5).[8] Backus and Zin (1993), Shea (1991) and Connolly and Guner (1999) show that long memory processes can also be helpful for the empirical modeling of interest rate processes. Therefore, we study a second class of models that assumes the FIGARCH(p,d,q) model of Baillie, Bollerslev and Mikkelsen (1996) for conditional volatility.[9]

[5] It is of course possible to derive results similar to the ones in this paper by modeling the pricing kernel instead of modeling an equilibrium setup with a representative investor.

[6] Note that the GARCH(p,q) specification is often represented as $q_{t+1} = \beta_0 + \sum_{j=1}^{p}\beta_j L^{j-1}q_t + \sum_{i=1}^{q}\alpha_i L^{i-1}u_t^2$ with $u_t = \sqrt{q_t}\varepsilon_t$ and $\varepsilon_t|\Omega_{t-1} \sim N(0,1)$. The representation in (5) is therefore completely standard.

[7] Although we limit ourselves to the linear GARCH(p,q) process here, our derivation can be easily generalized to GARCH(p,q) models with leverage effects such as the non-linear asymmetric GARCH(p,q) of Engle and Ng (1993), EGARCH(p,q) of Nelson (1991), or to more general models like the augmented GARCH(p,q) process proposed by Duan (1997).

[8] See the overview paper by Baillie (1996) for examples of long memory in the natural and social sciences and Baillie and Bollerslev (1999) for the importance of long memory for the foreign exchange market.

[9] Other forms of fractionally integrated GARCH process such as Bollerslev and Mikkelsen (1996) and McCurdy and Michaud (1996) can also be used to derive alternative long memory interest rate dynamics.

$$q_{t+1} = \beta_0 + \sum_{j=1}^{p} \beta_j L^{j-1} q_t + [1 - \sum_{j=1}^{p} \beta_j L^j - (1 - \sum_{i=1}^{\max(p,q)} \phi_i L^i)(1-L)^d] q_{t+1} \varepsilon_{t+1}^2 \tag{6}$$

where the parameters in (5) and (6) are subject to the usual restrictions. It should be noted that the FIGARCH(p, d, q) dynamic in (6) nests the GARCH(p, q) dynamic in (5) by setting $d = 0$. We present them as two separate specifications for the sake of clarity.[10]

Following the approach of Duan (1996) and Duan and Jacobs (1996), we use the Euler equation for the interest rate in (2) and the consumption dynamic in (3) and (4) to arrive at $r_t = \delta + \gamma\mu + (\gamma(2\kappa - \gamma)/2) q_{t+1}$.[11] This relationship between the interest rate and the conditional volatility is the cornerstone for the interest rate models derived below. To simplify notation we define an adjusted interest rate, $r_t^* \equiv r_t - \delta - \gamma\mu$ and $\beta_0^* \equiv (\beta_0 \gamma(2\kappa - \gamma))/2$.[12] This yields the following models

Model 1. Short-memory interest rate dynamic,

$$r_t^* = \beta_0^* + \sum_{j=1}^{p} \beta_j L^{j-1} r_{t-1}^* + \sum_{i=1}^{q} \alpha_i L^{i-1} r_{t-1}^* \varepsilon_t^2, \tag{7}$$

if the GARCH(p, q) consumption dynamic in (5) is employed; and

Model 2. Long-memory interest rate dynamic,

$$r_t^* = \beta_0^* + \sum_{j=1}^{p} \beta_j L^{j-1} r_{t-1}^* + [1 - \sum_{j=1}^{p} \beta_j L^j - (1 - \sum_{i=1}^{\max(p,q)} \phi_i L^i)(1-L)^d] r_t^* \varepsilon_{t+1}^2, \tag{8}$$

if the FIGARCH(p, d, q) consumption dynamic in (6) is used.

In both cases, the interest rate dynamic inherits the properties of the conditional volatility process for the aggregate consumption. The long-memory interest rate dynamic of Model 2 differs

[10] It has been noted in the literature that the long memory specification in (6) implies $\beta_0 = 0$ (e.g. see Chung (1999)) if a finite unconditional variance is desired. We have not as yet found a satisfactory long memory specification that resolves this issue. Whereas this is a problem for these models, it affects the estimate for β_0 but not the estimates for other parameters of interest in a significant way.

[11] Note that a similar relationship between the interest rate and the conditional volatility always holds. We can therefore generate arbitrary processes for the interest rate by an appropriate choice of volatility dynamic. However, the choice of volatility dynamic is of substantial interest when deriving the implications of the model under the risk-neutral probability measure. As will be shown in Section 5, the choice of the GARCH(p, q) and FIGARCH(p, d, q) dynamics is critical in that respect because the innovations are simple functions of the normal distribution, which allows us to characterize the interest rate dynamic explicitly under the risk-neutral measure.

[12] Notice that in the relationship $r_t = \delta + \gamma\mu + (\gamma(2\kappa - \gamma)/2) q_{t+1}$ between the short interest rate and volatility, it is the presence of the GARCH-in-mean parameter κ that makes it possible for positivity of interest rates to be imposed, specifically for $2\kappa - \gamma > 0$, under the reasonable assumption that $\delta + \gamma\mu > 0$. The interpretation of this is that (up to a constant μ) we only model the conditional mean through the GARCH-in-mean effect. In a more realistic model one could also specify ARMA-type effects in the conditional mean. These effects would interact with the conditional volatility dynamic to produce equilibrium interest rates. However, in such a more realistic model one cannot ascertain that interest rates will stay positive.

substantially from the long-memory interest rate model investigated by Backus and Zin (1993). The innovation in this model is of the chi-square type and the dynamic is derived endogenously from a necessary condition of the equilibrium. The interest rate in Model 2 can be made positive with proper parameter restrictions, whereas there is always a positive probability for the interest rate in the long-memory interest rate model of Backus and Zin (1993) to take on negative values.

For estimation purposes it is more convenient to express these interest rate dynamics with an innovation that has a zero conditional mean. Let $v_{t+1} \equiv (1/\sqrt{2})(\varepsilon_{t+1}^2 - 1)$. It is clear that v_{t+1} has mean 0 and variance 1, conditional on Ω_t. Model 1 can thus be rewritten as

$$r_t^* = \beta_0^* + \sum_{j=1}^{\max(p,q)} (\alpha_j + \beta_j) L^{j-1} r_{t-1}^* + \sqrt{2} \sum_{i=1}^{\max(p,q)} \alpha_i L^{i-1} r_{t-1}^* v_t^2. \tag{9}$$

Note that if $p > q$, then $\beta_j = 0$ for $j = q+1, \cdots, p$. Similarly, $\alpha_i = 0$ for $i = p+1, \cdots, q$ if $q > p$. Model 2, on the other hand, can be written as

$$r_t^* = \beta_0^* + [1 - (1 - \sum_{i=1}^{\max(p,q)} \phi_i L^{i-1})(1-L)^d] r_t^* + \sqrt{2}[1 - \sum_{j=1}^{p} \beta_j L^j - (1 - \sum_{i=1}^{\max(p,q)} \phi_i L^i)(1-L)^d] r_t^* v_{t+1}. \tag{10}$$

Expressing the fractional difference operation as an infinite sum can also be a useful alternative expression for the long memory interest rate model. An alternative expression is given below when equation (10) is derived based on the FIGARCH$(1, d, 1)$ process:

$$r_t^* = \beta_0^* + \sum_{k=0}^{\infty} [\theta \pi_k(d) - \pi_{k+1}(d)] L^k r_{t-1}^* + \sqrt{2}\{\sum_{k=0}^{\infty} [\theta \pi_k(d) - \pi_{k+1}(d)] L^k - \beta_1\} r_{t-1}^* v_t. \tag{11}$$

We have used in the above expression the fact that $(1-L)^d = \sum_{k=0}^{\infty} \pi_k(d) L^k$ with $\pi_0(d) = 1$ and $\pi_k(d) = (-1)^k \prod_{i=1}^{k} \frac{d-i+1}{i}$ for $k \geq 1$.

3 Heteroskedasticity in Equilibrium Interest Rates

Inspection of expressions (7) and (8) illustrates that the equilibrium processes derived in Section 2 allow for heteroskedasticity, because the innovations depend on the level of the lagged interest rate. In this section we show that the class of equilibrium interest rate processes derived in Section 2 can be easily extended to accommodate more general heteroskedasticity patterns. This extension is motivated by empirical studies supporting more general heteroskedasticity patterns. It must be noted that the empirical studies that made this observation start by assuming the existence of an *ad hoc* short interest rate process, without deriving this interest rate process in an equilibrium context. In this paper we show that interest rate processes with heteroskedasticity patterns similar to the ones in these *ad hoc* models can be derived and analyzed within an equilibrium framework.

Chan, et al. (1992) study the following specification for the short interest rate

$$r_t = \beta r_{t-1} + r_{t-1}^{\psi} \nu_t \tag{12}$$

where ν_t is a disturbance term with conditional mean zero. Their empirical results show that the data indicate point estimates of ψ around 1.5, and that these estimates are statistically different from 1. Therefore, the data seem to indicate support for heteroskedasticity patterns that are different from the ones modeled in Section 2.[13]

It is straightforward to incorporate this type of heteroskedasticity into the equilibrium model derived in Section 2. If we change the consumption dynamic to

$$q_{t+1} = \beta_0 + \sum_{j=1}^{p} \beta_j L^{j-1} q_t + \sum_{i=1}^{q} \alpha_i L^{i-1} q_t^{\psi} \varepsilon_t^2, \tag{13}$$

the corresponding interest rate dynamic becomes

Model 3. Short-memory interest rate dynamic with general heteroskedasticity:

$$r_t^* = \beta_0^* + \sum_{j=1}^{p} \beta_j L^{j-1} r_{t-1}^* + \sum_{i=1}^{q} \alpha_i L^{i-1} (r_{t-1}^*)^{\psi} \varepsilon_t^2. \tag{14}$$

To obtain the long-memory version of the model, we simply adopt the following consumption dynamic:

$$q_{t+1} = \beta_0 + \sum_{j=1}^{p} \beta_j L^{j-1} q_t + [1 - \sum_{j=1}^{p} \beta_j L^j - (1 - \sum_{i=1}^{\max(p,q)} \phi_i L^i)(1-L)^d] q_{t+1}^{\psi} \varepsilon_{t+1}^2. \tag{15}$$

The resulting interest rate model is

Model 4. Long-memory interest rate dynamic with general heteroskedasticity:

$$r_t^* = \beta_0^* + \sum_{j=1}^{p} \beta_j L^{j-1} r_{t-1}^* + [1 - \sum_{j=1}^{p} \beta_j L^j - (1 - \sum_{i=1}^{\max(p,q)} \phi_i L^i)(1-L)^d] (r_t^*)^{\psi} \varepsilon_{t+1}^2. \tag{16}$$

The incorporation of this type of heteroskedasticity in the context of our model is technically trivial. Inspecting the issue from a different perspective, however, it shows the flexibility of the equilibrium approach when focused on modeling the conditional volatility dynamic. The essence of the standard Lucas type of asset pricing equilibrium is that the dynamic for the equilibrium interest rate is driven by the behavior of aggregate consumption. The most straightforward way of getting an internally consistent interest rate model is via modeling the conditional volatility of the

[13]Subsequent studies, significantly differing in empirical methodology and model specification, have confirmed these findings (see for example Ait-Sahalia (1996), Campbell, et al. (1997), Conley, Hansen, Luttmer and Scheinkman (1997) and Ahn and Gao (1999)). Andersen and Lund (1997), on the other hand, do not find evidence for these type of heteroskedasticity effects.

aggregate consumption. The current literature on equilibrium models of the term structure does not follow this approach. Consequently, existing equilibrium models face difficulties in incorporating this type of heteroskedasticity into the specification of the interest rate process.

