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OUTSIDE OPTIONS AND CENTRALITY IN COOPERATIVE GAME THEORY

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Dedicated to Marlies, Bernhard, Brigitte and Heinz.

Keκsijä κeκsi κeκsi.

ABSTRACT

This thesis deals with outside options and centrality in cooperative games. First, the question whether outside options actually matter is investigated by a pilot experiment using glove games to model simple negotiation situations. The finding that outside options *do* matter is followed by the question of whether cooperative allocation rules can be supported by the data. Motivated by this experimental finding, a probabilistic cooperative forecasting model is used in theoretical support of the importance of outside options. Outside options are formalized and suitable axioms for a categorization into outside-option-sensitive and -insensitive allocation rules are suggested and applied. During this categorization it turns out that there is only one outside-option-sensitive allocation rule within the framework of networks. This and further issues arising from the analysis of the forecasting model lead to a deeper anaylsis of networks, especially the issue of centrality and power indices. A new approach for centrality measures and power indices is suggested which is based on the idea that not only the failure of a whole node is of interest, but rather the failure of a certain link. Axiomatic characterizations, a political application analyzing the performance for forecasting government formation and an application on centrality analyzing identification of top key nodes are provided. While existing cooperative allocation rules either lack sensitivity to outside options or ignore the difference in centralities of agents, a combination possibility of the previously analyzed issues follows. A new (axiomatically characterized) allocation rule for network structures accounting for both outside options and centrality is provided, enriched with an (axiomatically characterized) alternative variant being more suitable for applications in political networks due to moderate relative proportions and applicability in presence of incompatibilities. By the analysis of the explicit effect of outside options on these new allocation rules it is found that this effect is more complex for link-based allocation than for player-based allocation which finally provides a deep and detailed understanding of outside options and their effects.

PUBLICATIONS AND PRESENTATIONS

Publications

Earlier versions of Chapters 3.2.1, 4.3 and the first parts of Chapter 6 appeared in the *Ruhr Economic Papers* series as Numbers 326, 236 and 456. The first part of Chapter 4 is now published in *International Game Theory review* (see Belau [2011] in the Bibliography), and the first parts of Chapter 6 are now published in *Economic Theory Bulletin* (see Belau [2013b] in the Bibliography).

Presentations

Presentations based on Chapters 3, 4 and 6 were given at the RGS Jamboree from 2011 to 2013 and a presentation based on Chapter 5 was given in the Brown Bag Seminar series in Dortmund 2014. The following is a summary of national and international conference presentations.
Chapter 4 was presented at

- 7th Spain-Italy-Netherlands Meeting on Game Theory (SING7), Paris, France (2011)
- 5th RGS Doctoral Conference in Economics, Duisburg, Germany (2012)

Chapter 5 was presented at

- 7th RGS Doctoral Conference in Economics, Dortmund, Germany (2014)
- 10th Spain-Italy-Netherlands Meeting on Game Theory (SING10), Krakow, Poland (2014)

and will be presented at

- 29th Annual Congress of the European Economic Association (EEA), Toulouse, France (2014)
- Jahrestagung 2014 des Vereins für Socialpolitik (VfS), Hamburg, Germany (2014)

Chapter 6 was presented at

- 8th Spain-Italy-Netherlands Meeting on Game Theory (SING8), Budapest, Hungary (2012)
- 11th Meeting of the Society for Social Choice and Welfare, New Delhi, India (2012)

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Part I

INTRODUCTION

1 THE GOLDEN THREAD

The theoretical models and approaches used throughout this thesis are embedded in the theory of cooperative games. Therefore, it seems to be important to mention right at the outset that cooperative game theory is entirely different from non-cooperative game theory. While the latter focuses on the actors with their individual preferences and strategies, the focus of the cooperative approach is more on the outcome of a whole group of actors and its allocation. With the words of Robert Aumann:

> Yesterday we were talking about cooperative and noncooperative game theory and I said that perhaps a better name for cooperative would be "outcome oriented" or "coalitional" and for noncooperative "strategically oriented". [Robert Aumann in an Interview with Eric Damme (1998, p. 196)]

One should have in mind this different direction/objective from which problems, examples and motivations in this thesis are seen and analyzed. The focus of this thesis is on this coalitional/outcome oriented perspective as this direction, especially in times of globalization, offers new possibilities to model economic or monetary unions and its outcomes, for example for applications as forecasting relative power in political networks.

The outline of this thesis is as follows: An introduction to the basic frameworks, definitions and notations of cooperative games and a first insight to outside options by the example of glove games complete Part I. Part II experimentally and theoretically analyzes outside options. First, Chapter 3 investigates the question "Do outside options matter?". In the theoretical part of this chapter, formulas for different cooperative allocation rules for the special case of glove games are provided.[1] In the experimental part[2], an experimental case study

1 This part is based on Belau [2013a].
2 This part is based on Belau and Garmann [2013], joint work with Sebastian Garmann.

is provided which uses glove games to model simple negotiation situations in a double auction market. The finding that outside options *do* matter, that is, do affect negotiation, is followed by the question of a favorable allocation rule, that is, whether one of the theoretical allocation rules can be supported by our data. Furthermore, we discuss the relation to the ultimatum game, learning effects and price interdependency in the designed market.

Chapter 4 provides the formal, theoretical part of analyzing outside options: Motivated by arising differences in the generalization of cooperative allocation rules in a probabilistic forecasting model[3], outside-option-sensitivity axioms are suggested which turn out to be suitable for a formal, axiomatic categorization of all allocation rules discussed in Chapter 2 into outside-option-sensitive and -insensitive ones. During this categorization we shed some more light on the structural effect arising by outside options. Motivated by the fact that there is only one outside-option-sensitive allocation rule in the framework of network structures, a more general discussion of networks emphasizes the issues raised and analyzed in Part III: the analysis of centrality and link-based allocation rules in combination with outside-option-sensitivity.

As discussed in Chapter 4, network structures seem to be more suitable to model economic (or social or political) situations than coalitional models without an inner structure. Hence, before further analyzing the outside-option-issue, the other main topic of this thesis is addressed: centrality. Starting with having in mind political networks (i. e., small networks), Chapter 5 suggests a new cooperative allocation rule which is especially applicable in use as a power index or centrality measure. This approach is based on the idea that, in contrast to existing literature on centrality, not only consequences of failure of a whole node are of interest but rather consequences of failure of a certain link. Using cooperative allocation rules, link-based centrality measures (i. e., accounting for relative importance of links) are suggested. Accounting for failure consequences of connections is of recent importance considering for example the current Euro crisis and the resulting monetary flows. Axiomatic characterizations of the new allocation rule, a political application of the new rule used as a

3 This part of Chapter 4 is based on Belau [2011].

power index on the performance for forecasting government formation and an application of the new rule used as a centrality measure on identification of top key nodes are provided. Beside discussions on moderate relative proportions ("Banzhaf vs. Shapley"), the more convincing and appropriate performance of the new approach in applications for measuring political power or centrality emphasizes the general advantages of link-based values.

After the analysis of outside options and centrality with respect to links seperately, Chapter 6 combines these issues. The analysis in Chapter 5 was restricted to connected networks, that is, situations where (from a coalitional point of view) no outside options exist. In Chapter 6 it is shown that existing allocation rules for general network structures (i. e., also unconnected ones) either lack outside-option-sensitivity or bear other drawbacks, namely, the ignorance of the differences in the position of an agent within the network, that is, differences in centrality. A new link-based allocation rule for network structures is provided and axiomatically characterized which both accounts for outside options and centrality (i. e., accounts for the position of an agent within the network in place). Herewith the issues of the previous chapters are combined.[4] An (axiomatically characterized) alternative variant of this rule which is more moderate in terms of relative proportions and applicable in presence of incompatibilities is suggested and the effect of outside options on these new allocation rules is discussed. It is found that this effect is actually more complex than originally anticipated. This provides a detailed understanding of the structural effect of outside options on allocation. Moreover, it reopens the disucssion on the axioms formalizing outside-option-sensitivity from Chapter 4 and explains why the suggested "weak" axiom in fact might not be that weak and the "normal" axiom actually could be seen as sort of strong.

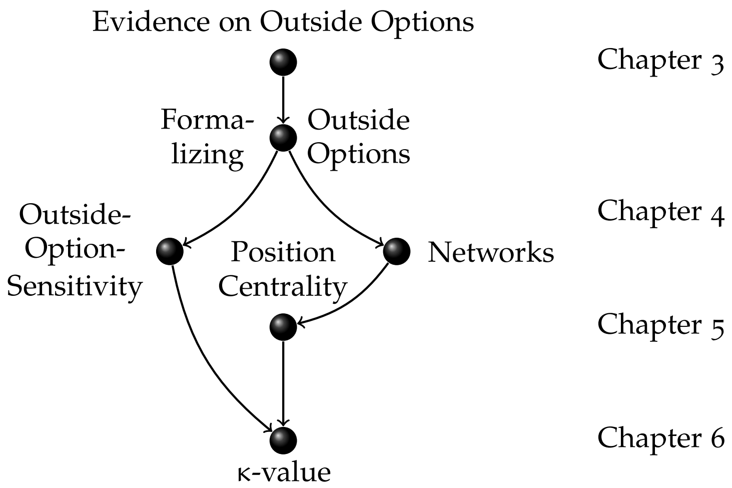
Chapter 7 concludes with a summary and discussion of open questions and an outlook on future work.

In order not to "lose the thread" on the way throughout this thesis, consider "the Golden Thread" displayed in Figure 1 which will ap-

4 This part is based on Belau [2013b].

pear again in the beginning of each chapter highlighting the specific issue of the chapter.

Figure 1: The Golden Thread



2

FRAMEWORKS AND ALLOCATION RULES IN COOPERATIVE GAME THEORY

This chapter provides definitions, notations, findings and motivations of the three main approaches in cooperative game theory which will be used throughout this thesis: games with transferable utility, the pointwise allocation rules and the two partitative approaches of cooperative coalition structures and cooperative network structures. Notations of the first two approaches are mainly based on Wiese [2005], notions of the third approach on Jackson and Wolinsky [1996] and Myerson [1977].

Furthermore, this chapter introduces glove games (introduced by Shapley and Shubik, 1969) which we use to discuss the performance of some of the suggested allocation rules and differences between them as well as an introductive motivation for the issue of outside options.

Note that this chapter does not provide a complete overview of the above-mentioned approaches or cooperative game theory in general. First of all, we only analyze *values* as solution concepts for allocation, that is, singleton-valued allocation rules assigning exactely one distribution of worth to each game by an *allocation formula*. We do not analyze *set-valued solutions* as the core (Edgeworth, 1881 and Gillies, 1959) or the kernel (Davis and Maschler, 1965), which may be empty or correspondences assigning multiple possible distributions. Beside the focus on values, we will for example not discuss games with non-transferable utility, Pareto-optimality, the nucleolus (Schmeidler, 1969), the Solidarity solution or the Nash-Bargaining-solution. This chapter is meant to provide notions and motivations of the aspects of cooperative game theory which will be used in the following chapters and can be reconsulted to facilitate recalling definitions and notations. Furthermore, especially in the axiomatization parts, we discuss definitions in more detail than in the following chapters.

2.1 GAMES WITH TRANSFERABLE UTILITY (TU-GAMES)

One of the main assumptions used in cooperative game theory is that utility is *transferable*, that is, agents can interchange utility between each other without losses. This assumption has been widely discussed in the literature, some works analyze conditions on this assumption (cf. Kaneko, 1976) as well as testability (Chiappori, 2010) or a revealed preference approach (Cherchye et al., 2011). With this assumption in place we model economic, social or political situations via a cooperative game. More precisely, we have a set of actors and a function which assigns to each subset of actors a numerical outcome based on the specific situation we like to model, the so-called characteristic or coalition function. Here, the assumption of utility being transferable from the group to the agents is crucial as outcome is assigned to a *set* of agents and not the agents *individually*. Popular coalition functions used to model various economic, political or social situations are for example weighted voting games, assigning a fixed worth whenever a coalition reaches a certain quorum (for example winning coalitions after an election), simple market games as the glove game where worth is obtained whenever a certain kind of matching was successfull or cost allocation games. Note that cost allocation differs from the other applications in the sense that here, "more is worse" in contrast to "more is better".

One of the main tasks in cooperative game theory is to answer the question of the allocation of this commonly created outcome among the individuals. Here, not only the definition of a certain allocation rule is of interest but also its properties. Therefore, most allocation rules come along with an *axiomatization*. These axiomatizations uniquely characterize the allocation rules, hence, one could say "if you support all of the following axioms, you have to use the corresponding allocation rule".[1]

Definition 2.1 (Player set and coalition function). *Let the set* $N = \{1, ..., n\}$ *be the nonempty and finite set of players. The set* $\mathbb{V}_N := \{v : 2^N \longrightarrow \mathbb{R} | v(\emptyset) = 0\}$ *is the set of all* coalition functions*, that is, a function* $v \in \mathbb{V}_N$ *describes the underlying game and assigns to any coalition* $K \subseteq N$ *its worth* $v(K)$*.*

1 Note that this might imply other axioms that are also satisfied by the rule one might not like to support.

The value $v(K)$ can be interpreted as representing the economic possibilities of coalition K. Note that a coalition function is sometimes also called *characteristic function*.

Definition 2.2 (Game with transferable utility (TU-game)). *A* game with transferable utility (TU-game) *is a tupel* (N, v).

For notational convenience, we sometimes only provide the coalition function v and refer to a game.

Definition 2.3 (Unanimity game). *For* $T \subseteq N, T \neq \emptyset$*, the game* (N, u_T) *with*

$$u_T(K) = \begin{cases} 1 & \text{, if } T \subseteq K \\ 0 & \text{, otherwise} \end{cases} \quad , K \subseteq N$$

is called a unanimity game *(or also* T*-unanimity game).*

Since the unanimity games form a basis of $\mathbb{V}_N$ (see Shapley, 1953), we can write any $v \in \mathbb{V}_N$ as

$$v(K) = \sum_{T \in 2^N \setminus \{\emptyset\}} \lambda_T(v) u_T(K) \quad , K \subseteq N$$

where the scalars $\lambda_T(v)$ are called *Harsanyi dividends* (see Harsanyi, 1959 and 1963).

Definition 2.4 (Zero-Normalization). *A coalition function is called* zero-normalized *if* $v(\{i\}) = 0 \; \forall \, i \in N$. *Denote by* $\mathbb{V}_N^0$ *the set of all zero-normalized coalition functions. Any* $v \in \mathbb{V}_N$ *can be zero-normalized: for any* $v \in \mathbb{V}_N$ *define the corresponding* $v_0 \in \mathbb{V}_N^0$ *as follows:*

$$v_0(K) := v(K) - \sum_{i \in N} v(\{i\}) u_{\{i\}}(K) = v(K) - \sum_{i \in N} \lambda_{\{i\}}(v) u_{\{i\}}(K).$$

Definition 2.5 ((Feasible) Allocation Rule). *An* allocation rule $Y : \{N\} \times \mathbb{V}_N \longrightarrow \mathbb{R}^{|N|}$ *distributes the worth of any TU-game among the players. An allocation rule is called* feasible *if* $\sum_{i \in N} Y_i(N, v) \leqslant v(N)$.

Remark 2.1 (Zero-Normalization and the Allocation Rule). *If one zero-normalized a coalition function* v *to* v_0*, one has to adapt the allocation rule. For this, set* $Y_i(N, v) = Y_i(N, v_0) + v(\{i\}) \; \forall \, i \in N$.

Definition 2.6 (Order of the Player set). *An* order *of* N *is a bijection* $\sigma : N \longrightarrow \{1, ..., |N|\}$ *where* $\sigma(i)$ *denotes player* i*'s position in order*

σ. *We denote by* $\Sigma(N)$ *the set of all orders of* N *and by* $K_i(\sigma) := \{j \in N | \sigma(j) \leqslant \sigma(i)\}$ *the set of players that do not come after player* i *under order* σ.

Shapley [1953] introduced one of the most popular allocation rules:

Definition 2.7 (Shapley value)**.** *For any TU-game* (N, v)*, the* Shapley value Sh *is for all* $i \in N$ *given by:*

$$Sh_i(N, v) := \frac{1}{|\Sigma(N)|} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma)$$

where $MC_i^v(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$ *is the* marginal contribution *of player* i *in* σ*. Alternatively, one can calculate the Shapley value for all* $i \in N$ *by*

$$Sh_i(N, v) = \sum_{K \subseteq N \setminus \{i\}} \frac{|K|!(|N| - |K| - 1)!}{|N|!} [v(K \cup \{i\}) - v(K)] .$$

Here, $MC_i^v(K) := v(K \cup \{i\}) - v(K)$ is called the *marginal contribution* of player i to coalition K in the TU-game (N, v), that is, the surplus player i creates when entering the coalition K, given the game (N, v). The Shapley value assigns to any player i the average marginal contribution over all orders of N. Note that the original definition of the Shapley value is based on sequences, that is, takes into account all possible sequences in which the players can enter a coalition. We will focus on the alternative formula which is sometimes called the *probabilistic method* as this is more applicable in use when the number of players increases. The two methods are equivalent in their outcome.

Axiom 2.1 (Additivity (**A**))**.** *An allocation rule* Y *satisfies* Additivity **A***, if for any coalition functions* v, w *we have:* $Y(N, v + w) = Y(N, v) + Y(N, w)$.

A is a standard axiom[2] and satisfied by all allocation rules referred to. Note that in presence of **A** we have for any $v \in \mathbb{V}_N$

$$Y(N, v) = \sum_{T \in 2^N \setminus \{\emptyset\}} Y(N, \lambda_T(v) u_T)$$

which means that allocation of worth can be decomposed in allocation of the worth corresponding to the basis elements.

2 For a motivation of **A** see Roth, 1977.

Axiom 2.2 (Efficiency (**E**))**.** *An allocation rule* Y *satisfies* Efficiency **E** *if for any TU-game* (N, v) *we have*

$$\sum_{i \in N} Y_i(N, v) = v(N).$$

This axiom states that no worth should be wasted. Note that this axiom implicitly assumes that the worth of the grand coalition $v(N)$ is actually produced. There are applications where this is not assured which will be discussed in the following sections.

Axiom 2.3 (Symmetry (**S**))**.** *Players* $i, j \in N$ *are called* symmetric *in* (N, v) *if* $v(K \cup \{i\}) = v(K \cup \{j\}) \forall K \subseteq N \setminus \{i, j\}$*. An allocation rule* Y *satisfies* Symmetry **S***, if* $Y_i = Y_j$ *for all symmetric Players* $i, j \in N$*.*

Symmetric players have the same productivity. One could argue that players with equal productivity in a game should obtain the same payoff from this game.

Axiom 2.4 (Null Player Axiom (**N**))**.** *A player* $i \in N$ *is called a* Null player *in* (N, v) *if* $v(K \cup \{i\}) = v(K) \forall K \subseteq N \setminus \{i\}$*. An allocation rule* Y *satisfies the* Null Player Axiom **N** *if* $Y_i = 0$ *for all Null players* $i \in N$*.*

This axiom excludes solidarity with unproductive players. One could argue that unproductive players should obtain some worth due to solidarity issues. There are alternatives to the Shapley value addressing this fact as for example the Solidarity value (Nowak and Radzik, 1994b) or the outside option values which will be disussed in the following sections.

Theorem 2.1 (Axiomatization of the Shapley value (Shapley, 1953))**.** *The Shapley value is the unique allocation rule that satisfies **A**, **N**, **E** and **S**.*

As an alternative to the Shapley value, Banzhaf [1952] introduced another popular allocation rule which was generalized by Owen [1975]:

Definition 2.8 (Banzhaf value)**.** *For any TU-game* (N, v)*, the* Banzhaf value Ba *is for all* $i \in N$ *given by*

$$Ba_i(N, v) := \sum_{K \subseteq N \setminus \{i\}} \frac{1}{2^{|N|-1}} \left[v(K \cup \{i\}) - v(K) \right].$$

The idea of the Banzhaf value is based on *simple games*, that is, a TU-game where $v(K) \in \{0, 1\}$ for all $K \subseteq N$. For any simple game, a coalition $K \subseteq N \setminus \{i\}$ is called *swing* for player $i \in N$, if $v(K \cup \{i\}) - v(K) = 1$, that is, if player i is needed to create worth. Then, the Banzhaf value counts the swings of player i proportional to the total number of potential swings ($2^{|N|-1}$): For any simple game $v \in \mathbb{V}_N$, the Banzhaf value is given by

$$Ba_i(N, v) := \frac{\text{\# of swings for } i}{\text{\# of potential swings for } i}.$$

The Banzhaf value does not satisfy **E**, but (in contrast to the Shapley value) the following axiom:

Axiom 2.5 (Banzhaf Efficiency (**BaE**)). *An allocation rule* Y *satisfies* Banzhaf Efficiency **BaE** *if*

$$\sum_{i \in N} Y_i = \sum_{i \in N} Ba_i.$$

The idea behind this axiom could be interpreted following the original definition for simple games using the ratio of swings and potential swings. The number of potential swings is the same for any player $i \in N$, therefore, **BaE** states that the total power of the grand coalition should be proportional to the total number of all swings (which is aimed to represent the sum of individual power). For general games, swings are replaced by marginal contributions.

Theorem 2.2 (Axiomatization of the Banzhaf value (Owen, 1975))**.** *The Banzhaf value is the unique allocation rule that satisfies **A**, **N**, **BaE** and **S**.*

Note that the Banzhaf value may distribute more worth than actually available/produced (i. e., might not be a feasible allocation rule). The Banzhaf value should rather be seen as a power measure and is, in this sort of applications, sometimes also called *Banzhaf power index*.

We see the main difference between the Shapley and the Banzhaf value in the answer to the question how marginal contributions (for simple games: swings) should be weighted. While the Shapley value weights a marginal contribution with respect to it's "strength", that is, taking into account the cardinality which one could interpret as robustness, the Banzhaf value weights each marginal contribution by the same factor. Depending on the application, each answer could be more or less favorable than the other. We will discuss this issue in more detail in Chapter 5.

2.2 COALITION STRUCTURES

A central problem in economics, political science or sociology is the analysis of coalition formation of individuals. Economic, political or social structures mostly contain more information than just about the set of actors and the underlying game. Individuals do not have to build the grand coalition, that is, *all* actors cooperate with each other, they could also build several (disjoint) coalitions which leads to the partition approach which we will call *coalition structure* approach. Another approach is that certain coalitions between individuals may a priori be fixed which refers to the *a priori unions* approach. One could argue that the actual constellation of the coalition (partition) or the a priori unions matters; one could for example question the reasoning of the efficiency axiom that implicitly assumes that the grand coalition has formed (or at least the worth of the grand coalition has been produced).

Coalition formation can be found in various economic, political or social situations we also meet in daily life. Almost everyone experiences coalition formation in one's social life by group formation in school classes, sport clubs, among collegues at work or joining online communities. Here, the "worth" of cooperation is simply the positive effect to one's social environment. Customers form coalitions in order to obtain group discounts, especially facilitated by electronic market places (see for example Tsvetovat et al., 2000). As an illustrating example consider the recent advertisement slogan of the German mobile phone operator *mobilcom-debitel* "together we achieve more", promoting discounts for mobile phone contracts[3]. Another popular application in economics and political science is the formation of coalitions between political parties to build the government after an election which is one of the main application of weighted voting games and power indices.

3 Promotion "Gemeinsam geht mehr", cf. http://www.mobilcom-debitel.de/unser-prinzip/

Note that there exist further approaches on coalition formation as for example von Neumann and Morgenstern [1944]'s analysis of stable sets or Aumann and Maschler [1964]'s analysis of the bargaining set which will not be part of this thesis.

Definition 2.9 (Coalition structure). *A partition $\mathcal{P}$ of N is called* coalition structure *and we denote by $\mathcal{P}(i)$ the coalition $P \in \mathcal{P}$ such that $i \in P$ (i. e., the coalition which contains player $i \in N$) and by $\mathbb{P}_N$ the set of all coalition structures of N.*

Definition 2.10 (Allocation rule for coalition structures). *A* TU-game with a coalition structure *is a tupel $(N, v, \mathcal{P})$ and an* allocation rule for coalition structures *is a function $Y : \{N\} \times \mathbb{V}_N \times \mathbb{P}_N \longrightarrow \mathbb{R}^{|N|}$, distributing worth of any TU-game with a coalition structure among the players.*

One of the most popular allocation rules for coalition structures is the component-restricted Shapley value, introduced by Aumann and Drèze [1974]:

Definition 2.11 (Aumann-Drèze value). *For any TU-game with a coalition structure $(N, v, \mathcal{P})$, the component-restricted Shapley value, denoted by* Aumann-Drèze value AD*, is for all $i \in N$ given by*

$$AD_i(N, v, \mathcal{P}) := Sh_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}).$$

Here, the whole game (player set and coalition function) is restricted to the coalition of a player. That is, once a coalition is formed, the worth produced by the coalition is distributed among the players within the coalition independently on the outside of the coalition.

Axiom 2.6 (Component Efficiency (**CE**)). *An allocation rule for coalition structures Y satisfies* Component Efficiency **CE***, if*

$$\sum_{j \in \mathcal{P}(i)} Y_j(N, v, \mathcal{P}) = v(\mathcal{P}(i)) \,\forall\, i \in N.$$

The motivation of this axiom is that players within a component, seen as a productive unit, cooperate to create the component's worth. **CE** stands in contrast to **E**, where the worth of the grand coalition is distributed among all players without coalitional restrictions. On the one hand, the use of **CE** instead of **E** avoids the problem of potentially distributing worth which is not actually produced while, on the other hand, it excludes the possibility of transferring worth between coalitions, that is, compensations or side payments.

Axiom 2.7 (Symmetry within Components (**CS**) for Coalition Structures)**.** *An allocation rule for coalition structures* Y *satisfies* Symmetry within Components **CS***, if for all* $\mathcal{P} \in \mathbb{P}(N)$ *we have*

$$Y_i(N, v, \mathcal{P}) = Y_j(N, v, \mathcal{P}) \, \forall \textit{ symmetric players } i, j \in N, j \in \mathcal{P}(i).$$

CS states that allocation rules should provide the same payoff for players with equal productivity that are in the same component, since there is nothing like an inner structure which could be responsible for a different treatment of these players. **CS** is a relaxation of **S**, where players with equal productivity are treated equally independently of whether they are in coalitions with potentially different coalitional productivity.

Theorem 2.3 (Original Axiomatization of the Aumann-Drèze value (Aumann and Drèze, 1974)**.** *The AD-value is the unique allocation rule for coalition structures that satisfies* ***A****,* ***N****,* ***CE*** *and* ***CS****.*

This characterization shows the connection to the (unrestricted) Shapley value. However, we will make use of another characterization of the AD-value using the following elegant axiom, introduced by Myerson [1977]:

Axiom 2.8 (Balanced Contributions (**BC**))**.** *An allocation rule for coalition structures* Y *satisfies* Balanced Contributions **BC***, if*

$$\begin{aligned} Y_i(N, v, \mathcal{P}) &- Y_i(N \setminus \{j\}, v|_{N \setminus \{j\}}, \mathcal{P}|_{N \setminus \{j\}}) \\ &= Y_j(N, v, \mathcal{P}) - Y_j(N \setminus \{i\}, v|_{N \setminus \{i\}}, \mathcal{P}|_{N \setminus \{i\}}) \end{aligned}$$

for all $i, j \in N$*.*

The exit of a player j should hurt/benefit another player i by the same amount as the exit of i hurts/benefits j. While Slikker [2000] used a component restricted version of **BC**, we apply the axiom directly as this is more in line with our further analysis.

Theorem 2.4 (Axiomatization of the Aumann-Drèze value (Slikker, 2000 & Belau, 2011))**.** *The AD-value is the unique allocation rule for coalition structures that satisfies* ***CE*** *and* ***BC****.*

Proof. Slikker [2000] shows that the AD-value is characterized by **CE** and the component-restricted version of **BC**. **BC** implies the component-restricted version, hence, it is sufficient to show that the AD-value satisfies **BC**: For $i, j \in N$ we have

$$\begin{aligned}
&AD_i(N \setminus \{j\}, v|_{N\setminus\{j\}}, \mathcal{P}|_{N\setminus\{j\}}) - AD_j(N \setminus \{i\}, v|_{N\setminus\{i\}}, \mathcal{P}|_{N\setminus\{i\}}) \\
&= Sh_i(\mathcal{P}|_{N\setminus\{j\}}(i), v|_{N\setminus\{j\}}|_{\mathcal{P}|_{N\setminus\{j\}}(i)}) - Sh_j(\mathcal{P}|_{N\setminus\{i\}}(j), v|_{N\setminus\{i\}}|_{\mathcal{P}|_{N\setminus\{i\}}(j)}) \\
&= Sh_i(\mathcal{P}(i) \setminus \{j\}, v|_{\mathcal{P}(i)\setminus\{j\}}) - Sh_j(\mathcal{P}(j) \setminus \{i\}, v|_{\mathcal{P}(j)\setminus\{i\}}) \\
&\overset{(*)}{=} Sh_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}) - Sh_j(\mathcal{P}(j), v|_{\mathcal{P}(j)}) \\
&= AD_i(N, v, \mathcal{P}) - AD_j(N, v, \mathcal{P})
\end{aligned}$$

$(*)$: If $j \in \mathcal{P}(i)$ use that Sh satisfies **BC**. If $j \notin \mathcal{P}(i)$ one also has $i \notin \mathcal{P}(j)$ and therefore $\mathcal{P}(i) \setminus \{j\} = \mathcal{P}(i)$ and $\mathcal{P}(j) \setminus \{i\} = \mathcal{P}(j)$. □

In contrast to the AD-value, one of the first allocation rules for a priori unions (we use it for coalition structures) was defined by Owen [1977]:

Definition 2.12 (Owen value)**.** *For any TU-game with a coalition structure* $(N, v, \mathcal{P})$*, the* Owen value Ow *is for all* $i \in N$ *given by*

$$Ow_i(N, v, \mathcal{P}) := \frac{1}{|\Sigma(N, \mathcal{P})|} \sum_{\sigma \in \Sigma(N,\mathcal{P})} [v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})]$$

where $\Sigma(N, \mathcal{P})$ *is the set of all orders* σ *over the player set that are* compatible *with the coalition structure* $\mathcal{P}$*, that is,* $\forall\, i, j \in P \in \mathcal{P}$ *we have* $|\sigma(i) - \sigma(j)| < |P|$*).*

As already mentioned, the original approach of Owen [1977] differs from the (partitional) understanding of a coalition structure we use here. Owen uses *a priori unions*, that is, players organize each other in fixed a priori unions (or bargaining blocks) and then the game is "played on the grand coalition" accounting for these unions. This approach differs in the sense that we interpret coalitions as productive unions where players cooperate in order to produce worth within their coalition and not within the grand coalition. However, the Owen value can still be seen as an allocation rule for coalition structures if we interpret the fixed a priori unions as productive unions. Note that the Owen value satisfies the efficiency axiom while the other allocation rules discussed in this section do not.

Axiom 2.9 (Symmetry of Components (**SC**)). *An allocation rule for coalition structures* Y *satisfies* Symmetry of Components **SC** *if*

$$\sum_{i \in P} Y_i(N, v, \mathcal{P}) = \sum_{i \in P'} Y_i(N, v, \mathcal{P})$$

for $P, P' \in \mathcal{P}$ *being symmetric in the* intermediate game $v^{int} : \mathcal{P} \longrightarrow \mathbb{R}$ *(the game in which the components of a partition are the players).*

SC can be seen as **S** for the intermediate game. The combination of **CS**, where players with the same productivity are treated equally if they are in the same coalition, and **SC**, where coalitions with the same coalitional productivity are treated equally, closes the gap arising if one eleminates **S**.

Theorem 2.5 (Axiomatization of the Owen value (Owen, 1977)). *The Owen value is the unique allocation rule for coalition structures that satisfies* ***E***, ***N***, ***CS***, ***SC*** *and* ***A***.

More recently, among the first explicitly emphasizing outside options, Wiese [2007] defined an allocation rule that is sensitive to outside options:

Definition 2.13 (Wiese value). *For any TU-game with a coalition structure* $(N, v, \mathcal{P})$, *the* Wiese value *W is for all* $i \in N$ *given by*

$$W_i(N, v, \mathcal{P}) := \sum_{\sigma \in \Sigma(N, \mathcal{P})} \frac{1}{|\Sigma(N)|} \begin{cases} v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i) \setminus \{i\}} MC_j^v(\sigma), & \text{if } \sigma \in \Sigma_i(N, \mathcal{P}) \\ MC_i^v(\sigma) & , \text{if } \sigma \notin \Sigma_i(N, \mathcal{P}) \end{cases}$$

where $\Sigma_i(N, \mathcal{P}) \subseteq \Sigma(N)$ *is the set of all orders* σ *of* N *where all players from* i*'s component* $\mathcal{P}(i)$ *except* i *come before player* i, *that is,* $|K_i(\sigma) \cap \mathcal{P}(i)| = |\mathcal{P}(i)|$.

We will not provide the characterization and will not analyze the the Wiese value in detail as

> The Wiese value has some drawbacks. Most notably, it lacks a "nice" axiomatization. In essence, there is a non-intuitive ad-hoc specification of the payoffs for unanimity games which is expanded by linearity to the whole class of games. Further, it is not yet clear whether there are stable coalition structures (in the sense of Hart and Kurz, 1983) with respect to the Wiese value for all TU-games. [Casajus, 2009b, p. 50]

We will only briefly analyze the Wiese value in Chapter 4 where we find that the structural effect of an outside option on allocation by the Wiese value is equal to the effect on allocation by the Shapley value, that is, there is no further insight in terms of structural effects of outside options.

As an alternative, Casajus [2009b] (also explicitly emphasizing outside options) suggested an allocation rule for coalition structures that is sensitive to outside options:

Definition 2.14 (χ-value)**.** *For any TU-game with a coalition structure* $(N, v, \mathcal{P})$*, the* χ-value *is for all* $i \in N$ *given by*

$$\chi_i(N, v, \mathcal{P}) := Sh_i(N, v) + \frac{v(\mathcal{P}(i)) - Sh_{\mathcal{P}(i)}(N, v)}{|\mathcal{P}(i)|},$$

where $Sh_{\mathcal{P}(i)}(N, v) = \sum_{i \in \mathcal{P}(i)} Sh_i(N, v).$

The χ-value can be seen as "the Shapley value made component efficient". It values outside options by a redistribution of the difference between the worth actually created by the coalition to what the coalition would obtain according to the Shapley value.

The essential characterizing axiom of the χ-value is the following:

Axiom 2.10 (Splittingaxiom (**SP**))**.** *A partition* $\mathcal{P}' \subseteq 2^N$ *is called* finer *than* $\mathcal{P} \subseteq 2^N$ *if* $\mathcal{P}'(i) \subseteq \mathcal{P}(i)$ *for all players* $i \in N$*. An allocation rule for coalition structures* Y *satisfies the* Splittingaxiom **SP***, if for* $\mathcal{P}'$ *being finer than* $\mathcal{P}$ *we have for all* $i \in N$ *and* $j \in \mathcal{P}'(i)$*:* $Y_i(N, v, \mathcal{P}) - Y_i(N, v, \mathcal{P}') = Y_j(N, v, \mathcal{P}) - Y_j(N, v, \mathcal{P}')$.

One could argue that gains or losses of splitting a coalition structure should be distributed equally on players staying together in the new coalition structure.

Furthermore, to allow for solidarity, the following axiom is introduced:

Axiom 2.11 (Grand Coalition Null Player (GN)). *An allocation rule for coalition structures* Y *satisfies the* axiom of Grand Coalition Null Player **GN**, *if* $Y_i(N, v, \{N\}) = 0$ *for all Null players* $i \in N$.

GN is less restrictive than **N**. While **N** generally excludes solidarity with unproductive players, **GN** only demands the exclusion of solidarity if all players cooperate with each other which could be seen as the absence of a coalition structure. One could argue that, if one decides to cooperate with an unproductive player while one could have excluded him (which is not possible if no coalition structure exists), the cooperating players should "pay" for this solidarity with a side payment towards the Null player. This stands in contradiction with **N** while it is possible when using **GN**.

Remark 2.2. *Note that for* $\mathcal{P} = \{N\}$, *the axioms* ***GN****,* ***CE*** *and* ***CS*** *become* ***N****,* ***E*** *and* ***S*** *for TU-games without further structure.*

Theorem 2.6 (Axiomatization of the χ-value (Casajus, 2009b)). *The* χ*-value is the unique allocation rule for coalition structures that satisfies* ***CE****,* ***CS****,* ***A****,* ***GN*** *and* ***SP****.*

2.3 NETWORK STRUCTURES

The use of graph theory to model social structures already started more than a century ago while the analysis of economic networks only became more and more important during the last decades (cf. Jackson, 2006). The key idea is the analysis of bilateral relations between the actors. This contains much more information than the coalition structure approach. Also on the application point of view, allocation rules working on networks might be more suitable than others.

We model network structures based on the idea of Myerson [1977] who interpreted networks as *communication structures*. Until now, we did not assume any restrictions on how and/or if actors can cooperate, that is, cooperation between *any* set of actors was allowed. Following Myerson's spirit, we interpret links/connections of a network as (direct) communication channels and argue that cooperation is only possible between actors that can directly or indirectly (i. e., via path) communicate with each other. One can interpret this approach as a generalization of both the approaches we used before: we enrich the underlying TU-game with a coalition structure where the coalitions contain those actors who are able to communicate with each other. This coalition structure is enriched with an inner structure, namely, the bilateral communication channels between the actors. In contrast to this approach which is based on a specific transformation of the coalition function, Jackson and Wolinsky [1996] model network structures by network functions, not assuming this explicit transformation. In the Jackson and Wolinsky model, the underlying function assigns worth to every network, independently on assumptions on the coalition function. We use the Myerson approach as we find it more in line with our applications. As we find the notation used by Jackson and Wolinsky [1996] more convenient than the (very abstract) one of Myerson [1977], we use the Jackson and Wolinsky notation, but in Myerson's spirit of communication structures.

Definition 2.15 (Network structure)**.** *We model a network as an undirected graph where the nodes represent the players and the edges represent the (bilateral) links between the players. Let* $N = \{1, ..., n\}$*, non-empty and finite, be the player set. The* complete network *(in which a link between any two players exists) is defined as* $g^N := \{ij := \{i, j\} | i, j \in N, i \neq j\}$*. Given that, we can define the set of all possible networks on* N*:* $\mathbb{G}_N := \{g | g \subseteq g^N\}$*. Given such a* $g \in \mathbb{G}_N$*, we call* (N, g) *a* network structure*.*

Remark 2.3 (Network vs. graph)**.** *Note that our definition of a network differs from the definition of a graph in the graph theoretical sense. In mathematical graph theory, a graph consists of edges* and *nodes, that is, a network structure in our notation. Furthermore, links in a network are bilateral connections* ij*,* $i \neq j$*, while a link* ii *is allowed in a general graph.*

Definition 2.16 (Allocation rule for network structures)**.** *A* TU-game with a network structure *is a tupel* (N, v, g) *and an* allocation rule for network structures $Y : \{N\} \times \mathbb{V}_N \times \mathbb{G}_N \longrightarrow \mathbb{R}^{|N|}$ *distributes the arising worth among the players, that is, assigns a payoff to each player.*

Definition 2.17 (Network-induced Partition, Connected Components and Connected Graph)**.** *We say that players* i *and* j*,* $i \neq j$*, are* connected *to each other in the network* g *if there exists a path* $ii_1, i_1i_2, ..., i_kj \in$

$g, i_1, ..., i_k \in N$. *Sets of connected players are components of a network* g *and these connected components build a partition on the player set* N. *We denote this partition by* $\mathcal{C}(N, g)$ *where* $\mathcal{C}_i := \mathcal{C}_i(N, g) \in \mathcal{C}(N, g)$ *is the component of all players connected with player* $i \in N$. *If* $\mathcal{C}(N, g) = \{N\}$, *we call* g *a* connected graph.

As mentioned before, Myerson [1977] introduced a transformation of the coalition function which accounts for the restricted cooperation possibilities arising due to the network structure:

Definition 2.18 (Graph-restricted game)**.** *Given a coalition function* $v \in \mathbb{V}_N$ *and a network structure* (N, g), *the corresponding* graph-restricted game (N, v^g) *is given by*

$$v^g(K) := \sum_{S \in \mathcal{C}(K, g|_K)} v(S) \quad \forall K \subseteq N,$$

where $g|_K = \{ij \in g | i, j \in K\}$.

If no confusion arises, we will denote the graph-restricted game simply by v^g.

Myerson [1977] used this game to define and characterize one of the first and most popular allocation rules for network structures:

Definition 2.19 (Myerson value)**.** *For any TU-game with a network structure* (N, v, g), *the* Myerson value μ *is for all* $i \in N$ *given by*

$$\mu_i(N, v, g) := Sh_i(N, v^g).$$

To characterize the Myerson value, **CE** is generalized for network structures and combined with an axiom which accounts for (what he calls) *fairness*.