To emphasize this important point, we demonstrate in our discrete time environment the problems of incorporating heteroskedasticity when using a model that displays similarities with the well-known class of exponential affine term structure models.[14] In the simplest of these exponential affine term structure models, the consumption dynamic is given by

$$\ln \frac{C_{t+1}}{C_t} = \phi \ln \frac{C_t}{C_{t-1}} + \eta_{t+1} \tag{17}$$

$$\eta_{t+1}|\Omega_t \sim N(0, \sigma^2), \tag{18}$$

which yields the following AR(1) equilibrium short-term interest rate dynamic:

$$r_t = \delta(1-\phi) + (\sigma^2/2)(1-\phi) + \phi r_{t-1} + \phi \eta_t. \tag{19}$$

This equilibrium interest rate model is similar to the continuous-time Vasicek model. At the first glance, it seems easy to insert heteroskedasticity into the equilibrium interest rate dynamic by changing the consumption dynamic to

$$\ln \frac{C_{t+1}}{C_t} = \phi \ln \frac{C_t}{C_{t-1}} + (\ln \frac{C_t}{C_{t-1}})^{\psi} \eta_{t+1}. \tag{20}$$

It can be easily verified that this consumption dynamic yields an analytical solution for the equilibrium interest rate only when $\psi = 0.5$. The nature of heteroskedasticity in the resulting model is similar to the continuous-time Cox-Ingersoll-Ross (1985) model, even though a comparison with this continuous-time model is tenuous. It nevertheless shows that the potential for building heteroskedasticity into equilibrium models of this class is very limited. In contrast, our approach allows for a wider range of heteroskedasticity patterns.

4 Estimation and Test Results

The parameters in Models 1-4 can in principle be estimated in several ways. We estimate the parameters using the quasi-maximum likelihood technique proposed in White (1982) and Bollerslev and Wooldridge (1992). This technique is used instead of exact maximum likelihood estimation, because the innovation in these systems is governed by a χ^2 distribution with one degree of freedom, which has an unbounded likelihood at the origin. To the best of our knowledge it is not possible to restrict the parameter space to deal with this problem.

We present estimation and test results using two different data sets. The first data set is the one analyzed in Ait-Sahalia (1996). This data set contains daily observations on one-week Eurodollar rates from June 1, 1973 to February 25, 1995, totaling 5505 observations. It is presented in Figure

[14] A comparison between discrete and continuous-time models is often tenuous. However, the characteristic of the model under study here has similar effects in both modeling approaches.

1. The second data set contains daily observations on overnight Eurodollar rates from January 1, 1981 to May 30, 1997, for a total of 4280 observations. This dataset is presented in Figure 2. In much of the term structure literature, there is a trade-off between using overnight rates versus one-week rates as the short-term interest rates. When estimating and testing continuous-time models, for instance, conceptually there is a clear advantage to using the overnight rates, because these data are a closer approximation to the theoretical construct of the model (see Chapman, Long and Pearson (1999) for an extensive analysis of this issue). In this paper, the theory is formulated in discrete time. If the data has daily frequency, the overnight rate is, by the definition of the model, the unambiguous choice for the short-term interest rate. Overnight rates are, however, more prone to microstructure effects, which can considerably complicate the interpretation of the empirical findings. We therefore also estimate the models using the one-week rates. The importance of the microstructure effects for the overnight data can clearly be seen from Figure 2, which contains far more outliers than Figure 1.[15] To illustrate the potential importance of long-memory specifications for the data, Figures 3 and 4 provide the sample autocorrelations up to lag 600 for the weekly and overnight data respectively. For the weekly yields in Figure 3, the autocorrelation at lag 600 is 0.21. For the daily data, the autocorrelation at lag 600 is even higher at 0.28. It is clear that long memory specifications are very useful to capture this kind of empirical phenomena, because it may be hard to fit these characteristics of the data with standard short memory models.

Tables 1 through 8 present the estimation results. Table 1 presents results for Model 1 (the short memory interest rate dynamics) obtained using the data on the one-week Eurodollar rates. Table 2 also uses the one-week Eurodollar rates but presents results for Model 2 (the long memory interest rate dynamic). Tables 3 and 4 present results for the short and long memory dynamics respectively, this time using data on the overnight Eurodollar rates. Tables 5 and 6 present results for the one-week rates using Models 3 and 4 respectively (the short and long memory interest rate dynamics with general heteroskedasticity). Finally, Tables 7 and 8 present results for the overnight rates using Models 3 and 4. In each table, a number of entries present estimates of parameters with the robust asymptotic standard error indicated in parentheses. Please note that in some cases parameters and standard errors are multiplied by 10, to improve the presentation of the results. Also note that the number of observations used in the estimation of the weekly data is 4505 and the number of observations for the daily data is 3280. The number of observations available in the two data sets are 5505 and 4280, respectively, but the first 1000 observations are reserved for setting the initial values of the interest rate dynamic implied by the FIGARCH consumption process. Inspection of expression (11) shows that one actually needs an infinite number of data points for the initial condition. In practice, one is forced to truncate the infinite sum in (11) at a sufficiently high number. In our empirical analyses, the data required for the initial condition are truncated at 1000 lags. Implicit in this practice is our assumption that the first 1000 data points are enough to bring the system to its steady state so that the resulting likelihood function for either Model 2 or 4 is a good proxy for the true likelihood function. For Models 1 and 3, the initial condition does

[15] To neutralize the effects of these outliers, one can filter the data to remove microstructure effects. However, Figure 1 shows that the one-week rates also contain significant outliers, and the analysis of these rates is standard. Therefore, we decided to analyze the raw data. Interestingly, we find that the estimation results for the overnight rates and one-week rates are not dramatically different along some dimensions, perhaps suggesting that the outliers are important for both series or for neither.

not require as many data points. In order to ensure meaningful comparisons between the results of Model 1 (3) and Model 2 (4), however, we also discard the first 1000 data points. Besides point estimates and standard errors for the parameters, columns 14 through 20 of each table also contain a variety of robust Wald tests that test the joint significance of extra coefficients for higher order specifications. The notational convention used in these columns is as follows: for a given row, α_k denotes the highest order α coefficient for that specification and β_k denotes the highest order β coefficient. For example, in the row corresponding to the GARCH$(2,2)$ model, α_k stands for the α_2 coefficient. Similarly, in the row corresponding to the GARCH$(6,6)$ model β_{k-1} stands for the β_5 coefficient. This notation explains the interpretation of the robust Wald statistics that are listed: for example, for the row corresponding to the GARCH$(5,5)$ specification the column with $\beta_{k-1} = \beta_k = 0$ presents the robust Wald statistic that $\beta_5 = \beta_4 = 0$. The last column in each table presents the value of the objective function at the optimum. It must be noted that because we estimate using quasi-maximum likelihood, the value of the objective function cannot be used to construct test statistics and we have to resort to the robust Wald statistics reported in the tables.

Table 1 presents the results for Model 1 using the one-week Eurodollar rates. The first row presents results corresponding to the GARCH$(1,1)$ consumption dynamic. This model is of particular interest because the resulting interest rate model displays similarities with a number of models in the literature. First note that the interest rate process inherits the characteristics of the conditional volatility process. Since the GARCH process has its conditional volatility behaving like an ARMA process, we can think of the equilibrium interest rate dynamic as an ARMA process with heteroskedastic features. In the case of the GARCH$(1,1)$ model, the resulting equilibrium interest rate process is Markovian and bears resemblance to the models belonging to the exponential affine class of interest rate models. Our reference to the ARMA process is tenuous, because the positivity requirement on the conditional volatility of the consumption process imposes restrictions on the parameters of the interest rate process. These restrictions do not apply if one directly writes down an ARMA process for the short-term interest rate without deriving it via an equilibrium argument.

Bollerslev (1986) gave the conditions on β_i and α_i of equation (5) under which the conditional volatility is always positive. His conditions require β_i and α_i to be non-negative. These conditions can actually be relaxed when one goes beyond the GARCH$(1,1)$ model. Nelson and Cao (1992) provided a weaker set of requirements that can ensure positive conditional volatility under GARCH(p,q). Our estimation results require invoking Nelson and Cao's (1992) conditions because some parameter estimates turn out to be negative. We have verified that the parameter values presented in Table 1 do satisfy the conditions for positive conditional variances.

Estimation results for the interest rate process corresponding to the GARCH$(1,1)$ consumption dynamic indicate that all coefficients are very precisely estimated. The estimate for β_1 is large, indicating a slow mean reversion (large persistence). The estimate for α_1 is small and the sum of α_1 and β_1 is smaller than one, satisfying the stationarity requirement. These results are consistent with the standard empirical findings in the literature when Markovian processes are used in estimation. The interesting question is therefore whether extensions to non-Markovian processes are required by the data. In rows 2 and 4, we present the results for the GARCH$(1,2)$ and GARCH$(2,2)$ dynamics, respectively. The results in rows 2 and 4 clearly support extensions to non-Markovian models. In row 2, the estimate for the parameter α_2 is clearly significant, based on the robust standard error.

A similar conclusion applies to the parameter β_2 in column 4. These results are supportive of the results established by Backus and Zin (1994), who reject the Markovian assumption in exponential affine models.

The other results for Model 1 presented in Table 1 are now summarized briefly. The table contains results for GARCH(p, q) processes with GARCH$(6, 6)$ the highest order that is reported. Higher order GARCH models were investigated but are not reported to keep the tables manageable. The main result is that for high orders of p and q (higher than 8), the robust Wald statistics indicate that the inclusion of extra coefficients does not improve the statistical fit of the model. This result is not surprising given the results reported in the table: the results for the GARCH$(5, 5)$ and GARCH$(6, 6)$ specifications indicate that for these models, t-statistics are much smaller than t-statistics for lower-order models, such as the GARCH$(2, 2)$ model. Not surprisingly, therefore, the robust Wald statistics for joint significance are much lower and in some cases indicate nonrejection of the null hypothesis that these extra coefficients are jointly equal to zero.

Table 2 presents results for Model 2, which also uses the weekly data but is based on the FIGARCH consumption dynamic. We start by inspecting the first row, which shows that the FIGARCH$(1, d, 1)$ dynamic yields an estimate of the long memory parameter d of 0.0122. Even though the estimated value for d is rather small, it is statistically different from zero as indicated by its robust standard error. The data seem therefore supportive of long memory in the interest rate dynamic. This finding of course confirms that the Markovian assumption is quite a poor assumption for the one-week interest rate data. The remaining rows in the table indicate what happens if we increase p and q in the specification of the FIGARCH(p, d, q) process. Importantly, they confirm the findings of Table 1: as we increase p and q, the standard errors increase rapidly and for the FIGARCH$(6, d, 6)$ process, the robust Wald statistics indicate in several cases that extra coefficients are not supported by the data. This is also the case when we investigate specifications for p and q larger than 6 (not reported). Table 2 also allows us to answer the following important question: as p and q increase, what happens to the estimate of the long-memory parameter d? This question is important because of the finding in row 1 that the long-memory parameter is significantly estimated. Whereas long-memory processes are of interest because they allow us to capture important empirical phenomena with a rather parsimonious parameterization, their implementation in equilibrium term structure models is far from obvious. It would therefore be interesting if we could capture nonzero autocorrelation at long horizons by increasing p and q in the specification of the FIGARCH(p, d, q) process. Inspection of Table 2 indicates that the estimates of d differ substantially between the different rows. However, for the FIGARCH$(1, d, 1)$, FIGARCH$(3, d, 3)$, FIGARCH$(4, d, 4)$ and FIGARCH$(5, d, 5)$ processes the estimates are all slightly larger than 0.01, and are very precisely estimated. The interesting outlier is the estimate of d for the FIGARCH$(6, d, 6)$ process, which is negative and insignificant. When investigating p and q higher than 6, we also found negative point estimates of d that were insignificantly estimated (not reported). This seems to indicate that by increasing p and q in the FIGARCH(p, d, q) process, it is possible to capture the empirical phenomena that give rise to a long memory parameter that is significantly different from zero in the FIGARCH$(1, d, 1)$ model.[16] The extra coefficients introduced in the FIGARCH$(6, d, 6)$ model

[16]It must be noted that these statements are based on in-sample fitting. In terms of predictive accuracy, it is likely that one will have to pay a price for estimating large numbers of MA parameters. Ray (1991) provides an analysis

capture the correlation patterns captured by the long memory parameter in more parsimonious models. The remaining correlation captured by the long-memory parameter is not statistically significant.

Tables 3 and 4 present the results for Models 1 and 2 using the overnight Eurodollar rates. To some extent, the estimates support the general conclusions drawn from those presented in Table 1. Row 1 in Table 3 indicates that the data support deviations from Markovian models. Row 1 in Table 4 indicates that the data support the presence of long memory. Again, the estimate for the coefficient that captures long memory is small but significantly different from zero. Finally, as we increase p and q in the FIGARCH(p, d, q) model in Table 4, it becomes possible to capture the empirical phenomena that yield a significantly positive long memory parameter in row 1 with short memory parameters. The last row of Table 4 indicates that for the FIGARCH$(6, d, 6)$ model, the estimated d parameter is negative. This result is different from the one obtained in Table 2, where the point estimate is also negative but statistically insignificant. It must be noted that investigation of p and q larger than 6 confirms the presence of a negative long-memory parameter, which in many cases is insignificantly estimated (not reported).

However, a comparison of Tables 1 and 3 indicates that there is an interesting difference between the empirical regularities for the weekly rates as opposed to the overnight rates. Whereas in Table 1 the standard errors increase significantly as we increase p and q in the GARCH(p, q) specification, this is not the case in Table 3. The same can be said about the comparison between Tables 2 and 4. Not surprisingly therefore, the robust Wald statistics in Tables 3 and 4 indicate that the inclusion of extra coefficients is supported by the data. Investigation of p and q higher than 6 indicated that very high orders are needed to provide an adequate statistical fit of the overnight rates (not reported). The highest order that we investigated is $p = q = 15$, and the test statistics indicated that including α_{15} and β_{15} improves the statistical fit of the model. Perhaps this finding is not surprising: comparison of Figures 1 and 2 shows that the time series of overnight rates contains many outliers which result from microstructure effects, and is consequently harder to fit with a parsimonious parameterization. These findings are of interest for the study of term-structure models. They seem to indicate that the performance for a given model may be critically affected by the choice of the short-term interest rate.

Tables 5 and 6 present the estimation results for Models 3 and 4, using data on one-week Eurodollar rates. The result corresponding to the modified GARCH$(1, 1)$ dynamic is presented in row 1 of Table 5. The point estimate for ψ is 1.340. This heteroskedasticity parameter is very precisely estimated. The result based on the robust standard error shows that it is significantly different from 1. This result is largely consistent with the finding by Chan, et al. (1992), although our estimate for ψ is slightly lower than theirs. The other results in Table 5 and 6 can be summarized very briefly, because they largely confirm the findings of Tables 1 and 2. The data favor larger values for p and q, and therefore support departures from the Markovian model. However, for the modified GARCH$(5, 5)$ and GARCH$(6, 6)$ specifications the standard errors are fairly large and the robust Wald statistics in some cases favor a more parsimonious parameterization. The estimated values for ψ in rows 2 to 11 are very similar to the one in row 1. The estimation using the long-memory model in Table 6 also yields a value of ψ similar to those in Table 5. The estimates for the α_i

of the predicitve performance of long-memory and competing short memory models, but analyzes AR models only.

parameters in Tables 5 and 6 are very different from those in Tables 1 and 2. The estimates for the β_i parameters are relatively similar, however. Point estimates of the long-memory parameter d in Table 6 are lower than those in Table 2. For the modified FIGARCH$(6, d, 6)$ specification, the estimated d parameter is negative and insignificantly estimated.

Tables 7 and 8 present our empirical results for Models 3 and 4 using the overnight Eurodollar rates. Interestingly, the point estimates for the parameter ψ are very different from those in Tables 5 and 6. Moreover, the robust standard errors suggest that the more general heteroskedasticity pattern is not supported by the data. This result indicates that the heteroskedasticity properties of the overnight Eurodollar rates are different from the one-week Eurodollar rates. Given that the data do not support a more general heteroskedasticity pattern, it is not surprising that the other parameter estimates for Models 3 and 4 presented in Tables 7 and 8 do not differ greatly from those for Models 1 and 2 presented earlier in Tables 3 and 4.

5 The Economic Significance of Long Memory

Whereas the estimates of the long memory parameter d are statistically significant for a parsimonious parameterization of the short memory component, they are fairly small. It is therefore of critical importance to investigate whether these small estimates of the long-memory parameter are economically significant. To investigate this issue, we present the results of a small-scale simulation experiment that investigates the implications of the long-memory estimates obtained in the previous section for the equilibrium term structure. The implications for the term structure can be investigated by exploiting the pricing result using the equilibrium price measure Q, which was earlier adopted in this context in Duan (1996) for the GARCH term structure model. The expression for the yield to maturity at time t for a default-free zero-coupon bond of maturity τ is

$$R_t(\tau) = -\frac{1}{\tau} \ln\{E^Q(e^{-\sum_{i=t}^{t+\tau-1} r_i} | \Omega_t). \tag{21}$$

To implement this pricing formula we use the expression for the interest rate dynamic under the equilibrium pricing measure Q. We present results for model 2, without the general heteroskedasticity built into model 4. Even though the innovation in our interest rate model is nonstandard, it is possible to characterize the interest rate dynamic (see Duan, 1996). It is important to note that this becomes possible because of the choice of a GARCH specification for the volatility dynamic. This choice of volatility dynamic implies that the driving process for interest rate is a noncentral chi-squared. We are therefore able to characterize the risk-neutral distribution because this noncentral chi-squared is a simple transformation of the normal distribution. Specifically, using the expressions for the short memory dynamic (7) and the long memory dynamic (8) under the data generating probability measure, the short memory dynamic under the equilibrium pricing measure Q is therefore

$$r_t^* = \beta_0^* + \sum_{j=1}^{p} \beta_j L^{j-1} r_{t-1}^* + \sum_{i=1}^{q} \alpha_i L^{i-1} r_{t-1}^* (\varepsilon_t - \sqrt{\eta r_{t-1}^*})^2 \tag{22}$$

and the long memory dynamic under the equilibrium pricing measure is

$$r_t^* = \beta_0^* + \sum_{j=1}^{p} \beta_j L^{j-1} r_{t-1}^* + [1 - \sum_{j=1}^{p} \beta_j L^j - (1 - \sum_{i=1}^{\max(p,q)} \phi_i L^i)(1-L)^d] r_t^* (\varepsilon_t - \sqrt{\eta r_{t-1}^*})^2. \quad (23)$$

In both equations the parameter η, which represents the price of risk, is equal to $\gamma(2\kappa - \gamma)/2$. In our framework the price of risk is therefore not only determined by the rate of relative risk aversion, but also by the specification of the GARCH-in-mean equation in (3). Moreover, η cannot be determined from the time series dimension of interest rates alone. Whereas our estimates in Tables 1 through 4 identify the regression intercept β_0^*, this intercept is equal to $\beta_0\gamma(2\kappa - \gamma)/2$ and therefore does not identify η. In order to implement (21) using the time-series parameters in Tables 2 and 4, we therefore need to make an assumption on the parameter η. In the simulation experiments below, η is set arbitrarily at 150. It must also be noted that whereas we have analytical expressions for the interest rates under the equilibrium pricing measure, we do not have analytical results for the yields to maturity in (21). Therefore, we have to evaluate expression (21) using simulation. In the simulation experiment below, the number of simulations used to compute bond yields is set at 5000. Increasing the number of simulations above 5000 does not change the results.

One can conceive of several experiments to demonstrate the impact of long memory. The purpose of this section is not to present an exhaustive analysis of all possible experiments. Rather, we want to present a limited set of results to convince the reader of the economic importance of long memory. This experiment also serves to demonstrate that our analysis is a full-fledged general equilibrium approach, and not ad hoc. To achieve this, we set up the following experiment. We use the estimates for the model parameters obtained in Table 4, using overnight Eurodollar rates. To focus on the importance of long memory, we limit our attention to the FIGARCH$(1, d, 1)$ specification. The parameter estimates used are therefore $\beta_1 = 0.987$, $\beta_0^* = 0.0132$, $\phi_1 = 0.998$ and $d = 0.0129$. We then generate term structures by computing the yields starting for day $t = 1$ to $t = 3650$ (10 years), using different sets of initial conditions. Now examine Figure 5. The yield curves in this figure are obtained by setting yesterday's interest rate (the interest rate on day $t = 0$) at an annualized 8%. The difference between the yield curves in Figure 5 is due to different values for the interest rates on days $t = -1$ to $t = -1000$.[17] For the solid line, the value for the interest rates on days $t = -1$ to $t = -1000$ is set at 2%. For the broken line, those rates are set at 8%. Over a wide range of maturities, the difference between the simulated yields exceeds 20 basis points. If the long memory parameters are economically insignificant, we would not expect to see any differences between these term structures. The results would be indistinguishable from the results under a Markovian system with yesterday's interest rate at 8%. Our results suggest to the contrary and illustrate the importance of long memory.[18]

[17] In the simulation experiment, we implement the long-memory specification in the same way as in the estimation, namely by truncating the infinite sum in (11) at 1000.

[18] It may at first seem counterintuitive that the long yield is higher in the experiments where lagged interest rates are low. However, this is simply a consequence of the finding that was discussed in the previous section. Short-memory parameters can generate significant autocorrelation at long horizons, provided that the process is sufficiently persistent. The AR(1) coefficient in the dynamic used for the simulation experiment is actually $\phi_1 + \pi_1(d) = 0.998$ - (-0.0129) = 1.0109. Therefore, this "short memory" component by itself would actually generate an explosive process.