Axiom 2.12 (Component Efficiency (CE) for Network Structures)**.** *An allocation rule for network structures* Y *satisfies* Component Efficiency **CE**, *if*

$$\sum_{i \in \mathcal{C}_i} Y_i(N, v, g) = v(\mathcal{C}_i) \, \forall \, i \in N.$$

Axiom 2.13 (Fairness (**F**))**.** *An allocation rule for network structures* Y *satisfies* Fairness **F**, *if for all* $ij \in g$:

$$Y_i(N, v, g) - Y_j(N, v, g) = Y_i(N, v, g - ij) - Y_j(N, v, g - ij).$$

F states that removing the link between two players should affect these two players in the same way.

Theorem 2.7 (Axiomatization of the Myerson value (Myerson, 1977))**.** *The Myerson value is the unique allocation rule for network structures that satisfies* ***CE*** *and* ***F****.*

In contrast to Myerson's transformation of restricted communication, Meessen [1988] and Borm et al. [1992] introduced an alternative transformation emphasizing links:

Definition 2.20 (Link-game (arc game))**.** *Given a coalition function* $v \in \mathbb{V}_N$ *and a network structure* (N, g)*, the corresponding* link-game *(or* arc game*)* (g, v^N) *or simply* v^N*, in which the links in the network* g *are the players, is given by:*

$$v^N(g') := v^{g'}(N) \quad \forall g' \subseteq g.$$

Now we have to restrict ourselves to zero-normalized games, otherwise v^N might not be a coalition function because $v^N(\emptyset) = v^{\emptyset}(N) \neq 0$. Originally introduced by Meessen [1988] and further analyzed by Borm et al. [1992] and Slikker [2005], this arc game is used to define the following allocation rule for network structures:

Definition 2.21 (Position value (for zero-normalized games only))**.** *For any zero-normalized TU-game with a network structure* (N, v, g)*, the* Position value π *is for all* $i \in N$ *given by*

$$\pi_i(N, v, g) := \sum_{\lambda \in g_i} \frac{1}{2} Sh_\lambda(g, v^N),$$
$$\text{where } g_i = \text{set of links including player } i \text{ and } v \in \mathbb{V}_N^0.$$

The Position value takes into account the role of links in which a player is (directly) involved in which is the basic idea of one of the most popular measures in the analysis of network centrality, the degree measure (cf. Freeman, 1978). The Position value will be discussed in more detail in Chapter 5 and Chapter 6.

While Borm et al. [1992] only provided a characterization of the Position value on cycle-free graphs (which nevertheless provides nice interpretations and properties which we will disuss in Chapter 5), Slikker [2005] introduced the following axiom for a characterization for general networks:

Axiom 2.14 (Balanced Link Contributions (**BLC**)). *An allocation rule for network structures* Y *satisfies* Balanced Link Contributions **BLC** *if for all* $i, j \in N$, $i \neq j$ *and* $v \in \mathbb{V}_0$ *we have*

$$\sum_{\lambda \in g_j} [Y_i(N, v, g) - Y_i(N, v, g - \lambda)] = \sum_{\lambda \in g_i} \left[Y_j(N, v, g) - Y_j(N, v, g - \lambda)\right].$$

Slikker [2005] argues that **BLC** "deals with the loss players can inflict on each other. The total threat of a player towards another player is defined as the sum over all links of the first player of the payoff differences the second player experiences if such a link is broken." **BLC** states that the total threat of a player towards another player should be equal to the reverse total threat.[4]

Theorem 2.8 (Axiomatization of the Position value (Slikker, 2005)). *The Position value is the unique allocation rule for network structures that satisfies* ***CE*** *and* ***BLC****.*

Following the basic structure of the χ-value, Casajus [2009a] introduced an outside-option-sensitive allocation rule for network structures. To reflect all (productive) outside options, he defined a network that captures the alternatives/outside options the players have:

Definition 2.22 (Lower Outside Option Graph (LOOG)). *For every network* g*, the corresponding* lower outside option graph (LOOG) *is given by*

$$g(i, N) := g|_{\mathcal{C}_i} \cup \{jk \in g^N | j \in \mathcal{C}_i, k \in N \setminus \mathcal{C}_i\}.$$

For notational convenience, if the player set is fixed, we will only write $g(i)$*.*

The LOOG reflects all alternative links a player and her coalitional players might have outside their actual coalition (which stays fixed). To provide these alternatives, links outside the own coalition are broken.

Using the LOOG, Casajus [2009a] defined an allocation rule for network structures accounting for outside options.

4 It might be worth to mention that Slikker [2005] originally called this axiom "balanced total threats".

Definition 2.23 (Graph-χ-value)**.** *For any TU-game with a network structure* (N, v, g)*, the* graph-χ-value $\chi^{\#}$ *is for all* $i \in N$ *given by*

$$\chi_i^{\#}(N, v, g) := \mu_i(N, v, g(i)) + \frac{v(\mathcal{C}_i) - \mu_{\mathcal{C}_i}(N, v, g(i))}{|\mathcal{C}_i|},$$
$$\textit{where } \mu_{\mathcal{C}_i} = \sum_{j \in \mathcal{C}_i} \mu_j$$

The graph-χ-value values outside options by adding/substracting some share of the Myerson value of her component to the Myerson value of the player (where the Myerson value is the Shapley value of the graph-restricted game), taking into account the outside option graph. This is, a redistribution of the difference between the worth actually created by the coalition to what the coalition would have obtained in the LOOG.

For the characterization, the axiom **CS** is generalized to network structures, an axiom accounting for outside options is defined as well as a weaker version of **F**.

Axiom 2.15 (Symmetry within Components (CS) for Network Structures)**.** *An allocation rule for network structures* Y *satisfies* Symmetry within Components **CS***, if*

$$Y_i(N, v, g) = Y_j(N, v, g) \; \forall \textit{ symmetric players } i, j \in N, j \in \mathcal{C}_i.$$

Axiom 2.16 (Outside Option Consistency (**OO**))**.** *An allocation rule for network structures* Y *satisfies* Outside Option Consistency **OO** *if for all* $i, j \in C \in \mathcal{C}(N, g)$ *we have*

$$Y_i(N, v, g) - Y_j(N, v, g) = Y_i(N, v, g(i)) - Y_j(N, v, g(j)).$$

OO could be interpreted as follows: the gains or losses of materialized outside options (difference between payoff in original network and the LOOG) should be equal for all players that are in the same coalition and hence obtain the same LOOG.

Axiom 2.17 (Weak Fairness 2 (**WF2**))**.** *An allocation rule for network structures* Y *satisfies* Weak Fairness 2 **WF2** *if for all connected* g *and all* $i, j \in N$ *we have*

$$Y_i(N, v, g) - Y_i(C_i(N, g - ij), v|_{C_i(N, g-ij)}, g|_{C_i(N, g-ij)})$$

$$= Y_j(N, v, g) - Y_j(C_j(N, g - ij), v|_{C_j(N,g-ij)}, g|_{C_j(N,g-ij)})$$

WF2 restricts **F** to a special situation where outside options are absent: the link ij is removed from a connected network and in a connected network, there are no outside options.

Remark 2.4. *Note that on connected graphs, the axioms **CE** and **CS** become **E** and **S**, respectively.*

Theorem 2.9 (Axiomatization of the graph-χ-value (Casajus, 2009a)). *The graph-χ-value is the unique allocation rule for network structures that satisfies **CE**, **OO** and **WF2**.*

2.4 GLOVE GAMES AND OUTSIDE OPTIONS

There are various examples in which an allocation rule should take into account *outside options*, that is, agreements *outside* the present agreements (fixed by the coalition or network structure) that could *alternatively* have been formed. Tutic et al. [2011] show in their experimental work about social interchange that outside options significantly affect negotiations. This connection has also been mentioned by Maschler [1992] as

> the need to let the players know what to expect from each coalition structure so that they can then make up their mind about the coalitions they want to join, and in what configuration. (p. 595)

In a similar spirit, von Neumann and Morgenstern [1944] state that any formed coalition between individuals "only describes one particular consideration". The result of negotiation between the individuals will be

> decisively influenced by the other alliances which each one might alternatively have entered. [...] Even if [...] one particular alliance is actually formed, the others are present in virtual existence: although they have not materialized, they have contributed essentially to shaping and determining the actual reality. (p. 36)

In the following we give a simple but in various ways applicaple example which motivates outside options and will be recalled several times throughout the following chapters: a *glove game*. In a glove game, introduced by Shapley and Shubik [1969], we assume to have a number of left and right gloves and a pair of gloves (one left and one right glove) produces worth x per pair. The agents/players are the glove holders, more precisely, the set of agents/players is split into left- and right-glove holders. This game is not only simple but also has a nice economic interpretation and it is used to analyze simple markets (cf. Shapley and Shubik [1969]).

We now formally introduce glove games. The efficient formulas for the Shapley value, the AD-value in general and the χ-value and the Owen-value for efficient partitions are provided in Chapter 4 as well as a discussion on the performance and differences of these solution concepts in general.

Definition 2.24 (Glove Game)**.** *In a general* glove game*,* N *is split such that* $N = L \cup R$ *with* $L \cap R = \emptyset$ *where* L *is the set of* left-glove holders *and* R *the set of* right-glove holders*. The characteristic function* v_{gg} *assigns a payoff of 1 to every pair, that is, the worth of a coalition* $K \subseteq N$ *is the number of matching pairs available in* K*. Formally, it is given by*

$$v_{gg}(K) := \min\{|L|_K|, |R|_K|\},$$

where $|L|_K| := |L \cap K|$ *and* $|R|_K| := |R \cap K|$*.*

Remark 2.5 (Normalization in the Glove Game)**.** *Generally, a matching pair could also generate an amount different to 1, but all analyzed allocation rules are additive which ensures that* $Y(N, x \cdot v) = x \cdot Y(N, v)$ *for all* $x \in \mathbb{N}$*. As all discussed allocation rules are also linear in the coalition functions, this even holds for all* $x \in \mathbb{R}$*.*

As one of the most popular applications of the glove game is the analysis of simple markets, we provide the following usefull definitions:

Definition 2.25 (Balanced/Imbalanced market)**.** *Let the glove game be interpreted to model a simple market.*[5] *The market is called* balanced *if* $|R| = |L|$ *and* imbalanced *otherwise.*

Definition 2.26 (Strong and Weak Players)**.** *In an imbalanced market, the players from the smaller set are called* strong players *(due to their scarceness) and the players from the larger set* weak players*. For notational reasons, denote by* $S := \min(|L|, |R|)$ *the number of strong players in the market and by* $W := \max(|L|, |R|)$ *the number of weak players in the market.*

As a leading and motivating example (see Casajus, 2009b and Belau, 2010), consider a glove game with two left-glove holders ($L = \{l_1, l_2\}$) and four right-glove holders ($R = \{r_1, r_2, r_3, r_4\}$), that is, an imbalanced market. Assume that l_1 and r_1 as well as l_2 and r_2 build a matching pair and r_3 as well as r_4 stay alone, that is, we face the (efficient) coalition structure

$$\mathcal{P} = \{\{l_1, r_1\}, \{l_2, r_2\}, \{r_3\}, \{r_4\}\}.$$

Following the allocation formulas, worth is distributed according to Table 1.

Table 1: Payoffs for the Glove Game

glove holder	Sh_i	Ba_i	AD_i	Ow_i	χ_i	W_i
l_1, l_2	0.7333	0.8125	0.5	0.8333	0.8	0.7167
r_1, r_2	0.1333	0.1875	0.5	0.1667	0.2	0.2833
r_3, r_4	0.1333	0.1875	0	0	0	0

Source: Own calculations and Casajus [2009b].

The Shapley value and the Banzhaf value do not take into account the coalition structure or, in other words, the unproductivity of the weak players outside the pair-building coalitions: all weak players obtain the same payoff. The AD-value does not take into account the imbalancedness of the market (bargaining power of the strong players against the weak players), that is, the outside options of the

5 For example, left- and right-glove holders are interpreted as sellers and buyers, respectively.

strong players: equal share of the worth among the pair-building players. The Owen value as well as the χ-value and the Wiese value take into account both the actual coalition structure and the existence of outside options in the imbalanced market.

Note that the Banzhaf value allocates more worth than actually available, that is, is not feasible. One could normalize the payoffs (which will lead to a distribution of 0.6842 and 0.1579 for the left- and right-glove holders, respectively. However, the normalization is problematic in axiomatization.

Remark 2.6 (Normalization, Justification by Properties and the Banzhaf value)**.** *Note that normalizations do not change ranks and relative distances of allocations and, hence, one can still use the original axiomatization for justification in terms of properties if one is not interested in the absolute numbers of allocation themselves. Therefore, the normalization of the Banzhaf value is unproblematic if one is mainly interested in ranks and relative distances as for example in applications for centrality measures or power indices as in Chapter 5. However, if absolute numbers are of interest (as in Chapter 3), the normalization of the Banzhaf value cannot be axiomatically justified.*

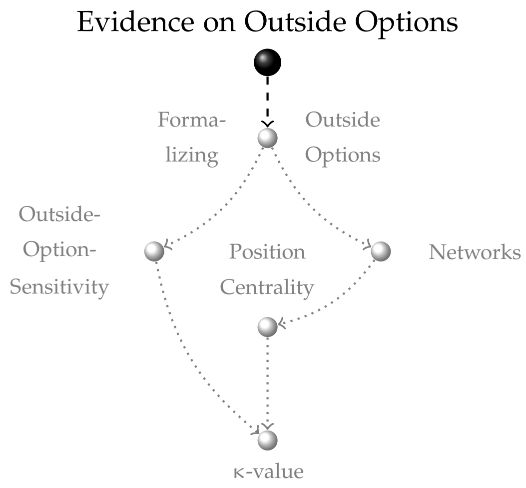
We have just seen an example which provided a brief idea of a certain difference between the allocation rules: some account for what is called *outside options*, some do not. In the next part of this thesis we will go deeper into the analysis of this difference by an experimental investigation (due to the axiomatic drawbacks of the normalized Banzhaf value and the Wiese value, these values are not analyzed here) and the analysis of a probabilistic forecasting model. We will find that outside options indeed affect bargaining processes and also applicability in the forecasting model and, hence, seem worthwile to be further analyzed and formalized. Beside that, we will also find applicability problems within the forecasting model for network structures which leads to the other main issue of this thesis: a deeper analysis of networks.

Part II

OUTSIDE OPTIONS

3

EVIDENCE ON OUTSIDE OPTIONS - A PILOT EXPERIMENT[1]



3.1 INTRODUCTION

In the game theoretic literature one finds many examples for the analysis of the process of negotiation (cf. for example Riker, 1967, Spector, 1977 or Maschler, 1978). Some of these works go deeper into the role of outside options and its influence on the solution of bargaining, for example through the analysis of changes in the solutions. Most bargaining models assume that the resulting outcome of an outside option is independent of the strategies of the actors during the negotiation process. But what happens if the outside option is again a negotiation?

To analyze the effect of outside options in a bargaining situation, we run an experimental case study using a glove game (Shapley and Shubik, 1969). In a glove game, the players are split into left-glove and right-glove holders which have to form matching pairs (one left

1 This chapter is based on Belau and Garmann [2013], joint work with Sebstian Garmann.

and one right glove) to create a payoff. This game has a nice economic interpretation as it is used to model markets in a simple way. Hence, our experimental results are economically interpretable. We ask the glove holders to form coalitions (i. e., matching pairs) by bargaining about the distribution of the outcome per pair. To model an imbalanced market, we use a glove game with an unequal number of left and right gloves.

Recall that we have many different approaches to predict how the worth of a coalition should be allocated among the players of the coalition: as one of the most popular approaches there is the Shapley value (Shapley, 1953) and as modifications there are the Owen value (Owen, 1977) and the AD-value (Aumann and Drèze, 1974) as well as the more recent χ-value (Casajus, 2009b).[2]

In our case study, we model the situation of the example from Chapter 2.4: There are two left-glove holders (l_1, l_2), due to their scarceness called strong players, and four right-glove holders ($r_1 - r_4$), called weak players. These players have to form matching pairs by bargaining about the distribution of the payoff from a matching pair. We take more than one left-glove holder to avoid monopoly and we hold the level of scarceness (difference between number of left and right gloves) fixed.

For the theoretical background, recall the distribution according to the different allocation rules (in the case that the coalitions (l_1, r_1) and (l_2, r_2) have been built and with the worth of a matching pair normalized to 100), given in Table 2. Recall that the Shapley value

Table 2: Payoffs for the Glove Game with $x = 100$

glove holder	Shapley value	AD-value	χ-value	Owen value
l_1, l_2	73.33	50	80	83.33
r_1, r_2	13.33	50	20	16.67
r_3, r_4	13.33	0	0	0

Source: Own calculations and Casajus [2009b].

2 Alternatively, there are also the Banzhaf value (Banzhaf, 1952) and the Wiese value (Wiese, 2007) which will not be analyzed in this chapter due to the feasibility issue of the Banzhaf value and the axiomatic drawbacks of the Wiese value and the normalized Banzhaf value (cf. Chapter 2). Furthermore, the Banzhaf value, as the Shapley value, is an impossible outcome in our case study. This will be discussed later on.

does not take into account the unproductivity of the weak players outside the pair-building coalitions (coalition structure) and the AD-value does not take into account the imbalancedness of the market (outside options) while the Owen value as well as the χ-value take into account both the actual coalition structure and the existence of outside options in the imbalanced market.

In the theoretical part of this chapter, we give the differences between the allocation rules for the glove game by analyzing the allocation formulas. For this, we derive the corresponding formulas in the special case of glove games that already allow us to precise the structural differences between the allocation rules in terms of outside options. Then, in the experimental part, we use the data from our case study first to test whether outside options influence bargaining situations at all. For that, we test the hypothesis if players with outside options (the strong players in the glove game) obtain significantly higher payoffs as the other players.

As we find that outside options affect negotiation, we further investigate which outside-option-sensitive allocation rule might be a favorable predictor for outcomes. As candidates we have the χ-value and the Owen value (as the Shapley value as an ex ante expected payoff is an impossible outcome in our case study). Both are characterized by an additivity-axiom, efficiency-axioms, symmetry-axioms and Null-player axioms[3]. While the characterization of the Owen value uses two symmetry-axioms, the one of the χ-value uses the so-called Splittingaxiom: if a coalition is split, arising gains or losses are distributed equally among the players that remain together. We implemented this in the experiment in order to test for this equal sharing as a hypothesis. Since all other axioms are widely accepted, the χ-value can be suggested as a solution concept if the Splittingaxiom is valid. However, we find that our data does not support this axiom. To the best of our knowledge there is no literature on this topic testing for axioms.

Existing studies about the effect of outside options on bargaining (as for example Tutic et al., 2011) ignore the Owen value (which turns out to be as consistent with our data as the χ-value and in fact is quite close in outcome for our special case). Furthermore, the bargaining situation is modeled by bilateral communication, that is, an offer is

3 cf. Owen [1977] and Casajus [2009b]

only observed by the participant the offer is made to. In contrast to that we model the bargaining process via multilateral communication using a double auction market, that is, all offers made are observable by all participants that are involved. Hereby, we are closer to the necessary experimental conditions for coalition games given by Selten [1972] (p. 142). Selten also suggests one-shot-plays; contrary to existing studies, we model the experiment in a way that we observe sort of a one-shot-play.

Pope et al. [2009] illustrate in their work the impact of prominent numbers (cf. Albers, 1998) in experimental research. They analyze the "Stages of Knowledge Ahead Theory" and find that the impact of prominent numbers especially occurs in the stage of evaluating the alternatives by the actor. While other studies ignore this fact, we use a prominent number assigned to each glove-pair that has to be distributed.

It is notable that the underlying game in our experiment could be seen as an extension of the popular (non-cooperative) ulimatum game: In the ultimatum game, two players must divide a positive amount of money which is often referred to as a "cake" which the two players can "eat". The first player proposes an allocation of the money which the second player can either accept or reject. If the second player accepts the proposal, the money is allocated according to the first player's proposal, if the second player rejects, no player receives any payment. One could argue that our experiment studies an "imbalanced-multiplayer-multiposition-multioffer ultimatum game". Our finding is in line with the finding of numerous experimental studies on ultimatum games and various extensions of this game that the experiments do not support the theoretical predicition (i. e., the subgameperfect Nash equilibrium predicition). To the best of our knowledge there is no, or if only few, experimental literature on imbalanced ultimatum games (different number of different positions) and multiposition-multioffer ultimatum games (*all* players can propose and except offers and new offers can unconditionally be made). Therefore, we contribute to the experimental literature on ultimatum games under competition.

A further contribution to the literature is our finding on price interdependency: We find evidence that, in contrast to existing literature, there is no interdependency between the outcomes within a market, that is, no evidence for price signals.

This chapter is organized as follows: The next section provides the theoretical background. We first shortly recall the cooperative framework and allocation rules and provide corresponding formulas for the case of glove games. We then use the explicit formulas to compare these allocation rules with respect to our main patterns outside-option-sensitivity and sensitivity to the coalition structures and describe the underlying game of our experiment and resulting theoretical predicions according to the analyzed rules. This is followed by a discussion of the relation to the non-cooperative ultimatum game. In Section 3, we explain our experiment and Section 4 provides the results for outside-option-sensitivity, the Splittingaxiom, learning effects and price interdependency. Section 5 discusses the relation of our results to the ultimatum game and, finally, Section 6 concludes.

3.2 THEORETICAL BACKGROUND

3.2.1 *Explicit Formulas for Glove Games*

We recall the framework of cooperative coalition structures introduced in Chapter 2: A *TU-game* is a tupel (N, v) where $N = \{1, ..., n\}$ denotes the (nonempty and finite) set of players and $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$ denotes the characteristic function, allocating to each coalition $K \subseteq N$ its worth $v(K)$. Then, an *allocation rule* $Y : \{N\} \times \mathbb{V}_N \longrightarrow \mathbb{R}^{|N|}$ distributes the worth of any TU-game among the players. Recall that the *Shapley value* (Shapley, 1953) is for all $i \in N$ given by

$$Sh_i(N, v) := \sum_{K \subseteq N \setminus \{i\}} \frac{|K|!(|N| - |K| - 1)!}{|N|!} [v(K \cup \{i\}) - v(K)]$$

where $MC_i^v(K) := v(K \cup \{i\}) - v(K)$ is the *marginal contribution* of player i to coalition K in the TU-game (N, v).

A partition $\mathcal{P}$ of N is called *coalition structure* and we denote by $\mathcal{P}(i)$ the coalition which contains player $i \in N$ and by $\mathbb{P}_N$ the set of all coalition structures of N. Then, a *TU-game with a coalition structure* is a tupel $(N, v, \mathcal{P})$ and an *allocation rule for coalition structures* is a function $Y : \{N\} \times \mathbb{V}_N \times \mathbb{P}_N \longrightarrow \mathbb{R}^{|N|}$. Aumann and Drèze [1974] defined the component restricted Shapley value (denoted by *Aumann-Drèze value* AD) as follows:

$$\forall i \in N: \ AD_i(N, \nu, \mathcal{P}) := Sh_i(\mathcal{P}(i), \nu|_{\mathcal{P}(i)}).$$

Casajus [2009b] defined an allocation rule that is sensitive to outside options, the χ-*value*, by

$$\forall i \in N: \ \chi_i(N, \nu, \mathcal{P}) := Sh_i(N, \nu) + \frac{\nu(\mathcal{P}(i)) - Sh_{\mathcal{P}(i)}(N, \nu)}{|\mathcal{P}(i)|}.$$

and in contrast, Owen [1977] defines the (also outside-option-sensitive) Owen value by

$$\forall i \in N: \ Ow_i(N, \nu, \mathcal{P}) := \frac{1}{|\Sigma(N, \mathcal{P})|} \sum_{\sigma \in \Sigma(N, \mathcal{P})} [\nu(K_i(\sigma)) - \nu(K_i(\sigma) \setminus \{i\})]$$

where $\Sigma(N, \mathcal{P})$ is the set of all orders σ over the player set that are compatible with the coalition structure $\mathcal{P}$ (i. e., $\forall\, i, j \in P \in \mathcal{P}$ we have $|\sigma(i) - \sigma(j)| < |P|$) and $K_i(\sigma)$ is the set of players that come before player i and including i under order σ.

In our experiment, the underlying TU-game is given by a *glove game* (Shapley and Shubik, 1969): Recall that in glove game, N is split into the left-glove holders L and the right-glove holders R (i. e., $L \cup R = N$ and $L \cap R = \emptyset$) and the worth of a coalition $K \subseteq N$ is the number of possible simultaneous matchings in K times the worth per matching pair x, that is, $\nu_{gg}(K) := x \cdot \min(|R \cap K|, |L \cap K|)$. In the theoretical analysis, we normalize the payoff per matching pair to $x = 1$ while we use $x = 100$ in the experiment as this facilitates division.

The market is called *balanced* if $|R| = |L|$ and *imbalanced* otherwise. In an imbalanced market, the players from the smaller set are called *strong players* and the players from the larger set *weak players*. For notational reasons, denote by $S := \min(|L|, |R|)$ the number of strong players in the market and by $W := \max(|L|, |R|)$ the number of weak players in the market.

Now we compare the different allocation rules by comparing the outcome for glove games. For Shapley-based values, computational effort is generally high[4] and Bachrach et al. [2010] state that calcu-

4 For example Deng and Papadimitriou [1994] show that already the computation for weighted majority games is #P-complete while Prasad and Kelly [1990] show that this is NP-hard.

lating the Shapley value in polynomial time (w. r. t. the number of agents) is only possible "in very specific and restricted domains". One of these specific domains indeed is the glove game. To facilitate computations, we first derive more applicable formulas for glove games which, as a special case, are more efficient from an computational point of view than the general ones (as this result is unsurprising we only discuss the computational effort issue in the appendix). Furthermore, these formulas provide better insight in the differences between the allocation rules.

Shapley and Shubik [1969] show that the Shapley value for glove games is given by

$$Sh_i(N, v_{gg}) = \begin{cases} \frac{1}{2} + \frac{W-S}{2 \cdot S} \sum_{k=1}^{S} \frac{W!S!}{(W+k)!(S-k)!} & \text{, if } i \text{ is strong player} \\ \frac{1}{2} - \frac{W-S}{2 \cdot W} \sum_{k=0}^{S} \frac{W!S!}{(W+k)!(S-k)!} & \text{, if } i \text{ is weak player} \end{cases} \quad (1)$$

Now we analyze the allocation rules for coalition structures. For all $i \in N$, set $S_i := \min(|L \cap \mathcal{P}(i)|, |R \cap \mathcal{P}(i)|) \geqslant 0$ (i. e., S_i denotes the number of strong players in player i's coalition) and $W_i := \max(|L \cap \mathcal{P}(i)|, |R \cap \mathcal{P}(i)|) \geqslant 1$ (i. e., W_i denotes the number of weak players in player i's coalition).

Consider the interesting and economically important case of minimal winning coalitions, that is, the coalition structure consists of matching pairs and singletons only. We call this case *efficient* and formally define:

Definition 3.1 (Efficient coalition structure)**.** *We call a coalition structure $\mathcal{P}$* efficient*, if only minimal winning coalitions are build and no strong player stays alone: $\forall\, P \in \mathcal{P} : P \subseteq \{l_i, r_j\}$ and if for some $l_i \in L$ we have $\{l_i\} \in \mathcal{P}$, then $\nexists\, r_j \in R$ such that $\{r_j\} \in \mathcal{P}$ and if for some $r_j \in R$ we have $\{r_j\} \in \mathcal{P}$, then $\nexists\, l_i \in L$ such that $\{l_i\} \in \mathcal{P}$.*

Theorem 3.1 (Aumann-Drèze value for (efficient) glove games)**.** *For all efficient coalition structures $\mathcal{P}$, the AD-value for the glove game is given by*

$$AD_i(N, v_{gg}, \mathcal{P}) = \begin{cases} \frac{1}{2} & \text{, if } i \text{ builds a pair} \\ 0 & \text{, if } i \text{ stays alone} \end{cases}$$

Proof. Restrict the Shapley allocation to the coalition of a player i: If $S_i = 0$ (i. e., either i stays alone as a singleton or is joined by the same type of gloves only), no matching pair exists in this coalition and the player obtains a payoff of zero. For $S_i > 0$ we have

$$AD_i(N, v_{gg}, \mathcal{P}) = \begin{cases} \frac{1}{2} + \frac{W_i - S_i}{2 \cdot S_i} \sum_{k=1}^{S_i} \frac{W_i! S_i!}{(W_i+k)!(S_i-k)!} & , \text{if } i \text{ is strong player in } \mathcal{P}(i) \\ \frac{1}{2} - \frac{W_i - S_i}{2 \cdot W_i} \sum_{k=0}^{S_i} \frac{W_i! S_i!}{(W_i+k)!(S_i-k)!} & , \text{if } i \text{ is weak player in } \mathcal{P}(i) \end{cases}. \quad (2)$$

Now let $\mathcal{P}$ be efficient, that is, $P = \{l_j, r_k\} \vee |P| = 1 \; \forall \; P \in \mathcal{P}$, then we have $AD_i(N, v, \mathcal{P}) = 0$ if $|\mathcal{P}(i)| = 1$, that is, i does not build a matching pair, and $AD_i(N, v, \mathcal{P}) = 1/2$ if $\mathcal{P}(i) = \{l_j, r_k\}$ as in this case we have $W_i = S_i = 1$. □

Theorem 3.2 (χ-value for (efficient) glove games)**.** *For all efficient coalition structures $\mathcal{P}$, the χ-value for the glove game is given by*

$$\chi_i(N, v_{gg}) = \begin{cases} Sh_i + \frac{1}{2}\left[1 - Sh_{strong} - Sh_{weak}\right] & , \textit{if } i \textit{ builds a matching pair} \\ 0 & , \textit{if } i \textit{ stays alone} \end{cases} \quad (3)$$

where Sh_{strong} denotes the Shapley value for the glove game of a strong player and Sh_{weak} of a weak player, respectively.

Proof. Note that in every coalition of an efficient coalition structure there is at most one left- and one right-glove holder, that is, within a winning coalition, every glove holder only faces a complementary glove holder. □

Theorem 3.3 (Owen value for (efficient) glove games)**.** *For all efficient coalition structures $\mathcal{P}$, the Owen value for the glove game is given by*

$$Ow_i(N, v_{gg}, \mathcal{P}) = \begin{cases} 1 - \frac{(S-1)!}{2 \cdot W!} \sum_{k=0}^{S-1} \frac{(W-(k+1))!}{(S-(k+1))!} & , \textit{if } i \textit{ is a strong player} \\ \frac{(S-1)!}{2 \cdot W!} \sum_{k=0}^{S-1} \frac{(W-(k+1))!}{(S-(k+1))!} & , \textit{if } i \textit{ is a weak matching-pair-player} \\ 0 & , \textit{if } i \textit{ stays alone} \end{cases} \quad (4)$$

Proof. Due to the form of an efficient coalition structure, we have for all $P \in \mathcal{P}$: $|P| \leqslant 2$ and $\Sigma(N, \mathcal{P})$ only contains orders where pairs (l, r) are next to each other (lr or rl). Hence, to analyze $\Sigma(N, \mathcal{P})$, we only have to consider orders of the components of $\mathcal{P}$, having in mind that each matching-pair-component has two possibilities. Therefore, $|\Sigma(N,\mathcal{P})|$ is the number of possibilities to order the components of $\mathcal{P}$ times the possibilities within each pair to be ordered, that is, times $2^{\#\text{ of matching pairs}}$ (each pair has two possibilities to be ordered).

Due to the form of efficient coalition structures, the number of components of $\mathcal{P}$ is equal to W and the number of matching pairs is equal to S. Hence, we have $|\Sigma(N,\mathcal{P})| = W!2^S$.

Since $|P| \leqslant 2$ for all $P \in \mathcal{P}$, we have

$$MC_i^{\nu_{gg}}(\sigma) \leqslant 1 \,\forall\, i \in N.$$

Consider $P \in \mathcal{P}$ such that $P = \{i\}$. If there is any matching candidate before i in order σ, the pair-partner of this candidate will be before i, too. Therefore, we have $MC_i^{\nu_{gg}}(\sigma) = 0$ and

$$Ow_i(N, \nu_{gg}, \mathcal{P}) = 0 \,\forall\, i \text{ such that } \{i\} \in \mathcal{P}.$$

For any weak player i who forms a matching pair we note: matching pairs before i in order σ do not affect $MC_i^{\nu_{gg}}(\sigma)$ since the worth created by this pair is created independently of using $K_i(\sigma)$ or $K_i(\sigma) \setminus \{i\}$. As all strong players (= matching candidates) before i in order σ appear with their matching partner, we have $MC_i^{\nu_{gg}}(\sigma) = 0$ whenever i is before his matching partner in order σ. If i's matching partner is before i in order σ and there is a singleton weak player before i's matching pair, we also have $MC_i^{\nu_{gg}}(\sigma) = 0$, because in $K_i(\sigma) \setminus \{i\}$, i's matching partner already creates worth with this singleton weak player. Hence,

$$MC_i^{\nu_{gg}}(\sigma) = 1 \Leftrightarrow \begin{cases} i\text{'s matching partner is before } i \text{ in order } \sigma \\ \text{and there are at most other matching pairs} \\ \text{before } i\text{'s matching partner.} \end{cases}$$

This happens how many times? There can be $k = 0, ..., S-1$ matching pairs before i's matching pair in σ. For each such k we have

$$\underbrace{(S-1)}_{\text{for } 1^{\text{st}} \text{ pair}} \cdot \underbrace{(S-2)}_{\text{for } 2^{\text{nd}} \text{ pair}} \cdot \ldots \cdot \underbrace{(S-k)}_{\text{for } k^{\text{th}} \text{ pair}} \cdot \underbrace{(|\mathcal{P}| - (k+1))}_{\text{remaining pairs and singletons}} \cdot 2^{S-1}$$

possibilities, where the 2^{S-1} follows from the fact that all matching pairs but i's can occur with two orders. This can be rewritten as

$$\frac{(S-1)!}{(S-(k+1))!}\cdot (W-(k+1))!\cdot 2^{S-1}.$$

Hence, $MC_i^{\nu_{gg}}(\sigma) = 1$ for

$$\sum_{k=0}^{S-1}\left(\frac{(S-1)!(W-(k+1))!}{(S-(k+1))!}\cdot 2^{S-1}\right)$$
$$= (S-1)!\cdot 2^{S-1}\cdot \sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!}$$

different $\sigma \in \Sigma(N, \mathcal{P})$. Therefore,

$$Ow_i(N, \nu_{gg}, \mathcal{P}) = \frac{(S-1)!}{2\cdot W!}\sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!}$$

for all weak players i who form a matching pair.

Since the Owen value is efficient (i. e., $\sum_{i\in N} Y_i = \nu(N)$), we have that

$$\sum_{i\in N} Ow_i = \#\text{ of matching pairs } = S.$$

Using this and that $Ow_i(N, \nu_{gg}, \mathcal{P}) = 0\ \forall\ i$ such that $\{i\} \in \mathcal{P}$, we have

$$\sum_{\substack{i \text{ builds} \\ \text{matching pair}}} Ow_i = S$$

Furthermore, the Owen value assigns equal payoffs to symmetric components. All components of the form $P = \{l, r\}$ are symmetric and hence, $Ow_l + Ow_r = \frac{S}{S} = 1$ for each matching pair (l, r). Using this, we get the Owen allocation for strong players: $Ow_{\text{strong player}} = 1 - Ow_{\text{weak player in matching pair}}$. And hence, finally,

$$Ow_i(N, \nu_{gg}, \mathcal{P}) = \begin{cases} 1 - \frac{(S-1)!}{2\cdot W!}\sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!} & , \text{if } i \text{ is a strong player} \\ \frac{(S-1)!}{2\cdot W!}\sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!} & , \text{if } i \text{ is a weak matching-pair-player} \\ 0 & , \text{if } i \text{ stays alone} \end{cases}$$

□

3.2.2 *The Underlying Game and Theoretical Predicition*

Now we compare the allocation rules using the explicit formulas for glove games to motivate the game underlying our experiment and explain their theoretical predictions.

Considering the formula given by Equation (1), we see that the Shapley value does not distinguish whether a weak player actually builds a matching pair (i. e., is productive or unproductive) as, beside the total number of strong and weak players, it only matters which kind of player one is, independently of whether one builds a matching pair or not.

In Equation (2), we already see that the AD-value generally only accounts for imbalancedness *within* a player's own coalition. In the case of minimal winning coalitions, the AD-value splits the worth equally among the matching-pair-players, that is, as if we had a balanced market.

In the formula given by Equation (3), we see that the χ-value distinguishes between pair-building and non-pair-bilding players. Furthermore, due to the use of the Shapley value, imbalancedness of the market is taken into account.

Finally, considering the formula in Equation (4), we see that also the Owen value accounts for both the coalition structure (matching vs. no matching) and the level of imbalancedness (S and W).

The theoretical comparison is summarized in Table 3.

Table 3: Comparison of Allocation Rules for the Glove Game

Property	Accounting for Coalition Structure	Outside-Option-Sensitivity
Indicator	matching vs non-matching player	Strong vs weak player
Shapley value	-	✓
AD-value	✓	-
χ-value	✓	✓
Owen value	✓	✓

Now, in order to use these theoretical predictions, we have to design a game for our experiment with the theoretical interpretation of a glove game. Furthermore, in order to be able to clearly distinguish between sensitivity and insensitivity to outside options, we have to take an unequal number of strong and weak players and have to ensure that only minimal winning coalitions will occur as in this case, no alternatives *inside* a coalition can affect the outcome. The underlying game in our experiment can be summarized as follows:[5]

As a basis, we take the glove game from the introduction where the player set N consists of two left-glove holders l_1 and l_2 (the strong players) and four right-glove holders $r_1 - r_4$ (the weak players). These players have to form matching pairs by bargaining about the distribution of the worth per matching pair which we set to $x = 100$ tokens. The bargaining and matching process is designed by a double auction market: all players simultanously place their demand of the 100 tokens as an offer on a list and each offer from this list can then be accepted by a matching player, that is, left-glove holders can accept offers from right-glove holders and vice versa. Players are allowed to make as many new offers as they like. As soon as an offer is accepted, the accepting player and the corresponding offering player form a matching pair. All open offers are observable by all players and accepted offers disappear from the list to indicate that a pair has been formed (complete information). If either all strong players have built a matching pair or 120 seconds have passed, the game ends and matching pair players obtain payoffs according to the corresponding offer (the offering player receives her demand, the accepting player receives the residual) and non-matching pair players do not receive any payoff (i. e., zero). Both cases imply that only minimal winning coalitions occur and that unproductive players obtain a payoff of zero by design.

Note that the first case implies an efficient coalition structure. As the second case never occured during the entire experiment, we indeed observe an efficient glove game with $S = 2$ and $W = 4$ and theoretical predictions are given by the formulas we derived before. Recall that the Shapley value does not distinguish between productive and unproductive players and, hence, is an impossible outcome in the framework of our experiment. Theoretical predictions for our

5 The actual experiment will be explained in detail in the next section.

experiment according to the remaining three allocation rules for the distribution within a matching pair are given in Table 4.[6]

Table 4: Theoretical Prediction for the Experiment

Type of Player	AD-value	χ-value	Owen value
Strong player	50	80	83.33
Weak player	50	20	16.67

If we find that outside options matter, that is, if strong players obtain a significantly higher payoff than weak players, only the χ-value and the Owen value would be appropriate predictors from the set of theoretical allocation rules. As our number of observations is rather small and the theoretical predictions by these two rules are very similar, a direct test on distributions is fairly hard. Hence, we will further test for axiomatic differences of the two remaining allocation rules.

Both the χ-value and the Owen value are characterized by an additivity-axiom, efficiency-axioms, symmetry-axioms and Null-player axioms[7]. While the characterization of the Owen value uses two symmetry-axioms, the crucial axiom of the characterization of the χ-value is the so-called splitting-axiom: A partition $\mathcal{P}' \subseteq 2^N$ is called *finer* than $\mathcal{P} \subseteq 2^N$ if $\mathcal{P}'(i) \subseteq \mathcal{P}(i)$ for all players $i \in N$. An allocation rule Y satisfies *Splitting* **SP** if for $\mathcal{P}'$ being finer than $\mathcal{P}$ we have for all $i \in N$, $j \in \mathcal{P}'(i)$:

$$Y_i(N, v, \mathcal{P}) - Y_i(N, v, \mathcal{P}') = Y_j(N, v, \mathcal{P}) - Y_j(N, v, \mathcal{P}').$$

The motivation of **SP** is the following: One could argue that gains or losses of splitting a coalition structure should be distributed equally on players staying together in the new coalition structure. Splitting a given coalition structure should affect all players that remain together in the new coalition structure by the same way.

We implemented a test for **SP** in our experiment to provide a potential further indicator.

6 We obviously obtain the same distributions as in Table 2 from the introduction.
7 cf. Owen [1977] and Casajus [2009b]

3.2.3 *Relation to the Ulimatum Game*

From the non-cooperative point of view, one could interpret the underlying game of our experiment as sort of an ultimatum game. In this section we will briefly present the definition, theoretical prediction and experimental findings on this topic.[8]

Definition 3.2 (Ultimatum Game)**.** *In the* ultimatum game *two players* P1 *and* P2 *must divide a positive amount of money which we denote by* x*.*[9] P1*, called the* proposer*, proposes a demand* y*,* $0 \leqslant y \leqslant x$*, for herself and* P2*, called the* responder*, can either accept or reject. If* P2 *accepts,* P1 *receives* y *and* P2 *receives* $x - y$ *and if* P2 *rejects, no player receives any payment (i. e., zero).*

The (non-cooperative) theoretical predicition of the outcome of this game (i. e., the subgame perfect Nash equilibrium prediction) is quite obvious: If both players are rational utility maximizers, P2 must accept all proposals $y < x$ since she receives $x - y > 0$ in this case and therefore, the best for P1 is to propose $y^* = x - \epsilon$ where ϵ is the smallest monetary unit available (which P2 will accept). Hence, the theoretical prediction is a divison $(x - \epsilon, \epsilon)$. The first experiment on the ultimatum game was conducted by Güth et al. [1982], followed by numerous subsequent studies[10]. Güth [1995] summarizes the main point as follows:

> Imagine an ultimatum game with $x = 120$ EUR: Would you really dare to demand EUR 119 for yourself? Very few participants do and those who dare to do so fail nearly always, i.e. their proposal is rejected. [...] The main tendencies observed were that responders are willing to sacrifice substantial amounts to punish a greedy proposer and that this is well anticipated by most proposers. (p. 331)

There have been numerous extensions of the original ultimatum game. Just to name a few, Ochs and Roth [1989] investigated multiperiod ultimatum bargaining, Hoffman and Spitzer [1985] and Hoffman et al.

8 For further reference we refer to Güth [1995] and Bearden [2001].
9 Often, x is referred to be a “cake” which P1 and P2 can “eat”.
10 For a survey on literature on experiments for the ulimatum game see for example Güth [1995].

[1994] analyzed effects of position (proposer or responder) determination, Roth et al. [1991] examine cultural differences, Oppewal and Tougareva [1992] study a three-person ultimatum game where the responders have to share the residual of the proposer's offer and Gneezy et al. [2003] analyze what they call a *"reverse"* ultimatum game in which the proposer is allowed to propose another (strictly lower) offer if the first offer has been rejected. What these studies have in common is that none of them found support for the theoretical predicition.

The underlying game of the experiment provided in this chapter could be seen as contributing to the aforementioned literature of the analysis of ultimatum games: As in the ultimatum game we ask players to devide an amount of money where proposals (offers) can be made and accepted and in case that no proposal is accepted, none of the players receives any payment. However, our underlying game embeds several extensions of the original ultimatum game: First of all, there are not only more than just two players (which makes it a *"multiplayer"* game), there is an unequal number of players corresponding to the two positions which makes the game *"imbalanced"*. Furthermore, our positions differ in the way that there are no proposers and responders as all players are allowed to offer proposals (proposer role) and to accept proposals (responder role) which makes the game a *"multiposition"* game. Note that there is no explicit rejection of an offer and remember that we still have two different positions, namely, the left-glove and the right-glove holders. One could however argue that proposers are sort of strong players while responders are sort of weak players. Moreover, players are allowed to make more than one offer. We could refer to the term "reverse ultimatum game" as in Gneezy et al. [2003], however, they called their game "reverse" as the proposer was only allowed to make strictly lower new offers, that is, offers that lead to a strictly higher payoff for the responder. We do not restrict new offers. Hence, we rather call it a *"multioffer"* game. To summarize, the underlying game of our experiment could be seen as an "imbalanced-multiplayer-multiposition-multioffer ultimatum game".

To the best of our knowledge, there are no experimental studies on either

- imbalanced multiplayer ultimatum games in the sense that weak players (or responders) have to bargain with each other as one (or more) of them will stay alone (without payment) or
- multiposition and multioffer ultimatum games where *every* player can be both proposer and responder and new offers can be made unconditionally

Note that the experiment discussed in this chapter has been a pilot experiment (or case study), we did not have different treatments to infer one or the other extension separately and moreover, we in fact have sort of two imbalanced ultimatum games simultanously as we even have more than one strong player. However, also for our experiment, the non-cooperative theoretical prediction would still be the same as for the basic ultimatum game which is $(99, 1)$ in our case (remember that we set the worth of a matching pair to $x = 100$).[11] This outcome could be argued to be even "more reasonable" due to the imbalancedness: the "punishing thread" of weak players is different to the one in the basic ultimatum game as if one weak player "rejects" (i. e., does not accept) an offer, the strong player does not receive zero, she could still find an agreement with another weak player. One could argue that the outcome might be different to $(99, 1)$ as long as there are several strong players that "compete" against each other but even if we take this into account, as soon as one strong player found a partner, the second matching pair would obtain $(99, 1)$ due to the (non cooperative) theoretical prediction.

Remark 3.1 (Ultimatum Game, Fairness and Allocation Rules)**.** *Note that the most popular explanation for the discrepancy of observed outcome to the theoretic prediction is the assumption that* fairness *strongly affects the outcome. Interestingly, all cooperative allocation rules we discussed in Chapter 2 are based on some sort of "fairness-like" axiom(s) however it is not clear how to explicitly formalize "the fairness axiom": Myerson [1977] called the driving axiom of the Myerson value "Fairness", this term has been criticized and changed later on, for example Jackson and Wolinsky [1996] call it "equal bargaining power" (which Jackson [2006] himself criticized again). Hence, an*

11 Actually the smallest monetary amount in our experiment has been 0.01 tokens but none of the participants ever used decimal places different from zero throughout the entire experiment.

analysis of "fairness" itself would not yield to a cooperative theoretical predictor for the ultimatum game. However it is notable that none of the discussed allocation rules suggests a division of $(99, 1)$ *for our experiment.*

3.3 THE EXPERIMENT

We ran six sessions, in each session having two independent groups of six participants each, hence, 12 participants per session and 72 participants in total. Participants were male and female students from the University of Duisburg-Essen, Germany, who were paid for participating in a session lasting no longer than 60 minutes. Each participant received a show-up fee of EUR 4 and earned additional money during the session, depending on her decisions and the decisions made by the other participants of the session. In addition to the show-up fee paid to each participant, we paid EUR 100 in each session, distributed among the participants depending on their decisions. Hence, on average, each participant earned EUR 100/(2*6) + EUR 4 = EUR 12.33[12]. To preserve anonymity, participants were paid out one after the other.

The experiment was conducted computer-based and took place in the "Essen laboratory for experimental economics (elfe)" at the University of Duisburg-Essen, Germany, in June 2012. Participants were recruited using ORSEE (Greiner, 2004) and the attached subject pool. The experiment was programmed and conducted with the software z-Tree (Fischbacher, 2007).

To ensure anonymity during the experiment, the participants were placed in seperated sound booths. The participants entered the laboratory one after the other and each participant drew a ball on which she found the number of her sound booth. The experiment consisted of two parts. Inside the sound booths the participants found the instructions for the first part of the experiment as well as a simple calculator, blank paper and a pen. They were given some time to read the instructions (the doors of the sound booths were let open), after that, an experimenter came to every participant separately to

12 Minimum was EUR 4, Maximum was EUR 25.20.

ask whether there are open questions[13]. After all doors of the sound booths were closed, no questions were allowed anymore. Part 1 of the experiment started after six comprehension questions. The instructions stated that the participants were going to be paid depending on their decisions. After the first part of the experiment, instructions for the second part were distributed to the participants. Again, they were given time to read them and an experimentator came to answer open questions. After this, no further questions were allowed. Part 2 of the experiment started after two comprehension questions. The instructions stated that not every round of part 2 was going to be paid and the participants were informed that in the end of the treatment, one round was going to be drawn randomly for payment. This guaranteed that the motivation of the participants was equally high in all rounds, that is, avoided "waiting for a better position". Instructions included example screens for higher understanding. Find the instructions and comprehension questions for both parts as well as example screens in the Appendix.

The participants were split into groups of six participants each and stayed in the same group all over the session. This allows us to study behavior over time. In each group we assigned the positions of left- and right-glove holders to the participants and the participants were asked to build matching pairs. We always had two left-glove holders and four right-glove holders, hence there could have been at most two matching pairs and at least two participants were not able to find a partner. A matching pair was build by agreeing on a certain distribution of 100 tokens (where 1 token is EUR 0.125). We chose the value of a matching pair to be 100 referring to the theory of prominent numbers: we assume participants to be able to distribute 100 more in line with their actual preferences than non-prominent values (as for examble 0.125). The experiment consisted of two parts, in the first part participants were introduced to the experiment and exactely *one* round was played. In the second part, participants were told to play the same experiment as before, just that there were multiple rounds (five) now. To avoid last round effects, it was not known by the participants how many rounds are played. We split the experiment in order

13 For comparability, the experimenter who answered the questions was the same for all sessions of the experiment.

to observe data as in a One-shot-play from the first part and data for behavior over time from the second part.

The experiment was modeled by a double auction: The participants were allowed to make an offer about the distribution of tokens per pair or accept an offer of another participant in the group. Every offer that was made was observable for all other participants in the group and every offer from a left-glove holder could have been accepted by every right-glove holder of the group and vice versa. If an offer had been accepted, a matching pair was formed. Accepting an offer was binding as well as making an offer, that is, already made offers could not be taken back. Players were allowed to make as many offers as they wanted. However, every former offer they made could still have been accepted leading to a matching pair with the distribution of this offer. All offers were listed and once an offer was accepted, this offer and all other offers from the two involved participants disappeared from the list. The round ended if either two matching pairs have been built or the time of 120 seconds has run out.

In each round, the positions were randomly assigned to the participants. Hence, a participant could have been a right-glove holder in one round and a left-glove holder in another round. The participants were not able to identify the other players. In order to make the groups comparable to each other, the path of assigned positions was equal in each group, that is, participant 1 in group 1 had the same path of positions over the treatment as participant 1 in the other groups and so on. To define the path of assigned positions, we drew a set of 6x6 random integers online at www.random.org[14].

After the second part of the experiment, the participants were informed that the payment-relevant part of the experiment was over. It followed a short payment-irrelevant part. In this part, participants faced the following situation: A distribution of 200 tokens among 6 individuals was given. Participants got informed that individual 6 left and how many tokens remain. They were asked to distribute the remaining tokens among the remaining 5 individuals. Find an example

14 For each of the six rounds we needed six integers to assign the positions among the six participants per group. The first column of the integer set corresponds to participant 1 in a group, the second column to participant 2 and so on. The first row of the integer set assigns the positions of the two left-glove holders in round 1 to the participants with the highest and second highest integer of this row, the other participants are right-glove holders in round 1. The second row corresponds to the second round and so on.

screen in the appendix. The distribution of the 200 tokens slightly varied for each participant: The basic distribution was $50, 25, 50, 25, 25, 25$ and we created variance by adding a zero-sum random vector. The tokens remaining after individual 6 left depended on how many tokens this individual had observed in the first distribution. The payment-irrelevant part was followed by questions about personal details.

3.4 RESULTS

3.4.1 *Outside Options*

In this section, we provide descriptives of the experiment and evaluate which of the allocation rules summarized and compared in section 3.2.2 is consistent with our experimental data. A first discriminatory feature of these value is the prediction of whether outside options matter at all: the Shapley value, the Owen value and the χ-value are sensitive to outside options while the AD-value disregards alternatives outside one owns coalition. Therefore, we first test whether outside options had an effect on bargaining outcomes in our experiment. If the existence of outside options had an effect on bargaining outcomes, we would expect that left-glove holders have significantly larger payoffs than right-glove holders that are part of a matching pair. In other words, left-glove holders would receive significantly more than 50 tokens when a matching pair is formed.

Two points deserve further discussion.

1. In our basic approach, we restrict this test to the results from the *first experimental round*.

The reason is that we had designed the first round as a *one-shot game* such that the behavior of the participants was not biased by the possibility of future rounds; participants did simply not know that further rounds will be played. Furthermore, the focus on the first round results also ensures that the results are not biased by *learning effects*. We have checked whether our results would differ if we consider all six rounds, but found that this is not the case.

2. We restrict our attention to the results of the *first matching pair* in each group.

We do this because the results of the first and the second matching pair in a group are likely to be *not independent* observations. Once one pair is built, the ratio of the number of left- and right-glove holders changes. This is problematic as theory predicts the price of a negotiation to be dependent on this ratio. Furthermore, participants might see the results of the first negotiation as a *price signal* on which they base the second negotiation. In Section 3.4.4, we test whether this is really the case.

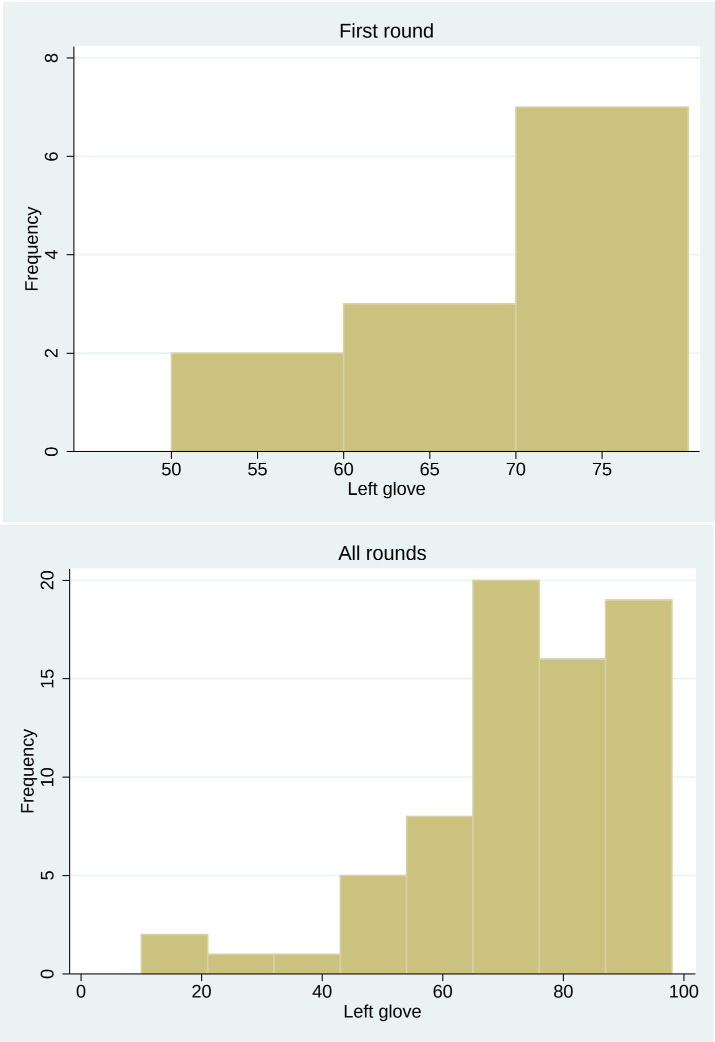
Find the observations of the first round from all groups in Table 5 (the results for all rounds can be found in Table 20 in the appendix) and Figure 2 displays the histograms showing the frequency of outcomes of the left-glove holders for both the one-shot game (first round) and for all rounds.

Table 5: Distribution of Shares in 1^{st} Round

Round	Group	Distribution 1st pair		Distribution 2nd pair	
		strong	weak	strong	weak
1	1	65	35	66	34
1	2	80	20	79	21
1	3	50	50	55	45
1	4	75	25	73	27
1	5	50	50	80	20
1	6	60	40	90	10
1	7	75	25	75	25
1	8	80	20	80	20
1	9	70	30	70	30
1	10	80	20	51	49
1	11	60	40	70	30
1	12	70	30	90	10
Mean:		67.92	32.08	73.25	26.75

Note that due to our experimental design, the whole worth of a matching pair has been distributed within a pair. Hence, the outcomes of the corresponding right-glove holders are omitted. The histograms already provide some evidence in favor of the impact of outside options (clustering above the 50 tokens mark). However, a formal test shall confirm this finding.

Figure 2: Histograms Outside Options



Formally, let y_{ilt} and y_{irt} be the payoff of the left- and right-glove holders that have formed the first matching pair in group i and round t, respectively.

Hypothesis 1 (Outside Options)**.** *Whether outside options matter can be tested via the null hypothesis*

$$H_0 : y_{il1} \leqslant 50$$

against the alternative

$$H_1 : y_{il1} > 50$$

We test the above null hypothesis with a one-sided t-test. We acknowledge that the number of observations could be quite low to defend the normality assumption that is needed. In robustness checks, we have therefore also used a non-parametric test. We have used the *Wilcoxon Signed Rank Test* (Wilcoxon, 1945[15]) which can be regarded as the non-parametric analogue to the t-test. However, our results do not differ which is why we restrict the presentation to the results from the t-test. The test statistic of the t-test yields a value of 5.66 which has to be compared with a t-distribution with 11 degrees of freedom. Accordingly, the null hypothesis that outside options do not matter is rejected at the 1%-level. In other words, left-glove holders obtain significantly more than 50 tokens.

Observation 1: *Outside options do affect negotiation.*

3.4.2 *Allocation Rules and the Splittingaxiom*

What does the finding of the previous subsection imply for the validity of different allocation rules proposed by cooperative game theory? As shown in Table 3 and Table 4, the AD-value proposes to distribute the same amount of worth to both left- and right-glove holders of a matching pair, whereas players that are not part of a matching pair do not receive any payoff. Thus

Observation 2: *Our experimental observations are inconsistent with the AD-value.*

Our experimental observations are, however, consistent with the Shapley value, the χ-value and the Owen value. Recall that the Shapley value, however outside-option-sensitive, is designed for cooperative games without an inner structure, that is, does not take into account the actual coalition (matching pair). Note that with our design we cannot evaluate the appropriateness of the Shapley value, as the Shapley value predicts positive payoffs for non-productive players (i. e., non-matching-pair players), whereas in our experiments, payoffs for these players are by design zero. The Shapley value could be seen as an expected payoff *before* an actual matching pair is build. In our experiment, however, we model the negotiation process to build the pair.

Observation 3: *The Shapley value is an impossible outcome due to the coalition structure and experimental design.*

15 For further details also see Siegel [1956].

Because the theoretical predictions for the χ-value and the Owen value for the glove game used in the experiment are very similar[16], it thus is not expedient to use the payoffs observed in the experiment to discriminate between these values.

Observation 4: *The predicitions due to the χ-value and the Owen value are barely statistically discriminable in our experiment.*

Therefore, we design a test whether the Splittingaxiom **SP** is supported by the data. While the Owen value does not satisfy **SP**, the χ-value crucially hinges on this axiom by its characterization. Thus, if we find evidence in favor of the Splittingaxiom, we find evidence to support the χ-value and against the Owen value.[17]

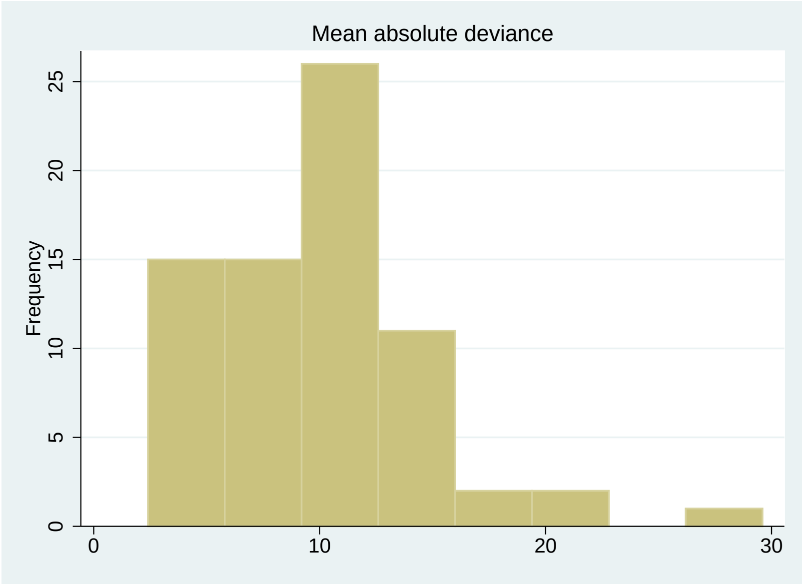
As described above, we have confronted the 72 participants of the experiment with a hypothetical payoff structure for two left- and four right-glove holders. After that, we have told the participants that one right-glove holder disappears, and that therefore the worth distributed to the whole group shrinks. We have then asked the participants to state how they would distribute the remaining worth to the remaining five players. This gives us a new payoff structure for each of the 72 participants. We can also calculate a payoff structure that the participants should have chosen if the Splittingaxiom **SP** were valid. Our test for **SP** is based on comparing the chosen payoff structure with those that would have resulted under the validity of **SP**. From that comparison, we calculate the *mean absolute deviance* of the chosen payoff structure from the calculated payoff structure. The intuition is that if this mean deviance is close to zero, then both distributions are equal and thus the participants would distribute worth according to the predictions of **SP**. Find the mean absolute deviances displayed in Figure 3.

A question that arises is how large this mean deviance should be to conclude that it is not close to zero. It may well happen that some

16 Recall here the general structural difference between the χ-value and the Owen value: While the χ-value considers productive unions (component efficiency), the Owen value considers bargaining blocks (efficiency). We discussed this difference in Chapter 2 where we explained that the Owen value can still be seen as an allocation rule accounting for productive unions if we interpret the fixed bargaining blocks as such productive unions. The theoretical predicition due to the Owen value has been calculated using exactly the productive union as a bargaining block. Hence, there is no reason for this structural difference affecting our analysis.

17 However note that the reverse does not hold. We will discuss this in the end of this section.

Figure 3: Mean Absolute Deviance



participants make small calculation errors such that the mean absolute deviance is slightly positive. We have chosen a cutoff value of 5, that is, if the mean absolute deviance is larger than 5, then we would reject the null hypothesis that **SP** is supported by the data.

Hypothesis 2 (Splittingaxiom)**.** *Whether the Splittingaxiom **SP** is supported by our data can be tested via the null hypothesis*

$$H_0 : \textit{mean absolute deviance} \leqslant 5$$

against the alternative

$$H_1 : \textit{mean absolute deviance} > 5$$

A t-test rejects the null hypothesis that the absolute mean deviance is smaller than or equal to 5 (p-value: 0.000). Thus, there is very strong evidence that **SP** does not hold. An interesting question is also how high the threshold should have been to be able to not reject the null hypothesis that the absolute mean deviance is small. We have tested different thresholds in turn, and have found that the threshold must have been set at 10 in order to be not able to reject the null hypothesis. Given the basic underlying distribution faced by the participants, this is a very huge mean deviance. Once more, this is strong evidence against **SP**. Note that findings from the data for this part of the experiment have to be interpreted with caution as this part was payment irrelevant and took place at the very end of each session.

Observation 5: *We cannot support the χ-value by evidence in favor for its crucial axiom.*

To conclude, our experiments have shown that

1. outside options do indeed matter and
2. the Splittingaxiom **SP** cannot be supported by the data.

At first glance one could argue that these results imply that, since - from the proposed allocation rules - only the χ-value and the Owen value are consistent with outside options and our experiment, the Owen value is most consistent with the data since the χ-value's crucial axiom cannot be supported. In fact, we did not test for any axiom characterizing the Owen value and, therefore, cannot find any explicit support to select the Owen value as the "best" predictor. Moreover, we cannot conclude that we found evidence *against* the χ-value, we just did not find evidence in favor of it. Also remember that the predicitions according to the χ-value and Owen value are very similar in our special case (which however does not generally hold).

Remark 3.2 (Axiomatization: Direction of Reasoning)**.** *Note that the story of axiomatizations "if you support these axioms, take that rule" (or an* axiomatic method *in general) basically is about finding reasonable and convincing properties to uniquely describe an allocation rule (or any theoretic model). The direction of reasoning is positive in the sense that there is support* in favor *of axioms. The direction of reasoning is not meant to be negative in the sense of finding evidence* against *axioms as "if there is an axiom that you do not support, do not take a rule which satisfies it".*

3.4.3 *Learning Effects*

Due to the multiple-round-setup that we have used, our experiment allows shedding some cautious light on the existence of learning effects. We pool the bargaining outcomes for the left-glove holders of each group and round that have first signed an agreement together and estimate the following regression:

$$y_{ilt} = \alpha + \nu_i + \lambda_t + \epsilon_{it}$$

where α is a constant, ν_i a *group-fixed effect* that captures time-invariant group differences in bargaining outcomes and ϵ_{it} is a *standard error* that is robust to heteroskedasticity. The main parameters of interest are the *round-fixed effects* λ_t. We leave out the dummy for the first round which serves as the reference category. Thus, we analyze whether the outcomes of later rounds differ significantly from the first round outcome. Table 6 shows the results.

Table 6: Results for Learning Effects Estimation

	(1)
VARIABLES	y_{ilt}
Round 2	4.417
	(1.070)
Round 3	3.083
	(0.845)
Round 4	0.083
	(0.014)
Round 5	5.583
	(0.726)
Round 6	14.333***
	(3.231)
Constant	67.917***
	(27.788)
Observations	72
Number of group	12
R-squared	0.114

Heteroskedasticity-Robust t-statistics in parentheses
*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$

Only the round 6 effect is significantly different from zero which suggests that the share that left-glove holders receive is significantly larger in the last round compared to the first round. Thus, there is evidence for some learning effects. However, we acknowledge that the number of rounds in our experiment is rather low. It may be possible that even stronger learning effects would have occurred at later rounds.

3.4.4 *Price Interdependency*

An interesting question is also whether there is any interdependency between the price of the first and the second matching pair in a group. As argued above, this may be the case if individuals see the price of the first matching pair as a signal of which price is "fair" and appropriate for a left-glove holder in a matching pair. To investigate whether this hypothesis is supported by the data, we run the following regression:

$$y_{ilt}^{2nd} = \alpha + \beta_1 y_{ilt}^{1st} + \nu_i + \lambda_t + \epsilon_{it}$$

A significant coefficient for β_1 would indicate that there is indeed an effect of the price of the first matching pair in a group-round combination on the price of the second matching pair. However, as shown in Table 7, we did not find any significant effect of the first price on the second price. Thus, price expectations of the remaining four participants were not influenced by the first negotiation in a group-round pair.

3.5 DISCUSSION OF RESULTS W.R.T. THE ULTIMATUM GAME

We will now briefly discuss our data and previous results in relation to the ultimatum game. As discussed in Section 3.2.3, the non-cooperative theoretical predicition of our experiment interpreted as an extended ultimatum game would be a share of 99 and 1 for the strong players and the weak matching-pair-players, respectively. Consulting our data from the one shot play (first round of the experiment, cf. Table 5), we see that the distribution with most divergent shares observed is (share strong player, share weak player) $= (80, 20)$ for the first pair and $(90, 10)$ for the second pair while the mean distrubtion is about $(68, 32)$ and about $(73, 27)$ for the first and second pair, re-

Table 7: Results for Price Interdependency Estimation

	(1)
VARIABLES	y^{2nd}_{ilt}
y^{1st}_{ilt}	0.059
	(0.260)
Round 2	-3.844
	(-0.361)
Round 3	-1.599
	(-0.151)
Round 4	0.912
	(0.086)
Round 5	1.420
	(0.133)
Round 6	3.570
	(0.322)
Constant	69.238***
	(4.038)
Observations	72
Number of group	12
R-squared	0.013

Heteroskedasticity-Robust t-statistics in parentheses
*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$

spectively. This obviously does not seem to support the theoretical predicition of $(99, 1)$.

Recall that one could argue that, due to the change of imbalancedness, the theoretical predicition is even more reasonable for the second matching pair in a group (decrease of punishing threads). However we found evidence against this difference in Section 3.4.4: according to our data there is no evidence that a decrease of punishing threads changes behavior.

An explanation used to save theoretical predictions of the ultimatum game has initially been suggested by Binmore et al. [1985] (p. 1180):

> Subjects, faced with a new problem, simply choose "equal division" as an "obvious" and "acceptable" compromise

> [...]. We suspect [...] that such considerations are easily displaced by calculations of strategic advantage, once players fully appreciate the structure of the game.

In other words, if participants are inexperienced with the game, their decisions are just the result of misunderstanding but as soon as they are experienced enough, the theoretical predicition will occur. Note that in this explanation the authors suggest equal devision, namely $(50, 50)$ in our case, as an outcome for the first round as this distribution is often observed in experimental studies on the one-shot ultimatum game. Note that we find strong evidence for outside options affecting the outcome, that is, strong evidence exactely against equal devision.

There have been experimental studies on the ultimatum game investigating learning effects which found that distributions, even though changing over time, do not converge to the non-cooperative equilibrium (cf. Bearden, 2001). In Section 3.4.3 we also observed some evidence for this change over time, that is, learning effects, occuring in the last round of our experiment. Consulting our data we will now take a brief look at the results from this last round which are presented in Table 8.

We see that the non-cooperative equilibrium indeed occured 3 times in this last round (while it never occured in previous rounds) and an almost equilibrium outcome of $(98, 2)$ occured two more times (which beside the last round only occured in the previous 2 rounds for the second pair in Group 8). Hence, there might be some evidence for the equilibrium outcome. However, the mean distribution of this last round is still clearly different from this.

Notably, the mean distributions $(82.25, 17.75)$ and $(77.67, 22.04)$ within pairs and $(79.96, 20.04)$ across pairs seem to be remarkably close to the theoretical predicitions of the χ-value and the Owen value. Unfortunately, we do not have data on further rounds to deeper investigate convergence.

3.6 CONCLUSION

In the theoretical part of this chapter we analyzed glove games, which are used to model simple markets, and corresponding outcomes with

Table 8: Results for Left-glove Holders of last Round

Group	1st Pair	2nd Pair
1	90	**99**
2	95	90
3	70	70
4	84	20
5	90	87
6	50	**99**
7	95	95
8	**98**	**98**
9	75	10
10	90	**99**
11	90	95
12	60	70
Mean	82.25	77.67
Pair Independent Mean: 79.96		

respect to different cooperative allocation rules. Corresponding formulas for the special case of glove games have been derived which we used to show the differences between the allocation rules with respect to whether outside options or coalitional structures are taken into account. In the economic interpretation of modeling a market, sensitivity to outside options refers to whether imbalancedness of the market is taken into account while the the question of accouning for coalition structures could be interpreted as the question of discriminating between productive and unproductive market agents.

In the experimental part we investigated the question whether outside options do have an impact on bargaining outcome. We ran an experimental case study using a glove game which showed that outside options significantly affect the outcomes. Among the theoretical allocation rules we studied, there are two possible outside-option-sensitive predictors left which are compatible with our experimental design, namely, the χ-value and the Owen value. For our experiment, theoretical predictions due to these values are very similar which did not allow for a direct statistical discrimination. We hence implemented a test on a certain axiom which could have distinguished between these two allocation rules if our data had supported this axiom. Unfortunately, the axiom cannot be supported by our data and hence, we do not find evidence in favor of one or the other allocation rule. Note that existing studies on this topic ignore the Owen value.

We discuss the relation of our experiment to the ultimatum game (in fact one could interpret our glove game as sort of an extended ultimatum game). As we do not find evidence for the corresponding non-cooperative equilibrium to occur in our experiment (in fact it seems that we even might have evidence against it), our investigation is in line with existing experimental literature on this topic. Note that the sort of ultimatum game present in our experiment has not been experimentally investigated before.

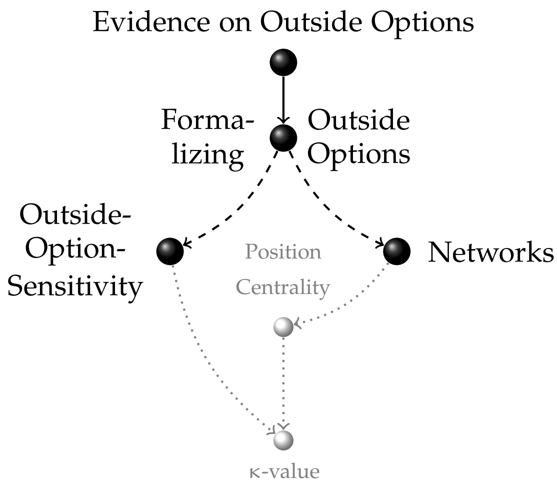
Furthermore, we find evidence that within a market, price expectations are not influenced by the outcome of negotiations which took place before. This finding is in contrast to existing the literature (for example Falk et al. [2006] on minimum wages).

Note that our experiment has been a pilot experiment. There are some issues on the experimental design that should be improved. To relate our experiment to the literature on ultimatum games, different treatments for one or the other extension should be included. An unfortunate issue is the number of rounds in the second part of our experiment. We found some evidence for learning effects and convergence for our last round and mean distributions of this last round seem to be remarkably close to the theoretical predicitions of the χ-value and the Owen value. Therefore, an investigation of convergence in more rounds seems highly interesting. Generally, we only observed a very small number of observations in the pilot experiment (strictly speaking, we only have 12 independent observations) which did not allow for the use of numerous statistical methods that rely on at least sort of large numbers of observation and our results and analysis would clearly benefit from more observations.

Therefore, beside the findings on the ultimatum game and price interdependency, we see this pilot experiment as a starting point of our analysis of outside options as we found strong evidence on their effect. This leads to experimental evidence for the importance of outside options and we will continue to analyze whether we can theoretically support this evidence.

4

FORMALIZING OUTSIDE OPTIONS - AN AXIOMATIC CATEGORIZATION



4.1 INTRODUCTION

In the previous chapter we found experimental evidence for the importance of "outside options" which will now be theoretically supported. Furthermore, as it is not yet clear from the literature what an *outside option* or *outside-option-sensitivity* in the context of cooperative allocation rules formally means, we will provide a theoretical formalization and analysis.

While some research on cooperative game theory assumes that some certain social or economic structure has already materialized, that is, individuals have already formed certain coalitions (or a priori unions) or networks, Gómez et al. [2008] and Belau [2010] provide probabilistic forecasting models for network structures and coalition structures that consider a point *before* any structure is fixed. Take for example elections: at the time of election, it is not known which parties will form coalitions in the end, but there might be some beliefs or assumptions about all possible coalitions that could occur. Here it is not reasonable to take certain structures as a priori given, one should include uncertainty and start the analysis at some point *before* coalitions are actually formed. To address this uncertainty, the generalized models of Gómez et al. [2008] and Belau [2010] take into account the likelihood of possible network or coalition structures. Referring to the likelihood assumption, Belau [2010] calls these settings probabilistic (instead of generalized) and we will follow this terminology.

For these probabilistic models, extensions of cooperative allocation rules have been defined and characterized: Gómez et al. [2008] gave the extension and characterization of the Myerson value (Myerson, 1977), the Position value (Meessen, 1988 and 1992, Borm et al.) was generalized by Ghintran et al. [2012] and Belau [2010] generalized the Aumann-Drèze value (Aumann and Drèze, 1974) and the χ-value (Casajus, 2009b). All these values are expected payoffs of the deterministic analogs. Interestingly, all aforementioned probabilistic extensions but the one of the χ-value used the "direct probabilization approach" (i. e., directly generalizing axioms of the original characterization of an allocation rule). For the χ-value, Belau [2010] shows that this approach fails. While the main finding of Belau's investigation was to finally characterize the probabilistic χ-value (introducing a new, fully probabilistic axiom), we are now interested in a claim raised (but not formally investigated) in this work: that the direct probabilization approach generally fails for "outside-option-sensitive" allocation rules. This would provide a theoretical support of the experimental finding in Chapter 3.

Following this claim, we first show that it indeed holds for the Owen value (Owen, 1977), another allocation rule which seems to account for outside options (cf. Chapter 2.4) and has been supported by our data in Chapter 3: the "direct probabilization approach" leads to a value that is different from the expected payoff of the deterministic Owen value (and the value obtained will turn out to be sort of "too risk averse"). The incompatibility of the direct approach and "outside-option-sensitivity" supports the importance of outside options as there indeed seems to be a theoretical difference between allocation rules accounting for outside options and those that do not.

However, outside-option-sensitivity of cooperative allocation rules has not yet been explicitly formalized in the literature so far (and, therefore, there have not been used formal arguments for the failure of the direct probabilization approach). We formally define outside options for both coalition and network structures and axioms for *(weak) outside-option-sensitivity*. We show that these axioms are incompatible with *component decomposability*, an axiom which is satisfied by all allocation rules where the direct probabilization approach was suitable. Therefore, our new axioms are suitable for a formal categorization of allocation rules into outside-option-sensitive and -insensitive ones. We show that our axioms are indeed satisfied by the Shapley value, the Banzhaf value, the Wiese value, the χ-value, the Owen value and, as the only allocation rule in the setting of network structures, the graph-χ-value. We further provide formal arguments for Belau [2010]'s claim.

This chapter is organized as follows: in the next section we provide the formal notion of the probabilistic forecasting model for coalition structures and Section 3 will discuss the direct probabilization approach of the Owen value. In Section 4, we formally define outside options and the outside-option-sensitivity axioms and use these axioms to categorize all allocation rules for coalition structures discussed so far. Section 5 provides the same definitions and categorizations for the framework of network structures. Finally, Section 6 concludes and is followed by a section that discusses the probabilistic forecasting model for networks to motivate our further investigations in this thesis.

4.2 PROBABILISTIC COALITION STRUCTURES

Recall that a (deterministic) TU-game with a coalition structure is a tuple $(N, v, \mathcal{P})$ with $N = \{1, ..., n\}$ being the (finite, non-empty) player set, $v \in \mathbb{V}_N$ the coalition (or characteristic) function describing the game and $\mathcal{P}$ the partition on N defining the coalition structure. Now, probabilistic coalition structures are modeled as follows (cf. Belau, 2010): consider a probability distribution over all coalition structures $\mathcal{P}$ of N, that is, a probability distribution on $\mathbb{P}_N$. Define the set of all probability distributions on $\mathbb{P}_N$:

$$\Delta(\mathbb{P}_N) := \left\{ p : \mathbb{P}_N \longrightarrow [0,1], \sum_{\mathcal{P} \in \mathbb{P}_N} p(\mathcal{P}) = 1 \right\}.$$

An element $p \in \Delta(\mathbb{P}_N)$ can be interpreted as $p(\mathcal{P})$ being the probability that the coalition structure $\mathcal{P}$ occurs. A *TU-game with a probabilistic coalition structure* is a tupel (N, v, p) and an *allocation rule for probabilistic coalition structures* is a function $Y : \{N\} \times \mathbb{V}_N \times \Delta(\mathbb{P}_N) \longrightarrow \mathbb{R}^{|N|}$, distributing the worth of any TU-game with a probabilistic coalition structure among the players. An example of how the probabilistic forecasting model can be applied can be found in the appendix.

Further, for any $p \in \Delta(\mathbb{P}_N)$, denote by $\mathbb{P}(p) := \{\mathcal{P} \in \mathbb{P}_N | p(\mathcal{P}) > 0\}$ the set of all coalition structures whose probability to occur under probability distribution p is not zero (i. e., the carrier of p).

A probability distribution $p \in \Delta(\mathbb{P}_N)$ is called *degenerated* if there exists a $\mathcal{P}^* \in \mathbb{P}_N$ such that $p(\mathcal{P}^*) = 1$ (i. e., $p(\mathcal{P}) = 0 \; \forall \mathcal{P} \neq \mathcal{P}^*$). As a notation, we write $p_{\mathcal{P}^*}$ for the degenerated probability distribution corresponding to the partition $\mathcal{P}^*$. Identifying $p_{\mathcal{P}}$ with the corresponding partition $\mathcal{P}$, define for every allocation rule for probabilistic coalition structures Y the corresponding (deterministic) allocation rule for coalition structures via $Y^{det}(N, v, \mathcal{P}) := Y(N, v, p_{\mathcal{P}})$.

An extension of (deterministic) axioms for coalition structures into probabilistic ones on degenerated probability distributions is quite straightforward: to obtain the degenerated pendant to deterministic axiom · (**d**·), we replace $Y(N, v, \mathcal{P})$ by $Y(N, v, p_{\mathcal{P}})$. Note that if an allocation rule for probabilistic coalition structures satisfies a degenerated axiom, this implies that the corresponding (deterministic) allocation rule for coalition structures satisfies the deterministic axioms.

Belau [2010] introduces a probabilistic axiom which fills the gap between degenerated probability distributions and general ones.

Axiom 4.1 (Linearity on Probability Distributions (**pL**))**.** *An allocation rule for probabilistic coalition structures* Y *satisfies* Linearity on Probability Distributions **pL** *if we have*

$$Y(N, v, \alpha p + (1-\alpha)q) = \alpha Y(N, v, p) + (1-\alpha) Y(N, v, q)$$

for all probability distributions $p, q \in \Delta(\mathbb{P}_N)$ *and all* $\alpha \in [0, 1]$.

This axiom states that mixing probabilities should lead to the same mix for the corresponding payoffs. Note that convex combinations of probability distributions are again probability distributions. Mixing probability distributions in a non-convex way would not make sense in this setting.

With the following Theorem we give a more applicable version of the one in Belau [2010].