6 Conclusion

This paper analyzes a large class of potential dynamics for the short-term interest rate. These dynamics are derived in a discrete-time equilibrium setting in which interest rates are determined by the specification for the conditional volatility of the aggregate consumption process, instead of the conditional mean of consumption. By modeling the conditional volatility of consumption as a nonnegative process, it is then possible to model nonnegative interest rates under certain parameter restrictions. In this paper such an approach is implemented by modeling the conditional volatility of consumption as a GARCH-type process, implying a chi-square innovation for interest rates. The class of interest rate models that we derive is very large and includes Markovian and non-Markovian interest rate dynamics. The interest rate dynamic can be of short or long memory. Because the conditional volatility of the consumption is the driving force behind the interest rate dynamic, it is straightforward to incorporate more general heteroskedasticity patterns that are known to be more consistent with interest rate data.

The empirical analysis, using overnight and one-week Eurodollar rates, indicates that the data support deviations from the Markovian assumption. The long memory parameter is small but very precisely estimated. A simulation experiment shows that the impact of long memory on the term structure is not negligible. For one-week Eurodollar rates, the data support heteroskedasticity patterns different from those within the exponential affine class. For the overnight Eurodollar rates, the result is opposite. Interestingly, the overnight and one-week rates also differ in other dimensions. When the interest rate model is restricted to be Markovian, however, the two interest rate series produce similar outcomes. Our empirical results seem to indicate that the class of equilibrium interest rate models proposed in this paper can parsimoniously capture some important aspects of the data. The ability of this class of equilibrium models to describe the cross-sectional variations of the term structure and to price term structure derivatives will be studied in future work. Another finding that deserves attention is the apparent ability of richly parameterized short memory models to capture long memory phenomena. It is clear that parsimony in parameterization may be a goal in itself. To get a clearer view of the performance of the long memory model versus richly parameterized short memory models, it will be necessary to evaluate the predictive (out of sample) performance of these models as opposed to the descriptive (in-sample) performance.

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Combined with the other AR coefficients, which are determined by the long memory parameter, the process is not explosive, because the long memory component by itself induces negative correlation at long horizons. Therefore, in the simulation experiment, the higher the initial interest rates, the lower future interest rates and therefore the lower the yields on the long end of the term structure. However, even though the stylized fact that we set out to explain is positive correlation at long horizons (see Figures 3 and 4), this does not mean that the long memory specification does not fulfill its goal. On the contrary, it adds to the complexity of the correlation function, which leads to a better fit and economically significant differences in yields as documented by the simulation experiment.

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Table 1: Parameter estimates for the equilibrium short-term rate dynamics (Model 1-short memory)

using one-week Eurodollar rates (4505 observations)

Model 1: consumption dynamic	β_1 (s.e.)	β_2 (s.e.)	β_3 (s.e.)	β_4 (s.e.)	β_5 (s.e.)	β_6 (s.e.)	β_0^* x 10 (s.e.x 10)	α_1x 10 (s.e.x 10)	α_2x 10 (s.e.x 10)	α_3x 10 (s.e.x10)	α_4x 10 (s.e.x10)	α_5x 10 (s.e.10)	α_6x 10 (s.e.x10)	$\alpha_{k-1}=$ $\alpha_k=0$ (sig.level)	$\alpha_{k-2}=\alpha_{k-1}$ $=\alpha_k=0$ (sig.level)	$\alpha_2,...,$ $\alpha_k=0$ (sig.level)	$\beta_{k-1}=$ $\beta_k=0$ (sig.level)	$\beta_{k-2}=\beta_{k-1}$ $=\beta_k=0$ (sig.level)	$\beta_2,...,\beta_k$ $=0$ (sig.level)	$\alpha_k=$ $\beta_k=0$ (sig.level)	Quasi Log Likelihood
GARCH (1,1)	0.975	—	—	—	—	—	0.214	0.238	—	—	—	—	—	—	—	—	—	—	—	—	3790.9
	(0.001)	—	—	—	—	—	(0.084)	(0.011)	—	—	—	—	—	—	—	—	—	—	—	—	—
GARCH (1,2)	0.984	—	—	—	—	—	0.124	0.224	-0.078	—	—	—	—	—	—	—	—	—	—	—	4074.1
	(0.002)	—	—	—	—	—	(0.120)	(0.007)	(0.013)	—	—	—	—	—	—	—	—	—	—	—	—
GARCH (1,3)	0.985	—	—	—	—	—	0.119	0.223	-0.076	-0.010	—	—	—	57.64	—	—	—	—	—	—	4078.5
	(0.001)	—	—	—	—	—	(0.078)	(0.007)	(0.013)	(0.008)	—	—	—	(0.000)	—	—	—	—	—	—	—
GARCH (2,2)	1.173	-0.185	—	—	—	—	0.116	0.223	-0.111	—	—	—	—	—	—	—	—	—	—	79.85	4089.4
	(0.041)	(0.040)	—	—	—	—	(0.057)	(0.007)	(0.012)	—	—	—	—	—	—	—	—	—	—	(0.000)	—
GARCH(3,3)	1.229	-0.311	0.069	—	—	—	0.121	0.223	-0.122	0.018	—	—	—	49.37	—	—	9.49	—	—	0.57	4091.4
	(0.096)	(0.177)	(0.093)	—	—	—	(0.069)	(0.007)	(0.022)	(0.028)	—	—	—	(0.000)	—	—	(0.009)	—	—	(0.751)	—
GARCH(4,4)	1.435	-0.925	0.576	-0.100	—	—	0.123	0.221	-0.167	0.129	-0.056	—	—	45.27	128.72	—	37.34	51.71	—	39.99	4130.2
	(0.325)	(0.628)	(0.165)	(0.156)	—	—	(0.083)	(0.008)	(0.067)	(0.101)	(0.014)	—	—	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.000)	—
GARCH (5,5)	1.400	-0.756	0.171	0.195	-0.024	—	0.120	0.220	-0.159	0.096	0.015	-0.033	—	7.12	41.71	89.01	3.96	47.30	57.53	12.62	4149.4
	(0.100)	(0.341)	(0.511)	(0.423)	(0.163)	—	(0.117)	(0.007)	(0.021)	(0.056)	(0.063)	(0.043)	—	(0.028)	(0.000)	(0.000)	(0.138)	(0.000)	(0.000)	(0.002)	—
GARCH(6,6)	1.272	-0.858	0.709	-0.591	0.480	-0.033	0.119	0.219	-0.129	0.130	-0.065	0.090	-0.050	4.79	4.85	81.33	6.23	12.49	65.49	10.47	4177.2
	(0.090)	(0.267)	(0.534)	(0.608)	(0.423)	(0.099)	(0.442)	(0.007)	(0.022)	(0.057)	(0.073)	(0.084)	(0.034)	(0.091)	(0.183)	(0.000)	(0.044)	(0.006)	(0.000)	(0.005)	—

Table 2: Parameter estimates for the equilibrium short-term rate dynamics (Model 2-long memory)

using one-week Eurodollar rates (4505 observations)

Model 2: consumption dynamic	β_1 (s.e.)	β_2 (s.e.)	β_3 (s.e.)	β_4 (s.e.)	β_5 (s.e.)	β_6 (s.e.)	β_0^* x 10 (s.e. x 10)	ϕ_1 (s.e.)	ϕ_2 (s.e.)	ϕ_3 (s.e.)-	ϕ_4 (s.e.)	ϕ_5 (s.e.)	ϕ_6 (s.e.)	d (s.e.)	$\phi_{k-1}=\phi_k=0$ (sig.level)	$\phi_{k-2}=\phi_{k-1}=\phi_k=0$ (sig.level)	$\phi_2,\ldots,\phi_k=0$ (sig.level)	$\beta_{k-1}=\beta_k=0$ (sig.level)	$\beta_{k-2}=\beta_{k-1}=\beta_k=0$ (sig.level)	$\beta_2,\ldots,\beta_k=0$ (sig.level)	$\phi_k=\beta_k=0$ (sig.level)	Quasi Log Likelihood
FIGARCH (1,1)	0.989	—	—	—	—	—	0.106	0.999	—	—	—	—	—	0.0122	—	—	—	—	—	—	—	4058.0
	(0.001)	—	—	—	—	—	(0.071)	(0.001)	—	—	—	—	—	(0.0014)	—	—	—	—	—	—	—	—
FIGARCH (1,2)	0.987	—	—	—	—	—	0.112	1.004	-0.005	—	—	—	—	0.0051	—	—	—	—	—	—	—	4082.9
	(0.001)	—	—	—	—	—	(0.077)	(0.002)	(0.002)	—	—	—	—	(0.0024)	—	—	—	—	—	—	—	—
FIGARCH (1,3)	0.987	—	—	—	—	—	0.112	1.003	-0.005	0.0003	—	—	—	0.0059	5.85	—	—	—	—	—	—	4083.1
	(0.001)	—	—	—	—	—	(0.075)	(0.003)	(0.002)	(0.001)	—	—	—	(0.0033)	(0.054)	—	—	—	—	—	—	—
FIGARCH (2,2)	1.140	-0.152	—	—	—	—	0.119	1.159	-0.161	—	—	—	—	0.0029	—	—	—	—	—	—	7.62	4090.1
	(0.056)	(0.056)	—	—	—	—	(0.064)	(0.059)	(0.058)	—	—	—	—	(0.0039)	—	—	—	—	—	—	(0.022)	—
FIGARCH(3,3)	1.427	-0.912	0.472	—	—	—	0.115	1.437	-0.917	0.479	—	—	—	0.0114	74.07	—	—	74.59	—	—	75.21	4136.7
	(0.069)	(0.112)	(0.055)	—	—	—	(0.077)	(0.070)	(0.114)	(0.056)	—	—	—	(0.0016)	(0.000)	—	—	(0.000)	—	—	(0.000)	—
FIGARCH(4,4)	1.357	-0.744	0.182	0.190	—	—	0.123	1.369	-0.749	0.188	0.191	—	—	0.0108	124.04	191.09	—	123.74	191.42	—	2.10	4142.9
	(0.090)	(0.202)	(0.293)	(0.171)	—	—	(0.082)	(0.092)	(0.204)	(0.295)	(0.173)	—	—	(0.0024)	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.350)	—
FIGARCH (5,5)	1.360	-1.012	0.726	-0.445	0.353	—	0.120	1.372	-1.017	0.734	-0.446	0.356	—	0.0101	4.60	152.78	354.87	4.671	153.56	355.14	2.623	4156.9
	(0.155)	(0.175)	(0.654)	(0.776)	(0.393)	—	(0.089)	(0.157)	(0.174)	(0.655)	(0.779)	(0.397)	—	(0.0023)	(0.100)	(0.000)	(0.000)	(0.097)	(0.000)	(0.000)	(0.269)	—
FIGARCH(6,6)	1.283	-0.792	0.623	-0.519	0.466	-0.083	0.140	1.317	-0.815	0.640	-0.530	0.478	-0.092	-0.0120	3.47	13.30	101.92	3.69	14.18	94.71	0.81	4179.6
	(0.111)	(0.146)	(0.407)	(0.489)	(0.304)	(0.104)	(0.203)	(0.099)	(0.143)	(0.416)	(0.505)	(0.319)	(0.114)	(0.0210)	(0.176)	(0.004)	(0.000)	(0.158)	(0.003)	(0.000)	(0.667)	—