Theorem 4.1 (Characterization via deg. prob. distr.)**.** *If an allocation rule for probabilistic coalition structures is defined as an expected payoff of a deterministic one, it is characterized by* ***pL*** *and the degenerated versions of the characterizing axioms of the deterministic one.*

Proof. Existence follows by

$$\begin{aligned} Y^p(N, v, \alpha p + (1-\alpha)q) &= \sum_{\mathcal{P} \in \mathbb{P}_N} (\alpha p + (1-\alpha)q)(\mathcal{P}) Y(N, v, \mathcal{P}) \\ &= \alpha Y^p(N, v, p) + (1-\alpha) Y^p(N, v, q) \end{aligned}$$

and since $Y^p(N, v, p_{\mathcal{P}}) = Y(N, v, \mathcal{P})$ by the deterministic allocation rule satisfying the deterministic axioms. Uniqueness follows by the fact that every probability distribution $p \in \Delta(\mathbb{P}_N)$ can be written as a convex combination of degenerated probability distributions (cf. Belau, 2010):

$$p(\mathcal{P}) = \sum_{\mathcal{P}' \in \mathbb{P}_N} p(\mathcal{P}') p_{\mathcal{P}'}(\mathcal{P})$$

and the use of **pL**. □

Note that every allocation rule for probabilistic coalition structures which satisfies **pL** can be written as an expected payoff of a deterministic allocation rule (which can always be derived via degenerated probability distributions)[1]. Hence, we already have a definition and characterization for generalized versions of all mentioned allocation rules. But Linearity on Probability Distributions is a very strong axiom. Hence, in order to avoid (or at least relax) **pL**, probabilistic axioms defined for general probability distributions are needed.

Belau [2010] shows that while the probabilistic AD-value can be characterized by probabilistic versions of the determinstic characterizing axioms, this is not possible for the probabilistic χ-value. It is claimed that the "direct probabilization approach" is problematic whenever an allocation rule is *outside option sensitive* (which has not been

1 see Belau [2010]

formally defined so far). We will first investigate if the claim also holds for the Owen value (Owen, 1977) which also seems to account for outside options as we have seen in Chapter 2.4.

4.3 THE PROBABILISTIC OWEN VALUE

Recall the definition of the Owen value:

$$Ow_i(N, v, \mathcal{P}) := \frac{1}{|\Sigma(N, \mathcal{P})|} \sum_{\sigma \in \Sigma(N, \mathcal{P})} [v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})]$$

where $\Sigma(N, \mathcal{P})$ is the set of all orders σ over the player set that are compatible with the coalition structure $\mathcal{P}$ (i. e.$\forall\ i, j \in P \in \mathcal{P}$ we have $|\sigma(i) - \sigma(j)| < |P|$) and $K_i(\sigma)$ is the set of players that come before player i and including i under order σ.

In line with the probabilistic extensions of the other allocation rules, we define the probabilistic version as the expected payoff of the determinstic Owen value:

Definition 4.1 (probabilistic Owen value)**.** *For any TU-game with a probabilistic coalition structure* (N, v, p)*, the* probabilistic Owen-value *is given by*

$$Ow^p(N, v, p) := \sum_{\mathcal{P} \in \mathbb{P}_N} p(\mathcal{P}) Ow(N, v, \mathcal{P}).$$

Recall that the determinsitic Owen value is characterized by **E**, **N**, **CS**, **SC** and **A** (Theorem 2.5). We will now follow the "direct probabilization approach" of finding probabilistic versions of characterizing axioms of the deterministic value and using these versions to determine the probabilistic value.

Belau [2010] already gives a probabilistic version of **CS**: Denote by $\mathcal{P}^p$ the *coarsest common refinement* of all $\mathcal{P} \in \mathbb{P}(p)$. One can interpret a component $\mathcal{P}^p(i) \in \mathcal{P}^p$ as the set of all players that are in the same component as player i for sure, that is, for any player $j \in \mathcal{P}^p(i)$ we have that $j \in \mathcal{P}(i)\ \forall\ \mathcal{P} \in \mathbb{P}(p)$. Using this, a probabilistic version of **CS** is defined:

Axiom 4.2 (probabilistic Symmetry within Components (**pCS**))**.** *An allocation rule for probabilistic coalition structures* Y *satisfies* probabilistic Symmetry within Components **pCS** *if we have for all* $p \in$

$\Delta(\mathbb{P}_N)$ *that* $Y_i(N, v, p) = Y_j(N, v, p)$ *for all players* i, j *being symmetric in* (N, v)*,* $j \in \mathcal{P}^p(i)$.

pCS states that players with the same productivity that are in the same component *for sure* should not be treated differently in the distribution of the payoff. Note that this axiom, due to the use of $\mathcal{P}^p$, is not very restrictive as it only demands to treat those players with the same productivity equally who are in the same coalition for *all* coalition structures that could occur.

We now define probabilistic versions of the missing axioms (these definitions are quite straigth forward):

Axiom 4.3 (probabilistic Efficiency (**pE**)). *An allocation rule for probabilistic coalition structures* Y *satisfies* probabilistic Efficiency **pE** *if we have*

$$\sum_{i \in N} Y_i(N, v, p) = v(N).$$

Note that we do not have to use any sort of probabilistic coalition function for **pE** since the worth of the grand coalition $v(N)$ does not depend on the coalition structure $\mathcal{P}$.

Axiom 4.4 (probabilistic Null Player Axiom (**pN**)). *An allocation rule for probabilistic coalition structures* Y *satisfies the* probabilistic Null Player Axiom **pN** *if we have* $Y_i(N, v, p) = 0$ *for all Null players* $i \in N$.

Axiom 4.5 (probabilistic Symmetry of Components (**pSC**)). *An allocation rule for probabilistic coalition structures* Y *satisfies* probabilistic Symmetry of Components **pSC** *if we have*

$$\sum_{i \in P} Y_i(N, v, p) = \sum_{i \in P'} Y_i(N, v, p)$$

for $P, P' \in \mathcal{P}^p$ *being symmetric in the intermediate game.*

The aforementioned probabilistic axioms are indeed sufficient to uniquely characterize an allocation rule:

Theorem 4.2. *The allocation rule for probabilistic coalition structures given by*

$$Y_i(N, v, p) := Ow(N, v, \mathcal{P}^p)$$

for all $i \in N$ *is the unique allocation rule for probabilistic coalition structures that satisfies* ***pE****,* ***pN****,* ***pCS****,* ***pSC*** and ***A****.*

Proof. For uniqueness, we mimic the original proof of the deterministic characterization. Let Y satisfy the axioms. Since the unanimity games $(u_T)_T$ form a basis of $\mathbb{V}_N$, by **A**, it is sufficient to show that $Y(N, \lambda u_T, p)$ is uniquely determined. In the following we will use the following properties of unanimity games:

1. All players $i \in N \setminus T$ are Nullplayers in u_T, hence $Y_i(N, \lambda u_T, p) = 0$ by **pN**.

2. All $i, j \in T$ are symmetric in u_T, hence all $P, P' \in \mathcal{P}^p|_T$ are symmetric in the intermediate game $u_T^{int} : \mathcal{P}^p \longrightarrow \mathbb{R}$.

Further we will use that for all $P \in \mathcal{P}^p \setminus \mathcal{P}^p|_T = \mathcal{P}^p|_{N\setminus T}$ we have $P \subseteq N \setminus T$, that is, all $i \in P$ are Nullplayers.
For all $i \in T$ consider $P := \mathcal{P}^p(i) \cap T \in \mathcal{P}^p|_T$. We have:

$$\begin{aligned}
\lambda u_T(N) &\overset{pE}{=} \sum_{j \in N} Y_j(N, \lambda u_T, p) \\
&= \sum_{\substack{j \in P', \\ P' \in \mathcal{P}^p|_T}} Y_j(N, \lambda u_T, p) + \sum_{\substack{j \in P', \\ P' \in \mathcal{P}^p|_{N\setminus T}}} Y_j(N, \lambda u_T, p) \\
&\overset{pSC, pN}{=} |\mathcal{P}^p|_T| \sum_{j \in P} Y_j(N, \lambda u_T, p) + 0
\end{aligned}$$

$$\Leftrightarrow \sum_{j \in P} Y_j(N, \lambda u_T, p) = \frac{\lambda}{|\mathcal{P}^p|_T|}$$

All players in $P = \mathcal{P}^p(i) \cap T$ are symmetric in u_T since for $P \in \mathcal{P}^p|_T$ we have $P \subseteq T$. Hence, by **pCS**, $\sum_{j \in P} Y_j(N, \lambda u_T, p) = |\mathcal{P}^p(i) \cap T| Y_i(N, \lambda u_T, p)$ and together

$$Y_i(N, \lambda u_T, p) = \begin{cases} \frac{\lambda}{|\mathcal{P}^p|_T| \cdot |\mathcal{P}^p(i) \cap T|} & , i \in T \\ 0 & , i \in N \setminus T \end{cases} \tag{5}$$

which is uniquely determined.

The value given by (5) corresponds to $Ow(N, v, \mathcal{P}^p)$ (here we use the basis-element representation of the Owen value for unanimity games). It is straightforward to show that $Ow(N, v, \mathcal{P}^p)$ satisfies the axioms **A**, **pN**, **pE** and **pCS** just by Ow satisfying the deterministic analogs and the definition of $\mathcal{P}^p$. By the so called intermediate game property (see Peleg and Sudhölter, 2007) one can show that $Ow_P(N, v, \mathcal{P})$ is equal to $Ow(N, v^{int}, \{\mathcal{P}\})$ for any $\mathcal{P} \in \mathbb{P}_N$ and $P \in \mathcal{P}$.

Using that $Ow(N, v^{int}, \{\mathcal{P}\}) = Sh(\mathcal{P}, v^{int})$ we have **pSC** by Sh satisfying **S**. □

While in the "direct probabilization approach" for generalizing the χ-value a uniqueness proof simply failed (cf. Belau, 2010), it is possible to follow this approach for the Owen value to characterize a unique value. However, we obtain another problem: the allocation rule from Theorem 4.2 does not coincide with the probabilistic Owen value.

Example 4.1. *Consider* $N = \{1,2,3\}$, $\mathcal{P}_1 = \{\{1,2\},\{3\}\}$, $\mathcal{P}_2 = \{\{1\},\{2,3\}\}$, $v = u_N$ *and* $p(\mathcal{P}_1) = p(\mathcal{P}_1) = \frac{1}{2}$. *We get* $\mathcal{P}^p = \{\{1\},\{2\},\{3\}\}$ *and*

$$Ow_1(N, v, \mathcal{P}^p) = \frac{1}{3}, \qquad \sum_{\mathcal{P} \in \mathbb{P}_N} p(\mathcal{P}) Ow_1(N, v, \mathcal{P})) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8} \neq \frac{1}{3},$$

that is, the values do not coincide.

The application of the "direct probabilization approach" for the χ-value failed as the probabilistic extensions of the original axioms, even though satisfied by the probabilistic χ-value, have been insufficient for uniqueness (there exists another value which also satisfies the axioms, see Belau, 2010). In contrast to that, the probabilistic extensions of the Owen-axioms, even though leading to a unique value, are not all satisfied the expected payoff extension: It is straightforward to prove that the probabilistic Owen value satisfies **pE**, **pN**, **pCS** and **A** by Ow^{det} satisfying the deterministic versions and the definition of $\mathcal{P}^p$. Hence, by Theorem 4.2 and Example 4.1, the probabilistic Owen value cannot satisfy **pSC**.

One could argue that $Ow(N, v, \mathcal{P}^p)$ should be taken as the probabilistic Owen value but this value does not take into account the actual probability of coalitions, it only takes into account whether a coalition occurs with a probability not being zero. Furthermore, $\mathcal{P}^p$ defines coalitions of players which are in the same coalition for sure; in a lot of cases we have $\mathcal{P}^p = \{\{i\}, i \in N\}$. One could say that using $\mathcal{P}^p$ is "too risk averse".

On the other hand, one could argue that another probabilistic generalization of **SC** should be used. The only other possibility that is in line with the previous extensions of probabilistic axioms is the use of $\mathcal{P}_p$, which is the finest common coarsening of all $\mathcal{P} \in \mathbb{P}(p)$ (cf. Belau, 2010), employed in the definition of the probabilistic analog of **CE**. However, it turned out that, due to the use of this $\mathcal{P}_p$ instead of $\mathcal{P}^p$, the probabilistic analog of **CE** was too weak to characterize the probabilistic χ-value. Therefore, we assume to obtain the same problem if we would redefine **pSC** with the use of $\mathcal{P}_p$.

4.4 OUTSIDE-OPTION-SENSITIVITY IN COALITION STRUCTURES

Following Belau [2010] and our investigation in the previous section, we have seen that both the χ-value and the Owen value cannot be generalized for the probabilistic model using the "direct probabilization approach" while this is possible for the AD-value (see Belau, 2010). For the χ-value, Belau [2010] claimed that this might be due to the fact that it does not satisfy the following property while the AD-value does:

Axiom 4.6 (Component Decomposability (**CD**) for Coalition Structures)**.** *An allocation rule for coalition structures* Y *satisfies* Component Decomposability **CD** *if*

$$Y_i(N, v, \mathcal{P}) = Y_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}, \{\mathcal{P}(i)\})$$

CD states that the player's outside world does not affect payoffs within a component. Neither the potential coalitions between players in and outside the component, nor the coalition structure can affect a component decomposable allocation rule. Therefore, Belau [2010] states that **CD** stands in contradiction to outside-option-sensitivity. However, outside-option-sensitivity has not been explicitly formalized so far. We will now suggest an axiom which accounts for outside options and indeed is incompatible with **CD**.

Definition 4.2 (Outside Option in Coalition Structures)**.** *Let* $(N, v, \mathcal{P})$ *be a TU-game with a coalition structure. A coalition* $K \subseteq N$ *is called* outside option *in* $(N, v, \mathcal{P})$ *for player* $i \in N$ *if*

$$K \subseteq N \setminus \mathcal{P}(i) \text{ and } v(K \cup \{i\}) - v(K) > 0.$$

Player i has an outside option if she could create a surplus by joining a group of other players *outside* her coalition $\mathcal{P}(i)$. Note that this definition only specifies outside options of a single player and not of a group of players.

Definition 4.3 (Outside-option-reduced Game)**.** *Let* $\tilde{K}$ *be an outside option in* $(N, v, \mathcal{P})$ *for player* $i \in N$*. Then, the* outside-option-reduced game *w.r.t.* $\tilde{K}$ *and* i *is given by*

$$v_{\tilde{K},i}(K) := \begin{cases} v(K) & , \textit{if } K \neq \tilde{K} \cup \{i\} \\ v(\tilde{K}) & , \textit{if } K = \tilde{K} \cup \{i\} \end{cases}$$

In the outside-option-reduced game w.r.t. $\tilde{K}$ and i, player i's outside option $\tilde{K}$ is neutralized: the surplus created by player i joining $\tilde{K}$ is set to zero. The first step towards an axiom for outside-option-sensitivity is given by the following axiom:

Axiom 4.7 (Weak Outside-Option-Sensitivity (**WOOS**) for Coalition Structures)**.** *Let* $(N, v, \mathcal{P})$ *be a TU-game with a coalition structure. An allocation rule for coalition structures* Y *satisfies* Weak Outside-Option-Sensitivity **WOOS** *if for all* $i \in N$ *and for all outside options* $\tilde{K}$ *in* $(N, v, \mathcal{P})$ *for player* $i \in N$ *there exists a coalition structure* $\mathcal{P}'$ *with* $\tilde{K}$ *also being outside option in* $(N, v, \mathcal{P}')$ *for player* $i \in N$ *such that*

$$Y_i(N, v, \mathcal{P}') \neq Y_i(N, v_{\tilde{K},i}, \mathcal{P}').$$

WOOS states that in at least one possible coalition formation the neutralization of a non-isolated player's outside options changes this players payoff. One could argue that neutralizing an outside option should always have an effect (for every coalition structure) and that this effect should be strictly negative. This is a convincing argument for coalition structures and any sort of player-based values.[2] Hence, we define:

Axiom 4.8 (Outside-Option-Sensitivity (**OOS**) for Coalition Structures)**.** *Let* $(N, v, \mathcal{P})$ *be a TU-game with a coalition structure. An allocation rule for coalition structures* Y *satisfies* Outside-Option-Sensitivity **OOS** *if for all* $i \in N$ *such that* $|\mathcal{P}(i)| > 1$ *we have*

$$Y_i(N, v, \mathcal{P}) > Y_i(N, v_{\tilde{K},i}, \mathcal{P})$$

2 The weak axiom will only become important in a special case of Chapter 6. Here, we will also discuss why **OOS** indeed might be seen as sort of "strong".

for all $\tilde{K}$ *being an outside option in* $(N, v, \mathcal{P})$ *for* i.

OOS states that a non-isolated player's payoff should strictly decrease if her outside options are neutralized which implies that outside options have a positive impact on a player's payoff. It is obvious that **OOS** implies **WOOS**. Note that **OOS** still is not very strong as the size or strength of this impact is not specified. Furthermore, we only analyze outside options of *one* player and not of a group of players.

Although **WOOS** seems to be a weak axiom at first glance, it is sufficient to state that an allocation rule is somehow affected, that is, sensitive to outside options. It turns out that it is already incompatible with **CD**:

Lemma 4.1 (**CD** vs. **WOOS** and **OOS**)**.** *If an allocation rule for coalition structures* Y *satisfies* ***CD****, it cannot satisfy* ***WOOS*** *or* ***OOS*** *and vice versa.*

Proof. Let $\tilde{K}$ be an outside option for some $i \in N$. We have that $\tilde{K} \subseteq N \setminus \mathcal{P}(i)$, that is, $\tilde{K} \cap \mathcal{P}(i) = \emptyset$ and hence, the restriction of v on $\mathcal{P}(i)$ is not affected by a neutralization of $\tilde{K}$. In other words, we have $v_{\tilde{K},i}|_{\mathcal{P}(i)} = v|_{\mathcal{P}(i)}$. Therefore, on the one hand, if an allocation rule satisfies **CD**, we obtain

$$\begin{aligned} Y_i(N, v_{\tilde{K},i}, \mathcal{P}) &\overset{\text{CD}}{=} Y_i(N|_{\mathcal{P}(i)}, v_{\tilde{K},i}|_{\mathcal{P}(i)}, \mathcal{P}|_{\mathcal{P}(i)}) \\ &= Y_i(N|_{\mathcal{P}(i)}, v|_{\mathcal{P}(i)}, \mathcal{P}|_{\mathcal{P}(i)}) \\ &\overset{\text{CD}}{=} Y_i(N, v, \mathcal{P}) \end{aligned}$$

for all outside option $\tilde{K}$ and all coalition structures $\mathcal{P}$. On the other hand, if an allocation rule satisfies **WOOS**, there exists a coalition structure $\mathcal{P}$ such that

$$Y_i(N, v_{\tilde{K},i}, \mathcal{P}) \neq Y_i(N, v, \mathcal{P}).$$

Hence, an allocation rule can either satisfy **CD** or **WOOS** but not both of them and as **OOS** implies **WOOS**, the same holds for **OOS**. □

Lemma 4.1 shows independence of **CD** and **OOS** (neither axiom can imply the other) and, by means of outside-option-sensitivity w.r.t. **OOS**, Lemma 4.1 implies that all component decomposable allocation rules are outside-option-insensitive. Hence, to categorize allocation rules into outside-option-sensitive and -insensitive ones by means of **OOS**, **CD** implies outside-option-insensitivity. However note that the reverse does not hold: if an allocation rule does not satisfy **CD**, this does not imply that **OOS** is automatically satisfied, that is, there are allocation rules that are neither component decomposable nor outside-option-sensitive.

Example 4.2 (Neither **CD** nor **(W)OOS**)**.** *Consider for example*

$$Y_i(N, v, \mathcal{P}) := \frac{v(N)}{|N|} \quad \forall i \in N$$

*and a glove game with one left glove and two right gloves. For any $\mathcal{P}$, the allocation rule above allocates $1/3$ to each player. Consider $\mathcal{P}$ such that one pair is build and the other right-glove holder stays alone. Reduced to the component of the pair, every pair-building player obtains $1/2$ and hence, **CD** is not satisfied. For every coalition structure $\mathcal{P}$ with at least two components, the singelton glove holder is an outside option for at least one other glove holder but, since the corresponding outside-option-reduced game still assigns the same value to the grand coalition (as there is still a matching pair), payoffs do not change. Hence, also **WOOS** and **OOS** are not satisfied.*

Lemma 4.2 (Outside-Option-Sensitivity of the Shapley and the Banzhaf value)**.** *The Shapley value and the Banzhaf value are outside-option-sensitive.*

Proof. Let $(N, v, \mathcal{P})$ be a TU-game with a coalition structure, $i \in N$ with $|\mathcal{P}(i)| > 1$ and let $\tilde{K}$ be an outside option in $(N, v, \mathcal{P})$ for i. Note that $Sh(N, v, \mathcal{P}) = Sh(N, v)$ and $Ba(N, v, \mathcal{P}) = Ba(N, v)$ as both values are allocation rules that do not account for any inner structure. By the definition of $v_{\tilde{K},i}$ we have for the Shapley value

$$\begin{aligned}
Sh_i(N, v_{\tilde{K},i}) = &\sum_{K \subseteq N \setminus \{i\}} \frac{|K|!(|N| - |K| - 1)!}{|N|!} \left[v_{\tilde{K},i}(K \cup \{i\}) - v_{\tilde{K},i}(K) \right] \\
&= Sh_i(N, v) + \frac{|\tilde{K}|!(|N| - |\tilde{K}| - 1)!}{|N|!} \underbrace{\left[v_{\tilde{K},i}(\tilde{K} \cup \{i\}) - v_{\tilde{K},i}(\tilde{K}) \right]}_{=0} \\
&\quad - \frac{|\tilde{K}|!(|N| - |\tilde{K}| - 1)!}{|N|!} \underbrace{\left[v(\tilde{K} \cup \{i\}) - v(\tilde{K}) \right]}_{>0}
\end{aligned}$$

$$< Sh_i(N, v)$$

and analogously we have for the Banzhaf value

$$\begin{aligned} Ba_i(N, v_{\tilde{K},i}) &= \sum_{K \subseteq N \setminus \{i\}} \frac{1}{2^{|N|-1}} \left[v_{\tilde{K},i}(K \cup \{i\}) - v_{\tilde{K},i}(K) \right] \\ &= Ba_i(N, v) + \frac{1}{2^{|N|-1}} \underbrace{\left[v_{\tilde{K},i}(\tilde{K} \cup \{i\}) - v_{\tilde{K},i}(\tilde{K}) \right]}_{=0} \\ & \quad - \frac{1}{2^{|N|-1}} \underbrace{\left[v(\tilde{K} \cup \{i\}) - v(\tilde{K}) \right]}_{>0} \\ &< Ba_i(N, v) \end{aligned}$$

□

Remark 4.1 (Outside-Option-Insensitivity of the AD-value)**.** *The AD-value is component decomposable (by definition), hence, it cannot be outside-option-sensitive.*

Lemma 4.3 (Outside-Option-Sensitivity of the Owen value)**.** *The Owen value is outside-option-sensitive.*

Proof. Let $(N, v, \mathcal{P})$ be a TU-game with a coalition structure, $i \in N$ with $|\mathcal{P}(i)| > 1$ and let $\tilde{K}$ be an outside option in $(N, v, \mathcal{P})$ for i. Note that $v(K_i(\sigma) \setminus \{i\}) \neq \tilde{K} \cup \{i\}$ for all $\sigma \in \Sigma(N, \mathcal{P})$. Therefore we have

$$Ow_i(N, v_{\tilde{K},i}, \mathcal{P}) = \frac{1}{|\Sigma(N, \mathcal{P})|} \sum_{\sigma \in \Sigma(N,\mathcal{P})} \underbrace{\left[v_{\tilde{K},i}(K_i(\sigma)) - v_{\tilde{K},i}(K_i(\sigma) \setminus \{i\}) \right]}_{\leqslant v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})}$$

with strict inequality for all σ such that $K_i(\sigma) = \tilde{K} \cup \{i\}$ which implies

$$Ow_i(N, v_{\tilde{K},i}, \mathcal{P}) < Ow_i(N, v, \mathcal{P}).$$

□

Lemma 4.4 (Outside-Option-Sensitivity of the Wiese-value)**.** *The Wiese-value is outside-option-sensitive.*

Proof. Let $(N, v, \mathcal{P})$ be a TU-game with a coalition structure, $i \in N$ with $|\mathcal{P}(i)| > 1$ and let $\tilde{K}$ be an outside option in $(N, v, \mathcal{P})$ for i. As before we have

$$v_{\tilde{K},i}(K_i(\sigma)) - v_{\tilde{K},i}(K_i(\sigma) \setminus \{i\}) \leqslant v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$$

with strict inequality for all σ such that $K_i(\sigma) = \tilde{K} \cup \{i\}$. Now consider $j \in \mathcal{P}(i) \setminus \{i\}$ and $\sigma \in \Sigma_i(N, \mathcal{P})$, that is, i is the last player of $\mathcal{P}(i)$ under σ. Hence we have $i \notin K_j(\sigma)$ which implies that $K_j(\sigma) \neq \tilde{K} \cup \{i\}$ and, hence, we have

$$v_{\tilde{K},i}(K_i(\sigma)) - v_{\tilde{K},i}(K_i(\sigma) \setminus \{i\}) = v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$$

for all $\sigma \in \Sigma_i(N, \mathcal{P})$ which proves **OOS**. □

Lemma 4.5 (Outside-Option-Sensitivity of the χ-value)**.** *The χ-value is outside-option-sensitive.*

Proof. Let $(N, v, \mathcal{P})$ be a TU-game with a coalition structure, $i \in N$ with $|\mathcal{P}(i)| > 1$ and let $\tilde{K}$ be an outside option in $(N, v, \mathcal{P})$ for i. For all $j \notin \tilde{K}$, $j \neq i$ we have that $j \notin \tilde{K} \cup \{i\}$ and hence, $K \cup \{j\} \neq \tilde{K} \cup \{i\}$ for all $K \subseteq N \setminus \{j\}$. Using this and setting

$$\tilde{k} := \frac{|\tilde{K} \cup \{i\}|!(|N| - |\tilde{K} \cup \{i\}| - 1)!}{|N|!} > 0$$

we obtain

$$\begin{aligned}
Sh_j(N, v_{\tilde{K},i}) &= Sh_j(N, v) + \tilde{k} \underbrace{\left[v_{\tilde{K},i}(\tilde{K} \cup \{i\} \cup \{j\}) - v_{\tilde{K},i}(\tilde{K} \cup \{i\})\right]}_{=v(\tilde{K} \cup \{i\} \cup \{j\}) - v(\tilde{K})} \\
&\quad - \tilde{k}\left[v(\tilde{K} \cup \{i\} \cup \{j\}) - v(\tilde{K} \cup \{i\})\right] \\
&= Sh_j(N, v) + \tilde{k} \underbrace{\left[v(\tilde{K} \cup \{i\}) - v(\tilde{K})\right]}_{>0} \\
&> Sh_j(N, v)
\end{aligned}$$

Therefore and since $\mathcal{P}(i) \cap \tilde{K} = \emptyset$ we have for the χ-value of player i

$$\begin{aligned}
\chi_i(N, v_{\tilde{K},i}, \mathcal{P}) &= Sh_i(N, v_{\tilde{K},i}) + \frac{v_{\tilde{K},i}(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i)} Sh_j(N, v_{\tilde{K},i})}{|\mathcal{P}(i)|} \\
&= \left(1 - \frac{1}{|\mathcal{P}(i)|}\right) \underbrace{Sh_i(N, v_{\tilde{K},i})}_{<Sh_i(N,v)} + \frac{v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i) \setminus \{i\}} \overbrace{Sh_j(N, v_{\tilde{K},i})}^{>Sh_j(N,v)}}{|\mathcal{P}(i)|} \\
&< \left(1 - \frac{1}{|\mathcal{P}(i)|}\right) Sh_i(N, v) + \frac{v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i) \setminus \{i\}} Sh_j(N, v)}{|\mathcal{P}(i)|} \\
&= \chi_i(N, v, \mathcal{P})
\end{aligned}$$

which proves **OOS**. □

Note that the previous proofs of **WOOS** provided more than "just" outside-option-sensitivity of the analyzed allocation rules as we can explicitly determine the structural effect of an outside option on allocation. Fix player $i \in N$, outside option $\tilde{K}$ and TU-game (N, v). The structural effect arising due to outside option $\tilde{K}$ on allocation by allocation rule Y can be expressed by

$$E(Y) := Y_i(N, v, \mathcal{P}) - Y_i(N, v_{\tilde{K},i}, \mathcal{P})$$

Denote by $A := v(\tilde{K} \cup \{i\}) - v(\tilde{K})$ player i's contribution when joining outside option $\tilde{K}$ and find the expressions for $E(Y)$ in Table 9[3]

Table 9: Structural Effects of an Outside Option on Allocation

Y	Structural Effect $E(Y)$
Sh	$\frac{\lvert\tilde{K}\rvert!(\lvert N\rvert - \lvert\tilde{K}\rvert - 1)!}{\lvert N\rvert!}A$
Ba	$\frac{1}{2^{\lvert N\rvert-1}}A$
Ow	$\left\lvert\left\{\sigma \mid K_i(\sigma) = \tilde{K} \cup \{i\}\right\}\right\rvert A$
W	$\frac{\lvert\tilde{K}\rvert!(\lvert N\rvert - \lvert\tilde{K}\rvert - 1)!}{\lvert N\rvert!}A$
χ	$\left(\frac{\lvert\tilde{K}\rvert!(\lvert N\rvert - \lvert\tilde{K}\rvert - 1)!}{\lvert N\rvert!} + \underbrace{B}_{>0}\right)A$

Remark 4.2 (Comparison of Structural Effects of an Outside Option on Allocation)**.** *Note that all structural effects are multiples of player* i*'s contribution when joining outside option* $\tilde{K}$*. The effects for the Banzhaf value, the Shapley value and the Owen value generally differ in their structure: the effect for the Banzhaf value only depends on* A *and* $\lvert N\rvert$ *while the effect for the Shapley value also depends on the size of* $\tilde{K}$ *and the one for the Owen value on the number of orders and*

3 We used the expressions of the Wiese value and the χ-value given by Casajus [2007], p.81, to obtain that $W_i(N, v, \mathcal{P}) = Sh_i(N, v) + F_W$ and $\chi_i = Sh_i(N, v) + F_\chi$ where F_W does not change for the outside-option-reduced game as only $\sigma \in \Sigma_i(N, \mathcal{P})$ occurs while F_χ increases for the outside-option-reduced game. The explicit proofs are omitted.

$\tilde{K}$. *Note that the effect for the χ-value is strictly larger than the one of the Shapley value which strongly emphasizes the outside-option-issue while the effect for the Wiese value is equal to the one for the Shapley value. One could argue that, in terms of structural effects, the analysis of the Wiese value does not provide new insights as this is covered by an analysis of the Shapley value.*

We have categorized all dicussed allocation rules for coalition structures and without inner structure. We found that, due to Lemma 4.1, the χ-value, the Wiese value and the Owen value are not component decomposable. Note that the "direct probabilization approach" has to use some sort of probabilistic component which could be the probabilistic components $\mathcal{P}^p$ or $\mathcal{P}_p$. For the outside-option-sensitive allocation rules for coalition structures the step from these probabilistic components to the deterministic components did not work out: for the χ-value, a version using $\mathcal{P}_p$ also satisfies the probabilistic axioms (cf. Belau, 2010) and for the Owen value we obtained a version using $\mathcal{P}_p$. We claim that it is the absence of component decomposability which causes these problems:

Remark 4.3 (Component Decomposability and the "Direct Probabilization Approach")**.** *To obtain a deterministic component out of a probabilistic one there has to be a point where the situation can be reduced to an allocation rule without inner structure (like the Shapley or the Banzhaf value) since in this case, probabilistic differences are absent and equivalence between probabilistic and deterministic components occurs. This reduction has to be sort of general, that is, a reduction to components. If, however, an allocation rule can be reduced to components of the underlying structure, it is component decomposable which means that this allocation rule cannot be outside-option-sensitive.*

Recall that the network approach for modeling social or economic situations captures more information about the structure of the society or economy than the coalition structure approach (cf. Chapter 2). Hence, it seems to be a more adequate approach for various applications. We will now investigate the same idea of categorization for network structures where we will also obtain further issues leading to the following investigations of this thesis.

4.5 OUTSIDE-OPTION-SENSITIVITY IN NETWORK STRUCTURES

Recall that a (deterministic) TU-game with a network structure is a tuple (N, v, g) that consists of a (non-empty and finite) player set $N = \{1, ..., n\}$, a coalition function $v \in \mathbb{V}_N := \{v : 2^N \longrightarrow \mathbb{R} | v(\emptyset) = 0\}$ and a network structure (N, g) with nodes $N = \{1, ..., n\}$ and a network $g \subseteq \{ij | i, j \in N\}$. Nodes i and j are called connected in network g if there exists a path $ih_1, ..., h_k j \in g$, $h_1, ..., h_k \in N$ and the resulting connected components build a partition on N denoted by $\mathcal{C}(N, g)$ where $\mathcal{C}_i$ is the component of all players connected with player $i \in N$.

Gómez et al. [2008] define the probabilistic model for network structures and characterize the probabilistic Myerson value via the "direct probabilization approach". Ghintran et al. [2012] define the probabilistic Position value via a probabilistic extension of the coalition function but it turned out that this definition is equivalent to the expected payoff of the deterministic Position value. They further find that the probabilistic Position value can be characterized via the "direct probabilization approach" using probabilistic analogs of the characterizing axioms of the deterministic Position value. Belau [2010] claims that the "direct probabilization approach" for the graph-χ-value will fail, already the definitions of the probabilistic extensions of the characterizing axioms turn out to be problematic in application and reasonability. This claim is in line with our discussion in Remark 4.3 for coalition structures as the graph-χ-value indeed is not component decomposable

Axiom 4.9 (**CD** for Network Structures)**.** *An allocation rule for network structures* Y *satisfies* Component Decomposability **CD** *if*

$$Y_i(N, v, g) = Y_i(\mathcal{C}_i(g), v|_{\mathcal{C}_i(g)}, g|_{\mathcal{C}_i(g)})$$

We follow the steps of formalizing outside options for coalition structures.

Definition 4.4 (Outside Option in Network Structures)**.** *Let* (N, v, g) *be a TU-game with a network structure. A coalition* $K \subseteq N$ *is called* outside option *in* (N, v, g) *for player* $i \in N$ *if*

$$K \subseteq N \setminus \mathcal{C}_i(g) \text{ and } v(K \cup \{i\}) - v(K) > 0.$$

Player i has an outside option if she could create a surplus by joining a group of other players *outside* her connected component $\mathcal{C}_i(g)$.

Axiom 4.10 ((Weak) Outside-Option-Sensitivity for Network Structures)**.** *Let (N, v, g) be a TU-game with a network structure. An allocation rule for network structures Y satisfies* Weak Outside-Option-Sensitivity **WOOS** *if for all $i \in N$ and for outside options $\tilde{K}$ in (N, v, g) for player $i \in N$ such that $|\mathcal{C}_i(g)| > 1$ there exists a network structure (N, g') with $\tilde{K}$ also being outside option in (N, v, g') for player $i \in N$ such that*

$$Y_i(N, v, g') \neq Y_i(N, v_{\tilde{K},i}, g').$$

An allocation rule for network structures Y satisfies Outside-Option-Sensitivity **OOS** *if for all $i \in N$ such that $|\mathcal{C}_i(g)| > 1$ we have*

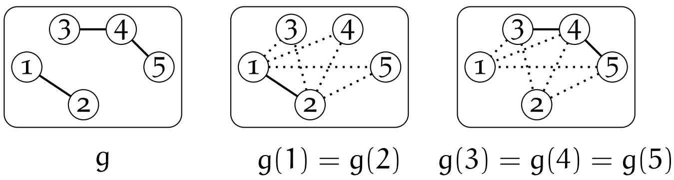
$$Y_i(N, v, g) < Y_i(N, v_{\tilde{K},i}, g)$$

for all $\tilde{K}$ being an outside option in (N, v, g) for i.

Obviously, the analogue of Lemma 4.1 also holds in the framework of network structures. Hence, we obtain the corresponding categorization possibilities as in the previous section.

Before we start to categorize the allocation rules we discussed so far, let us briefly emphasize the use of the lower outside option graph (LOOG), given by $g(i) = g|_{\mathcal{C}_i} \cup \{jk \in g^N | j \in \mathcal{C}_i, k \in N \setminus \mathcal{C}_i\}$. Let $N = \{1, 2, 3, 4, 5\}$ and $g = \{12, 34, 45\}$, the LOOGs are displayed in Figure 4.

Figure 4: Network and corresponding LOOGs



Remark 4.4 (Outside Options, Games on Networks and the LOOG)**.** *Note that an outside option $\tilde{K}$ is a group of players* outside *a player i's connected component. Therefore, no network-resricted game as the graph-restricted game or the arc-game can be affected by the difference between v and and $v_{\tilde{K},i}$ as $\tilde{K} \cup \{i\}$ is always unconnected and,*

hence, split up into connected subsets. Recall that the lower outside option graph (LOOG) for any player $i \in N$ is given by the connected component of player i plus a link between any player inside this component to every player outside this component (and there are no other connections between players outside player i's connected component). Therefore, the LOOG ensures that all outside options of player i for the original structure are taken into account in the graph-restricted game as well as in the arc-game.

Remark 4.5 (Outside-Option-Insensitivity of the Myerson value and the Position value)**.** *Both the Myerson value and the Position value are component decomposable (cf. van den Nouweland, 1993), hence, cannot be outside-option-sensitive.*

Lemma 4.6 (Outside-Option-Sensitivity of the Graph-χ-value)**.** *The graph-χ-value is outside-option-sensitive.*

Proof. Let (N, v, g) be a TU-game with a network structure, $i \in N$ with $|\mathcal{C}_i(g)| > 1$ and let $\tilde{K}$ be an outside option in (N, v, g) for i. We first analyze

$$\mu_i(N, v_{\tilde{K},i}, g(i)) = \sum_{K \subseteq N \setminus \{i\}} f(K) \left[v_{\tilde{K},i}^{g(i)}(K \cup \{i\}) - v_{\tilde{K},i}^{g(i)}(K) \right]$$
$$\text{where } f(K) := \frac{|K|!(|N| - 1 - |K|)!}{|N|!}, \quad K \subseteq N$$

Note that $v^g(K)$ is not affected by neutralizing the outside option $\tilde{K}$ for all $K \subseteq N$ such that $\tilde{K} \not\subseteq K$ for any network g. Using this, the special form of the LOOG $g(i)$ and since $\tilde{K} \subseteq N \setminus \mathcal{C}_i(g)$ we have

$$\begin{aligned}
\mu_i(N, v_{\tilde{K},i}, g(i)) &= \mu_i(N, v, g(i)) + \sum_{\substack{K \subseteq N \setminus \{i\} \\ \tilde{K} \subseteq K}} f(K) \left[v_{\tilde{K},i}^{g(i)}(K \cup \{i\}) - v_{\tilde{K},i}^{g(i)}(K) \right] \\
&\quad - \sum_{\substack{K \subseteq N \setminus \{i\} \\ \tilde{K} \subseteq K}} f(K) \left[v^{g(i)}(K \cup \{i\}) - v^{g(i)}(K) \right] \\
&= \mu_i(N, v, g(i)) + \sum_{\substack{K \subseteq N \setminus \tilde{K} \\ i \notin K}} f(K \cup \tilde{K}) \Big[v_{\tilde{K},i}^{g(i)}(K \cup \tilde{K} \cup \{i\}) - \underbrace{v_{\tilde{K},i}^{g(i)}(K \cup \tilde{K})}_{\substack{= v^{g(i)}(K \cup \tilde{K}) \\ \text{as } i \notin K}} \Big] \\
&\quad - \sum_{\substack{K \subseteq N \setminus \tilde{K} \\ i \notin K}} f(K \cup \tilde{K}) \left[v^{g(i)}(K \cup \tilde{K} \cup \{i\}) + v^{g(i)}(K \cup \tilde{K}) \right] \\
&= \mu_i(N, v, g(i))
\end{aligned}$$

$$+ \sum_{\substack{K \subseteq N \setminus \tilde{K} \\ i \notin K}} f(K \cup \tilde{K}) \Big[\underbrace{v_{\tilde{K},i}^{g(i)}(K \cup \tilde{K} \cup \{i\}) - v^{g(i)}(K \cup \tilde{K} \cup \{i\})}_{\leqslant 0 \, \forall \, K \subseteq N \setminus \tilde{K} \text{ with } i \notin K \text{ and } <0 \text{ e.g. for } K=\emptyset} \Big]$$
$$< \mu_i(N, v, g(i))$$

Consider any $j \in \mathcal{C}_i(g) \setminus \{i\}$. Analogously to the case of i we first have

$$\begin{aligned}
\mu_j(N, v_{\tilde{K},i}, g(i)) = \mu_j(N, v, g(i)) & \\
+ \sum_{\substack{K \subseteq N \setminus \tilde{K} \\ j \notin K}} f(K \cup \tilde{K}) \Big[v_{\tilde{K},i}^{g(i)}(K \cup \tilde{K} \cup \{j\}) - v_{\tilde{K},i}^{g(i)}(K \cup \tilde{K}) \Big] & \\
- \sum_{\substack{K \subseteq N \setminus \tilde{K} \\ j \notin K}} f(K \cup \tilde{K}) \Big[v^{g(i)}(K \cup \tilde{K} \cup \{j\}) - v^{g(i)}(K \cup \tilde{K}) \Big] &
\end{aligned}$$

Note that here, in contrast to the case of i, we have

$$v_{\tilde{K},i}^{g(i)}(K \cup \tilde{K} \cup \{j\}) = v^{g(i)}(K \cup \tilde{K} \cup \{j\})$$

as $j \notin \tilde{K} \cup \{i\}$ and therefore $K \cup \tilde{K} \cup \{j\} \neq \tilde{K} \cup \{i\}$ for all $K \subseteq N \setminus \tilde{K}$ such that $j \notin K$. Hence, we get

$$\begin{aligned}
\mu_j(N, v_{\tilde{K},i}, g(i)) = \mu_j(N, v, g(i)) & \\
+ \sum_{\substack{K \subseteq N \setminus \tilde{K} \\ j \notin K}} f(K \cup \tilde{K}) \Big[\underbrace{v^{g(i)}(K \cup \tilde{K}) - v_{\tilde{K},i}^{g(i)}(K \cup \tilde{K})}_{\substack{\geqslant 0 \, \forall \, K \subseteq N \setminus \tilde{K} \text{ with } j \notin K \\ \text{and } >0 \text{ e.g. for } K=\{i\}}} \Big] & \\
> \mu_j(N, v, g(i)) &
\end{aligned}$$

Using these findings for i and all $j \in \mathcal{C}_i(g) \setminus \{i\}$ we obtain **OOS** for the graph-χ-value:

$$\begin{aligned}
\chi_i^{\#}(N, v_{\tilde{K},i}, g) &= \mu_i(N, v_{\tilde{K},i}, g(i)) + \frac{v_{\tilde{K},i}(\mathcal{C}_i(g)) - \sum_{j \in \mathcal{C}_i(g)} \mu_j(N, v_{\tilde{K},i}, g(i))}{|\mathcal{C}_i(g)|} \\
&= \left(1 - \frac{1}{|\mathcal{C}_i(g)|}\right) \underbrace{\mu_i(N, v_{\tilde{K},i}, g(i))}_{<\mu_i(N,v,g(i))} + \frac{v(\mathcal{C}_i(g)) - \sum_{\substack{j \in \mathcal{C}_i(g) \\ j \neq i}} \overbrace{\mu_j(N, v_{\tilde{K},i}, g(i))}^{>\mu_j(N,v,g(i))}}{|\mathcal{C}_i(g)|} \\
&< \left(1 - \frac{1}{|\mathcal{C}_i(g)|}\right) \mu_i(N, v, g(i)) + \frac{v(\mathcal{C}_i(g)) - \sum_{\substack{j \in \mathcal{C}_i(g) \\ j \neq i}} \mu_j(N, v, g(i))}{|\mathcal{C}_i(g)|}
\end{aligned}$$

$$= \chi_i^{\#}(N, v, g)$$

□

Due to the intuition of the LOOG, Remark 4.4 and the use of the outside option consistency axiom, the result itself that the graph-χ-value is outside-option-sensitive is not very surprising. However, a finding that occured within the proof is notable:

Remark 4.6. *If an allocation rule is outside-option-sensitive and component efficient, it is quite clear that the loss player i experiences by neutralizing her outside option is distributed as a surplus among the other players in i's connected component (as the value of the coalition/connected component does not change). We have also seen this effect within the proofs of the outside-option-sensitive allocation rules for coalition structures and without inner structure.*

We found that, among the allocation rules for network structures analyzed, the graph-χ-value is the only outside-option-sensitive allocation rule. Following Remark 4.4, we suspect that allocation rules not using any sort of outside-option-graph as the LOOG will in general not be outside-option-sensitive.

4.6 CONCLUSION

Gómez et al. [2008] and Belau [2010] provide probabilistic forecasting models for network structures and coalition structures and, among others, analyze the generalization of cooperative allocation rules in this setting. While the "direct probabilization approach" (directly generalizing axioms of the original characterization of an allocation rule) turns out to be suitable for characterizing most probabilistic extensions, this approach failed for the χ-value. We were interested in a claim raised (but not formally investigated) in Belau [2010]: that the direct probabilization approach generally fails for "outside-option-sensitive" allocation rules. Following this claim, we first show that it indeed holds for the Owen value, another allocation rule which seems to account for outside options. As outside-option-sensitivity of cooperative allocation rules has not been explicitly formalized in the literature so far, we formally define outside options for both coalition and network structures and suggest *outside-option-sensitivity-axioms*. We have shown that these axioms are incompatible with *component decomposability* (which is satisfied by all allocation rules where the direct probabilization approach was suitable) and, therefore, our new axioms are suitable for a formal categorization of allocation rules into outside-option-sensitive and -insensitive ones. We show that our axioms are indeed satisfied by the Shapley value, the Banzhaf value, the Wiese value, the χ-value, the Owen value and, as the only allocation rule in the setting of network structures, the graph-χ-value. We briefly analyzed the explicit (numerical) structural effects of an outside option on allocation with respect to the outside-option-sensitive allocation rules for coalition structres where we found that this effect is equal for the Shapley value and the Wiese value. We further provide formal arguments for the claim of Belau [2010].

4.7 RECAP AND OUTLOOK

Note that the probabilistic forecasting model we discussed in this chapter in fact raises many more questions than "just" the formalization of outside-option-sensitivity. Let us shortly recap our findings so far: We have found experimental and theoretical support for outside options and during the categorization by our outside-option-sensitivity axioms we further found that there is only one outside-option-sensitive allocation rule for network structures so far. Recall that the network approach captures more information about the structure of a society or economy than the coalition structure approach (cf. Chapter 2). This motivates the analysis of this allocation rule and whether there might be some drawbacks such that there is a need for another outside-option-sensitive approach for networks. Before we can start with such an analysis, we have to specify which potential drawbacks might exist.

Let us take a brief look at the probabilistic model for network structures: As the model we analyzed in the beginning of this chapter, the probabilistic model for network structures uses the likelihood of each network structure that could possibly occur. Note that even if this probabilistic model might look useful for forecasting issues at first glance, it bears some general drawbacks in contrast to the probabilistic model for coalition structures. There is a huge difference between the number of elements in $\Delta(\mathbb{P}_N)$ and

$$\Delta(\mathbb{G}_N) := \left\{ p : \mathbb{G}_N \to [0,1], \sum_{g \in \mathbb{G}_N} p(g) = 1 \right\},$$

the set of all probability distributions over all possible networks g on N: Let $n := |N|$ be the number of players/nodes. It is quite known that there exist $\frac{n(n-1)}{2}$ possible (bilateral) links. From combinatorics we further know that there exist $\binom{n}{k}$ possibilities to choose k objects from a set of n objects. Hence, to find the number of all possible networks, we have to sum up the numbers of possible networks with k links for $k = 0, ..., \frac{n(n-1)}{2}$. Therefore, the number of all possible networks is

$$\sum_{k=0}^{\frac{n(n-1)}{2}} \binom{\frac{n(n-1)}{2}}{k} = 2^{\frac{n(n-1)}{2}}$$

The number of partitions of a set with n players is given by the n^{th} *Bell number* where the Bell numbers are defined recursively via

$$B_{k+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

with $B_0 = 0$.[4] Obviously, the n^{th} Bell number is much smaller than the corresponding number of all possible networks. To see that the difference already occurs to be notable for a very small number of agents, consider the situation for $n = 4$: Let $N = \{1, 2, 3, 4\}$ and let m denote the number of connected components in a network. For possible partitions $\mathcal{P} = \{P_1, ..., P_m\}$ and networks corresponding to each m see Figures 11 and 12 in the appendix. While there are 15 possible partitions in the case of $n = 4$, the number of possible networks already reaches 64, that is, more than 4 times the number of partitions. Most probably, beliefs of subjects will not be sufficiently detailed to actually form a probability distribution over all possible networks, therefore, the probabilistic approach seems not to be applicable for forecasting and we would like to find another method.

4 As a reference, see sequence A000110 in the On-Line Encyclopedia of Integer Sequences (OEIS).

As "One of the open problems in the theory of cooperative games are power indices"[5], this leads to the motivation of our first further investigation

1. Can we find an allocation rule for (political) networks that is both a power index and suitable for forecasting?

Calvo et al. [1999] introduced a generalized setting for networks considering the probability of each bilateral relation between agents where the structure analyzed is seen as the result of independent relations. Recall that, in contrast to that, Gómez et al. [2008] consider the likelihood of all possible networks because "the importance of removing the independence assumption should not be underestimated". This is explained for example by incompatibilities: the presence of a certain relation between persons, enterprises, or political parties can exclude the possibility of a relation between one of them and a third actor (Gómez et al., 2008, p. 540). A coalition between two parties might exclude the possibility of a coalition between them and a third one due to insuperable conflicts between this third party and one of the first two. This brings us to a further question

2. Can we find an allocation rule for (political) networks that is suitable in the presence of incompatibilities?

We will find such an allocation rule for the last two questions in Chapter 5. Note that there is an important difference between this rule and other power indices: The index we propose is *link-based*, that is, accounts for links rather than whole nodes. Recall that we aimed to further analyze the graph-χ-value and whether there might be some drawbacks. In fact, the graph-χ-value is not link-based. As the index we propose is not outside-option-sensitive, this leads to the the following question

3. Can we find a link-based allocation rule for (political) networks that is outside-option-sensitive?

Recall that we analyzed the structural effects of outside options on allocation for the allocation rules for coalition structures. Having in mind the previous research question, the last question is obvious:

5 Gianfranco Gambarelli in the Plenary Session at the SING10 Conference, July 2014 over the Special Issue of the International Game Theory Review, also see for example Bertini et al. [2013].

4. Is there a difference in structural effects of outside options on link-based allocation compared to player-based allocation?

We will find a link-based allocation rule that is outside-option-sensitive and an extension for political networks (i. e., suitable for incompatibilities) in Chapter 6. There, we will further discuss the outside-option-sensitivity axioms and find that the structural effect of outside options on allocation is indeed more complex for link-based allocation rules than for player-based ones. This leads to the explanation why our "weak" axiom might not be that weak and our "normal axiom" might actually be sort of strong.

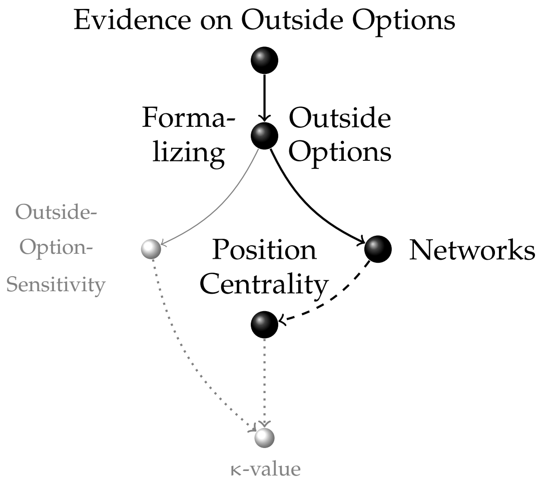
Part III

CENTRALITY AND LINK-BASED VALUES

5

CENTRALITY AND CONSEQUENCES OF CONNECTION FAILURE

WEIGHTED VOTING (STILL) DOESN'T WORK



5.1 INTRODUCTION

In this chapter we address the first two questions raised in Chapter 4, namely, whether we can find an allocation rule for networks which can be used as a power index and is suitable in the presence of incompatibilities. As we focus on networks, the analysis of *centrality* of an agent within a network seems worthwile, especially for the application to measure (relative) power. Before we connect this issue with outside-option-sensitivity in Chapter 6, this chapter focuses on connected networks (i. e., networks in which there exists a path between any two agents) as in this case, outside options are absent. This allows us to first analyze the centrality issue independently of outside-option-sensitivity.

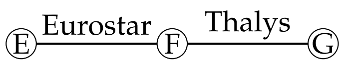
There is a large literature on centrality measures, mostly applied for social networks, economic networks and also political networks. Centrality is often used to identify top key nodes, those nodes in the network being most *important* for the network. But what means important? Mostly, we mean *importance for cohesion*, that is, how crucial or essential a node is for the network (or a set of nodes) to be connected.[1] A reasonable and convincing approach for the analysis of importance for cohesion is the analysis of consequences of failure. Application domains are for example energy networks or political networks: consequenses of failure are capacity overloads or blackouts for energy networks or a (partly or complete) breakdown of trading routes/networks or an Economic and Monetary Union. These examples are of recent importance considering increasing blackout probabilities due to outdated reactors as for example in France or the recent nuclear phaseout decision in Germany.

While the existing literature on centrality only analyzes failure consequences of whole nodes, already the failure of a certain connection can seperate a whole network into pieces. The application of analyzing failure consequences of connections seems even more relevant for political or economic networks: oil pipelines can break (connection failure) without a breakdown of the whole gas province (node failure), a bilateral trading agreement can be broken without a whole country leaving the trading union, political parties can stop bilateral coalitional negotiations due to imcompatibilities without leaving the political spectrum or a country might stop monetary flows for bail-out packages without leaving the European Union. Hence, consequences of connection failure should be considered in a measure for identifying top key nodes or (relative) coalitional power by means of relative importance for cohesion of the whole.

Existing centrality measures either generally ignore the importance of cohesion or bear other drawbacks. As an example, consider the following simple transit-country example: Imagine one wants to travel from London, England (E) to Cologne, Germany (G) and has a fear of flying and boats, hence, has to go by train. While London is connected to Paris, France (F) by the Eurostar train and Paris and Cologne are connected by the Thalys train, there is no direct train connection between England and Germany. Hence, France as a transit country connects the boundary countries England and Germany. The situation can be described by the train-network in Figure 5.

1 When talking about *cohesion* we basically refer to structural cohesion by means of essentiality for connectedness of a network.

Figure 5: Train Connections



Now, we are interested in the relative power of each country in this train-network. The most popular centrality measures (measuring how crucial a node is for connectedness) are Bonacich [1972]'s *eigenvector* centrality or Freeman [1978]'s *closeness*, *betweenness* and *degree* measures. Centrality according to these approaches (normalized for comparability issues) is presented in Table 10.

Table 10: Centrality in the Train-Network

Country	E	F	G
Eigenvector	29.29	41.42	29.29
Closeness	28.57	42.86	28.57
Betweenness	0	100	0
Degree	25.00	50.00	25.00

Eigenvector centrality (Bonacich, 1972) is based on the idea that relative power of a node depends to the relative power of the node's neighborhood and measures the *influence* of a node on the network by analyzing eigenvalues of the corresponding adjacency matrix (this idea is for example the basis of Google's PageRank). Due to use of the adjacency matrix, this approach accounts for connectedness of the whole network. However, for the train example above we see that this approach leads to relatively small distances between the relative power of the boundary countries (England (E) and Germany (G)) and the transit conutry (France (F)) which could be seen as unreasonable in terms of failure consequences: remember that the transit country connects the boundary countries and a failure would result in a complete breakdown of the train-network.

The closeness measure counts the *length of shortest paths*, that is, how many intermediate nodes have at least to be passed to get from one node to another. As paths between *any* two nodes are considered, also this measure accounts for connectedness of the whole network, but, as the eigenvector approach, the relative difference of centralities between the boundary countries and the transit country is very small. Furthermore, the closeness measure bears some problems for weighted networks as the "length" of a shortest path is difficult to measure correctly. For a further discussion on drawbacks of closeness, also see Gómez et al. [2003].

Usually, the betweenness measure which counts how often a node lies between any two other nodes on a shortest path is suggested to be the most suitable measure for cohesion/connectedness. In the train example we obtain a sort of an opposed problem to the cases before: the boundary countries obtain no power at all and the whole power is distributed to the transit country. One could argue that the boundary countries do have some power as their existence actually "creates" the transit country's power: if a boundary country fails (i. e., leaves the train network), the betweenness measure of the former transit country becomes zero.[2] These problems with the betweenness measure are generally quite likely to occur in small networks or large networks with a small number of connections. Note that our interest is more on small than large networks.

The degree measure counts the number of direct connections of a node and, beside generally demanding the lowest computational effort, in the example clearly differentiates between boundary and transit countries and accounts for the boundary countries creating the transit country's power. However, the degree measure only considers a node's direct links and not the structure of the whole network, that is, generally does not account for connectedness of the whole network. This is unproblematic in the train example and one could argue that in more complex networks this problem could be solved by normalization, but this turns out not to be the case.[3]

Beside the aforementioned drawbacks, there is another general problem in the use of "classic" centrality measures: centrality measures are independent of any characteristic function modelling the specific situation (i. e., the underlying game) which makes them barely applicable for specific economic or political applications where the un-

2 As an alternative to the original betweenness measure, Freeman et al. [1991] suggest a betweenness measure based on network flow. Still, in the example, the boundary countries would obtain a centrality of zero.

3 Find an example of a simple communication network and a discussion on drawbacks of the classic centrality measures for this case in the appendix.

derlying game matters. Especially for the application as a power index for political networks, centrality measures are not affected by a weighted voting game (modelling potential winning coalitions), that is, generally not affected by differences in seat shares (or vote shares) in a parliament.

Considering game theoretic approaches, there of course exist the popular Shapley-Shubik-Power-Index (Shapley and Shubik, 1969) and the Banzhaf-Power-Index (Banzhaf, 1952)[4] which apply the Shapley or Banzhaf value, respectively, to a majority game (weighted voting game). However, these indices are not designed to account for centrality as they do not consider any inner structure like a network. Accounting for the network structure, one could apply the communication restricted Shapley value (known as the Myerson value, cf. Myerson, 1977). Applied to the train example, power resulting from these approaches is reported in Table 11 where we used the so-called unanimity game with respect to whether England and Germany are connected.

Table 11: Game Theoretic Approaches ($u_{E,G}$) in the Train-Network

Country	E	F	G
Shapley/Banzhaf	50.00	00.00	50.00
Myerson	33.33	33.33	33.33

We see that the Shapley- or Banzhaf-Power-Index, respectively, completely ignores the existence of the transit country while the Myerson approach does not disinguish between boundary and transit countries at all.

There only exist few game theoretic investigations directly related to network centrality. Existing approaches are for example Gómez et al. [2003] or Suri and Narahari [2008] (using the Shapley value) or Grofman and Owen [1982] (using the Banzhaf value). These measures take into account consequences of failure by analyzing marginal contributions, that is, the surplus a certain node creates when entering a network (which could be seen as the negative of the failure of this node for each coalition). The network structure is taken into account

4 The Banzhaf-Power-Index is also known as Banzhaf-Coleman index (cf. Coleman, 1971) or Penrose-Banzhaf index (cf. Penrose, 1946).

by using functions for the underlying game which consider (shortest) paths present in the network depending on the presence or absence of a node. Note that these approaches account for centrality by fixing the underlying game and hence, would also not be applicable as power indices with respect to a weighted voting game. Furthermore, all these approaches analyze failure of a whole node with all its connections while already the failure of one connection influences (shortest) paths and even might split the whole network. We will present a political example which shows that these game theoretic approaches are not always suitable while the analysis of connection failure solves this problem. Moreover, these approaches have no extension for weighted networks.

Even though not directly applied to the centrality issue, the Position value (Meessen, 1988 and Borm et al., 1992) does account for the relative importance of links (instead of nodes). Note that the original characterization (Borm et al., 1992) indirectly includes the centrality issue: in fact, the Position value is a generalization of Freeman's degree measure. Remember that the degree measure has provided the most promising distribution of power in the train example. There are different extensions of the Position value: Ghintran et al. [2012] extend it to the probabilistic model (cf. Chapter 4), Ghintran [2013] provides a version using weights based on the weighting scheme introduced by Haeringer [2006] and Kamijo and Kongo [2009] provide non-symmetric generalizations.[5] However, all these approaches apply the Shapley value to measure the importance of a link. Remember that the Shapley approach uses different weights, depending on the size of the coalitions for which a node or player creates surplus when entering. This so-called "weighted voting" might be unreasonable if for example incompatibilities between actors are present because then, weights are not relatively balanced anymore. Beside the problem of incompatibilities, especially for an application of political networks, Banzhaf [1952] generally argued that "Weighted voting doesn't work": the Shapley weights could lead to implausible high or low distribution of power.[6]

5 There also exist generalizations in the Jackson and Wolinsky [1996] framework of networks as for example Kamijo [2009].

6 Also consider the communication network example in the appendix to see the difference between both approaches.

In this chapter, we suggest a new approach for (but not restricted to) centrality: a new (axiomatically characterized) cooperative allocation rule for network structures which accounts for a node's importance for connectedness of the network by the relative impact of its connections. For this, we analyze consequences of link failure using the Banzhaf value, generalized for a game which accounts for the links of a network, the so-called arc game. We assign the corresponding Banzhaf values to all links of the network to capture consequences of failure, that is, the importance for cohesion/connectedness of connections. Then, we apply the generalization of Freeman [1978]'s degree measure for weighted networks to obtain the *Banzhaf Position value*.

Moreover, we suggest an alternative centrality measure by means of Bonacich [1972]'s eigenvector measure, the *Shapley/Banzhaf-Eigenvector-value*, and a generalization to weighted networks by implementing an emphasis parameter which allows regulating emphasis for importance for connectedness or centrality by means of political, economic or social weights individually.

This chapter is organized as follows: the following section provides definitions and notations, in Section 3 we define our new allocation rule, the *Banzhaf Position value*, and provide axiomatic characterizations. Section 4 discusses properties and extensions. In Section 5 we apply our approach to the case of the state parliament elections (*Bürgerschaftswahl*) 2001 in Hamburg, Germany to use our allocation rule as a power index. We further provide an example where we use our allocation rule as a centrality measure to identify top key nodes which also discusses exclusiveness of links. Finally, Section 6 concludes.

5.2 PRELIMINARIES

Recall the framework of cooperative network structures. A *TU-game with a network structure* (N, v, g) consists of a (non-empty and finite) player set $N = \{1, ..., n\}$, a *coalition function* $v \in \mathbb{V}_N := \{v : 2^N \longrightarrow \mathbb{R} | v(\emptyset) = 0\}$ and a *(binary) network structure* (N, g) with nodes in $N = \{1, ..., n\}$ and a (binary) network $g \subseteq \{ij | i, j \in N, i \neq j\}$, that is, strength of all links is equal and normalized to 1. Let $\mathbb{G}_N^b$ denote the set of all binary networks g. Recall that nodes i and j, $i \neq j$, are called *connected* in the network g if there exists $k \in \mathbb{N}$ and a path $ih_1, ..., h_k j \in g$, $h_1, ..., h_k \in N$ and that the resulting connected components build a partition on N denoted by $\mathcal{C}(N, g)$ where $\mathcal{C}_i$ is the component of all players connected with player $i \in N$.

A TU-game is called *simple* if $v(K) \in \{0, 1\}$ for all $K \subseteq N$ and is called *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$. As every $v \in \mathbb{V}_N$ can be zero-normalized, we can restrict ourselves to the set of zero-normalized games, that is, $v \in \mathbb{V}_N^0$. Recall the definition of the *link-game* (or *arc game*) in which the links in the network are the players:

$$v^N : \{g' \subseteq g\} \longrightarrow \mathbb{R}, v^N(g') := v^{g'}(N) = \sum_{S \in \mathcal{C}(N,g')} v(S) \, \forall g' \subseteq g$$

where v^g is the *graph-restricted game* and v is zero-normalized.

An *allocation rule for (binary) network structures* is a function $Y : \{N\} \times \mathbb{V}_N \times \mathbb{G}_N^b \longrightarrow \mathbb{R}^n$, assigning payoffs $Y_i(N, v, g)$. Recall the definitions of the *Shapley value* Sh and the *Banzhaf value* Ba, for all $i \in N$ given by

$$Sh_i(N, v) = \sum_{K \subseteq N \setminus \{i\}} \frac{|K|!(|N| - 1 - |K|)!}{|N|!} [v(K \cup \{i\}) - v(K)]$$

$$Ba_i(N, v) = \sum_{K \subseteq N \setminus \{i\}} \frac{1}{2^{|N|-1}} [v(K \cup \{i\}) - v(K)]$$

which are designed for TU-games without inner structure. In contrast, recall the Position value π, for all $i \in N$ given by

$$\pi_i(N, v, g) = \sum_{\lambda \in g_i} \frac{1}{2} Sh_\lambda(g, v^N).$$

The Position value takes into account the role of links in which a player is (directly) involved.

For normalization issues, we will now introduce a new version of feasibility, designed for network structures.

Definition 5.1 (Feasibility in network structures)**.** *An allocation rule for network structures* $Y(N, v, g)$ *is called* feasible *if*

$$\sum_{i \in N} Y_i \leqslant \sum_{C \in \mathcal{C}(N,g)} v(C).$$

In the commonly used definition of feasiblity, the literature uses $v(N)$ instead of $\sum_{C \in \mathcal{C}(N,g)} v(C)$. But the use of $v(N)$ is only suitable if the cooperative game has no inner structure or if g is connected, because only then $\sum_{C \in \mathcal{C}(N,g)} v(C) = v(N)$ holds for sure.

In order no model potential differences between the links (for example different strengths), we use weighted graphs:

Definition 5.2 (Weighted Network Structure)**.** *A weighted (social, economic or political) network $g(N, w)$ is a binary network g endowed with a weight w_{ij} for every link $ij \in g$. If $w_{ij} = 0$, we could interpret the "dummy link" ij as not existing in network g. Hence, a binary network corresponds to a weighted network with weights being 0 or 1. Let $\mathbb{G}_N^w$ denote the set of all weighted networks $g(N, w)$. Note that $\mathbb{G}_N^b \subset \mathbb{G}_N^w$. Then, a weighted network structure is a tuple (N, g, w).*

Independently of an underlying game, centrality measures are used as sort of allocation rules on binary or weighted network structures.

Definition 5.3 (Centrality Measure)**.** *A centrality measure is a function $C : \mathbb{G}_N^w \longrightarrow \mathbb{R}^n$ which assigns an index to every node depending on how "central" this node is in network g.*

Most centrality measures are originally designed for binary networks and later generalized to weighted ones. The most popular centrality measures for binary networks are Freeman's centrality measures (Freeman, 1978). The first (and simplest) one is the *degree measure* which has been extended for weighted networks by Barrat et al. [2004], Newman [2004] and Opsahl et al. [2008]:

Definition 5.4 ((generalized) Degree Measure)**.** *The (generalized) degree measure C^d for weighted network $g(N, w)$ is for every $i \in N$ given by*

$$C_i^d(g(N, w)) = \sum_{\substack{j \in N \setminus \{i\}, \\ ij \in g}} w_{ij}.$$

The degree measure for binary networks counts the number of direct connections and in case of weighted networks, the weights of those connections are summed up.

As we do not explicitly analyze the other classic centrality measures, we will only provide a brief, less formal overview. Formal definitions are provided in the appendix. Freeman's *closeness* and *betweenness* are designed by so-called *shortest paths*: A *(binary) shortest path* between node i and j, $i \neq j$, is defined by the minimal number of links that have to passed from i to j. Closeness is measured by the lengths of the shortest paths from a node to all other nodes (closeness to other nodes) while betweenness counts how often a node lies on a shortest paths between two other nodes (betweenness of nodes) relative to all shortest paths.

Beside Freeman's centrality measures, Bonacich [1972] introduces the *Eigenvetor centrality*. The idea of this approach is that the centrality of a node should be proportional to the centralities of the node's neighbors. Formally, the Eigenvector centrality of a node i is given by the i's entry of the eigenvector corresponding to the largest eigenvalue of g' adjacency matrix.

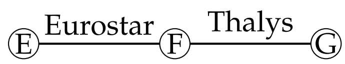
Closeness and Betweenness have been extended to weighted networks by Brandes [2001] and Newman [2001] using *fastest paths*. Note that Eigenvector centrality is originally applicable for weighted networks due to the use of the adjacency matrix.

5.3 THE BANZHAF POSITION VALUE

5.3.1 *Derivation and Definition*

We will now relate the issue of relative importance of links to the centrality measures. For this, we consider the train example from the introduction in a more formal way:

Example 5.1 (Transit Countries, Centrality and the Position value)**.** *Let* $N = \{$*England (E), France (F) , Germany (G)*$\}$ *and consider again the train-network with links* Eurostar:= $\{EF\}$ *and* Thalys:= $\{FG\}$. *The*



degree measure distributes 1 to England and Germany and 2 to France as France has twice as many connections as the other two countries. Now let $v(K) = 1$ *if* $K = N$ *and* $v(K) = 0$ *otherwise. Both links are needed to create worth and* Eurostar *is needed as much as* Thalys*, that is, they are symmetric in the link-game. Hence, they both obtain half* *of the worth:* $\mathrm{Sh}_{Eurostar} = \mathrm{Sh}_{Thalys} = \frac{1}{2}$*. The Position value assigns to every country half of the payoff of every link it is involved in, hence,* $\pi_E = \frac{1}{2}\mathrm{Sh}_{Eurostar} = \frac{1}{4}$*,* $\pi_F = \frac{1}{2}\mathrm{Sh}_{Eurostar} + \frac{1}{2}\mathrm{Sh}_{Thalys} = \frac{1}{2}$ *and* $\pi_G = \frac{1}{2}\mathrm{Sh}_{Thalys} = \frac{1}{4}$*. Note that this is proportional to the degree measure.*

The motivation for the use of degree centrality is that France is more central by means of trade, travel, ect. The motivation for the Position value by means of the cooperative game is that France is "needed more" to create the worth. Both ideas use that France, as a transit country, connects the three countries and hence, its position in the network is stronger which comes through the fact that it is involved in more links than the other two countries.

In fact, the Position value is a generalization of the degree measure. To see this, we will first generally introduce how we combine the cooperative approach with the centrality approach.

Let g be a (binary) network. For the moment, we will only consider connected networks (where outside options are absent). Note that this especially excludes empty networks (i. e., networks where there do not exist any connections) which is important for normalizations. Let $v \in \mathbb{V}_N^0$ be any coalition function.[7]

Transformation 5.1. *Define a new weighted network* $g(N, \tilde{w}(v, Y))$ *(*$Y = \mathrm{Sh}$ *or* $Y = \mathrm{Ba}$*) with*

$$\tilde{w}_{ij}(v, \mathrm{Sh}) := \mathrm{Sh}_{ij}(g, v^N) \text{ or}$$
$$\tilde{w}_{ij}(v, \mathrm{Ba}) := \mathrm{Ba}_{ij}(g, v^N), \text{ respectively}$$

Now we apply the generalized degree measure:

Definition 5.5 (Shapley/Banzhaf-Degree-value)**.** *For any* $v \in \mathbb{V}_N^0$ *and every network* $g \subseteq g^N$*, the Shapley-Degree-value* $\mathrm{CD}^{\mathrm{Sh}}$ *and the Banzhaf-Degree-value* $\mathrm{CD}^{\mathrm{Ba}}$ *are given by*

$$\mathrm{CD}_i^Y(g, v) := C_i^d\left(g(N, \tilde{w}(v, Y))\right) = \sum_{j \in N \setminus \{i\}} \tilde{w}_{ij}(v, Y)$$
$$\text{where } Y = \mathrm{Sh} \text{ or } Y = \mathrm{Ba}, \text{ respectively.}$$

7 If one is only interested in a centrality measure, v should only account for connectedness of the network or a set of nodes. As a simple example one could consider the *unanimity games* where the corresponding $T \subseteq N$ represents the set of essential nodes.

For comparability (and in order to define a feasible/efficient allocation rule), we can normalize the values. This is unproblematic in terms of axiomatic justification in our applications as we are mainly interested in ranks and relative distances (cf. Remark 2.6). Here, we follow the *multiplicative efficient normalization* approach used for normalization of the Banzhaf value due to Dubey and Shapley [1979][8] and normalize by multiplication with the scaling factor

$$\frac{v(N)}{\sum_{j\in N} CD_j^Y(g, v^N)}$$

while one could also use the *additive efficient normalization* approach used for the additive normalized Banzhaf value introduced by Hammer and Holzman [1992] which we find less applicable.[9]

Corollary 5.1 (Relation to the Position value)**.** *The normalized Shapley-Degree-value coincides with the Position value.*

Proof. Generally, it holds that

$$\begin{aligned}\sum_{j\in N} CD_j^Y(g, v^N) &= \sum_{j\in N}\sum_{k\in N\setminus\{j\}} \tilde{w}_\lambda(v, Y)\\ &= 2\cdot\sum_{\lambda\in g} \tilde{w}_\lambda(v, Y) = 2\cdot\sum_{\lambda\in g} Y_\lambda(g_b, v^N).\end{aligned}$$

For $Y = Sh$ we have $\sum_{\lambda\in g} Y_\lambda(g_b, v^N) = v(N)$, hence, the multiplicative scaling factor becomes $1/2$ in this case. □

Remark 5.1 (Shapley Position value)**.** *Due to Corollary 5.1 and to highlight the difference between the use of either the Banzhaf or the Shapley value, we will use the notion "(Shapley) Position value" instead of Position value throughout this chapter.*

Remark 5.2 (Relation to Closeness and Betweenness)**.** *Our transformation uses the corresponding Shapley/Banzhaf value of links, that is, we calculate how important a certain connection is for the cohesion of a network. This is done by taking into account all other connections*

8 An axiomatization of the multiplicative normalized Banzhaf value is for example proposed by van den Brink and van der Laan [1998].

9 Here, we would add the factor $\frac{v(N)-\sum_{j\in N} CD_j^Y(g,v^N)}{|N|}$. An axiomatization of the additive normalized Banzhaf value is for example proposed by Ruiz et al. [1996].

of the network, hence, these values capture the idea of (relative) necessity of connections which is the basic idea of betweenness. Taking into account the actual value captures the idea of closeness. Hence, one could argue that the Degree-values are sufficient as the position between other nodes and relative necessity are captured by the weights for importance of cohesion.

However, applying closeness and betweenness can be done analogously using the transformation above and the corresponding generalized measures for weighted networks. Eigenvector centrality can be applied directly using the transformation above. For completeness we provide the formal definitions in the appendix.

We will continue by providing axiomatic characterizations of the Banzhaf-Degree-value for binary networks. For motivating issues, we focused on connected networks. However, the following axioms and characterizations are defined and hold for general ones. In line with Corollary 5.1 and Remark 5.1, we define

Definition 5.6 (Banzhaf Position value). *For any TU-game with a network structure* (N, v, g)*, the Banzhaf Position value* π^{Ba} *is given by*

$$\pi^{Ba}(N, v, g)_i := \sum_{\lambda \in g_i} \frac{1}{2} Ba_\lambda(g, v^N).$$

To be precise, the Banzhaf Position value is the Banzhaf-Degree-value scaled by factor $1/2$.

Remark 5.3 (Link-Based and Player-Based Values). *Beside the difference in outside-option-sensitivity discussed and analyzed in Chapter 4, there is another notable difference between cooperative allocation rules we did not formally denote so far. Remember that we are interested in consequences of link failure (rather than node failure) in this chapter. Allocation rules accounting for consequences of link failure are also know as* link-based *values and allocation rules accounting for consequences of node failure are also known as* player-based *values. Note that the Position values (and its variants as given in the introduction of this chapter) are link-based while all other allocation rules analyzed in this thesis so far are player-based.*

5.3.2 *Axiomatization on Cycle-Free Networks*

We will first provide an axiomatization of the Banzhaf Position value which is closely related to the original axiomatization of the Shapley value by Shapley [1953]. Recall that the Shapley value is characterized by Additivity **A**, Efficiency **E**, Symmetry **S** and the Null Player Axiom **N** (cf. Chapter 2). Unfortunately, this characterization only holds on cycle-free networks and as Borm et al. [1992] for the (Shapley) Position value, we are not able to find a "Shapley-like" characterization for general networks.[10]

Definition 5.7 (Cycle-free Network)**.** *A network* g *is called* cycle-free *if there exists at most one path between any two nodes.*

Remark 5.4 (Relation to Graph-Theory: Trees and Forests)**.** *In graph theory, a cycle-free connected subnetwork is called tree, a cycle-free connected network (i. e., all nodes are connected) is called spanning tree and cycle-free network which consists of connected subnetworks is called forest.*

We first introduce an axiom which could be seen as the "original driving force" of the (Shapley) Position value (cf. Borm et al., 1992) and shows the relation to the degree measure (even though this had never been an application)[11]:

Axiom 5.1 (Degree Property (**DEG**))**.** *A game* (N, v, g) *is called* link anonymous *if* $\exists\, f : \{0, 1, ..., |g|\} \longrightarrow \mathbb{R}$ *such that* $v^N(g') = f(|g'|)$ *for all* $g' \subseteq g$*. An allocation rule for network structures satisfies the* Degree Property **DEG** *if for all link anonymous games* (N, v, g) *there exists* $\alpha \in \mathbb{R}$ *such that*

$$Y(N, v, g) = \alpha \cdot C^d(g).$$

In a link anonymous game, we have $v^N(g' \setminus \lambda) = v^N(g' \setminus \lambda')$ for all $\lambda, \lambda' \in g' \subseteq g$, hence, all links provide the same marginal contri-

10 van den Nouweland and Slikker [2012] show that such a Shapley-like characterization indeed generally holds for superadditive coalition functions within the framework of Jackson and Wolinsky [1996]. Note that this framework explicitly differs from the framework of communication situations we use in this thesis such that findings cannot be transferred. Furthermore, superadditivity is quite a restrictive property, especially for political applications and in presence of incompatibilities.

11 This axiom is also know as *arc anonymity*, see for example van den Nouweland [1993] for the framework of communication structures, or *link anonymity*, van den Nouweland and Slikker [2012] for the framework due to Jackson and Wolinsky, which might be more convincing terms if no relation to centrality is of interest. We stick to the original term due to Borm et al. [1992] to emphasize this relation.

butions and are therefore equally important within the network. One could argue that in this case, the strength of a node should be measured by its degree. If an allocation rule is used for centrality issues, **DEG** seems a favourable axiom.

Remark 5.5 (The Degree Property and Symmetry)**.** *Note that **DEG** is closely related to **S** (players that are not distinguishable but by name should obtain the same payoff), but in the context of a* link-based *value (instead of a* player-based *one as the Shapley value). In a link anonymous game, only the size of subnetworks matters, independently of the inner structure. Hence, in this case, players can only be distinguished by the number of their links and **DEG** implies that players with the same number of links should be treated equally in the allocation of worth.*

Example 5.2. *Consider $N = \{1, 2, 3\}$, $g = \{13, 23\}$ and the unanimity game $u_{\{1,2\}}$. Note that for*

$$f : \{0, 1, 2, 3\} \longrightarrow \mathbb{R}, f(x) := \begin{cases} 1, & \textit{if } x = 3 \\ 0, & \textit{else} \end{cases}$$

it holds that $u^N_{\{1,2\}}(g') = f(|g'|)$ for all $g' \subseteq g$, that is, (N, v, g) is link anonymous. For the Shapley value, the multiplicative efficient normalized Banzhaf value and the Myerson value we get

$$Sh(N, v) = (\frac{1}{2}, \frac{1}{2}, 0) = \overline{Ba}(N, v), \mu(N, v, g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

*while the degree measure is given by $C^d(g) = (1, 1, 2)$, hence, **DEG** is not satisfied by the Shapley value, the Banzhaf value and the Myerson value.*
On the other hand, the (Shapley) Position value and the Banzhaf Position value are given by

$$\pi^{Sh}(N, v, g) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), \pi^{Ba}(N, v, g) = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}).$$

that is, they are proportional to the degree measure.

Borm et al. [1992] show that the (Shapley) Position value satisfies **DEG**.

Lemma 5.1. *The Banzhaf Position value satisfies **DEG**.*

Proof. Let (N, v, g) be link anonymous, that is, there exists a function $f : \{0, 1, ..., |g|\} \longrightarrow \mathbb{R}$ such that $f(|g'|) = v^N(g')$ for all $g' \subseteq g$. For $g = \emptyset$, we have $\pi_i^{Ba}(N, v, g) = 0 = C^d(g)$. Let $g \neq \emptyset$ and $\lambda \in g$.

$$
\begin{aligned}
Ba_\lambda(g, v^N) &= \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \left(v^N(g' \cup \lambda) - v^N(g')\right) \\
&= \frac{1}{2^{|g|-1}} \sum_{i=0}^{|g \setminus \lambda|} \sum_{\substack{g' \subseteq g \setminus \lambda \\ |g'| = i}} (f(i+1) - f(i)) \\
&= \underbrace{\frac{1}{2^{|g|-1}} \sum_{i=0}^{|g|-1} \binom{|g|-1}{i} (f(i+1) - f(i))}_{:= A \in \mathbb{R} \text{ independent of } \lambda} \\
\Rightarrow \pi_i^{Ba}(N, v, g) &= \sum_{\lambda \in g_i} \frac{A}{2^{|g|}} = \frac{A}{2^{|g|}} |g_i| = \underbrace{\frac{A}{2^{|g|}}}_{:= \alpha \in \mathbb{R}} C^d(g)
\end{aligned}
$$

□

The next axiom is used to characterize the (Shapley) Position value on cycle-free networks (cf. Borm et al., 1992).