Table 3: Parameter estimates for the equilibrium short-term rate dynamics (Model 1-short memory)

using overnight Eurodollar rates (3280 observations)

Model 1: consumption dynamic	β_1 (s.e)	β_2 (s.e)	β_3 (s.e)	β_4 (s.e)	β_5 (s.e)	β_6 (s.e)	β_0^* x 10 (s.e. x 10)	α_1x 10 (s.e.x 10)	α_2x 10 (s.e.x 10)	α_3x 10 (s.e.x 10)	α_4x 10 (s.e.x 10)	α_5x 10 (s.e.x 10)	α_6x 10 (s.e.x 10)	$\alpha_{k-1}=$ $\alpha_k=0$ (sig.level)	$\alpha_{k-2}=\alpha_{k-1}$ $=\alpha_k=0$ (sig.level)	$\alpha_2,...,\alpha_k$ $=0$ (sig.level)	$\beta_{k-1}=$ $\beta_k=0$ (sig.level)	$\beta_{k-2}=\beta_{k-1}$ $=\beta_k=0$ (sig.level)	$\beta_2,...,\beta_k$ $=0$ (sig.level)	$\alpha_k=$ $\beta_k=0$ (sig.level)	Quasi Log Likelihood
GARCH (1,1)	0.971	—	—	—	—	—	0.538	0.245	—	—	—	—	—	—	—	—	—	—	—	—	3591.4
	(0.002)	—	—	—	—	—	(0.223)	(0.012)	—	—	—	—	—	—	—	—	—	—	—	—	—
GARCH (1,2)	0.977	—	—	—	—	—	0.368	0.242	-0.042	—	—	—	—	—	—	—	—	—	—	—	3629.9
	(0.002)	—	—	—	—	—	(0.172)	(0.012)	(0.013)	—	—	—	—	—	—	—	—	—	—	—	—
GARCH (1,3)	0.983	—	—	—	—	—	0.215	0.239	-0.050	-0.042	—	—	—	18.72	—	—	—	—	—	—	3671.8
	(0.003)	—	—	—	—	—	(0.146)	(0.012)	(0.012)	(0.015)	—	—	—	(0.000)	—	—	—	—	—	—	—
GARCH (2,2)	0.942	0.034	—	—	—	—	0.389	0.242	-0.033	—	—	—	—	—	—	—	—	—	—	21.14	3630.1
	(0.061)	(0.058)	—	—	—	—	(0.199)	(0.012)	(0.024)	—	—	—	—	—	—	—	—	—	—	(0.000)	—
GARCH(3,3)	0.819	0.158	0.003	—	—	—	0.267	0.239	-0.015	-0.046	—	—	—	8.63	—	—	6.72	—	—	9.33	3676.8
	(0.077)	(0.078)	(0.008)	—	—	—	(0.159)	(0.012)	(0.021)	(0.016)	—	—	—	(0.013)	—	—	(0.035)	—	—	(0.009)	—
GARCH(4,4)	0.796	-0.198	0.658	-0.270	—	—	0.156	0.235	-0.028	0.033	-0.115	—	—	34.76	59.12	—	23.59	27.91	—	18.82	3726.5
	(0.231)	(0.584)	(0.419)	(0.072)	—	—	(0.248)	(0.012)	(0.047)	(0.105)	(0.027)	—	—	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.000)	—
GARCH (5,5)	0.604	-0.050	0.442	0.099	-0.111	—	0.173	0.233	0.020	0.020	-0.070	-0.065	—	40.70	44.19	72.48	5.76	16.24	38.49	22.40	3755.0
	(0.203)	(0.365)	(0.237)	(0.110)	(0.049)	—	(0.240)	(0.012)	(0.040)	(0.054)	(0.025)	(0.015)	—	(0.000)	(0.000)	(0.000)	(0.056)	(0.001)	(0.000)	(0.000)	—
GARCH(6,6)	0.657	-0.058	0.022	0.063	0.793	-0.493	0.157	0.229	0.007	0.015	0.023	0.002	-0.138	141.52	145.31	206.91	338.40	414.95	706.92	240.80	3819.1
	(0.051)	(0.039)	(0.028)	(0.031)	(0.045)	(0.038)	(0.112)	(0.012)	(0.010)	(0.007)	(0.007)	(0.005)	(0.012)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	—

Table 4: Parameter estimates for the equilibrium short-term rate dynamics (Model 2-long memory)

using overnight Eurodollar rates (3280 observations)

Model 2: consumption dynamic	β_1 (s.e.)	β_2 (s.e.)	β_3 (s.e.)	β_4 (s.e.)	β_5 (s.e.)	β_6 (s.e.)	β_0^* x 10 (s.e. x 10)	ϕ_1 (s.e.)	ϕ_2 (s.e.)	ϕ_3 (s.e.)	ϕ_4 (s.e.)	ϕ_5 (s.e.)	ϕ_6 (s.e.)	d (s.e.)	$\phi_{k-1}=\phi_k=0$ (sig.level)	$\phi_{k-2}=\phi_{k-1}=\phi_k=0$ (sig.level)	$\phi_2,\ldots,\phi_k=0$ (sig.level)	$\beta_{k-1}=\beta_k=0$ (sig.level)	$\beta_{k-2}=\beta_{k-1}=\beta_k=0$ (sig.level)	$\beta_2,\ldots,\beta_k=0$ (sig.level)	$\phi_k=\beta_k=0$ (sig.level)	Quasi Log Likelihood
FIGARCH (1,1)	0.987	—	—	—	—	—	0.132	0.998	—	—	—	—	—	0.0129	—	—	—	—	—	—	—	3679.2
	(0.002)	—	—	—	—	—	(0.088)	(0.001)	—	—	—	—	—	(0.0022)	—	—	—	—	—	—	—	—
FIGARCH (1,2)	0.990	—	—	—	—	—	0.093	0.993	0.005	—	—	—	—	0.0207	—	—	—	—	—	—	—	3696.9
	(0.001)	—	—	—	—	—	(0.068)	(0.002)	(0.001)	—	—	—	—	(0.0024)	—	—	—	—	—	—	—	—
FIGARCH (1,3)	0.990	—	—	—	—	—	0.096	0.995	0.004	-0.001	—	—	—	0.0188	14.15	—	—	—	—	—	—	3697.4
	(0.001)	—	—	—	—	—	(0.059)	(0.004)	(0.002)	(0.002)	—	—	—	(0.0037)	(0.001)	—	—	—	—	—	—	—
FIGARCH (2,2)	0.663	0.325	—	—	—	—	0.111	0.669	0.330	—	—	—	—	0.0182	—	—	—	—	—	—	97.98	3721.6
	(0.079)	(0.079)	—	—	—	—	(0.088)	(0.079)	(0.079)	—	—	—	—	(0.0019)	—	—	—	—	—	—	(0.000)	—
FIGARCH(3,3)	0.664	-0.186	0.508	—	—	—	0.151	0.670	-0.182	0.510	—	—	—	0.0174	20.06	—	—	20.01	—	—	17.73	3755.1
	(0.230)	(0.241)	(0.121)	—	—	—	(0.099)	(0.211)	(0.242)	(0.121)	—	—	—	(0.0013)	(0.000)	—	—	(0.000)	—	—	(0.000)	—
FIGARCH(4,4)	0.863	-0.373	-0.100	0.593	—	—	0.185	0.871	-0.371	-0.099	0.596	—	—	0.0148	359.99	400.43	—	350.48	397.71	—	61.80	3778.2
	(0.101)	(0.138)	(0.127)	(0.085)	—	—	(0.171)	(0.101)	(0.138)	(0.126)	(0.085)	—	—	(0.0017)	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.000)	—
FIGARCH (5,5)	0.374	-0.187	0.020	0.234	0.531	—	0.310	0.383	-0.182	0.024	0.238	0.533	—	0.0148	297.32	845.59	806.06	290.25	810.93	828.89	11.09	3797.5
	(0.181)	(0.090)	(0.075)	(0.136)	(0.181)	—	(0.226)	(0.181)	(0.090)	(0.076)	(0.136)	(0.182)	—	(0.0016)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.004)	—
FIGARCH(6,6)	0.655	-0.084	0.039	0.053	0.774	-0.455	0.179	0.690	-0.085	0.042	0.055	0.774	-0.478	-0.0124	310.84	336.09	628.50	298.78	318.81	607.87	315.61	3825.4
	(0.080)	(0.046)	(0.040)	(0.059)	(0.054)	(0.029)	(0.121)	(0.082)	(0.049)	(0.041)	(0.060)	(0.055)	(0.029)	(0.0048)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	—

Table 5: Parameter estimates for the equilibrium short-term rate dynamics (Model 3-short memory) using one-week Eurodollar rates (4505 observations)

Model 3: consumption dynamic	β_1 (s.e.)	β_2 (s.e.)	β_3 (s.e.)	β_4 (s.e.)	β_5 (s.e.)	β_6 (s.e.)	β_0^* x 10 (s.e.x 10)	α_1x 10 (s.e.x 10)	α_2x 10 (s.e.x 10)	α_3x 10 (s.e.x 10)	α_4x 10 (s.e.x 10)	α_5x 10 (s.e.x 10)	α_6x 10 (s.e.x 10)	ψ (s.e)	$\alpha_{k-1}=$ $\alpha_k = 0$ (sig.level)	$\alpha_{k-2}=\alpha_{k-1}$ $=\alpha_k = 0$ (sig.level)	$\alpha_2,\ldots,\alpha_k$ $= 0$ (sig.level)	$\beta_{k-1}=$ $\beta_k = 0$ (sig.level)	$\beta_{k-2}=\beta_{k-1}$ $=\beta_k = 0$ (sig.level)	$\beta_2,\ldots,\beta_k$ $= 0$ (sig.level)	$\alpha_k=$ $\beta_k = 0$ (sig.level)	Quasi Log Likelihood
GARCH (1,1)	0.970	—	—	—	—	—	0.978	0.115	—	—	—	—	—	1.340	—	—	—	—	—	—	—	3933.4
	(0.002)	—	—	—	—	—	(0.161)	(0.019)	—	—	—	—	—	(0.066)	—	—	—	—	—	—	—	—
GARCH (1,2)	0.981	—	—	—	—	—	0.662	0.091	-0.033	—	—	—	—	1.415	—	—	—	—	—	—	—	4296.8
	(0.001)	—	—	—	—	—	(0.097)	(0.013)	(0.006)	—	—	—	—	(0.065)	—	—	—	—	—	—	—	—
GARCH (1,3)	0.982	—	—	—	—	—	0.640	0.091	-0.032	-0.002	—	—	—	1.414	33.42	—	—	—	—	—	—	4298.2
	(0.001)	—	—	—	—	—	(0.097)	(0.013)	(0.006)	(0.003)	—	—	—	(0.066)	(0.000)	—	—	—	—	—	—	—
GARCH (2,2)	1.092	-0.108	—	—	—	—	0.583	0.092	-0.040	—	—	—	—	1.408	—	—	—	—	—	—	26.77	4303.8
	(0.046)	(0.045)	—	—	—	—	(0.095)	(0.013)	(0.008)	—	—	—	—	(0.065)	—	—	—	—	—	—	(0.000)	—
GARCH(3,3)	1.161	-0.346	0.165	—	—	—	0.708	0.090	-0.043	0.015	—	—	—	1.416	20.18	—	—	2.46	—	—	1.33	4313.0
	(0.117)	(0.265)	(0.158)	—	—	—	(0.182)	(0.013)	(0.011)	(0.017)	—	—	—	(0.065)	(0.000)	—	—	(0.293)	—	—	(0.513)	—
GARCH(4,4)	1.221	-0.652	0.480	-0.069	—	—	0.702	0.091	-0.050	0.040	-0.018	—	—	1.411	20.45	25.54	—	44.99	48.87	—	13.53	4346.3
	(0.105)	(0.148)	(0.078)	(0.061)	—	—	(0.124)	(0.013)	(0.013)	(0.012)	(0.006)	—	—	(0.064)	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.001)	—
GARCH (5,5)	1.240	-0.571	0.149	0.180	-0.018	—	0.712	0.091	-0.053	0.033	0.006	-0.011	—	1.405	7.37	22.11	26.45	3.75	40.38	43.65	7.06	4358.9
	(0.092)	(0.188)	(0.242)	(0.193)	(0.064)	—	(0.116)	(0.013)	(0.012)	(0.013)	(0.013)	(0.008)	—	(0.064)	(0.025)	(0.000)	(0.000)	(0.153)	(0.000)	(0.000)	(0.029)	—
GARCH(6,6)	1.149	-0.617	0.437	-0.300	0.287	0.018	0.869	0.091	-0.044	0.040	-0.013	0.024	-0.014	1.402	5.39	5.40	26.10	20.48	24.71	57.62	17.10	4388.6
	(0.069)	(0.226)	(0.416)	(0.440)	(0.275)	(0.046)	(0.184)	(0.013)	(0.009)	(0.021)	(0.022)	(0.025)	(0.009)	(0.065)	(0.067)	(0.145)	(0.000)	(0.000)	(0.0000)	(0.000)	(0.000)	—