Axiom 5.2 (Superfluous Link Property (**SLP**))**.** *A link* λ *is called* superfluous *if* $v^N(g' \cup \lambda) = v^N(g')$ *for all* $g' \subseteq g$*, that is, if* λ *is a Nullplayer in* (g, v^N)*. An allocation rule for network structures* Y *satisfies the* Superfluous Link Property **SLP** *if* $Y(n, v, g) = Y(N, v, g \setminus \lambda)$ *for all superfluous links* λ*.*

Remark 5.6 (The Superfluous Link Property and the Null Player Axiom)**.** *Note that* ***SLP*** *is closely related to* **N** *(a null player, that is, a player that never creates any surplus, should obtain a payoff of zero). In the framework of network structures, a player can be seen as a Null player if all her links are superfluous and in this case* ***SFL*** *implies a payoff of zero.*

Borm et al. [1992] show that, on cycle-free networks, the (Shapley) Position value is uniquely determined by **DEG**, **SLP**, **A** and **CE**.[12]

As a difference to the (Shapley) Position value, the Banzhaf Position value does not satisfy component efficiency **CE**. Hence, we need a new axiom to close the emerging gap.

12 cf. Chapter 2 for **CE**

Recall that the Banzhaf value has originally been defined for simple games: If $v \in \mathbb{V}_N$ is simple, the Banzhaf value is given by

$$Ba_i(N,v) = \frac{\text{\# of swings for } i}{\text{\# of potential swings for } i} = \frac{\text{\# of swings for } i}{2^{|N|-1}}$$

where $K \subseteq N$ is a *swing* for i iff $v(K \cup \{i\}) - v(K) = 1$. As v is simple, a swing can be seen as coalition where player i is crucial for to become a winning coalition when joining. Hence, the basic idea of the Banzhaf value (used as a power measure) is that the worth of an agent is given by the number of this agent's swings (i. e., the number of coalitions this agent is crucial for to become a winning coalition when joining) relative to the number of potential swings (i. e., the numer of coalitions the agent is potentially crucial for to become a winning coalition when joining which equals the numer of possible coalitions without the agent). This is the agent's relative power. Combining this intuition with the idea of **CE** (productive units observe the worth created) and our aim of considering power of links rather than nodes, one could argue that the worth of a component should be equal to the relative power of this component, that is, the sum of all swings within this component relative to all potential swings within the component.

In other words, for an allocation rule Y and v being simple we should have

$$\begin{aligned}
\sum_{i \in C} Y_i(N,v,g) &= \frac{\begin{array}{c}\text{\#of all subnetworks a link in C is crucial}\\ \text{for becoming a winning subnetwork}\end{array}}{\begin{array}{c}\text{\# of all subnetworks a link in C is potentially}\\ \text{crucial for becoming a winning subnetwork}\end{array}} \\
&= \frac{\text{\# \textit{all} link-swings within a component}}{\text{\# of potential link-swings in the component}} \\
&= \sum_{\lambda \in g|_C} \frac{\text{\# of swings for } \lambda}{\text{\# of potential swings for } \lambda \text{ in } g|_C} \\
&= \frac{1}{2^{|g|_C|-1}} \sum_{\lambda \in g|_C} \text{\# of swings for } \lambda.
\end{aligned}$$

For general games, swings are generalized by marginal contributions, hence, counting link-swings can be generalized by marginal contributions in the arc game. This leads to the following axiom:

Axiom 5.3 (Component Link Banzhaf Efficiency (**CLBE**))**.** *An allocation rule for network structures* Y *satisfies* Component Link Banzhaf Efficiency **CLBE** *if we have for all* $C \in \mathcal{C}(N, g)$*:*

$$\sum_{i \in C} Y_i(N, v, g) = \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C).$$

CLBE could be interpreted as follows: the worth of a component is equal to the relative *link*-power of this component, that is, relative power measured by the links (contributions of links relative to their potential contributions). For connected networks, this is in the same spirit as *Banzhaf Efficiency*, the crucial axiom of the characterization of the Banzhaf value, just with respect to link contributions.

Theorem 5.1. *The Banzhaf Position value is uniquely determined by* ***A****,* ***DEG****,* ***SLP*** *and* ***CLBE*** *for all cycle-free networks.*

Proof. EXISTENCE: **A** is clear and **DEG** has been shown in Lemma 5.1.
SLP: Let λ be superfluous in game (N, v). By the Banzhaf value satisfying the Nullplayer axiom, we then have $Ba_\lambda(g, v^N) = 0$. For $\nu \neq \lambda \in g$ we have

$$\begin{aligned}
Ba_\nu(g, v^N) &= \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \nu} \left[v^N(g' \cup \nu) - v^N(g')\right] \\
&= \frac{1}{2^{|g|-1}} \left[\sum_{\substack{g' \subseteq g \setminus \nu \\ \lambda \in g'}} \underbrace{\left[v^N(g' \cup \nu) - v^N(g')\right]}_{= v^N((g' \setminus \lambda) \cup \nu) - v^N(g' \setminus \lambda)} \right. \\
&\qquad \left. + \sum_{g' \subseteq (g \setminus \lambda) \setminus \nu} \left[v^N(g' \cup \nu) - v^N(g')\right] \right] \\
&= \frac{2}{2^{|g|-1}} \sum_{g' \subseteq (g \setminus \lambda) \setminus \nu} \left[v^N(g' \cup \nu) - v^N(g')\right] \\
&= Ba_\nu(g \setminus \lambda, v^N)
\end{aligned}$$

and hence, $\pi^{Ba}(N, v, g) = \pi^{Ba}(N, v, g \setminus \lambda)$.
CLBE: Let $C \in \mathcal{C}(N, g)$.

$$\begin{aligned}
\sum_{i \in C} \pi_i^{Ba}(N, v, g) &= \sum_{i \in C} \sum_{\lambda \in g_i} \frac{1}{2} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \left(v^N(g' \cup \lambda) - v^N(g')\right) \\
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \left(v^N(g' \cup \lambda) - v^N(g')\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \underbrace{\left[v^N((g' \cap g|_C) \cup \lambda) - v^N(g' \cap g|_C)\right]}_{\substack{\text{for } \lambda \in g|_C \text{, marginal contributions are} \\ \text{not effected by connections outside } C}} \\
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \underbrace{\left[v^N(g'|C \cup \lambda) - v^N(g'|_C)\right]}_{\substack{\text{the same for all } g', g'' \subseteq g \setminus \lambda \\ \text{such that } g'|_C = g''|_C}} \\
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{\tilde{g} \subseteq (g|_C) \setminus \lambda} 2^{|g|_{N \setminus C}|} \left[v^N(\tilde{g} \cup \lambda) - v^N(\tilde{g})\right] \\
&\overset{|g| = |g|_{N\setminus C}| + |g|_C|}{=} \sum_{\lambda \in g|_C} \frac{1}{2^{|g|_C|-1}} \sum_{\tilde{g} \subseteq (g|_C) \setminus \lambda} \left[v^N(\tilde{g} \cup \lambda) - v^N(\tilde{g})\right] \\
&= \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C)
\end{aligned}
$$

UNIQUENESS[13]: Let Y satisfy **A**, **DEG**, **SLP** and **CLBE** and g be cycle free. By **A**, it is sufficient to show that

$$Y(N, \beta u_T, g) = \pi(N, \beta u_T, g)$$

for all $\beta \in \mathbb{R}$ and $T \in 2^N$ such that $|T| \geqslant 2$. Let such β, T be arbitrary but fixed.

CASE 1: $\nexists\, C \in \mathcal{G}(N, g)$ such that $T \subseteq C$.

That is, there exist $i, j \in T$ being unconnected in g and hence, $\beta u_T^N(g') = 0$ for all $g' \subseteq g$. Therefore, $Ba_\lambda(g, \beta u_T) = 0$ for all $\lambda \in g$ and

$$\pi_i^{Ba}(N, \beta u_T, g) = 0 \text{ for all } i \in N.$$

On the other hand, if $\beta u_T(g') = 0$ for all $g' \subseteq g$, every $\lambda \in g$ is superfluous, and hence, by **SLP**, we have $Y(N, \beta u_T, g) = Y(N, \beta u_T, g \setminus \lambda_1) = Y(N, \beta u_T, g \setminus \{\lambda_1, \lambda_2\}) = ... = Y(N, \beta u_T, \emptyset)$.

Trivially, the game $(N, \beta u_t, \emptyset)$ is link anonymous and hence, by **DEG**, there exists $\alpha \in \mathbb{R}$ such that

$$
\begin{aligned}
Y_i(N, \beta u_T, g) = Y_i(N, \beta u_t, \emptyset) = \alpha \cdot C_i^d(\emptyset) = 0 \quad \forall i \in N \\
\Rightarrow Y(N, \beta u_T, g) = \pi^{Ba}(N, \beta u_T, g)
\end{aligned}
$$

13 We follow the idea of the proof for the (Shapley) Position value of Borm et al. [1992].

CASE 2: $\exists\, C \in \mathcal{G}(N, g)$ such that $T \subseteq C$.
Consider the (unique) *connected hull* (cf. Owen, 1986) of T, $H(T)$, given by

$$H(T) := \bigcap \{S | T \subseteq S \subseteq C \text{ such that } g|_S \text{ is connected subgraph}\}$$

As g is cycle-free, $H(T)$ is the minimal set of nodes that are essential to connect T. Note that cycle-freeness is essential here: if there is more than one path connecting T, the intersection is empty on the disjoint parts of the connecting paths. We have

$$\beta u_T^N(g') = \begin{cases} \beta & \text{, if } g|_{H(T)} \subseteq g' \\ 0 & \text{, otherwise} \end{cases}$$

If $\lambda \notin g|_{H(T)}$, we have $Ba_\lambda(g, \beta u_T^N) = 0$. For $\lambda \in g|_{H(T)}$, we have

$$\begin{aligned} Ba_\lambda(g, \beta u_T^N) &= \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \left(\beta u_T^N(g' \cup \lambda) - \beta u_T^N(g')\right) \\ &\overset{\text{cf. existence CLBE}}{=} \frac{1}{2^{|g|_{H(T)}|-1}} \sum_{g' \subseteq g|_{H(T)} \setminus \lambda} \underbrace{\left(\beta u_T^N(g' \cup \lambda) - \beta u_T^N(g')\right)}_{= \begin{cases} \beta & \text{, if } g' = g|_{H(T)} \setminus \lambda \\ 0 & \text{, otherwise} \end{cases}} \\ &= \frac{\beta}{2^{|g|_{H(T)}|-1}} \end{aligned}$$

and therefore, it holds that

$$\begin{aligned} Ba_\lambda(g, \beta u_T^N) &= \begin{cases} \frac{\beta}{2^{|g|_{H(T)}|-1}} & \text{, if } \lambda \in g|_{H(T)} \\ 0 & \text{, otherwise} \end{cases} \\ &= Ba_\lambda(g|_C, \beta u_T^N) \\ &= Ba_\lambda(g|_{H(T)}, \beta u_T^N) \end{aligned}$$

$$\begin{aligned} \Rightarrow \pi_i^{Ba}(N, \beta u_T, g) &= \sum_{\lambda \in g_i \cap g|_{H(T)}} \frac{1}{2} \cdot \frac{\beta}{2^{|g|_{H(T)}|-1}} = \frac{|g_i|_{H(T)}| \cdot \beta}{2^{|g|_{H(T)}|}} \\ &= \frac{\beta}{2^{|g|_{H(T)}|}} \cdot C_i^d(g|_{H(T)}) \end{aligned}$$

On the other hand, all links $\lambda \notin g|_{H(T)}$ are superfluous in $(N, \beta u_T)$, hence, by **SLP**, $Y(N, \beta u_T, g) = Y(N, \beta u_T, g|_{H(T)})$.
The game $(N, \beta u_T, g|_{H(T)})$ is link anonymous (all links have the same number of swings, namely, one) with

$$f : \{0, 1, ..., |g|_{H(T)}|\} \longrightarrow \mathbb{R}, f(x) := \begin{cases} \beta & , \text{ if } x = |g|_{H(T)}| \\ 0 & , \text{ otherwise} \end{cases}$$

hence, by **DEG**, there exists $\alpha \in \mathbb{R}$ such that

$$Y_i(N, \beta u_T, g) \overset{\text{SLP}}{=} Y_i(N, \beta u_T, g|_{H(T)}) \overset{\text{DEG}}{=} \alpha \cdot C_i^d(g|_{H(T)}) \qquad (*)$$

It directly follows that

$$Y_i(N, \beta u_T, g) = 0 \quad \forall i \in N \setminus H(T) \qquad (**)$$

By (**) and **CLBE**, we have

$$\sum_{i \in H(T)} Y_i(N, \beta u_T, g) \overset{(**)}{=} \sum_{i \in C} Y_i(N, \beta u_T, g) \overset{\text{CLBE}}{=} \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, \beta u_T^N)$$
$$= \sum_{\lambda \in g|_{H(T)}} \frac{\beta}{2^{|g|_{H(T)}|-1}} = \frac{|g|_{H(T)}| \cdot \beta}{2^{|g|_{H(T)}|-1}}$$

On the other hand, by (*), we have

$$\sum_{i \in C} Y_i(N, \beta u_T, g) = \alpha \sum_{i \in C} C_i^d(g|_{H(T)}) = \alpha \cdot 2 \cdot |g|_{H(T)}|$$

Combining this, we get

$$\alpha \cdot 2 \cdot |g|_{H(T)}| = \frac{|g|_{H(T)}| \cdot \beta}{2^{|g|_{H(T)}|-1}} \Leftrightarrow \alpha = \frac{\beta}{2^{|g|_{H(T)}|}}$$

and hence

$$Y_i(N, \beta u_T, g) = \frac{\beta}{2^{|g|_{H(T)}|}} C_i^d(g|_{H(T)}) = \pi_i^{Ba}(N, \beta u_T, g)$$

which finishes the proof. □

Remark 5.7. *On cycle-free networks the following holds:*
1. *The Harsanyi-dividend representation of* π^{Ba} *is*

$$\pi_i^{Ba}(N, v, g) = \sum_{T \subseteq 2^N \setminus \{\emptyset\}} \frac{\lambda_T(v)}{2^{|g|_{H(T)}|}} C_i^d(g|_{H(T)}),$$

where $\lambda_T(v)$ *is the* Harsanyi dividend *of* v *corresponding to* T*. Hence, the Banzhaf Position value is given by the degree measures of the connected hulls of* v*'s basis representations, weighted accordingly.*
2. *The (multiplicative efficient) normalized Banzhaf Position value of any unanimity game is given by*

$$\overline{\pi_i^{Ba}}(N, \beta u_T, g) := \frac{\pi_i^{Ba}(N, \beta u_T, g)}{\sum_{j \in N} \pi_j^{Ba}(N, \beta u_T, g)} \cdot v(N) = \frac{|g_i|_{H(T)}|}{2 \cdot |g|_{H(T)}|} \cdot \beta$$

which coincides with the (Shapley) Position value $\pi_i^{(Sh)}(N, \beta u_T, g)$*. This is due to the fact, that on cycle-free networks, only the connected hull matters and there, all links are equally important.*

5.3.3 *General Axiomatization*

As mentioned before, there is no "Shapley-like" axiomatization of the (Shapley) Position value for general networks. For a general axiomatization of the Banzhaf Position value we make use of an axiom which, together with **CE**, characterizes the (Shapley) Position value for general networks (Slikker, 2005): An allocation rule for network structures Y satisfies *Balanced Link Contributions* **BLC** if we have for all $i, j \in N$, $i \neq j$, and $v \in \mathbb{V}_N^0$:

$$\sum_{\lambda \in g_j} [Y_i(N, v, g) - Y_i(N, v, g \setminus \lambda)] = \sum_{\lambda \in g_i} \left[Y_j(N, v, g) - Y_j(N, v, g \setminus \lambda)\right].$$

BLC states that the total threat of a player towards another player should be equal to the reverse total threat. Using **BLC**, we find the following general characterization of the Banzhaf Position value:

Theorem 5.2. *The Banzhaf Position value* π^{Ba} *is uniquely determined by **BLC** and **CLBE**.*

Proof. EXISTENCE: We have already shown that π^{Ba} satisfies **CLBE**. To see **BLC**, we follow Slikker [2005], who states that there exists a unique linear combination of link-unanimity games which represents any link game $v^N : 2^{|g|} \longrightarrow \mathbb{R}$: for any v^N exist $\beta_{g'}$, $g' \subseteq g$, such that

$$v^N(\tilde{g}) = \sum_{g' \subseteq g} \beta_{g'} u^N_{g'}(\tilde{g}).$$

For $(g, \beta_{g'} u^N_{g'})$ we get

$$Ba_\lambda(g, \beta_{g'} u^N_{g'}) = \begin{cases} \frac{\beta_{g'}}{2^{|g'|-1}} & \text{, if } \lambda \in g' \\ 0 & \text{, if } \lambda \notin g' \end{cases}$$

and using this, we have

$$\begin{aligned} \pi_i^{Ba}(N, v, g) &= \sum_{\lambda \in g_i} \frac{1}{2} Ba_\lambda(g, v^N) \\ &\overset{\text{Ba satisfies } \mathbf{A}}{=} \sum_{\lambda \in g_i} \frac{1}{2} \sum_{g' \subseteq g} Ba_\lambda(g, \beta_{g'} u^N_{g'}) \\ &= \sum_{g' \subseteq g} \sum_{\lambda \in g_i} \frac{1}{2} \underbrace{Ba_\lambda(g, \beta_{g'} u^N_{g'})}_{= \begin{cases} \frac{\beta_{g'}}{2^{|g'|-1}}, \text{ if } \lambda \in g' \cap g_i \\ 0, \text{ if } \lambda \notin g' \cap g_i \end{cases}} \\ &= \sum_{g' \subseteq g} \sum_{\lambda \in g'_i} \frac{\beta_{g'}}{2^{|g'|}} = \sum_{g' \subseteq g} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_{\lambda \in g_j} \left[\pi_i^{Ba}(N, v, g) - \pi_i^{Ba}(N, v, g \setminus \lambda)\right] \\ &\quad = \sum_{\lambda \in g_j} \left(\sum_{g' \subseteq g} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} - \sum_{g' \subseteq g \setminus \lambda} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} \right) \\ &\quad = \sum_{\lambda \in g_j} \sum_{\substack{g' \subseteq g \\ \lambda \in g'}} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} = \sum_{g' \subseteq g} \sum_{\lambda \in g'_j} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} \\ &\quad = \sum_{g' \subseteq g} \beta_{g'} \frac{|g'_j| \cdot |g'_i|}{2^{|g'|}} \\ &\quad \overset{\text{backwards}}{=} \sum_{\lambda \in g_i} \left[\pi_j^{Ba}(N, v, g) - \pi_j^{Ba}(N, v, g \setminus \lambda)\right] \end{aligned}$$

for all $i \in N$.

UNIQUENESS: Suppose Y, W being two allocation rules satisfying **BLC** and **CLBE**. We proceed by induction over $|g|$.

Induction basis [IB]: For $|g| = 0$, that is, $g = \{\emptyset\}$, we have that $\mathcal{C}(N, g) = \{i\}_{i \in N}$ and hence, by **CLBE**, we have

$$\begin{aligned} Y_i(N, v, g) &= \sum_{i \in C} Y_i(N, v, g) = \sum_{\lambda \in \{\emptyset\}} Ba_\lambda(g, v^N) = 0 \\ &= \sum_{i \in C} W_i(N, v, g) = W_i(N, v, g). \end{aligned}$$

Now suppose that $Y(N, v, g) = W(N, v, g)$ for all g such that $|g| = k$, $k \geqslant 0$ (induction hypothesis [IH]).

Consider g such that $|g| = k + 1$. As $k + 1 \geqslant 1$, there exists $i, j \in N$, $i \neq j$ such that $j \in C_i(N, g)$ (for all l with $|C_l(N, g)| = 1$, we have $Y_l(N, v, g) = W_l(N, v, g)$ by **CLBE**). By **BLC**, we have for all $i, j \in C_i(N, g)$:

$$\begin{aligned} &\sum_{\lambda \in g_j} Y_i(N, v, g) - \sum_{\lambda \in g_i} Y_j(N, v, g) \\ &\qquad \overset{\text{BLC}}{=} \sum_{\lambda \in g_j} Y_i(N, v, g \setminus \lambda) - \sum_{\lambda \in g_i} Y_j(N, v, g \setminus \lambda) \\ &\qquad \overset{\text{[IH]}}{=} \sum_{\lambda \in g_j} W_i(N, v, g \setminus \lambda) - \sum_{\lambda \in g_i} W_j(N, v, g \setminus \lambda) \\ &\qquad \overset{\text{BLC}}{=} \sum_{\lambda \in g_j} W_i(N, v, g) - \sum_{\lambda \in g_i} W_j(N, v, g) \end{aligned}$$

and hence

$$|g_j| Y_i(N, v, g) - |g_i| Y_j(N, v, g) = |g_j| W_i(N, v, g) - |g_i| W_j(N, v, g) \quad (6)$$

Summing up (6) over all $j \in C$ yields:

$$\begin{aligned} &\sum_{j \in C} \left[|g_j| Y_i(N, v, g) - |g_i| Y_j(N, v, g) \right] \\ &= \sum_{j \in C} \left[|g_j| W_i(N, v, g) - |g_i| W_j(N, v, g) \right] \\ \Leftrightarrow & Y_i(N, v, g) \underbrace{\sum_{j \in C} |g_j|}_{\substack{:= A(j) > 0 \\ \text{by } |C_i| \geqslant 1}} - |g_i| \sum_{j \in C} Y_j(N, v, g) \end{aligned}$$

$$= W_i(N,v,g) \underbrace{\sum_{j \in C} |g_j|}_{\substack{:=A(j)>0 \\ \text{by } |C_i| \geqslant 1}} - |g_i| \sum_{j \in C} W_j(N,v,g)$$

$$\overset{\text{CLBE}}{\Rightarrow} A(j) \cdot Y_i(N,v,g) - |g_i| \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C)$$

$$= A(j) \cdot W_i(N,v,g) - |g_i| \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C)$$

$$\Leftrightarrow Y_i(N,v,g) = W_i(N,v,g) \quad \forall i \in N$$

which finishes the proof. □

Remark 5.8 (Shapley vs. Banzhaf Positions)**.** *The main difference between the (Shapley) Position value and the Banzhaf Position value is how a certain position is actually valued. While the Shapley approach weights swings with respect to their "strength", the Banzhaf approach weights swings equally. The use of one or the other approach depends on the application domain and whether "weighted voting" is of interest: on the one hand, if differences in the strength of outcome matter, different weights might be more appropriate while on the other hand, "weighted voting" can lead to unreasonable allocations and is sometimes not applicable at all, especially in political applications with incompatibilities as in the political example in the applications section.*

5.4 CONSEQUENCES FOR OUTSIDE-OPTION-SENSITIVITY, PROPERTIES AND EXTENSIONS

In this section we will discuss properties of the Position values and and provide an extension of our approach to weighted networks and briefly discuss the eigenvector approach.

However uniquely characterizing the Position values on cycle-free networks only, all axioms but **CLBE** in Theorem 5.1 are generally satisfied by both Position values. Let us take a more detailed look on the crucial axioms in this axiomatization (remember that **A** is usually satisfied by allocation rules): Degree Property **DEG**: A game (N, v, g) is called link anonymous if $\exists\, f : \{0, 1, ..., |g|\} \longrightarrow \mathbb{R}$ such that $v^N(g') = f(|g'|)$ for all $g' \subseteq g$. An allocation rule for network structures satisfies the **DEG** if for all link anonymous games (N, v, g) there exists $\alpha \in \mathbb{R}$ such that

$$Y(N, v, g) = \alpha \cdot C^d(g).$$

Remark 5.9 (Degree Property and Outside-Option-Sensitivity)**.** *Note that link anonymity of a TU-game with a network structure (N, v, g) does not imply link anonymity of the corresponding game using the lower outside option graph $(N, v, g(i))$. Now consider a link anonymous (N, v, g) such that g consists of one connected component and singletons and suppose there exists an outside option for a player within the connected component. As **DEG** implies proportionality to the degree measure of g which will not change if this outside option is neutralized, **DEG** stands in contradiction to the outside-option-sensitivity axioms **WOOS** and **OOS**.*

Superfluous Link Property **SLP**: A link λ is called superfluous if $v^N(g' \cup \lambda) = v^N(g')$ for all $g' \subseteq g$, that is, if λ is a Nullplayer in (g, v^N). An allocation rule for network structures Y satisfies the **SLP** if $Y(n, v, g) = Y(N, v, g \setminus \lambda)$ for all superfluous links λ.

Remark 5.10 (Superfluous Link Property and Outside-Option-Sensitivity)**.** *If a link is superfluous in TU-game (N, v, g), its presence never creates any surplus. However, such a superfluous link might be involved in an outside option, that is, it might create a surplus in combination with unmaterialized links of the lower outside option graph. Hence, a link that is superfluous in (N, v, g) does not have to be superfluous in the corresponding $(N, v, g(i))$. Furthermore, deleting a superfluous link does not affect payoffs within a component but, as deleting such a link possibly changes connected compomponents, the resulting lower outside option graph differs. In other words, changing connected components might create outside options which have not been present before. An outside-option-sensitive allocation rule is however affected by* all *outside options, hence, **SLP** stands in contradiction to **WOOS** and **OOS**.*

Note that the two previous remarks already show us that the Position values cannot be outside-option-sensitive. This will be further analyzed in Chapter 6.

We will now briefly discuss computational complexity for calculating the Banzhaf and Shapley weights needed for the Position values. Taking the unanimity game of the grand coalition

$$u_N(K) = \begin{cases} 1 & \text{, if } K = N \\ 0 & \text{, otherwise} \end{cases}$$

as a basis, the calculation is mainly about checking whether a subnetwork is connected or not.

Theorem 5.3 (Computational Complexity of Weights)**.** *For $v = u_N$, computational complexity is at most*

$$|g| \cdot \sum_{k=|T|-1}^{|g|-1} \binom{|g|-1}{k}$$

connectivity checks for the Banzhaf value and for the Shapley value it is the same connectivity checks, just that for every "yes"-labeled item, one number has to be stored.

Proof. In general, the computational effort for calculating the Shapley or the Banzhaf value is high (at least P-complete) as it increases disproportionately with the number of nodes in the network of interest. But, as u_T^N is a simple game, the Shapley and the Banzhaf value can be calculated via swings: $g' \subseteq g \setminus \{\lambda\}$ is *swing* for $\lambda \in g$ iff $u_T^N(g' \cup \lambda) - u_T^N(g') = 1$. For the Shapley value, swings have to be weighted according to their cardinality, for the Banzhaf value, one only has to count the total number of swings and devide this by the number of potential swings, namely, $2^{|g|-1}$.

The effort of identifying swings in u_T^N further decreases as g' is a swing for $\lambda = ij$ iff $T \subseteq G \in \mathcal{G}(N, g' \cup \{ij\})$ while $\nexists G \in \mathcal{G}(N, g')$ with $T \subseteq G$. That is, iff $g' \cup \{ij\}$ connects T while g' does not. For all $\lambda \in g$, there are $2^{|g|-1}$ subnetworks of $g \setminus \{\lambda\}$. Of interest are those of the $2^{|g|-1}$ subnetworks, that do not connect the nodes in T but the nodes will be connected when adding λ, that is, we can restrict the analysis of the $2^{|g|-1}$ subnetworks to those with at least $|T| - 1$ links. The number of subnetworks that have to be checked is at most:

$$\sum_{k=0}^{|g|-1} \binom{|g|-1}{k} - \sum_{k=0}^{|T|-2} \binom{|g|-1}{k}$$

$$= 2^{|g|-1} - \sum_{k=0}^{|T|-2} \binom{|g|-1}{k} = \sum_{k=|T|-1}^{|g|-1} \binom{|g|-1}{k}$$

It is well known that, unfortunately, there is no closed formula for the number of all subsets of a certain set with cardinality greater than some border. Note that the number of subnetworks of interest further decreases as only those that include all nodes in T are of interest.

We end up with the fact that calculating the Shapley and the Banzhaf value are counting problems: For the Banzhaf value, one has to count for any $\lambda \in g$ how many of the subnetworks of interest do not connect T but will when adding λ. For the Shapley value, also the cardinality of these subnetworks is of interest. Computational complexity is at most checking

$$|g| \sum_{k=|T|-1}^{|g|-1} \binom{|g|-1}{k}$$

networks, where a connectivity check (g' does not connect T but $g' \cup \lambda$ does: yes/no) is sufficient; for the Shapley value, one further has to save the information about the cardinality of those g' with label "yes". □

Remark 5.11 (NP-Hardness). *Connectivity checks are made by depth-first search algorithms. It is well known, that worst case performance of a depth-first-search is* $O(|N|)$*, that is, not NP-hard. However, the number of connectivity checks needed might exceed polynomial order.*

Now we will suggest an extension of our approach for weighted networks. Let $g = g(N, w)$ be a weighted network. In this case, we consider the corresponding binary network g_b of g and combine the new weights as defined for the binary transformation with the original weights. For this, we need to match the size of the two different weights. As numerical size of indices or weights does not change ranks and relative distances, we normalize w as follows:

$$\overline{w}_{ij} := \frac{w_{ij}}{\sum_{\lambda \in g} w_\lambda} \cdot \sum_{\lambda \in g} Y_\lambda(g_b, \nu^N), \qquad Y = Sh, Ba$$

Note that for $Y = Sh$, this is normalization to $v(N)$ as due to efficiency of the Shapley value we have

$$\sum_{ij \in g} Sh_{ij}(g_b, v^N) = v^N(g_b) = v(N)$$

where the latter follows from the fact that we assumed g to be connected.

Transformation 5.2. *Define the new weighted network* $g(N, \tilde{w}(v, \alpha, Y))$ *(*$Y = Sh$ *or* $Y = Ba$*) with*

$$\tilde{w}_{ij}(v, \alpha, Sh) := \alpha \overline{w}_{ij} + (1 - \alpha) Sh_{ij}(g_b, v^N) \text{ or}$$
$$\tilde{w}_{ij}(v, \alpha, Ba) := \alpha \overline{w}_{ij} + (1 - \alpha) Ba_{ij}(g_b, v^N), \text{ respectively}$$

where $\alpha \in [0, 1]$*. We call* α *the* emphasis parameter *regulating the emphasis of social/economic/political weights and importance for cohesion.*

Note that

$$\sum_{\lambda \in g} \tilde{w}_\lambda(v, \alpha, Y) = \alpha \sum_{\lambda \in g} \overline{w}_\lambda + (1 - \alpha) \sum_{\lambda \in g} Y_\lambda(g_b, v^N) = \sum_{\lambda \in g} Y_\lambda(g_b, v^N)$$

Remark 5.12 (α for binary networks)**.** *Note that the definition of* $\tilde{w}$ *in the case of binary networks coincides with the definition for weighted networks for* $\alpha = 0$*.*

Now, Shapley/-Banzhaf-Centralities can be defined as in the binary approach, just using $\tilde{w}_{ij}(v, \alpha, Y)$ instead of $\tilde{w}_{ij}(v, Y)$.

Remark 5.13 (Relation to Freeman's Centrality measures)**.** *For* $\alpha = 1$*, the normalized Shapley/Banzhaf-Centrality-values coincide with Freeman's centrality measures.*

Note that Freeman's closeness and betweenness measures bear some problems in application to weighted networks as the extension of shortest paths by *fastest or weighted shortest* paths (see appendix) is problematic as this concept has been originally designed for networks in which weights represent costs. In cases where a "good" link has a high weight instead of low costs, Brandes [2001] and Newman [2001] invert weights to interpret them as costs. The potential problem is obvious as the sum of fractions is not equal to the sum of the denominators. However, this problem does not arise for Bonacich's eigenvector approach. Therefore, it might be of interest, at least for the application section, to consider this approach as well.

Definition 5.8 (Shapley/Banzhaf-Eigenvector-value)**.** *For every network* g*, the Shapley-Eigenvector-value* CEV^{Sh} *and the Banzhaf-Eigenvector-value* CEV^{Ba} *are given by*

$$CEV_i^Y(N, v, g) := C_i^{EV}\left(g(N, \tilde{w}(v, Y))\right)$$
$$\textit{where } Y = Sh \textit{ or } Y = Ba, \textit{ respectively.}$$

That is, CEV^Y *is given by the unique nonnegative solution of*

$$(\tilde{w}(v, Y)_{ij})_{ij} \cdot CEV^Y = \lambda \cdot CEV^Y.$$

Derivation and further details can be found in the appendix.

5.5 APPLICATIONS

We provided generally holding axiomatizations of the Banzhaf Position value as a cooperative allocation rule. The Banzhaf Position value can hence be applied for any sort of coalition function, that is, any sort of economic, social or political situation. In this section we will provide examples for the application as a power index and as a centrality measure.

5.5.1 *The Banzhaf Position value as a Power Index*

As an application for political networks and the use of our new allocation rule as a power index, let us consider the state parliament elections (*Bürgerschaftswahl*) in Hamburg, Germany in 2001. After the election, there were five parties obtaining seats in the parliament[14], namely the Social Democratic Party "SPD", the Christian Democratic Union "CDU", the Conservative Law and Order Party "Schill", the Green/Alternative Party "Grüne" and the Free Democratic Party "FDP". The distribution of seats was according to Table 12. To build the government, a coalition needs at least 50% of the seats[15]. In the end, the

14 Due to regulations, parties obtaining a vote share less than 5% are not going to be in the parliament.
15 Parties are assumed to vote en bloc

Table 12: Results of the State Parliament Elections in Hamburg, 2001

party	SPD	CDU	Schill	Grüne	FDP	$\sum$
seats	46	33	25	11	6	121
seat share	0.38	0.27	0.21	0.09	0.05	1

source: Statistical Office of Hamburg and Schleswig-Holstein

government was built by the coalition {CDU, Schill, FDP}[16]. We are now interested in the question if this outcome could have been forecasted by the use of centrality measures.

For notational reasons, let us denote the parties by $N = \{1, 2, 3, 4, 5\}$ (i. e., SPD is player 1, CDU is player 2,...). The situation can be modeled by a simple weighted voting game: assign to every player $i \in N$ the corresponding seat share s_i. Then, the coalition function is given by

$$v(K) := \begin{cases} 1 & \text{, if } \sum\limits_{i \in K} s_i \geqslant 0.5 \\ 0 & \text{, otherwise} \end{cases} .$$

Minimal winning coalitions are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$. Applying the Shapley value or the normalized Banzhaf value leads to a distribution of power and resulting coalitional power displayed in Table 13.

In fact, a coalition between "CDU" and "Grüne" as well as between "Schill" and "Grüne" was excluded by the parties due to ideological/political incompatibilities. This means, that (minimal winning) coalition $\{2, 3, 4\}$ would not materialize as well as all $K \subseteq N$ such that $\{2, 4\} \subseteq K$ or $\{3, 4\} \subseteq K$ would not be winning coalitions. To account for these incompatibilities, we restrict the space of potential swings[17]: any set including $\{2, 3\}$ or $\{3, 4\}$ is excluded as potential swing for $i \in \{1, 5\}$, any set including $\{4\}$ is excluded for $i \in \{2, 3\}$ and any set including $\{2\}$ or $\{3\}$ is excluded for $i = 4$. Due to this restriction, Shapley-based approaches are not suitable anymore as the weights of the marginal contributions would be unappropriate and not be relatively balanced

16 For completeness one should note that this coalition broke 2 years later due to upcoming personal issues between the leaders of "CDU" and "Schill". However, these issues have not been known after the elections and hence, can not be taken into account for forecasting issues.

17 Here, $K \subseteq N \setminus \{i\}$ is a swing for i iff $v(K \cup \{i\}) - v(K) = 1$.

Table 13: Gametheoretic Approaches, Unrestricted Case

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
Sh_i	40.00	23.33	23.33	6.67	6.67
$\overline{Ba}_i$	38.46	23.08	23.08	7.69	7.69
Resulting Coalitional Power					
coalition	$\{1,2\}$	$\{1,3\}$	$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$
$\sum Sh_i$	**63.33**	**63.33**	53.34	53.33	53.33
$\sum \overline{Ba}_i$	**61.54**	**61.54**	53.84	53.85	53.85

(Highest coalitional power is bolt face.)

anymore, hence, we will only use the Banzhaf-based approaches from now on[18]. Distribution of power and resulting coalitional power for restricted case is displayed in Table 14[19].

Table 14: Gametheoretic Approach, Restricted Case

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
$\overline{Ba}_i$	40.00	20.00	20.00	6.67	13.33
Resulting Coalitional Power					
coalition	$\{1,2\}$	$\{1,3\}$	$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$
$\sum \overline{Ba}_i$	**60.00**	**60.00**	**60.00**	-	53.33

(Highest coalitional power is bolt face.)

As we see, the coalitions obtaining the highest coalitional power are $\{1,2\}$ and $\{1,3\}$ in the unrestricted case (for both the Shapley and the normalized Banzhaf value) and $\{1,2\}$, $\{1,3\}$ and $\{1,4,5\}$ in the restricted case, which does not explain the resulting coalition $\{2,3,5\}$.

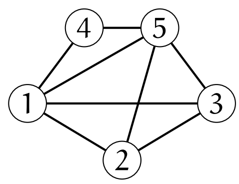
Now, take a look at the corresponding political network where we could interpret a link as a potential coalitional negotiation. We could argue that relative political power of a party should depend on the

18 Note that for voting games, one usually uses the Banzhaf approach to avoid the different weights of swings.

19 For further details consult the appendix, Table 24.

seats obtained *and* the position within the political network. Taking into account the incompatibilities, the network could be modeled as presented by Figure 6.

Figure 6: Political Network



Applying the classical centrality measures (which are independent on the weighted voting game) is not appropriate for the use of a power index as in this case, there is no difference between parties 1 and 5 while the relative power of a party should depend on the seats obtained.[20] The Banzhaf Position value accounts for both the seat share and the position within the political network. We restrict the set of potential swings for a connection containing 2 or 3 to the set of connections not containing 4 and for a connection containing 4 to the set of connections not containing 2 or 3.[21] Find the distribution of individual and coalitional power due to the (multiplicative efficient) normalized Banzhaf Position value and Banzhaf-Eigenvector-value displayed in Table 15.[22]

We see that, taking into account both centrality in the political network and relative importance in the voting game, the coalition $\{2,3,5\}$ is uniquely selected with respect to highest coalitional power. Hence, both our new approaches could have been considered for forecasting issues leading to the coalition which was actually built. Note that the other approaches either do not forecast the actual coalition or are not suitable as the seat share was not taken into account.

20 However, only Eigenvector centrality uniquely selects coalition $\{2,3,5\}$ obtaining the highest coalitional power. For completeness, find the values for the classic approaches in Table 25, appendix.

21 One could argue that this is too restrictive and connections containing 4 can still be part of a swing for a connection containing 2 or 3 as long as no connected component resulting in excluded coalitions is build (and correspondingly for swings for connections containing 2 or 3). Note that in this less restrictive case the order of highest individual and coalitional power does not change. For completeness, find details for this case in the appendix, Table 24 and Table 26.

22 For further details consult the appendix, Table 24.

Table 15: Banzhaf Position Power

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
$\overline{\pi^{Ba}}_i$	**30.00**	20.00	20.00	6,67	23.33
$\overline{CEV^{Ba}}_i$	**27.04**	21.81	21.81	7.28	22.05
Resulting Coalitional Power					
coalition	$\{1,2\}$	$\{1,3\}$	$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$
$\sum \overline{\pi^{Ba}}_i$	50.00	50.00	60.00	-	**63.33**
$\sum \overline{CEV^{Ba}}_i$	48.85	48.85	56.37	-	**65.67**

(Highest individual and coalitional power are bolt face.)

5.5.2 *The Banzhaf Position value as a Centrality Measure*

Recall that our anaylsis was motivated by the analysis of centrality measures. So far, we did not explicitly apply the Banzhaf Position value for this issue. For this, we first give examples of coalition functions for which the Banzhaf Position value could be used as a centrality measure. In order to calculate relative importance for cohesion of essential nodes, a simple and convincing coalition function would be the *unanimity game*

$$u_T(K) = \begin{cases} 1 & \text{, if } T \subseteq K \\ 0 & \text{, otherwise} \end{cases}$$

where $T \subseteq N$ is the set of essential nodes that one likes to stay connected.

Lemma 5.2 (Proportionality)**.** *Let* g *be a cycle-free connected binary network (i. e., a spanning tree) and consider the unanimity game* βu_T*,* $T \subseteq N$*,* $|T| > 1$*. Then, we have for* $Y = Sh$ *or* $Y = Ba$

$$\begin{aligned} \overline{CD}_i^Y(g, \beta u_T) &= \overline{C^d}_i(g|_{H(T)}), \\ \overline{CC}_i^Y(g, \beta u_T) &= \overline{C^c}_i(g|_{H(T)}), \\ \overline{CB}_i^Y(g, \beta u_T) &= \overline{C^b}_i(g|_{H(T)}) \text{ and} \\ \overline{CEV}_i^Y(g, \beta u_T) &= \overline{C^{EV}}_i(g|_{H(T)}) \end{aligned}$$

where $H(T)$ *is the connected hull of* T*, that is, set of nodes that are essential to connect* T.

Proof. Let g be a minimal connected binary network and $v = u_T$. Then, following the uniqueness proof of Theorem 5.1 (and similar for the (Shapley) Position value), we have

$$Y_\lambda(g, \beta u_T^N) = \begin{cases} \text{constant} & , \text{for } \lambda \in g|_{H(T)} \\ 0 & , \text{for } \lambda \notin g|_{H(T)} \end{cases}$$

for $Y = Sh$ or $Y = Ba$. Hence, in network $g_{H(T)}$, all weights $\tilde{w}_\lambda$ are constant. As the scale of weights does not matter for ranks and relative distances of the indices, weights can be rescaled to 1 which leads the original (binary) network, restricted on $H(T)$, that is, $g_{H(T)}$. This proves proportionality to Freeman's centrality measures as well as Eigenvector centrality. □

The unanimity game is simple in calculations and a natural and intuitive way to define a game accounting for cohesion of a *specific* set of nodes. Michalak et al. [2013] discuss other possibilities for the coalition function which are more general as they do not rely on a specific set of nodes; here, the number of nodes that are reachable by path from a certain coalition is counted:

$$v(K) = \#\text{ of nodes in } K \text{ and those (directly) connected to } K$$

where one could either analyze direct connections or paths with at most k steps.

In order to match this value function with our framework, we have to zero-normalize the function and transform the weights correspondingly. This can be done for any coalition function by the following transformation:

Transformation 5.3. *For any coalition function* $v \in \mathbb{V}_N$*, proceed as follows*

STEP 1: *Zero-normalization:*

$$v_0(K) := v(K) - \sum_{i \in N} v(\{i\}) \cdot u_{\{i\}}(K)$$

STEP 2: *Transformation of weights: Set*

$$Y_{ij}(g_b, v^N) := Y_{ij}(g_b, v_0^N) + v^N(\{ij\}), \textit{ where } Y = Sh, Ba.$$

Following the idea of reachability, we define another possible cohesion game which directly accounts for the network structure:

Definition 5.9 (Cohesion Game)**.** *Given a network* $g \in \mathbb{G}_N^b$*, we define the corresponding* cohesion game $c : \mathbb{G}_N \longrightarrow \mathbb{R}$ *as follows:*

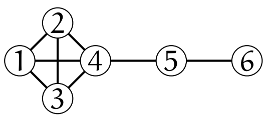
$$c(g, K) := \sum_{i \in K} c_i(g) \, \forall K \subseteq N$$

$$\textit{where } c_i(g) := \# \textit{ of nodes reachable from } i \textit{ by path in } g$$

Note, that we do not need a zero-normalization for the cohesion game (as $c(\emptyset) = 0$) or a transformation via the link-game as it directly works on networks (that is, $c \in \mathbb{V}_{\mathbb{G}_N}$).

Remark 5.14. *Note that the unanimity games form a basis of* $\mathbb{V}_N$ *and both the Shapley value and the Banzhaf value are linear in the coalition function, hence, we can express any* $v \in \mathbb{V}_N$ *by unanimity games. Therefore, we will focus on unanimity games.*

We will now discuss an example on the performance of our new approach used as centrality measures. For comparative reasons we do not restrict ourselves to the Banzhaf Position value and will discuss the Position and Eigenvector approach using both the Banzhaf and the Shapley value.
Let $N = \{1, 2, 3, 4, 5, 6\}$ and consider the binary network:
$g = \{12, 13, 14, 23, 24, 34, 45, 56\}$



Let $v = u_N$, that is, we are interested in connectedness of the whole network. Weights are displayed in Table 16.
Results for the degree measure and Eigenvector centrality and the corresponding Shapley/Banzhaf-Centralities are given in Table 17.
This example does not only show numerical differences, also the ranking between nodes changes. Consider the top 2 nodes: in contrast to the Degree or Eigenvector measure ($\alpha = 1$), node 5 is under the top 2 nodes for the Shapley/Banzhaf-Centralities ($\alpha = 0$) and even the

Table 16: Weights

λ	12, 13, 14, 23, 24, 34	45, 56
Sh_λ	$^{17}/_{420}$	$^{53}/_{140}$
$\overline{Ba}_\lambda$	$^{10}/_{136}$	$^{38}/_{136}$

Table 17: Classical Centrality Measures and Shapley/Banzhaf-Centralities (in share of 100%)

Y		1	2	3	4	5	6
Degree and Shapley/Banzhaf-Degree							
$\overline{C}^d$	$(\alpha = 1)$	**18.75**	**18.75**	**18.75**	**25.00***	12.50	6.25
$\overline{\pi}^{Sh}$	$(\alpha = 0)$	6.07	6.07	6.07	**25.00**	**37.86***	18.93
$\overline{\pi}^{Ba}$	$(\alpha = 0)$	11.03	11.03	11.03	**25.00**	**27.94***	13.97
Eigenvector and Shapley/Banzhaf-Eigenvector							
$\overline{C}^{EV}$	$(\alpha = 1)$	**21.65**	**21.65**	**21.65**	**23.74***	8.56	2.76
$\overline{CEV}^{Sh}$	$(\alpha = 0)$	2.43	2.43	2.43	**27.48**	**38,28***	26.93
$\overline{CEV}^{Ba}$	$(\alpha = 0)$	7.04	7.04	7.04	**25.38**	**31.87***	21.61

Top-2-nodes are bolt face,* identifies Top-node

top node changes. From an interpretative point of view, it is reasonable that nodes 4 and 5 should obtain the highest value in terms of centrality and connectedness as node 4 is the most central while 5 is essential for connectedness (the only node that connects 6 to the network).

One could argue that node 6 and node 1 (or 2 or 3) should obtain the same payoff as there is no structural reason (as connection node versus boundary node) to treat them differently. This can be regulated by the emphasis parameter α. If α decreases, one could argue that, as cohesion emphasis $(1 - \alpha)$ increases, node 6 should obtain a higher payoff than 1 (or 2 or 3) due to the "exclusiveness" of node 6: there is only one connection to node 6, hence an exclusive connection which could be seen as more important than other connections. If emphasis for centrality increases, 1 (or 2 or 3) should obtain a higher payoff than 6 due to a higher centrality in terms of connections (or original weights for weighted graphs).

Remark 5.15 (Moderate Relative Distances and the Banzhaf Approach). *Even though not affecting ranks of top-key-nodes, notice the differences in relative distances between using the Shapley- or the Banzhaf-based approach (especially for the Position values): Due to the weighted-voting approach, nodes 1, 2 and 3 obtain very low payoffs while nodes 5 and 6 obtain very high payoffs for the Shapley-based measures. One could argue that the relative distances due to the weighted-voting in the Shapley-based measures are disproportionately high / out-of-scale. Here we see that the Banzhaf Position approach might be seen as more moderate.*

Remark 5.16 (Gametheoretic Approaches). *Gómez et al. [2003] and Suri and Narahari [2008] suggest to identify top nodes by computing the Myerson value of the nodes (or its difference to the Shapley value) while no specific game is suggested. Considering the game* u_N *and our example, both the Shapley value and the Myerson value assign an equal share of* $1/6$ *to every node. This stands in contradiction to centrality (4 and 5 are more central than 1, 2 and 3 and all more central than 6) and furthermore, the difference between the connections 1-4 and 5-6 (exclusiveness) is ignored. One can argue that the drawbacks arise due to the too simple form of the unanimity game we use. However, independently of the characteristic function, the concepts suggested above only analyze failure of a whole node with all its connections at once.*

5.6 CONCLUSION

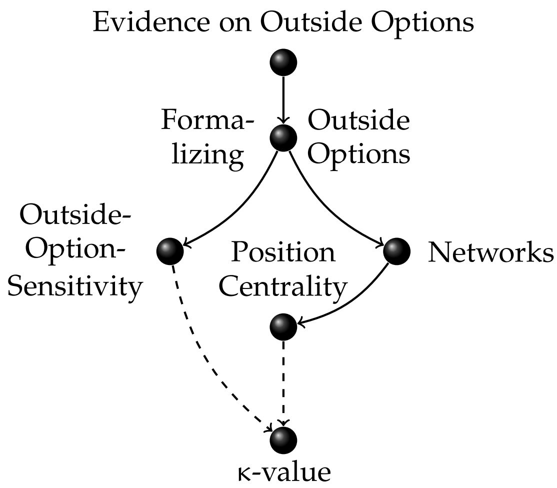
We introduced a cooperative allocation rule for network structures, the Banzhaf Position value, accounting for the importance for connectedness of the network and relative power of connections using cooperative game theory. This allocation rule could be seen as a centrality measure or power index. In contrast to existing (cooperative) game theoretic approaches on centrality or power indices, we analyzed consequenses of connection failures rather than failures of whole nodes. This makes our approach suitable in applications in for example energy networks or political networks: oil pipelines can break (connection failure) without a breakdown of the whole gas province (node failure), a bilateral trading agreement can be broken without a whole country leaving the trading union or, as in the political example we used, political parties can have bilateral incompatibilities without leaving the whole political spectrum.

We provided axiomatic characterizations of the Banzhaf Position value for both cycle-free and general (binary) networks. Furthermore, we suggested an extension to weighted networks or the eigenvector approach. Applying our approach to the state parliament elections in Hamburg, Germany 2001, we showed that, in contrast to existing power indices, the Banzhaf Position value used as a power index is able to forecast the actual government formation. Using our approach as a centrality measure, we discussed an example which emphasized the differences to classic centrality measures and the identification of top key nodes due to our approach might be seen as more plausible if one is interested in connectedness of a network, that is, exclusiveness of links or nodes for connectedness. These applications shed some more light on the difference between the Shapley and the Banzhaf approach and could be argued to support that "weighted voting (still) doesn't work".

For the examples in the application section we have seen that our extension using the eigenvector approach performed comparably well to the Banzhaf Position value. Therefore, for further research, a deeper analysis of this approach seems worthwile. Moreover, further investigation of the extension for weighted networks and an application for weighted networks might be of interest. Beside these possibilities, another issue occured within our analysis of the axiomatic characterization for cycle-free networks: our "old friend" outside-option-sensitivity. Remember that we focused on connected networks throughout this chapter, that is, situations where outside options are absent. However, both Position values are applicable for general networks and the discussion in Remark 5.9 and Remark 5.10 showed that the link-based Position values cannot be outside-option-sensitive. This leads to the investigation in the next chapter.

6

OUTSIDE-OPTION-SENSITIVE ALLOCATION RULES FOR NETWORKS



6.1 INTRODUCTION

In an economic or social situation where agents have to group in order to achieve common goals, how should we allocate the worth arising from the coalition formation among the agents? There are various frameworks and various allocation rules suggested in literature, each suitable for one or another application domain, dependent on the framework itself and the characterizing axioms. In this chapter we suggest a new allocation rule for network structures which, in contrast to existing allocation rules, accounts for both outside options and the role of an agent within the network.

We analyze situations where coalition formation is restricted by an undirected graph, that is, the economic or social structure is described by a network which captures the (bilateral) relations between agents and agents can only form a coalition if they are connected in such an interaction network via path (directly or indirectly). This approach was introduced by Myerson [1977] (for applications and further approaches on communication structures see for example van den Brink et al., 2007 and 2011). We use undirected networks to describe the relations between the agents while it would also be possible to use directed networks (i. e., interaction has a fixed direction) as analyzed by González-Arangüena et al. [2008]. They use generalized characteristic functions as introduced by Nowak and Radzik [1994a], that is, games where the worth of a coalition depends on the order in which agents enter the coalition. In our approach, interaction is not directed and the order of entry does not change the worth of the coalition.

There are various examples in which an allocation rule should take into account outside options. In Chapter 3 we found experimental evidence for outside options significantly affecting negotiations and we theoretically disussed the outside-option-issue in detail in Chapter 4. Recall the simple example from section 2.4, the glove game with more right gloves than left gloves where a pair of gloves produces worth which has to be distributed among the agents holding the gloves. Due to the bargaining position (outside options) of the left glove holder, this agent should obtain a higher payoff than the others. Recall "the need to let the players know what to expect from each coalition structure" (Maschler, 1992, p. 595) and that any formed coalition between individuals "only describes one particular consideration" while "the other [alliances] are present in virtual existence: although they have not materialized, they have contributed essentially to shaping and determining the actual reality" (von Neumann and Morgenstern, 1944, p. 36).

Furthermore, an allocation rule should take into account the specific position of agents within the network, that is, taking into account the path of information flow. We discussed this issue in detail in Chapter 5: Recall the simple example on train connections between England, France and Germany. The Eurostar train connects London and Paris and the Thalys train connects Paris and Cologne while there is no direct train connection between England and Germany. Hence, as a transit country, France has to be passed for any kind of flow via trains between England and Germany (travel, trade,...). As further examples consider cost allocation among the nodes in energy networks or social networks used for job offers (nodes with a lot of links should be treated differently to nodes with just a few links) or the political application in Chapter 5.5.1.

Recall the different popular allocation rules suggested in literature and presented in Chapter 2: The Shapley value (Shapley, 1953) as well as the component-restricted Shapley value (AD-value, Aumann and Drèze, 1974) and, as another modification of the Shapley value, the Owen value (Owen, 1977) do not take into account the network structure as they are designed for coalitional models without any interaction structure. Also more recent allocation rules that account for outside options, the Wiese value (Wiese, 2007) and the χ-value (Casajus, 2009a), do not consider the inner interaction structure of a coalition. In order to account for the position within the network, this chapter only analyzes allocation rules designed for networks (without further structure). These are the Position value (Meessen, 1988 and further analyzed by Borm et al., 1992 and Slikker, 2005), the Banzhaf Position value suggested in Chapter 5, the graph-χ-value (Casajus, 2009b) and the Myerson value (Myerson, 1977) or, as a generalization of the latter to the Jackson and Wolinsky [1996] approach, the equal bargaining rule (Jackson and Wolinsky, 1996)[1]. The Position values never account for structures outside the own coalition, hence, are outside-option-insensitive while we show that the graph-χ-value generally does not take into account the network structure within a coalition for weighted voting games with minimal winning coalitions. We further show that the Myerson value, and hence also the equal bargaining rule, even have both these drawbacks. Therefore, there is a need for a new allocation rule.

The contribution of this chapter is the definition and characterization of a new allocation rule, the kappa-value, that is outside-option-sensitive and takes into account the position of an agent within the network. To the best of our knowledge, there exists no other allocation rule combining these properties. The kappa-value has a nice axiomatization, we only need already known and approved axioms or weakened versions of them (which combine the ideas of known and approved axioms). The kappa-value combines the advantages of the graph-χ-value and the Position value while lacking their drawbacks and furthermore provides an elegant use of the quite intuitive

1 We do not analyze the equal splitting rule (Jackson and Wolinsky, 1996), since it does not account for any inner structure (neither coalitions nor interaction between agents).

concept of the Position value. We further show the independence of the characterizing axioms.

To motivate the need for an allocation rule accounting for both outside options and the specific position of agents within the network, consider the following weighted voting game: we have four agents $\{1, 2, 3, 4\}$ holding weights $(w_1, w_2, w_3, w_4) = (39, 30, 25, 6)$ and let the threshold be $T = 60$, that is, the worth of a coalition is 1 if the sum of weights of the coalitional agents is at least 60 and 0 otherwise. Hence, minimal winning coalitions are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3, 4\}$. We consider minimal winning coalitions without organized opposition, that is, agents outside the winning coalition stay as singletons. The networks that could occur are shown in Figure 7.

Figure 7: Networks for Minimal Winning Coalitions

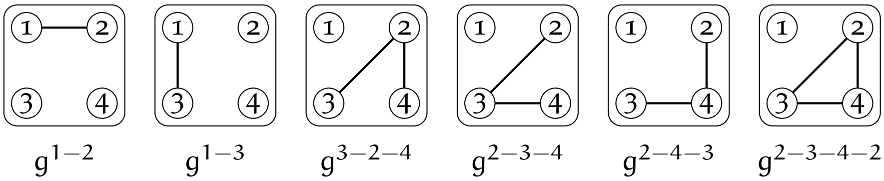


Table 18 reports the the distribution of worth assigned by the Position value[2], denoted by π and the graph-χ-value, denoted by $\chi^{\#}$.[3]

Table 18: Payoffs using π or $\chi^{\#}$

network	π_1	π_2	π_3	π_4	$\chi^{\#}_1$	$\chi^{\#}_2$	$\chi^{\#}_3$	$\chi^{\#}_4$
g^{1-2}	0.5	0.5	0	0	0.58	0.42	0	0
g^{1-3}	0.5	0	0.5	0	0.58	0	0.42	0
g^{3-2-4}	0	0.5	0.25	0.25	0	0.39	0.39	0.22
g^{2-3-4}	0	0.25	0.5	0.25	0	0.39	0.39	0.22
g^{2-4-3}	0	0.25	0.25	0.5	0	0.39	0.39	0.22
$g^{2-3-4-2}$	0	0.33	0.33	0.33	0	0.39	0.39	0.22

2 To avoid indices, we will omit the reference to Shapley we used in Chapter 5 for the Position value.

3 Note that the value function considers all possible winning coalitions, not only the minimal ones. We consider marginal contributions: $\{2, 3\}$ does not create worth but together with 1 it does. Hence, it has to be considered to assign the right distribution even though it ends up in a winning coalition not being minimal.

Consider the networks g^{3-2-4} and g^{2-3-4}: the graph-χ-value assigns the same payoff to agent 2 even though its position in the first case is stronger than in the second case while the Position value accounts for the different positions. On the other hand, the Position value does not take into account the existence of alternative winning coalitions, that is, outside options: consider the network g^{3-2-4} again and note that agents 3 and 4 obtain the same payoff even though 3 would have the outside option of a winning coalition with 1 while 4 has no outside option. The graph-χ-value accounts for outside options and assigns a higher payoff to agent 3.

Table 19 reports the distribution of worth by the new allocation rule suggested in this chapter, the kappa-value, denoted by κ.

Table 19: Payoffs using the new κ

network	κ_1	κ_2	κ_3	κ_4
g^{1-2}	0.56	0.44	0	0
g^{1-3}	0.56	0	0.44	0
g^{3-2-4}	0	0.42	0.35	0.23
g^{2-3-4}	0	0.35	0.42	0.23
g^{2-4-3}	0	0.36	0.36	0.28
$g^{2-3-4-2}$	0	0.37	0.37	0.27

The kappa-value takes into account both outside options and the position within the network: consider the coalition $\{2,3,4\}$, agents 2 and 3 always obtain a higher payoff than 4, because they could also cooperate with 1 instead, while 4 has no outside option. Agents 2 and 3 are symmetric in terms of outside options but 2 obtains a higher payoff than 3 if the coalition is connected through 2, a lower payoff than 3 if the coalition is connected through 3 and the same payoff if the coalition is connected through 4. An agent obtains the highest payoff within the coalition if she has the strongest position.

Note that not every outside option has the same value/impact: consider g^{1-2} and note that agents 1 and 2 have the same position in this network. Agent 1 obtains a higher payoff due to the fact that her outside options (building a coalition with 3) is valued higher/has a larger impact than the outside option of 2 (cooperating with $\{3,4\}$).

This can be explained by the fact that in smaller coalitions, the worth has to be distributed among less agents.

In contrast to defining a coalition as the group of agents that is connected via path, one could define a coalition by complete subnetworks, that is, as the group of agents that are directly connected to each other. The latter definition has the drawback of not taking into account different interaction paths within a coalition and the position of all agents within the coalition would be the same. This is why we use the more general definition of a coalition. But still it is notable that also in the case of equal positions, the kappa-value is not redundant; it differs from the graph-χ-value as shown by g^{1-2}, g^{1-3} and $g^{2-3-4-2}$.

As an application one could consider weighted voting games as simple trade agreements: countries hold a specific amount of an input good and have to form agreements in order to reach some required amount of these input goods to produce an output. There exist different possible trade routes (outside options) and furthermore, transit countries and other countries should be treated differently (position within the network). Using allocation rules, we can distribute the worth of the output good among the countries. This can also be seen as measuring the (relative) distribution of power of the countries. As other applications one could consider weighted voting games as defense agreements (taking into account rights of way) or political agreements: There is a specific vote distribution among parties in a parliament and the parties have to build agreements in order to reach some required quorum (for example to pass a bill). Here we can use allocation rules to measure the (relative) distribution of power of the parties as for example in the political application in Chapter 5.

This chapter is structured as follows: we start with the framework and analyze weighted voting games to show the drawbacks of the existing allocation rules. Following the idea of combining the properties of the analyzed allocation rules, Section 3 will give an axiomatic characterization of the the kappa-value and analyzes independence of the characterizing axioms. In Section 4, we provide a variant of the kappa-value using the Banzhaf approach which seems more adequate in application to political networks, again providing an axiomatic characterization and indepenedence of axioms. Section 5 shows that the kappa-values indeed are outside-option-sensitive w. r. t. the axioms we suggested in Chapter 4 and we discuss why the suggested "weak" axiom in fact is not that weak and the "normal" axiom actually could be seen as sort of strong, especially for link-based values. Section 6 concludes.

6.2 FRAMEWORK AND THE NEED OF A NEW ALLOCATION RULE

6.2.1 *Framework*

Recall the framework for network structures: we model a network as an undirected graph where the nodes represent the players and the edges represent the links between the players. Let $N = \{1, ..., n\}$, non-empty and finite, be the player set and $\mathbb{G}_N := \{g | g \subseteq \{ij := \{i,j\} | i,j \in N, i \neq j\}\}$ denotes the set of all possible networks on N. A TU-game (N, v) consists of the player set N and a coalition function $v \in \mathbb{V}_N := \{v : 2^N \longrightarrow \mathbb{R} | v(\emptyset) = 0\}$ and $\mathbb{V}_0(N)$ is the set of all *zero-normalized* coalition functions, that is, $v(\{i\}) = 0 \; \forall \; i \in N$. Without loss of generality, we will restrict ourselves to zero-normalized games since any $v \in \mathbb{V}_N$ can be zero-normalized. An *allocation rule* $Y : \{N\} \times \mathbb{V}_N \times \mathbb{G}_N \longrightarrow \mathbb{R}^N$ distributes the arising worth among the players, that is, assigns a payoff to each player. For notational convenience we define $\sum_{i \in K} Y_i =: Y_K$. An allocation rule is *feasible* if $Y_N(N, v, g) \leqslant v(N)$.

Recall the allocation formula of the *Shapley value* (Shapley, 1953):

$$\forall i \in N : Sh_i(N, v) := \sum_{K \subseteq N \setminus \{i\}} \frac{k!(n-1-k)!}{n!} [v(K \cup \{i\}) - v(K)],$$
$$\text{where } k = |K|, n = |N|.$$

The Shapley value assigns to every player her share of what she creates when entering a coalition, that is, her marginal contributions $[v(K \cup \{i\}) - v(K)]$.

We say that players i and j are *connected* in the network g if there exists a path in g connecting them. Recall that connected players form components of a network g and these components build a partition on the player set N, denoted by $\mathcal{C}(N, g)$ where $\mathcal{C}_i(N, g) \in \mathcal{C}(N, g)$ is the component of all players connected with player $i \in N$. If N is fixed, we will write $\mathcal{C}_i(g)$ or simply $\mathcal{C}_i$.

Further recall the *graph-restricted game* (Myerson, 1977)

$$v^g(K) := \sum_{S \in \mathcal{C}(K, g|_K)} v(S),$$
$$\text{where } g|_K = \{ij \in g | i, j \in K\}.$$

used to define the *Myerson value* $\mu(N, v, g) := Sh(N, v^g)$ and the *link-game* (Meessen, 1988 and Borm et al., 1992)

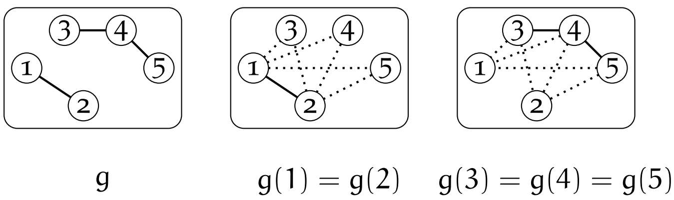
$$v^N(g') := v^{g'}(N) \quad , g' \subseteq g$$

used to define the *Position value* (for zero-normalized games)[4]:

$$\pi_i(N, v, g) := \sum_{\lambda \in g_i} \frac{1}{2} Sh_\lambda(g, v^N),$$
$$\text{where } g_i := \text{set of links including player } i.$$

As an alternative, Casajus [2009a] introduced an outside-option-sensitive allocation rule. To reflect all (productive) outside options, he defines for every network g the corresponding *lower outside option graph (LOOG)*, the graph that captures the alternatives/outside options the players have: $g(i, N) := g|_{\mathcal{C}_i} \cup \{jk \in g^N | j \in \mathcal{C}_i, k \in N \setminus \mathcal{C}_i\}$. For notational convenience, if the player set is fixed, we will only write $g(i)$. To recall the specific form of the LOOG consider again the example $N = \{1, 2, 3, 4, 5\}$ and $g = \{12, 34, 45\}$ (cf. Chapter 4). The LOOGs are displayed in Figure 8. The LOOG reflects all alternative links a player

Figure 8: Network and corresponding LOOGs



and her coalitional players might have outside their actual coalition (which stays fixed). To provide these alternatives, links outside the own coalition are broken.

4 Note that we restricted ourselves to zero-normalized games, otherwise v^N might not be a coalition function because $v^N(\emptyset) = v^\emptyset(N) \neq 0$.

Using the LOOG, Casajus [2009a] defines the the *graph-χ-value*:

$$\chi_i^{\#}(N, \nu, g) := \mu_i(N, \nu, g(i)) + \frac{\nu(\mathcal{C}_i) - \mu_{\mathcal{C}_i}(N, \nu, g(i))}{|\mathcal{C}_i|}.$$

The graph-χ-value values outside options by adding or substracting some share of the Myerson value of her component to the Myerson value of the player (where the Myerson value is the Shapley value of the graph-restricted game), taking into account the outside option graph.

6.2.2 *The Need of a new Allocation Rule*

Analyzing weighted voting games, we will now explain the drawbacks of the graph-χ-value, the Myerson value and the Position value and, therefore, the need of a new allocation rule which closes the arising gap.

Definition 6.1 (Weighted Voting Game)**.** *A* weighted voting game*, denoted by* $(N, \nu_{w,T})$*, is a TU-game that consists of the set of players* $N = \{1, 2, ..., n\}$*, each endowed with weights* $w = (w_1, w_2, ..., w_n)$*,* $w_i \in \mathbb{R}^+$ *and a threshold* $T \in \mathbb{R}$*. The worth of a coalition* $K \subseteq N$ *is given by*

$$\nu_{w,T}(K) = \begin{cases} 1 & \text{, if } \sum_{i \in K} w_i \geqslant T \\ 0 & \text{, otherwise} \end{cases}.$$

Without loss of generality, we normalize w *such that* $\sum_{i \in N} w_i = 100$*.*

Definition 6.2 (Minimal winning coalition and Opposition)**.** *A* minimal winning coalition *is a coalition* $K \subseteq N$ *such that* $\nu(K) = 1$ *and* $\nu(K \setminus \{i\}) = 0$ *for all* $i \in K$*. We say that there is* no organized opposition *if all players outside this coalition stay alone (i. e., as singletons).*

We will restrict ourselves to situations with $w_i < T$ for all $i \in N$ (i. e., we neglect the trivial case that one individual builds a minimal winning coalition by itself).

Remark 6.1 (Outside-Option-Insensitivity of the Position value and the Myerson value)**.** *Recall that the Position value and the Myerson* *value satisfy Component Decomposability **CD**, that is, are outside-option-insensitive (cf. Chapter 4).*

Lemma 6.1 (Invariance of the Myerson value). *Let $(N, v_{w,T})$ be a weighted voting game with $T > 50$ and consider a network g describing a unique minimal winning coalition without organized opposition. Then, we have*

$$\mu_i(N, v_{w,T}, g) = \begin{cases} \frac{1}{|C|} & , \text{ if } i \in C \\ 0 & , \text{ otherwise} \end{cases}$$

where C is the minimal winning coalition and u_C the corresponding unanimity game ($u_C(K) = 1$ if $C \subseteq K$ and zero otherwise).

Proof. Let v and g be of the required form. Then, $\mathcal{C}(N, g)$ consists of one connected component C and singletons, where C is a minimal winning coalition. It holds that $v^g_{w,T}(K) = v_{w,T}(K \cap C)$, independent of the network structure within C. Since C is a minimal winning coalition (i. e., $v_{w,T}(S) = 1$ if and only if $C \subseteq S$), this is the same as considering the unanimity game u_C. Since the Myerson value is component decomposable, we can, for every $i \in N$, restrict the player set (and the value function) to $\mathcal{C}_i$. Hence, $\mu_i(N, v_{w,T}, g) = Sh_i(\mathcal{C}_i, u_C|_{\mathcal{C}_i})$. For all $i \notin C$ (the minimal winning coalition), we have $\mathcal{C}_i = \{i\}$ (no organized opposition) and therefore $Sh_i(\mathcal{C}_i, u_C|_{\mathcal{C}_i}) = 0$. As $\mathcal{C}_i = C$, we have for $i \in C$

$$\begin{aligned}
Sh_i(\mathcal{C}_i, u_C|_{\mathcal{C}_i}) &= Sh_i(C, u_C) \\
&= \sum_{K \subseteq C\setminus\{i\}} \frac{|K|!(|C|-1-|K|)!}{|C|!} (u_C(K \cup \{i\}) - \underbrace{u_C(K)}_{=0\,\forall K}) \\
&= \sum_{K \subseteq C\setminus\{i\}} \frac{|K|!(|C|-1-|K|)!}{|C|!} \underbrace{u_C(K \cup \{i\})}_{=\begin{cases} 1, \text{ if } K = C\setminus\{i\} \\ 0, \text{ otherwise} \end{cases}} \\
&= \frac{|C \setminus \{i\}|!(|C|-1-|C \setminus \{i\}|)!}{|C|!} \\
&= \frac{(|C|-1)!(|C|-1-(|C|-1))!}{|C|!} \\
&= \frac{1}{|C|}
\end{aligned}$$

leading to

$$\mu_i(N, v_{w,T}, g) = \begin{cases} \frac{1}{|C|} & \text{, if } i \in C \\ 0 & \text{, otherwise} \end{cases} .$$

□

Lemma 6.2 (Invariance of the graph-χ-value)**.** *Let $(N, v_{w,T})$ be a weighted voting game with $T > 50$ and only consider networks describing minimal winning coalitions without organized opposition. Then, the graph-χ-value coincides with the χ-value for coalition structures (Casajus, 2009b) of the partition identified with this coalition:*

$$\begin{aligned} \chi_i^{\#}(N, v_{w,T}, g) &= \chi_i(N, v_{w,T}, \mathcal{C}(N, g)) \\ &= Sh_i(N, v_{w,T}) + \frac{v_{w,T}(\mathcal{C}_i) - Sh_{\mathcal{C}_i}(N, v_{w,T})}{|\mathcal{C}_i|} \end{aligned}$$

Proof. Let v and g be of the required form. Then, $\mathcal{C}(N, g)$ consists of one connected component C and singletons. Note that we excluded the case where one player obtains the threshold by herself without the need of cooperation. For any singleton it is obvious that its graph-χ-value is zero. We will show that for any $i \in C$ we have

$$v^{g(i)}(K \cup \{i\}) - v^{g(i)}(K) = v(K \cup \{i\}) - v(K) \quad \forall K \subseteq N \setminus \{i\}$$

because this leads to $\mu(N, v, g(i)) = Sh(N, v^{g(i)}) = Sh(N, v)$ and hence

$$\chi_i^{\#}(N, v, g) = \chi_i^{\#}(N, v, g') \quad \forall g, g' \text{ s.th. } \mathcal{C}_i(N, g) = \mathcal{C}_i(N, g')$$

which finishes the proof.
First note that whenever there exist $j_1, j_2 \in K$ such that $j_1 \in \mathcal{C}_i$ and $j_2 \in N \setminus \mathcal{C}_i$, then $g(i)|_K$ is connected, i. e.$\mathcal{C}(K, g(i)|_K) = \{K\}$ and therefore $v^{g(i)}(K) = v(K)$.
For $K = \mathcal{C}_i$, we have $v^{g(i)}(K) = 1$. If $K \subset \mathcal{C}_i$, we have $v^{g(i)}(K) = 0$ because the $\mathcal{C}_i$ is minimal winning coalition (i. e.all members of $\mathcal{C}_i$ are needed to create worth). If $K \subseteq N \setminus \mathcal{C}_i$, $\mathcal{C}(K, g(i)|_K) = \{\{j\}|j \in K\}$

and therefore $v^{g(i)}(K) = 0$.
Case 1: $K = \mathcal{C}_i \setminus \{i\}$

$$v^{g(i)}(\mathcal{C}_i) - v^{g(i)}(\mathcal{C}_i \setminus \{i\}) = v(\mathcal{C}_i) - 0 = v(\mathcal{C}_i) - v(\mathcal{C}_i \setminus \{i\})$$

Case 2: $K \subset \mathcal{C}_i \setminus \{i\}$

$$v^{g(i)}(K \cup \{i\}) - v^{g(i)}(K) = 0 - 0 = v(K \cup \{i\}) - v(K)$$

Case 3: $K \subseteq N \setminus \mathcal{C}_i$

$$\underbrace{v^{g(i)}(K \cup \{i\})}_{g(i)|_{K\cup\{i\}}\text{connected}} - v^{g(i)}(K) = v(K \cup \{i\}) - 0 = v(K \cup \{i\}) - v(K)$$

since the members of $\mathcal{C}_i$ achieve at least the threshold T and hence any group in $N \setminus \mathcal{C}_i$ can at most achieve $(100 - T) < T$ (since $T > 50$). Identify all g, g' such that $\mathcal{C}_i(N, g) = \mathcal{C}_i(N, g')$ with one representing element $\tilde{g}$. Using that $Sh(N, v^{g(i)}) = Sh(N, v)$, we obtain for any g in the same "equivalence class"

$$\chi_i^{\#}(N, v, g) = Sh_i(N, v) + \frac{v(\mathcal{C}_i) - Sh_{\mathcal{C}_i}(N, v)}{|\mathcal{C}_i|} = \chi_i(N, v, \mathcal{C}).$$

□

Remark 6.1 tells us that both the Position value and the Myerson value do generally not account for outside options. Lemma 6.1 and Lemma 6.2 tell us that, for an economically important and large class of games, the Myerson value and the graph-χ-value do not take into account the specific position of a player within the network, that is, the communication path. In other words, these allocation rules do not consider differences in the agents centralities and, therefore, are "centrality-invariant". Hence, to account for both outside options and the position of a player within the network, we need a new allocation rule.

6.3 A NEW OUTSIDE OPTION VALUE: THE KAPPA-VALUE

In this section we define a new allocation rule for network structures which takes into account both outside options and the position of an agent within the network/the path of information flow. In the previous section we have seen that (for a large class of games) the graph-χ-value does not differ within the class of networks referring to the same coalition but it takes into account outside options. While the latter is not true for the Position value, it takes into account the position of an agent within a network. We have seen that the Myerson value lacks both desired properties. Hence, we will further analyze the characterizing axioms of the graph-χ-value and the Position value.

Recall that an allocation rule satisfies *Component Efficiency* **CE** if for all $C \in \mathcal{C}(N, g)$ we have

$$\sum_{i \in C} Y_i(N, v, g) = v(C)$$

and that an allocation rule satisfies *Balanced Link Contributions* **BLC** if for all $i, j \in N$, $i \neq j$ and $v \in \mathbb{V}_0$ we have

$$\sum_{\lambda \in g_j} [Y_i(N, v, g) - Y_i(N, v, g - \lambda)] \\ = \sum_{\lambda \in g_i} \left[Y_j(N, v, g) - Y_j(N, v, g - \lambda)\right].$$

Further recall that the Position value is characterized by **CE** and **BLC** and also satisfies **CD** (i. e., for all $i \in C \in \mathcal{C}(N, g)$ we have $Y_i(N, v, g) = Y_i(C, v|_C, g|_C)$) which stands in contradiction to outside-option-sensitivity (see Remark 6.1 and Chapter 4). Hence, if we want an allocation rule to account for outside options but still to consider the role of a player within the network, we need to weaken **BLC** (in order to get rid of the allocation rule satisfying **CD** but still having **CE**). Note that a connected network lacks outside options (see Chapter 5). Having this in mind, we define the following weaker version of **BLC**:

Axiom 6.1 (Weak Balanced Link Contributions (**WBLC**))**.** *An allocation rule for network structures* Y *satisfies* Weak Balanced Link Contributions **WBLC** *if we have*

$$\sum_{\lambda \in g_j} \left[Y_i(N, v, g) - Y_i(\mathcal{C}_i(N, g - \lambda), v|_{\mathcal{C}_i(N, g-\lambda)}, g|_{\mathcal{C}_i(N, g-\lambda)})\right] \\ = \sum_{\lambda \in g_i} \left[Y_j(N, v, g) - Y_j(C_j(N, g - \lambda), v|_{C_j(N, g-\lambda)}, g|_{C_j(N, g-\lambda)})\right].$$

for all connected g *and all* $i, j \in N$, $i \neq j$, *and* $v \in \mathbb{V}_0$

WBLC combines the ideas underlying **BLC** and *Weak Fairness 2* **WF2**, the modification of *Fairness* **F** that characterizes the Myerson value.

Lemma 6.3. *If an allocation rule for network structures* $Y(N, v, g)$ *satisfies* ***CE*** *and* ***WBLC****, it coincides with the Position value for all connected networks.*

Proof. Note that in presence of **CD**, **WBLC** reduces to **BLC** on connected networks. Hence, since the Position value satisfies **CD** and is characterized by **CE** and **BLC**, it satisfies **CE** and **WBLC** (which provides existence). For uniqueness, we cannot use the presence of **CD** any longer, hence, we cannot use **BLC**. We follow the idea of the proof for the Myerson value of Casajus [2009b]. Let φ and ψ be allocation rules satisfying **CE** and **WBLC**. Suppose N is the minimal player set such that φ and ψ differ on a connected graph. We must have $|N| > 1$, because for $|N| = 1$ we would have a contradiction by **CE** due to the connectedness of the graph. Suppose that g is the minimal connected graph on N such that $\varphi \neq \psi$. Now let $i, j \in N$. By **WBLC** we have

$$
\begin{aligned}
|g_j|\varphi_i(N, v, g) - |g_i|\varphi_j(N, v, g) \\
&= \sum_{\lambda \in g_j} \varphi_i(\mathcal{C}_i(N, g-\lambda), v|_{\mathcal{C}_i(N,g-\lambda)}, g|_{\mathcal{C}_i(N,g-\lambda)}) \\
&\quad - \sum_{\lambda \in g_i} \varphi_j(C_j(N, g-\lambda), v|_{C_j(N,g-\lambda)}, g|_{C_j(N,g-\lambda)})
\end{aligned}
$$

Note that $g|_C$ is always connected on the connected component C. If now $C_k(N, g-\lambda) \neq N$ (for $k = i, j$), we have $C_k(N, g-\lambda) \subset N$ and hence

$$
\begin{aligned}
&\varphi_k(C_k(N, g-\lambda), v|_{C_k(N,g-\lambda)}, g|_{C_k(N,g-\lambda)}) \\
&= \psi_k(C_k(N, g-\lambda), v|_{C_k(N,g-\lambda)}, g|_{C_k(N,g-\lambda)})
\end{aligned}
$$

since N is the minimal player such that φ and ψ differ on a connected graph. If $C_k(N, g-\lambda) = N$ we have that $g-\lambda$ is connected on N and hence

$$
\begin{aligned}
&\varphi_k(C_k(N, g-\lambda), v|_{C_k(N,g-\lambda)}, g|_{C_k(N,g-\lambda)}) \\
&= \psi_k(C_k(N, g-\lambda), v|_{C_k(N,g-\lambda)}, g|_{C_k(N,g-\lambda)})
\end{aligned}
$$

since g is the minimal connected graph on N such that φ and ψ differ. Using that ψ satisfies **WBLC**, we get

$$\begin{aligned} |g_j|\varphi_i(N,v,g) - |g_i|\varphi_j(N,v,g) &= |g_j|\psi_i(N,v,g) - |g_i|\psi_j(N,v,g) \\ \Leftrightarrow |g_j|\left[\varphi_i(N,v,g) - \psi_i(N,v,g)\right] &= |g_i|\left[\varphi_j(N,v,g) - \psi_j(N,v,g)\right] \end{aligned}$$

Summing up over $j \in \mathcal{C}_i(N,g) = N$ (connected graph), we have by **CE**:

$$\underbrace{\left(\sum_{j\in N} |g_j|\right)\frac{|N|}{|g_i|}}_{>0}\left[\varphi_i(N,v,g) - \psi_i(N,v,g)\right] = v(N) - v(N) = 0$$

and hence $\varphi_i(N,v,g) = \psi_i(N,v,g)$. □

If we combine Lemma 6.3 with the presence of **CD**, we will have a characterization of the Position value. Hence, we need to weaken **CD**. We use the characterizing axiom of the graph-χ-value that accounts for outside options given by Casajus [2009b]: Recall that an allocation rule satisfies *Outside Option Consistency* **OO** if for all $i,j \in C \in \mathcal{C}(N,g)$ we have

$$Y_i(N,v,g) - Y_j(N,v,g) = Y_i(N,v,g(i)) - Y_j(N,v,g(j)).$$

Theorem 6.1 (The Kappa-value)**.** *Let $v \in \mathbb{V}_0$. There is a unique allocation rule for network structures that satisfies **CE**, **OO** and **WBLC**:*

$$\kappa_i(N,v,g) := \pi_i(N,v,g(i)) + \frac{v(\mathcal{C}_i) - \pi_{\mathcal{C}_i}(N,v,g(i))}{|\mathcal{C}_i|}$$

Proof. UNIQUENESS: We follow the idea of the uniqueness proof of the graph-χ-value (Casajus, 2009a). Let Y satisfy **CE**, **OO** and **WBLC**. For $i,j \in \mathcal{C}_i$ we have $g(i) = g(j)$. First, by **OO**, we get

$$Y_i(N,v,g) - Y_i(N,v,g(i)) = Y_j(N,v,g) - Y_j(N,v,g(i)).$$

Then, summing up over $j \in \mathcal{C}_i$ and using **CE** gives

$$|\mathcal{C}_i|\left[Y_i(N,v,g) - Y_i(N,v,g(i))\right] = v(\mathcal{C}_i) - Y_{\mathcal{C}_i}(N,v,g(i)).$$

Since $g(i)$ is connected, Lemma 6.3 implies

$$Y_i(N, v, g(i)) = \pi_i(N, v, g(i))$$

for all $i \in N$ and hence we have

$$Y_i(N, v, g) = \pi_i(N, v, g(i)) + \frac{v(\mathcal{C}_i) - \pi_{\mathcal{C}_i}(N, v, g(i))}{|\mathcal{C}_i|} \tag{7}$$

which uniquely determines Y.