Table 6: Parameter estimates for the equilibrium short-term rate dynamics (Model 4-long memory)

using one-week Eurodollar rates (4505 observations)

Model 4: consumption dynamic	β_1 (s.e.)	β_2 (s.e.)	β_3 (s.e.)	β_4 (s.e.)	β_5 (s.e.)	β_6 (s.e.)	β_0^* x 10 (s.e.x 10)	ϕ_1 (s.e.)	ϕ_2 (s.e.)	ϕ_3 (s.e.)	ϕ_4 (s.e.)	ϕ_5 (s.e.)	ϕ_6 (s.e.)	d (s.e.)	ψ (s.e.)	$\phi_{k-1}=$ $\phi_k=0$ (sig.level)	$\phi_{k-2}=\phi_{k-1}$ $=\phi_k=0$ (sig.level)	$\phi_2,\ldots,\phi_k$ $=0$ (sig.level)	$\beta_{k-1}=$ $\beta_k=0$ (sig.level)	$\beta_{k-2}=\beta_{k-1}$ $=\beta_k=0$ (sig.level)	$\beta_2,\ldots,\beta_k$ $=0$ (sig.level)	$\phi_k=$ $\beta_k=0$ (sig.level)	Quasi Log Likelihood
FIGARCH (1,1)	0.986	—	—	—	—	—	0.479	0.990	—	—	—	—	—	0.0054	1.397	—	—	—	—	—	—	—	4259.0
	(0.002)	—	—	—	—	—	(0.084)	(0.001)	—	—	—	—	—	(0.0009)	(0.065)	—	—	—	—	—	—	—	—
FIGARCH (1,2)	0.983	—	—	—	—	—	0.592	0.991	-0.003	—	—	—	—	0.0014	1.412	—	—	—	—	—	—	—	4300.9
	(0.002)	—	—	—	—	—	(0.105)	(0.002)	(0.001)	—	—	—	—	(0.0011)	(0.070)	—	—	—	—	—	—	—	—
FIGARCH (1,3)	0.983	—	—	—	—	—	0.583	0.990	-0.002	0.0002	—	—	—	0.0020	1.411	11.12	—	—	—	—	—	—	4301.4
	(0.002)	—	—	—	—	—	(0.103)	(0.002)	(0.001)	(0.0004)	—	—	—	(0.0013)	(0.069)	(0.004)	—	—	—	—	—	—	—
FIGARCH (2,2)	1.072	-0.088	—	—	—	—	0.565	1.080	-0.092	—	—	—	—	0.0008	1.408	—	—	—	—	—	—	4.34	4304.5
	(0.057)	(0.056)	—	—	—	—	(0.112)	(0.059)	(0.058)	—	—	—	—	(0.0017)	(0.058)	—	—	—	—	—	—	(0.114)	—
FIGARCH(3,3)	1.220	-0.638	0.399	—	—	—	0.633	1.225	-0.640	0.401	—	—	—	0.0040	1.408	178.33	—	—	179.78	—	—	85.76	4351.5
	(0.026)	(0.050)	(0.045)	—	—	—	(0.110)	(0.026)	(0.051)	(0.046)	—	—	—	(0.0007)	(0.040)	(0.000)	—	—	(0.000)	—	—	(0.000)	—
FIGARCH(4,4)	1.212	-0.573	0.175	0.166	—	—	0.675	1.217	-0.575	0.178	0.167	—	—	0.0039	1.409	79.16	209.71	—	79.22	209.96	—	2.81	4357.4
	(0.026)	(0.069)	(0.175)	(0.118)	—	—	(0.125)	(0.026)	(0.070)	(0.176)	(0.119)	—	—	(0.0011)	(0.038)	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.245)	—
FIGARCH (5,5)	1.203	-0.749	0.528	-0.281	0.276	—	0.778	1.208	-0.751	0.530	-0.281	0.277	—	0.0042	1.405	19.80	129.58	726.48	19.85	130.18	725.47	13.07	4368.2
	(0.084)	(0.035)	(0.198)	(0.215)	(0.099)	—	(0.131)	(0.084)	(0.035)	(0.198)	(0.216)	(0.100)	—	(0.0008)	(0.029)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	—
FIGARCH(6,6)	1.151	-0.592	0.417	-0.293	0.301	-0.009	0.862	1.162	-0.597	0.421	-0.295	0.304	-0.011	-0.0013	1.401	69.07	82.22	587.99	69.43	85.08	604.15	18.31	4388.8
	(0.090)	(0.056)	(0.228)	(0.262)	(0.113)	(0.009)	(0.153)	(0.090)	(0.056)	(0.228)	(0.264)	(0.115)	(0.009)	(0.0019)	(0.029)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	—

Table 7: Parameter estimates for the equilibrium short-term rate dynamics (Model 3-short memory)

using overnight Eurodollar rates (3280 observations)

Model 3: consumption dynamic	β_1 (s.e.)	β_2 (s.e.)	β_3 (s.e.)	β_4 (s.e.)	β_5 (s.e.)	β_6 (s.e.)	β_0^* x 10 (s.e.x 10)	α_1x 10 (s.e.x 10)	α_2x 10 (s.e.x 10)	α_3x 10 (s.e.x 10)	α_4x 10 (s.e.x 10)	α_5x 10 (s.e.x 10)	α_6x 10 (s.e.x 10)	ψ (s.e)	$\alpha_{k-1}= \alpha_k = 0$ (sig.level)	$\alpha_{k-2}= \alpha_{k-1} = \alpha_k = 0$ (sig.level)	$\alpha_2,\ldots,\alpha_k = 0$ (sig.level)	$\beta_{k-1}= \beta_k = 0$ (sig.level)	$\beta_{k-2}= \beta_{k-1} = \beta_k = 0$ (sig.level)	$\beta_2,\ldots,\beta_k = 0$ (sig.level)	$\alpha_k = \beta_k = 0$ (sig.level)	Quasi Log Likelihood
GARCH (1,1)	0.972	—	—	—	—	—	0.420	0.268	—	—	—	—	—	0.950	—	—	—	—	—	—	—	3592.7
	(0.005)	—	—	—	—	—	(0.412)	(0.051)	—	—	—	—	—	(0.127)	—	—	—	—	—	—	—	—
GARCH (1,2)	0.978	—	—	—	—	—	0.224	0.275	-0.049	—	—	—	—	0.927	—	—	—	—	—	—	—	3632.6
	(0.003)	—	—	—	—	—	(0.241)	(0.051)	(0.018)	—	—	—	—	(0.125)	—	—	—	—	—	—	—	—
GARCH (1,3)	0.985	—	—	—	—	—	0.046	0.291	-0.064	-0.054	—	—	—	0.888	9.49	—	—	—	—	—	—	3677.7
	(0.003)	—	—	—	—	—	(0.149)	(0.052)	(0.021)	(0.022)	—	—	—	(0.122)	(0.009)	—	—	—	—	—	—	—
GARCH (2,2)	0.976	0.002	—	—	—	—	0.226	0.275	-0.048	—	—	—	—	0.928	—	—	—	—	—	—	9.26	3632.6
	(0.013)	(0.013)	—	—	—	—	(0.299)	(0.052)	(0.020)	—	—	—	—	(0.128)	—	—	—	—	—	—	(0.010)	—
GARCH(3,3)	0.841	0.165	-0.024	—	—	—	0.085	0.282	-0.027	-0.064	—	—	—	0.905	1.81	—	—	3.73	—	—	8.20	3680.9
	(0.144)	(0.454)	(0.309)	—	—	—	(1.145)	(0.053)	(0.030)	(0.110)	—	—	—	(0.101)	(0.405)	—	—	(0.154)	—	—	(0.017)	—
GARCH(4,4)	0.830	-0.292	0.787	-0.337	—	—	-0.031	0.305	-0.047	0.061	-0.174	—	—	0.852	16.77	22.60	—	27.90	36.69	—	13.22	3736.7
	(0.239)	(0.645)	(0.500)	(0.096)	—	—	(0.126)	(0.052)	(0.063)	(0.158)	(0.051)	—	—	(0.117)	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.001)	—
GARCH (5,5)	0.629	-0.107	0.506	0.077	-0.118	—	0.020	0.283	0.018	0.035	-0.095	-0.081	—	0.890	11.10	11.27	16.08	1.57	15.66	22.36	4.78	3760.6
	(0.313)	(0.475)	(0.344)	(0.428)	(0.227)	—	(0.376)	(0.053)	(0.077)	(0.071)	(0.061)	(0.073)	—	(0.125)	(0.004)	(0.010)	(0.003)	(0.456)	(0.001)	(0.000)	(0.092)	—
GARCH(6,6)	0.657	-0.059	0.023	0.064	0.796	-0.495	0.075	0.254	0.008	0.017	0.026	0.003	-0.156	0.940	10.92	19.88	23.64	275.58	300.57	685.25	151.24	3820.8
	(0.051)	(0.099)	(0.146)	(0.109)	(0.054)	(0.043)	(1.152)	(0.090)	(0.013)	(0.017)	(0.031)	(0.012)	(0.067)	(0.222)	(0.004)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	—

Table 8: Parameter estimates for the equilibrium short-term rate dynamics (Model 4-long memory)

using overnight Eurodollar rates (3280 observations)

Model 4: consumption dynamic	β_1 (s.e.)	β_2 (s.e.)	β_3 (s.e.)	β_4 (s.e.)	β_5 (s.e.)	β_6 (s.e.)	$\beta_0^* \times 10$ (s.e. x 10)	ϕ_1 (s.e.)	ϕ_2 (s.e.)	ϕ_3 (s.e.)	ϕ_4 (s.e.)	ϕ_5 (s.e.)	ϕ_6 (s.e.)	d (s.e.)	ψ (s.e)	$\phi_{k-1}=\phi_k=0$ (sig.level)	$\phi_{k-2}=\phi_{k-1}=\phi_k=0$ (sig.level)	$\phi_2,\ldots,\phi_k=0$ (sig.level)	$\beta_{k-1}=\beta_k=0$ (sig.level)	$\beta_{k-2}=\beta_{k-1}=\beta_k=0$ (sig.level)	$\beta_2,\ldots,\beta_k=0$ (sig.level)	$\phi_k=\beta_k=0$ (sig.level)	Quasi Log Likelihood
FIGARCH (1,1)	0.989	—	—	—	—	—	0.002	1.002	—	—	—	—	—	0.0165	0.882	—	—	—	—	—	—	—	3686.1
	(0.003)	—	—	—	—	—	(0.131)	(0.004)	—	—	—	—	—	(0.0043)	(0.131)	—	—	—	—	—	—	—	—
FIGARCH (1,2)	0.992	—	—	—	—	—	-0.005	0.995	0.007	—	—	—	—	0.0260	0.879	—	—	—	—	—	—	—	3704.2
	(0.002)	—	—	—	—	—	(0.102)	(0.003)	(0.002)	—	—	—	—	(0.0052)	(0.130)	—	—	—	—	—	—	—	—
FIGARCH (1,3)	0.991	—	—	—	—	—	-0.008	0.998	0.005	-0.001	—	—	—	0.0233	0.876	10.01	—	—	—	—	—	—	3705.0
	(0.003)	—	—	—	—	—	(0.176)	(0.006)	(0.003)	(0.002)	—	—	—	(0.0075)	(0.141)	(0.007)	—	—	—	—	—	—	—
FIGARCH (2,2)	0.671	0.318	—	—	—	—	0.022	0.678	0.324	—	—	—	—	0.0217	0.906	—	—	—	—	—	—	63.82	3726.0
	(0.088)	(0.087)	—	—	—	—	(0.109)	(0.088)	(0.087)	—	—	—	—	(0.0044)	(0.128)	—	—	—	—	—	—	(0.000)	—
FIGARCH(3,3)	0.678	-0.223	0.533	—	—	—	0.036	0.685	-0.218	0.535	—	—	—	0.0206	0.901	19.13	—	—	19.15	—	—	18.64	3760.10
	(0.352)	(0.406)	(0.136)	—	—	—	(0.382)	(0.357)	(0.407)	(0.138)	—	—	—	(0.0052)	(0.200)	(0.000)	—	—	(0.000)	—	—	(0.000)	—
FIGARCH(4,4)	0.880	-0.403	-0.080	0.587	—	—	0.050	0.889	-0.401	-0.078	0.592	—	—	0.0175	0.911	392.90	406.39	—	389.75	411.02	—	42.93	3782.12
	(0.166)	(0.233)	(0.210)	(0.138)	—	—	(0.270)	(0.168)	(0.233)	(0.211)	(0.138)	—	—	(0.0049)	(0.160)	(0.000)	(0.000)	—	(0.000)	(0.000)	—	(0.000)	—
FIGARCH (5,5)	0.407	-0.217	0.034	0.240	0.510	—	0.161	0.416	-0.211	0.039	0.245	0.512	—	0.0166	0.940	217.84	796.57	799.80	212.53	820.95	825.41	6.99	3799.4
	(0.311)	(0.199)	(0.048)	(0.155)	(0.263)	—	(0.273)	(0.312)	(0.200)	(0.049)	(0.155)	(0.263)	—	(0.0044)	(0.141)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.030)	—
FIGARCH(6,6)	0.655	-0.087	0.041	0.054	0.776	-0.456	0.089	0.694	-0.089	0.044	0.056	0.776	-0.481	-0.0135	0.944	126.38	132.00	336.40	132.26	139.33	357.57	103.96	3826.9
	(0.078)	(0.078)	(0.069)	(0.042)	(0.069)	(0.047)	(0.802)	(0.083)	(0.080)	(0.070)	(0.043)	(0.069)	(0.052)	(0.0055)	(0.199)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	—

Figure 1: Annualized weekly yields from June 1, 1973 to February 25,1995 (5505 observations)

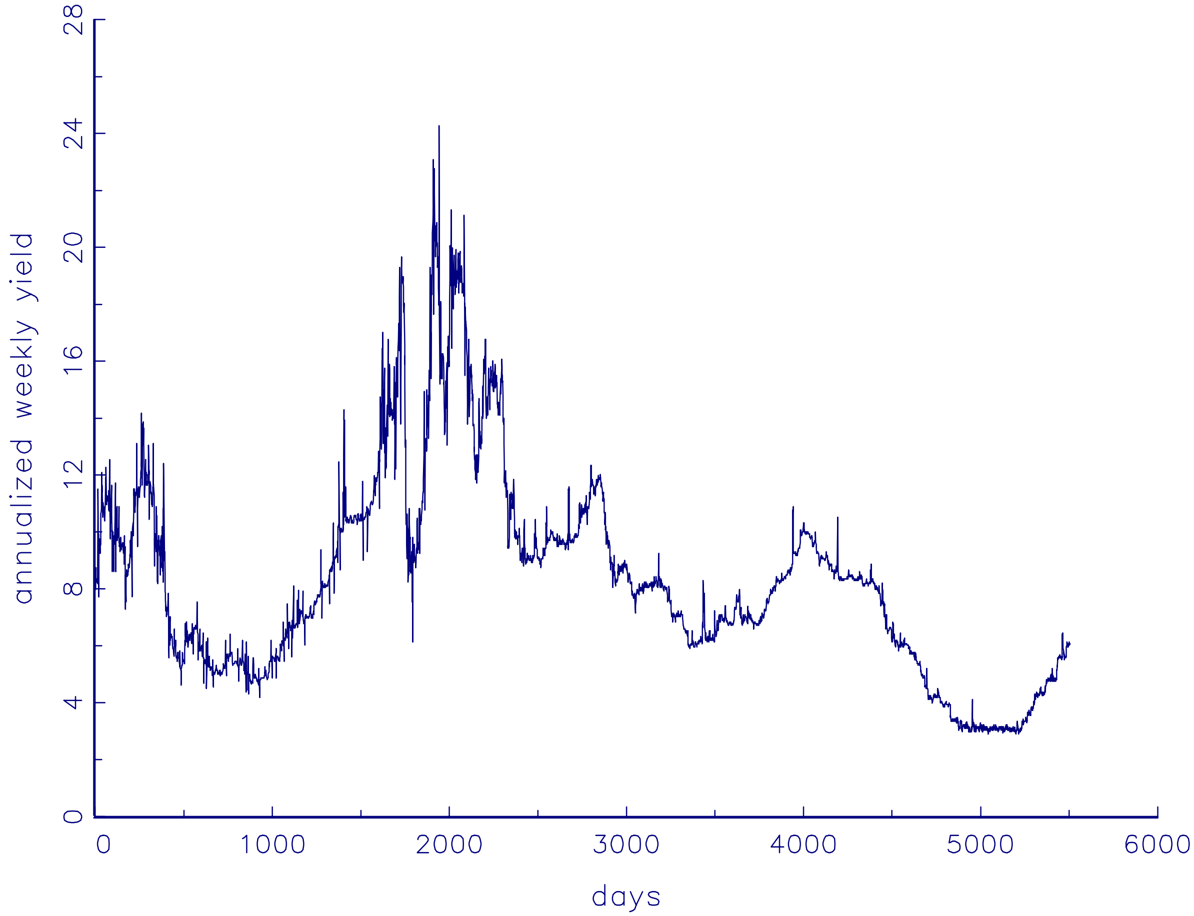


Figure 2: Annualized daily yields from January 1, 1981 to May 30, 1997 (4280 observations)

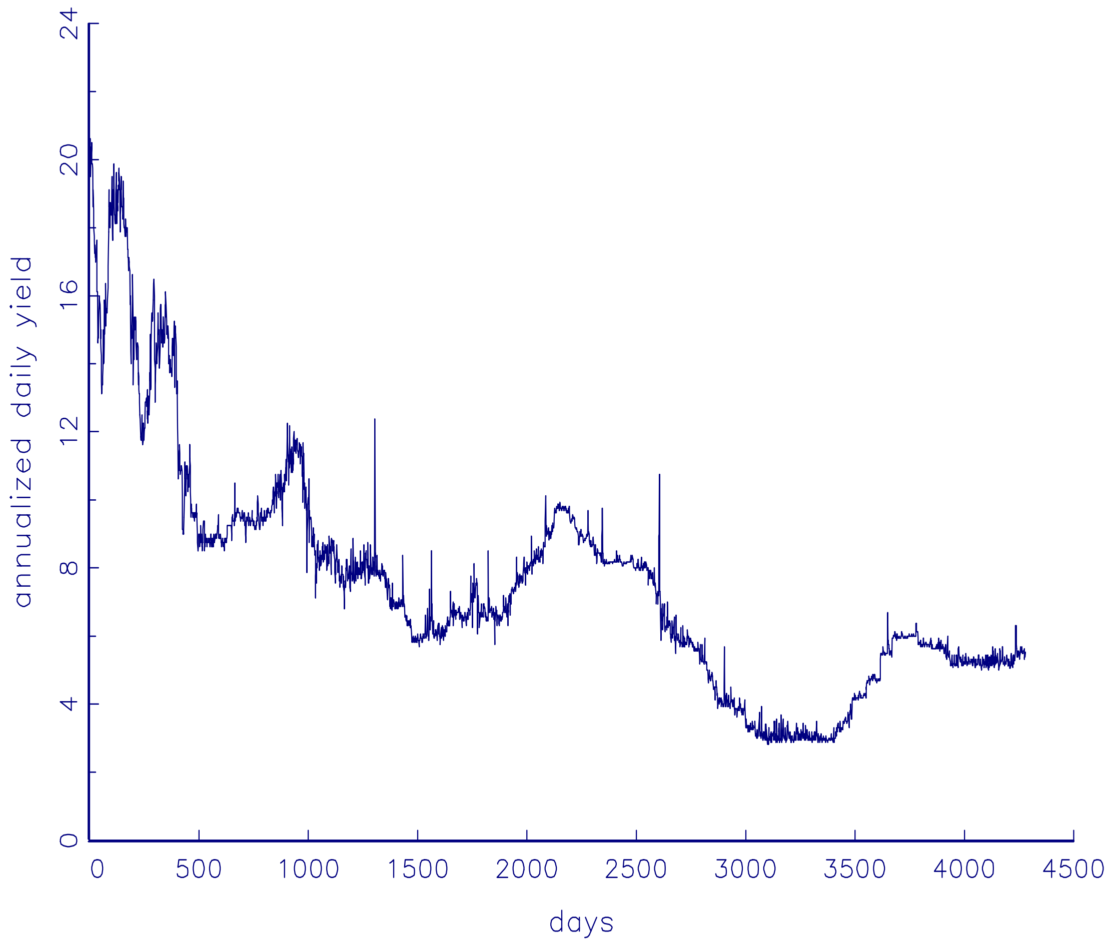


Figure 3: Autocorrelation function for annualized weekly yields

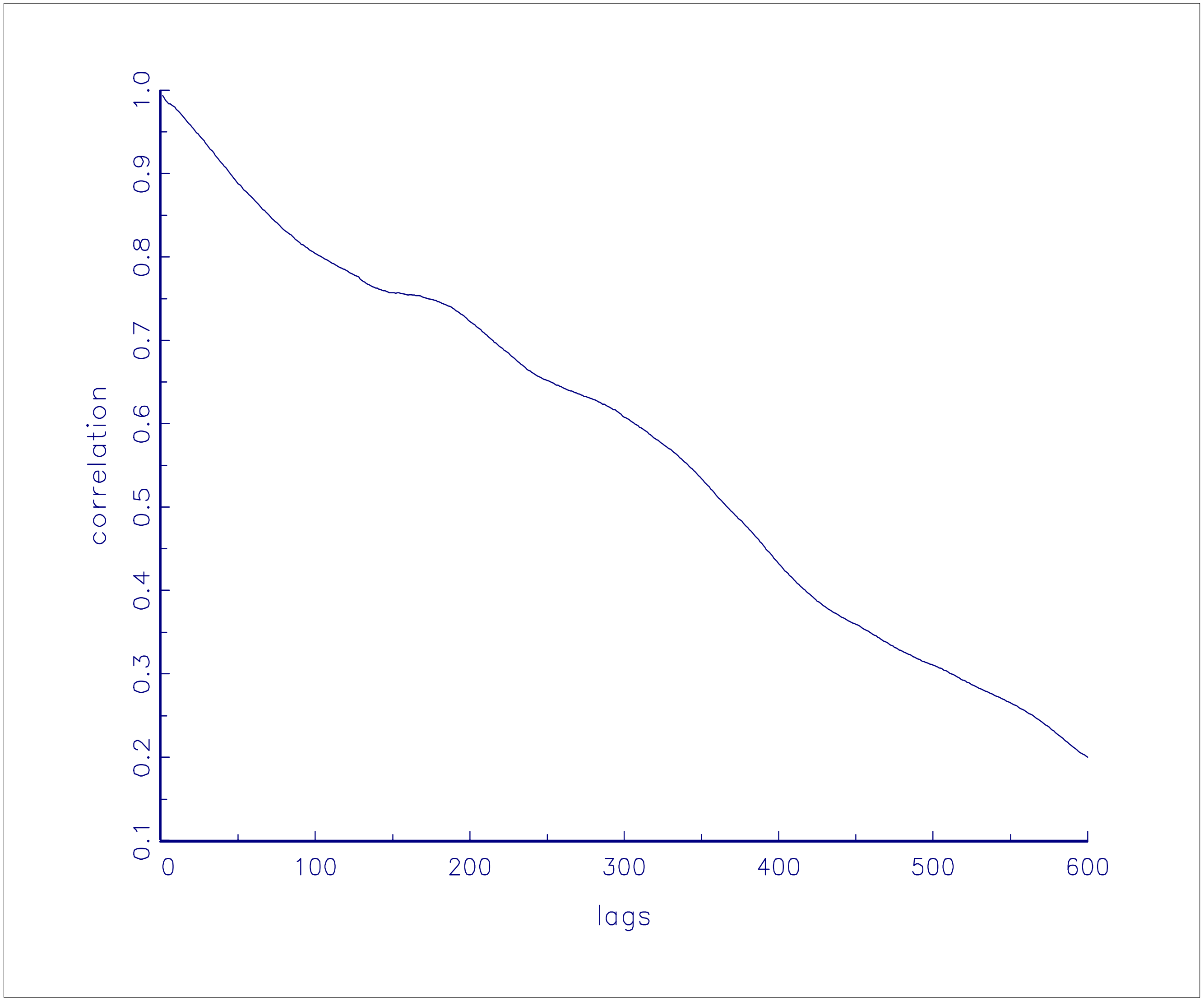


Figure 4: Autocorrelation function for annualized daily yields

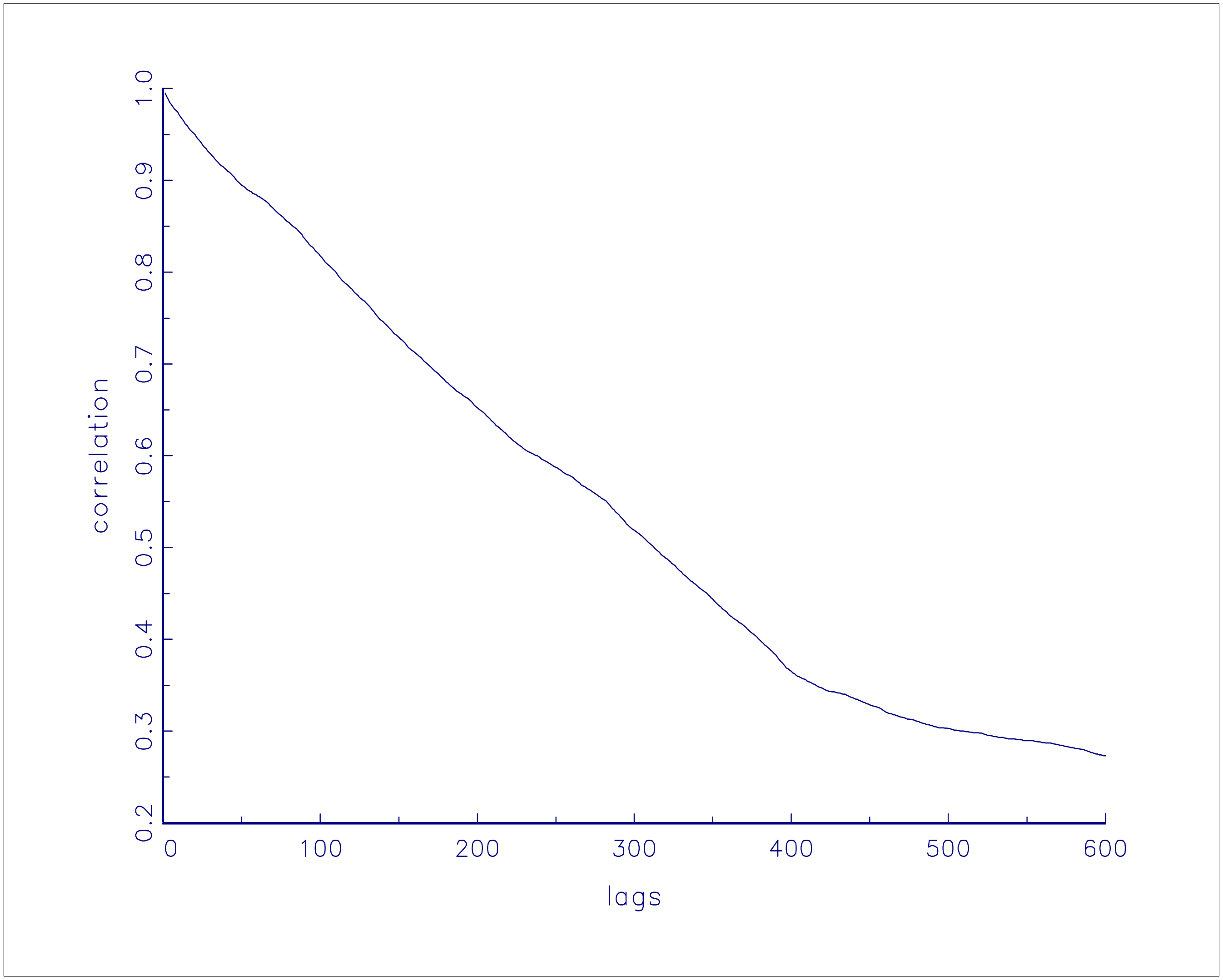
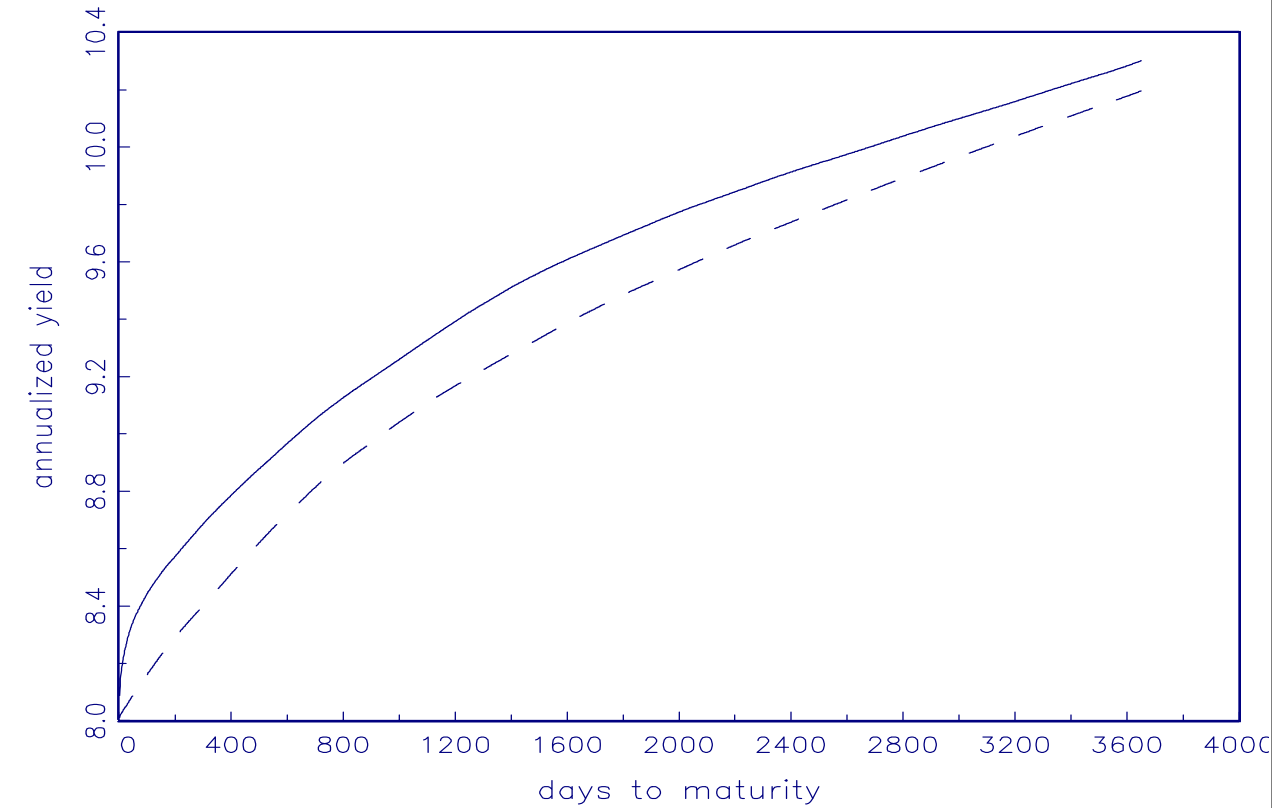


Figure 5: Simulation of Annualized Yields; Lagged Interest Rate Set at 8 %



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