EXISTENCE: It is easily shown that the value given by equation (7) satisfies **CE** and **OO** (note that $(g(i))(i) = g(i)$). To see **WBLC**, first note that the Position value satisfies **WBLC** by satisfying **BLC** and **CD**. Let g be connected and $i, j \in N$, then we have $\mathcal{C}_i = C_j = N$ and $g(i) = g(j) = g$. Hence,

$$\begin{aligned}
& \sum_{\lambda \in g_j} Y_i(N, v, g) - \sum_{\lambda \in g_j} Y_j(N, v, g) \\
= \quad & \sum_{\lambda \in g_j} \left[\pi_i(N, v, g) + \underbrace{\frac{v(N) - \pi_N(N, v, g)}{|N|}}_{=0 \text{ by CE}} \right] \\
& - \sum_{\lambda \in g_i} \left[\pi_j(N, v, g) + \underbrace{\frac{v(N) - \pi_N(N, v, g)}{|N|}}_{=0 \text{ by CE}} \right] \\
= \quad & \sum_{\lambda \in g_j} \pi_i(N, v, g) - \sum_{\lambda \in g_i} \pi_j(N, v, g) \\
\overset{WBLC}{=} \quad & \sum_{\lambda \in g_j} \pi_i(\mathcal{C}_i(N, g - \lambda), v|_{\mathcal{C}_i(N, g-\lambda)}, g|_{\mathcal{C}_i(N, g-\lambda)}) \\
& - \sum_{\lambda \in g_j} \pi_j(C_j(N, g - \lambda), v|_{C_j(N, g-\lambda)}, g|_{C_j(N, g-\lambda)}) \\
= \quad & \sum_{\lambda \in g_j} Y_i(\mathcal{C}_i(N, g - \lambda), v|_{\mathcal{C}_i(N, g-\lambda)}, g|_{\mathcal{C}_i(N, g-\lambda)}) \\
& - \sum_{\lambda \in g_j} Y_j(C_j(N, g - \lambda), v|_{C_j(N, g-\lambda)}, g|_{C_j(N, g-\lambda)})
\end{aligned}$$

where the last step follows from the fact that $g|_{C_k(N, g-\lambda)}$ is connected on $C_k(N, g - \lambda)$ and by **CE**. □

We call the value given by (7) "kappa-value" and denote it by κ. The kappa-value assigns to each player in a coalition the worth of the position she would obtain in the outside option graph plus her share of the worth players outside the actual coalition would obtain in the outside option graph. This share is equal for all players in the actual coalition and the worth of a position is given by the Shapley value of the arc game, the Position value.

The kappa-value provides a very elegant use of the quite intuitive Position value, lacking its drawbacks by using the outside option graph. Note that $\pi(N, v, g(i))$ also captures both positions in the network and outside options. Besides the fact that it is not feasible[5] which could be solved by (multiplicative or additive efficient) normalizations, note that the LOOG only reflects possible outside options that *might alternatively have been formed*, that is, these outside options have not *actually* materialized. Hence, $\pi(N, v, g(i))$ might distribute non-negative payoffs to agents being actually unproductive.

In order to avoid the mentioned drawbacks of the graph-χ-value, the kappa-value differs by using the Position value instead of the Myerson value. Note that the proof of Lemma 6.2 directly implies that the Myerson value of the LOOG ($\mu(N, v, g(i))$) also has the drawback of not accounting for the position of an agent within the network for a broad class of games. Hence, there is no simpler alternative to the kappa-value. However, in the following chapter we present an alternative to the kappa-value which is not simpler but uses the Banzhaf approach instead of the Shapley approach which might be more suitable for political applications.

Lemma 6.4 (Independence of κ-Axioms)**.** *The axioms* ***CE, OO*** *and* ***WBLC*** *are independent.*

Proof. Consider the weighted voting game from the introduction ($N = \{1, 2, 3, 4\}$, $w = (39, 30, 25, 6)$ and $T = 60$).

Independence 1 (CE+WBLC, ¬ OO). The Position value satisfies **CE** and **WBLC** (because of **BLC** and **CD**). Consider $g = \{12\}$. We have

$$\pi_1(N, v, g) - \pi_2(N, v, g) = \frac{1}{2} - \frac{1}{2} = 0 \text{ and}$$
$$\pi_1(N, v, g(1)) - \pi_2(N, v, g(2)) = \frac{49}{120} - \frac{34}{120} \neq 0$$

5 Consider the example from the introduction and $g = \{23, 24, 34\}$, then $g(2) = g(3) = g(4)$ while $g(1)$ differs and $\pi_1(N, v, g(1)) = 1/2 > 2/5 = \pi_1(N, v, g(2))$, hence $\sum_{i \in N} \pi_i(N, v, g(i)) > \sum_{i \in N} \pi_i(N, v, g(2)) = v(N)$ (by π satisfying **CE** and $\mathcal{C}_i = N$ on $g(i)$).

$\Rightarrow$ **OO** is violated.

Independence 2 (CE+OO, ¬ WBLC). The graph-χ-value satisfies **CE** and **OO**. Consider $g = \{12, 23, 34\}$. We have

$$\sum_{\lambda \in g_1} \left[\chi_2^{\#}(N, v, g) - \chi_2^{\#}(C_2(N, g-\lambda), v|_{C_2(N,g-\lambda)}, g|_{C_2(N,g-\lambda)})\right]$$
$$= \chi_2^{\#}(N, v, g) - \chi_2^{\#}(N \setminus \{1\}, v|_{N\setminus\{1\}}, g - 12) = \frac{7}{12} - \frac{1}{6} = \frac{5}{12}$$

and

$$\sum_{\lambda \in g_2} \left[\chi_1^{\#}(N, v, g) - \chi_1^{\#}(C_1(N, g-\lambda), v|_{C_1(N,g-\lambda)}, g|_{C_1(N,g-\lambda)})\right]$$
$$= 2\chi_1^{\#}(N, v, g) - \chi_1^{\#}(\{1\}, v|_{\{1\}}, \emptyset) - \chi_1^{\#}(\{1,2\}, v|_{\{1,2\}}, \{12\})$$
$$= 2 \cdot \frac{3}{12} - 0 - \frac{1}{2} = 0 \neq \frac{5}{12}$$

$\Rightarrow$ **WBLC** is violated.

Now consider the TU-game (N, v) with $N = \{1, 2, 3, 4\}$ and $v = u_N$.

Independence 3 (WBLC+OO, ¬ CE). $Y_i(N, v, g) := \pi_i(N, v, g(i))$ satisfies **OO** by $(g(i))(i) = g(i)$ and **WBLC** by $Y = \pi$ for connected networks. Consider $g = \{12, 23\}$. We have

$$Y_1(N, v, g) = Y_3(N, v, g) = \frac{13}{60},$$
$$Y_2(N, v, g) = \frac{17}{60} \text{ and } Y_4(N, v, g) = \frac{1}{2}$$
$$\Rightarrow Y_{\{4\}} = Y_4 = \frac{1}{2} \neq 0 = u_N(\{4\}) \text{ and}$$
$$Y_{\{1,2,3\}} = \frac{43}{60} \neq 0 = u_N(\{1, 2, 3\})$$

$\Rightarrow$ **CE** is violated.

□

6.4 THE BANZHAF-KAPPA-VALUE

In this section we give an alternative to the kappa-value which is in line with the investigations in Chapter 5 of emphazising the use of the Banzhaf Position value. For this, we first note that the Banzhaf Position value is outside-option-insensitive.

Lemma 6.5 (Outside-Option-Insensitivity of the Banzhaf Position value). *The Banzhaf Position value satisfies* **CD** *and hence, is outside-option-insensitive (cf. Chapter 4).*

Proof. Let $i \in C \in \mathcal{C}(N, g)$. Note that if $\lambda \in g_i$, we also have that $\lambda \in g|_C$. Recall from the proof of π^{Ba} satisfying **CLBE** that for $\lambda \in g|_C$, marginal contributions are not effected by connections outside C and are the same for all $g', g'' \subseteq g \setminus \lambda$ such that $g'|_C = g''|_C$. Hence we have

$$\begin{aligned}\pi_i^{Ba}(N, v, g) &= \sum_{\lambda \in g_i} \frac{1}{2} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \left(v^N(g' \cup \lambda) - v^N(g')\right) \\ &= \sum_{\lambda \in g_i} \frac{1}{2} \frac{1}{2^{|g|-1}} \sum_{\tilde{g} \subseteq (g|_C) \setminus \lambda} 2^{|g|_{N \setminus C}|} \left[v^N(\tilde{g} \cup \lambda) - v^N(\tilde{g})\right] \\ &\overset{|g| = |g|_{N \setminus C}| + |g|_C|}{=} \sum_{\lambda \in g_i} \frac{1}{2} \frac{1}{2^{|g|_C|-1}} \sum_{\tilde{g} \subseteq (g|_C) \setminus \lambda} \left[v^N(\tilde{g} \cup \lambda) - v^N(\tilde{g})\right]\end{aligned}$$

Note that for $\lambda \in g_i \subseteq g|_C$ and $\tilde{g} \subseteq (g|_C) \setminus \lambda \subset g|_C$ we have that $\tilde{g} \cup \lambda \subseteq g|_C$. Further, from the definition of v^N, we obtain for any $g' \subseteq g|_C$

$$v^N(g') = \sum_{S \in g|_N} v(S) = \sum_{S \in g|_C} v(S) = v^C(g)$$

and therefore, using $v^N|_C = v^C$, we get

$$\pi_i^{Ba}(N, v, g) = \pi_i^{Ba}(C, v|_C, g|_C).$$

□

Lemma 6.6. *If an allocation rule for network structures* $Y(N, v, g)$ *satisfies* **CLBE** *and* **WBLC***, it coincides with the Banzhaf Position value for all connected networks.*

Proof. We follow the arguments of Lemma 6.3: recall that in presence of **CD**, **WBLC** reduces to **BLC** on connected networks. Hence, since the Banzhaf-Position value satisfies **CD** by Lemma 6.5 and is characterized by **CLBE** and **BLC** by Theorem 5.2, it satisfies **CLBE** and **WBLC** (which provides existence). Recall that for uniqueness, we cannot use the presence of **CD** and therefore **BLC** any longer. Let φ and ψ be allocation rules satisfying **CLBE** and **WBLC**. Suppose N is the minimal player set such that φ and ψ differ on a connected graph. Again, by **CLBE**, we must have $|N| > 1$, because for $|N| = 1$ we would have a contradiction. Suppose that g is the minimal connected graph on N such that $\varphi \neq \psi$. Now let $i, j \in N$. By **WBLC** and the same arguments as in the proof of Lemma 6.3 we get

$$\begin{aligned} |g_j|\varphi_i(N,v,g) - |g_i|\varphi_j(N,v,g) &= |g_j|\psi_i(N,v,g) - |g_i|\psi_j(N,v,g) \\ \Leftrightarrow |g_j|\left[\varphi_i(N,v,g) - \psi_i(N,v,g)\right] &= |g_i|\left[\varphi_j(N,v,g) - \psi_j(N,v,g)\right]. \end{aligned}$$

Summing up over $j \in \mathcal{C}_i(N,g) = N$ (connected network), we have by **CLBE**:

$$\begin{aligned} &\underbrace{\left(\sum_{j\in N} |g_j|\right)\frac{|N|}{|g_i|}}_{>0}\left[\varphi_i(N,v,g) - \psi_i(N,v,g)\right] \\ &\qquad = \sum_{\lambda\in g|_C} Ba_\lambda(g|_C, v^N|_C) - \sum_{\lambda\in g|_C} Ba_\lambda(g|_C, v^N|_C) \\ &\qquad = 0 \end{aligned}$$

and hence $\varphi_i(N,v,g) = \psi_i(N,v,g)$. □

Theorem 6.2 (The Banzhaf-kappa-value)**.** *Let $v \in \mathbb{V}_0$. There is a unique allocation rule for network structures that satisfies **CLBE**, **OO** and **WBLC**:*

$$\kappa_i^{Ba}(N,v,g) := \pi_i^{Ba}(N,v,g(i)) + \frac{\pi_{\mathcal{C}_i}^{Ba}(N,v,g) - \pi_{\mathcal{C}_i}^{Ba}(N,v,g(i))}{|\mathcal{C}_i|}$$

Proof. We follow the steps from the proof of Theorem 6.1.
Uniqueness: Let Y satisfy **CLBE**, **OO** and **WBLC**. For $i, j \in \mathcal{C}_i$ we have $g(i) = g(j)$. First, by **OO**, we get

$$Y_i(N,v,g) - Y_i(N,v,g(i)) = Y_j(N,v,g) - Y_j(N,v,g(i)).$$

Then, summing up over $j \in \mathcal{C}_i$ and using **CLBE** gives

$$\begin{aligned} &|\mathcal{C}_i|\left[Y_i(N,v,g) - Y_i(N,v,g(i))\right] \\ &\quad = \sum_{\lambda\in g|_C} Ba_\lambda(g|_C, v^C) - Y_{\mathcal{C}_i}(N,v,g(i)) \\ &\quad = \pi_{\mathcal{C}_i}^{Ba}(N,v,g) - Y_{\mathcal{C}_i}(N,v,g(i)). \end{aligned}$$

Since $g(i)$ is connected, Lemma 6.6 implies

$$Y_i(N, v, g(i)) = \pi_i^{Ba}(N, v, g(i))$$

for all $i \in N$ and hence we have

$$Y_i(N, v, g) = \pi_i^{Ba}(N, v, g(i)) + \frac{\pi_{\mathcal{C}_i}^{Ba}(N, v, g) - \pi_{\mathcal{C}_i}^{Ba}(N, v, g(i))}{|\mathcal{C}_i|} \tag{8}$$

which uniquely determines Y.

EXISTENCE: It is easily shown that the value given by equation (8) satisfies **CLBE** and **OO** (note that $(g(i))(i) = g(i)$). Further note that the Banzhaf Position value satisfies **WBLC** by satisfying **BLC** and **CD**. Then, by the same chain of arguments as in the corresponding part of the proof of Theorem 6.1 and the fact that the use of **CLBE** instead of **CE** does not change these arguments, we obtain **WBLC**. □

We call the value given by (8) "Banzhaf-kappa-value" and denote it by κ^{Ba}. Similar as the kappa-value, the Banzhaf-kappa-value assigns to each player in a coalition the worth of the position she would obtain in the outside option graph plus an equal share of sort of the worth players outside the actual coalition would obtain in the outside option graph. It differs from the kappa-value in how positions are valued: while the Shapley approach weights according to the strength of swings, the Banzhaf approach weights equally. Recall that weighting according to strength of swings ("weighted voting") can lead to unreasonable allocations in terms of disproportionately high relative distances and, especially in political applications with incompatibilities, is not applicable (see for example Remark 5.8 or Remark 5.15).

Lemma 6.7 (Independence of κ^{Ba}-Axioms)**.** *The axioms **CLBE, OO** and **WBLC** are independent.*

Proof. Consider again the weighted voting game from the introduction ($N = \{1, 2, 3, 4\}$, $w = (39, 30, 25, 6)$ and $T = 60$).

Independence 4 (CLBE+WBLC, ¬ OO). The Banzhaf Position value satisfies **CLBE** and **WBLC**. Consider $g = \{12\}$. We have

$$\pi_1^{Ba}(N, v, g) - \pi_2^{Ba}(N, v, g) = \frac{1}{2} - \frac{1}{2} = 0 \text{ and}$$
$$\pi_1^{Ba}(N, v, g(1)) - \pi_2^{Ba}(N, v, g(2)) = \frac{11}{32} - \frac{9}{32} \neq 0$$

$\Rightarrow$ **OO** is violated.

Independence 5 (CLBE+OO, ¬ WBLC). The allocation rule given by

$$Y_i(N, v, g) := \frac{\sum_{\lambda \in g|_{\mathcal{C}_i}} Ba_\lambda(g|_{\mathcal{C}_i}, v^N|_{\mathcal{C}_i})}{|\mathcal{C}_i|}$$

satisfies **CLBE** by definition and **OO** as for $i, j \in C \in \mathcal{C}(N, g)$ we have $\mathcal{C}_i = \mathcal{C}_j$ and $g(i) = g(j)$. Consider $g = \{12, 23, 34\}$. g is connected and we have

$$\sum_{\lambda \in g_1} \left[Y_2(N, v, g) - Y_2(\mathcal{C}(2)(N, g - \lambda), v|_{\mathcal{C}(2)(N,g-\lambda)}, g|_{\mathcal{C}(2)(N,g-\lambda)})\right]$$
$$= \frac{5}{16} - Y_2(\{2,3,4\}, v|_{\{2,3,4\}}, \{23, 34\}) = \frac{5}{16} - \frac{1}{3} = -\frac{1}{48}$$

and

$$\sum_{\lambda \in g_2} \left[Y_1(N, v, g) - Y_1(\mathcal{C}(1)(N, g - \lambda), v|_{\mathcal{C}(1)(N,g-\lambda)}, g|_{\mathcal{C}(1)(N,g-\lambda)})\right]$$
$$= \frac{5}{16} - Y_1(\{1\}, v|_{\{1\}}, \emptyset) + \frac{5}{16} - Y_1(\{1,2\}, v|_{\{1,2\}}, \{12\})$$
$$= \frac{5}{16} - 0 + \frac{5}{16} - \frac{1}{2} = \frac{1}{8} \neq -\frac{1}{48}$$

$\Rightarrow$ **WBLC** is violated.

Now consider the TU-game (N, v) with $N = \{1, 2, 3, 4\}$ and $v = u_N$.

Independence 6 (WBLC+OO, ¬ CLBE). $Y_i(N, v, g) := \pi_i(N, v, g(i))$ satisfies **OO** by $(g(i))(i) = g(i)$ and **WBLC** by $Y = \pi$ for connected networks. Consider $g = \{12, 34\}$. We have

$$Y_i(N, v, g) = \pi_i(N, v, g(i)) = \frac{1}{4} \text{ and}$$
$$Y_{\{1,2\}} = \frac{1}{2} \neq 0 = \sum_{\lambda \in \{12\}} Ba_\lambda(g|_C, v^C)$$

$\Rightarrow$ **CLBE** is violated.

□

6.5 OUTSIDE-OPTION-SENSITIVITY AND THE KAPPA-VALUES

Recall the formalization of outside options and outside-option-sensitivity from Chapter 4: A coalition $K \subseteq N$ is called *outside option* in (N, v, g) for player $i \in N$ if

$$K \subseteq N \setminus \mathcal{C}_i(g) \text{ and } v(K \cup \{i\}) - v(K) > 0$$

and an allocation rule for network structures Y is called *outside-option-sensitive* (i.e., satisfies **OOS**) if

$$Y_i(N, v, g) > Y_i(N, v_{\tilde{K},i}, g)$$

for all $i \in N$ such that $|\mathcal{C}_i(g)| > 1$ and all $\tilde{K}$ being an outside option for i in (N, v, g) where

$$v_{\tilde{K},i}(K) := \begin{cases} v(K) & \text{, if } K \neq \tilde{K} \cup \{i\} \\ v(\tilde{K}) & \text{, if } K = \tilde{K} \cup \{i\} \end{cases}$$

is the *outside-option-reduced game* w.r.t. $\tilde{K}$ and i.

An allocation rule for network structures is called *weakly outside-option-sensitive* if for all outside options $\tilde{K}$ in (N, v, g) for player $i \in N$ such that $|\mathcal{C}_i(g)| > 1$ there exists a network structure (N, g') with $\tilde{K}$ also being outside option in (N, v, g') for player $i \in N$ such that

$$Y_i(N, v, g') \neq Y_i(N, v_{\tilde{K},i}, g').$$

Analyzing outside-option-sensitivity for the kappa-values will take quite some effort: For player-based values as the graph-χ-value it can be clearly determined whether the effect of neutralizing an outside option is positive or negative as marginal contributions are affected directly (failure of whole nodes). In contrast to that, for link-based values, we have to analyze marginal contributions of links and neutralizing an outside option affects a player's links into her outside option in a different way from this player's other links.

Remark 6.2 (The Kappa-values and **CD**). *Note that the restriction of* (N, v, g) *on a player's connected component leads to the fact that the component-restricted network is connected on the component-restricted* *set of players. In case of connected networks we know that the kappa-values coincide with the corresponding Position values. However, we also know that these values do not coincide in general, hence, the kappa-values cannot be component decomposable.*

We will now show that the (Shapley-) Position value becomes outside-option-sensitive w.r.t. a network g when applying the corresponding LOOG $g(i)$.

Lemma 6.8. *The allocation rule given by*

$$Y_i(N, v, g) := \pi_i(N, v, g(i))$$

is outside-option-sensitive.

Proof. Let (N, v, g) be a TU-game with a network structure, $i \in N$ with $|\mathcal{C}_i(g)| > 1$ and let $\tilde{K}$ be an outside option in (N, v, g) for i. Set $\alpha := v(\tilde{K} \cup \{i\}) - v(\tilde{K}) > 0$ and

$$f(|g'|) := \frac{|g'|!(|g(i)| - |g'| - 1)!}{|g(i)|!} \text{ for all } g' \subseteq g(i).$$

Note that for the arc game v^N we have $v^N(g' \cup \{ij\}) - v^N(g') \neq 0$ if and only if $\mathcal{C}(g' \cup \{ij\} \neq \mathcal{C}(g')$. Particularly, we obtain for $\mathcal{C}(g' \cup \{ij\} \neq \mathcal{C}(g')$:

$$v^N(g' \cup \{ij\}) - v^N(g') = v(\mathcal{C}_i(g' \cup \{ij\})) - v(\mathcal{C}_i(g')) - v(\mathcal{C}_j(g') \quad (9)$$

and, therefore, the outside-option-reduced arc-game w.r.t. $\tilde{K}$ and i, $v^N_{\tilde{K},i}$, only differs from v^N if one of the connected components above becomes $\tilde{K} \cup \{i\}$.

Due to the special form of the LOOG $g(i) = g|_{\mathcal{C}_i} \cup \{jj' | j \in \mathcal{C}_i, j' \in N \setminus \mathcal{C}_i\}$ we have for all $g' \subseteq g(i)$ that $\tilde{K} \cup \{i\} \in \mathcal{C}(N, g')$ if and only if we can write $g' = g^k \cup \tilde{g}$ with $g^k := \{ik | k \in \tilde{K}\}$ and

$$\tilde{g} \subseteq \tilde{G} := \left\{ \left\{ g|_{\mathcal{C}_i(g)} \setminus g_i \right\} \cup \left\{ jj' | j \in \mathcal{C}_i \setminus \{i\}, j' \in N \setminus \{\mathcal{C}_i(g) \cup \tilde{K}\} \right\} \right\}$$

For all $\tilde{k} \in \tilde{K}$ we have $g^k \not\subseteq g'$ for all $g' \subseteq g(i) \setminus \{i\tilde{k}\}$, that is

$$\tilde{K} \cup \{i\} \notin \mathcal{C}(N, g') \, \forall \, g' \subseteq g(i) \setminus \{i\tilde{k}\}.$$

On the other hand we have

$$\tilde{K} \cup \{i\} \in \mathcal{C}(N, g' \cup \{i\tilde{k}\}) \, \forall \, g' = g^k \setminus \{i\tilde{k}\} \cup \tilde{g}, \tilde{g} \in \tilde{G}$$

and we obtain

$$\begin{aligned} Sh_{i\tilde{k}}(g(i), v^N_{\tilde{K},i}) = {} & Sh_{i\tilde{k}}(g(i), v^N) + \sum_{\tilde{g} \subseteq \tilde{G}} f(|\tilde{g} \cup g^k \setminus \{i\tilde{k}\}|) \Big[v^N_{\tilde{K},i}(\tilde{g} \cup g^k) \\ & - \underbrace{v^N_{\tilde{K},i}(\tilde{g} \cup g^k \setminus \{i\tilde{k}\})}_{= v^N(\tilde{g} \cup g^k \setminus \{i\tilde{k}\})} - v^N(\tilde{g} \cup g^k) + v^N(\tilde{g} \cup g^k \setminus \{i\tilde{k}\}) \Big] \\ = {} & Sh_{i\tilde{k}}(g(i), v^N) + \sum_{\tilde{g} \subseteq \tilde{G}} f(|\tilde{g}| + |\tilde{K}| - 1) \Big[v^N_{\tilde{K},i}(\tilde{g} \cup g^k) - v^N(\tilde{g} \cup g^k) \Big] \\ \overset{\text{Equ. 9}}{=} {} & Sh_{i\tilde{k}}(g(i), v^N) + \sum_{\tilde{g} \subseteq \tilde{G}} f(|\tilde{g}| + |\tilde{K}| - 1) \underbrace{\Big[v_{\tilde{K},i}(\tilde{K} \cup \{i\}) - v(\tilde{K} \cup \{i\}) \Big]}_{= -\alpha} \\ = {} & Sh_{i\tilde{k}}(g(i), v^N) - \alpha \sum_{|\tilde{g}|=0}^{|\tilde{G}|} \binom{|\tilde{G}|}{|\tilde{g}|} f(|\tilde{g}| + |\tilde{K}| - 1) \end{aligned}$$

where the last equality follows from the fact that f is a function of the number of links of $\tilde{g}$ only (and not of the actual structure of $\tilde{g}$), hence, we only need to account for subsets of $\tilde{G}$ with the same number of links.

Consider any $j \in N \setminus \tilde{K} \cup \{i\}$ such that $ij \in g(i)$. Here we have

$$\tilde{K} \cup \{i\} \notin \mathcal{C}(N, g' \cup \{ij\}) \, \forall \, g' \subseteq g(i) \setminus \{ij\}.$$

Note that this affects the opposite part of the marginal contributions as before and we obtain

$$\begin{aligned} Sh_{ij}(g(i), v^N_{\tilde{K},i}) = {} & Sh_{ij}(g(i), v^N) + \sum_{\tilde{g} \subseteq \tilde{G}} f(|\tilde{g} \cup g^k|) \Big[\underbrace{v^N_{\tilde{K},i}(\tilde{g} \cup g^k \cup \{ij\})}_{= v^N(\tilde{g} \cup g^k \cup \{ij\})} \\ & - v^N_{\tilde{K},i}(\tilde{g} \cup g^k) - v^N(\tilde{g} \cup g^k \cup \{ij\}) + v^N(\tilde{g} \cup g^k) \Big] \\ = {} & Sh_{ij}(g(i), v^N) + \sum_{\tilde{g} \subseteq \tilde{G}} f(|\tilde{g}| + |\tilde{K}|) \Big[-v^N_{\tilde{K},i}(\tilde{g} \cup g^k) + v^N(\tilde{g} \cup g^k) \Big] \\ \overset{\text{Equ. 9}}{=} {} & Sh_{ij}(g(i), v^N) + \sum_{\tilde{g} \subseteq \tilde{G}} f(|\tilde{g}| + |\tilde{K}|) \underbrace{\Big[v(\tilde{K} \cup \{i\}) - v_{\tilde{K},i}(\tilde{K} \cup \{i\}) \Big]}_{= \alpha} \\ = {} & Sh_{ij}(g(i), v^N) + \alpha \sum_{|\tilde{g}|=0}^{|\tilde{G}|} \binom{|\tilde{G}|}{|\tilde{g}|} f(|\tilde{g}| + |\tilde{K}|) \end{aligned}$$

Note that there are $(|g_i| + |N| - |\mathcal{C}_i| - |\tilde{K}|)$ such j.

Using the formulas we derived above we obtain for $Y(N, _{\tilde{K},i}, g) = \pi_i(N, v_{\tilde{K},i}, g(i))$

$$\pi_i(N, v_{\tilde{K},i}, g(i)) = \pi_i(N, v, g(i)) - \frac{\alpha}{2} \underbrace{|\tilde{K}| \sum_{|\tilde{g}|=0}^{|\tilde{G}|} \binom{|\tilde{G}|}{|\tilde{g}|} f(|\tilde{g}| + |\tilde{K}| - 1)}_{:=B_1}$$
$$+ \frac{\alpha}{2} \underbrace{(|g_i| + |N| - |\mathcal{C}_i| - |\tilde{K}|) \sum_{|\tilde{g}|=0}^{|\tilde{G}|} \binom{|\tilde{G}|}{|\tilde{g}|} f(|\tilde{g}| + |\tilde{K}|)}_{:=B_2}$$
$$\Rightarrow \pi_i(N, v_{\tilde{K},i}, g(i)) = \pi_i(N, v, g(i)) + \frac{\alpha}{2}(-B_1 + B_2) \tag{10}$$

We now show that $B := -B_1 + B_2$ is strictly negative. Note that

$$|\tilde{G}| = |g|_{\mathcal{C}_i}| - |g_i| + (|\mathcal{C}_i| - 1)(|N| - |\mathcal{C}_i)| - |\tilde{K}|)$$
$$\text{and } |g(i)| = |g|_{\mathcal{C}_i}| + |\mathcal{C}_i|(|N| - |\mathcal{C}_i|)$$

leading to $|\tilde{G}| = |g(i)| - A$ with $A := |g_i| + |\mathcal{C}_i| \cdot |\tilde{K}| + |N| - |\mathcal{C}_i| - |\tilde{K}|$. Rearranging terms and because $\mathcal{C}_i$ contains at least 2 players[6] we get

$$A - (|g_i| + |N| - |\mathcal{C}_i|) = |\tilde{K}| \underbrace{(|\mathcal{C}_i| - 1)}_{\geqslant 1} \tag{11}$$
$$\Rightarrow A - \underbrace{(|g_i| + |N| - |\mathcal{C}_i|)}_{>0} \geqslant |\tilde{K}| \tag{12}$$

From Equ. 12 we get $A > |\tilde{K}|$ which allows us to apply the following combinatorical finding[7]: For all $k < A \leqslant G$ we have

$$\sum_{l=0}^{G-A} \binom{G-A}{l} \frac{(k+l)!(|G| - 1 - (k+l))!}{|G|!} = \frac{k!(A-1-k)!}{A!} \tag{13}$$

Using this we get

$$\sum_{|\tilde{g}|=0}^{|\tilde{G}|} \binom{|\tilde{G}|}{|\tilde{g}|} f(|\tilde{g}| + |\tilde{K}| - 1) = \sum_{l=0}^{|g(i)|-A} \binom{|g(i)| - A}{l} f(l + |\tilde{K}| - 1)$$
$$\overset{\text{Equ. 13}}{=} \frac{(|\tilde{K}| - 1)!(A - 1 - (\tilde{K} - 1))!}{A!}$$

6 $|\mathcal{C}_i| > 1$, that is, $|\mathcal{C}_i| \geqslant 2$ by requirement of **OOS**.
7 The proof follows by the same steps as in van den Nouweland [1993], page 29-30.

$$\Rightarrow B_1 = |\tilde{K}|\frac{(|\tilde{K}|-1)!(A-\tilde{K})!}{A!} = (A-\tilde{K})\frac{|\tilde{K}|!(A-1-\tilde{K})!}{A!} \tag{14}$$

$$\sum_{|\tilde{g}|=0}^{|\tilde{G}|}\binom{|\tilde{G}|}{|\tilde{g}|}f(|\tilde{g}|+|\tilde{K}|) = \sum_{l=0}^{|g(i)|-A}\binom{|g(i)|-A}{l}f(l+|\tilde{K}|)$$

$$\overset{\text{Equ. 13}}{=} \frac{|\tilde{K}|!(A-1-\tilde{K})!}{A!}$$

$$\Rightarrow B_2 = (|g_i|+|N|-|\mathcal{C}_i|-|\tilde{K}|)\frac{|\tilde{K}|!(A-1-\tilde{K})!}{A!} \tag{15}$$

By Equ. 14 and Equ. 15 we get for $B = B2 - B1$:

$$\begin{aligned} B =& \frac{|\tilde{K}|!(A-1-\tilde{K})!}{A!}\left[(|g_i|+|N|-|\mathcal{C}_i|-|\tilde{K}|-(A-\tilde{K}))\right] \\ =& \frac{|\tilde{K}|!(A-1-\tilde{K})!}{A!}\underbrace{[|g_i|+|N|-|\mathcal{C}_i|-A]}_{=-|\tilde{K}|(|\mathcal{C}_i|-1)\text{ by Equ. 11}} \\ =& -|\tilde{K}|\underbrace{(|\mathcal{C}_i|-1)}_{>0}\underbrace{\frac{|\tilde{K}|!(A-1-\tilde{K})!}{A!}}_{>0} < 0 \end{aligned}$$

Using this in Equ. 10, we finally get **OOS**:

$$Y_i(N, v_{\tilde{K},i}, g) = \pi_i(N, v_{\tilde{K},i}, g(i)) < \pi_i(N, v, g(i)) = Y_i(N, v, g)$$

□

Lemma 6.9 (Outside-Option-Sensitivity of the Kappa-value)**.** *The kappa-value is outside-option-sensitive.*

Proof. Let (N, v, g) be a TU-game with a network structure, $i \in N$ with $|\mathcal{C}_i(g)| > 1$ and let $\tilde{K}$ be an outside option in (N, v, g) for i. Consider $j \notin \tilde{K} \cup \{i\}$. For all $jj' \in g(i)$ we have

$$\mathcal{C}_j(g' \cup \{jj'\}) \not\subseteq \tilde{K}\,\forall g' \subseteq g(i) \setminus \{jj'\}$$

and hence

$$v^N(g' \cup \{jj'\}) - v^N(g') \leqslant v^N_{\tilde{K},i}(g' \cup \{jj'\}) - v^N_{\tilde{K},i}(g')\,\forall g' \subseteq g(i) \setminus \{jj'\} \tag{16}$$

which leads to

$$Sh_{jj'}(g(i), v^N_{\tilde{K},i}) \geqslant Sh_{jj'}(g(i), v^N)\,\forall jj' \in g(i) : j \notin \tilde{K} \cup \{i\}. \tag{17}$$

We obtain strict inequality in Equ. 16 for example for $g' = \{ik | k \in \tilde{K}\}$ and therefore, since for $j \in \mathcal{C}_i \setminus \{i\}$ we have that $jk \in g(i)$ for all $k \in \tilde{K}$, we obtain strict inequality in Equ. 17 for all jk, $k \in \tilde{K}$ which leads to

$$\pi_j(N, \nu_{\tilde{K},i}, g(i)) > \pi_j(N, \nu, g(i)) \, \forall j \in \mathcal{C}_i \setminus \{i\}.$$

Using this and by Lemma 6.8 we get **OOS** for the kappa-value:

$$\begin{aligned}
\kappa_i(N, \nu_{\tilde{K},i}, g) &= \pi_i(N, \nu_{\tilde{K},i}, g(i)) + \frac{\nu_{\tilde{K},i}(\mathcal{C}_i(g)) - \sum_{j \in \mathcal{C}_i(g)} \pi_j(N, \nu_{\tilde{K},i}, g(i))}{|\mathcal{C}_i(g)|} \\
&= \left(1 - \frac{1}{|\mathcal{C}_i(g)|}\right) \underbrace{\pi_i(N, \nu_{\tilde{K},i}, g(i))}_{\substack{< \pi_i(N,\nu,g(i)) \\ \text{by Lemma 6.8}}} + \frac{\nu(\mathcal{C}_i(g)) - \sum_{\substack{j \in \mathcal{C}_i(g) \\ j \neq i}} \overbrace{\pi_j(N, \nu_{\tilde{K},i}, g(i))}^{> \pi_j(N,\nu,g(i))}}{|\mathcal{C}_i(g)|} \\
&< \left(1 - \frac{1}{|\mathcal{C}_i(g)|}\right) \pi_i(N, \nu, g(i)) + \frac{\nu(\mathcal{C}_i(g)) - \sum_{\substack{j \in \mathcal{C}_i(g) \\ j \neq i}} \pi_j(N, \nu, g(i))}{|\mathcal{C}_i(g)|} \\
&= \kappa_i(N, \nu, g)
\end{aligned}$$

□

The result itself that the Position value of the LOOG (w.r.t. to the original network) and the kappa-value are indeed outside-option-sensitive seems not very surprising at first glance by the intuition of the LOOG. Also, we have already discussed in Remark 4.6, that in presence of outside-option-sensitivity and and component efficiency the loss player i experiences by neutralizing her outside option is distributed as a surplus among the other players in i's connected component. However, some notable finding occured within the proof of Lemma 6.8: We obtained that not the whole loss player i experienced has been distributed among the other players: Player i experienced a loss due to her links into her outside option, but this loss was partly redistributed onto player i's other links. Hence, not the whole loss caused by neutralization is shifted as a surplus towards the other players anymore. This difference occurs as we do not analyze contributions of whole nodes, but rather links.

This brings us back to the discussion and investigation in Chapter 5 about consequences of failure: Intuitively, the neutralization of i's outside option reduces overall failure consequences of player i (i.e., the payoff decreases). However, only consequences of the failure of player i's links into the outside option decrease as only these links actually connect player i to the outside option. The other links indeed become more important as the overall value does not change.

Note that this sheds some light on the difference between the weak axiom **WOOS** and **OOS** and why **OOS** is indeed sort of "strong". When we argued that **OOS** is not a very strong axiom, we analyzed failure consequences of whole nodes (i.e., player-based values) and only considered an increase of importance of other nodes. We did not take into account the aforementioned redistribution: neutralizing i's outside option leads to an increase of importance of i's other links and this effect could possibly overcompensate the loss of importance of links into the outside option. We have seen that this problem did not occur for the link-based values that used Shapley's weighted voting approach. However, this effect has an impact for equal voting: Remember that the Shapley weights might be seen as disproportionately and we argued in Remark 5.15 that Banzhaf's equal voting approach might be seen as more moderate. We will now see that this difference in weighting indeed might change the sign of the overall effect of neutralizing an outside option.

Lemma 6.10 (Weak Outside-Option-Sensitivity of the Banzhaf-Kappa-value)**.** *The Banzhaf-kappa-value is weakly outside-option-sensitive.*

Proof. Let (N, v, g) be a TU-game with a network structure, $i \in N$ with $|\mathcal{C}_i(g)| > 1$ and let $\tilde{K}$ be an outside option in (N, v, g) for i. We proceed as in the proof of Lemma 6.8 and Lemma 6.9. Note that the Banzhaf approach does not use the weighting factor which we defined by the funtion $f(|g'|)$ but a scalar only depending on $|g(i)|$. Following the same steps as in the proof of Lemma 6.8 we get for $k \in \tilde{K}$ and $j \notin \tilde{K} \cup \{i\}$

$$Ba_{i\tilde{k}}(g(i), v^N_{\tilde{K},i}) = Ba_{i\tilde{k}}(g(i), v^N) - \alpha \sum_{|\tilde{g}|=0}^{|\tilde{G}|} \binom{|\tilde{G}|}{|\tilde{g}|} \frac{1}{2^{|g(i)|-1}} \text{ and}$$

$$Ba_{ij}(g(i), v^N_{\tilde{K},i}) = Ba_{ij}(g(i), v^N) + \alpha \sum_{|\tilde{g}|=0}^{|\tilde{G}|} \binom{|\tilde{G}|}{|\tilde{g}|} \frac{1}{2^{|g(i)|-1}}$$

which, after some calculations, leads to

$$\pi_i^{Ba}(N, v_{\tilde{K},i}, g(i)) = \pi_i^{Ba}(N, v, g(i)) + \frac{\alpha 2^{|\tilde{G}|}}{2^{|g(i)|}} \left(|g_i| + |N| - |\mathcal{C}_i| - 2|\tilde{K}|\right)$$

which does not always have to be smaller than $\pi_i^{Ba}(N, v, g(i))$, hence, we cannot proceed as for the (Shapley-) kappa-value.

Analogue to how we calculated $\pi_i^{Ba}(N, v, g(i))$, one can show that for $j \in \mathcal{C}_i \setminus \{i\}$ we have

$$\pi_j^{Ba}(N, v_{\tilde{K},i}, g(i)) = \pi_j^{Ba}(N, v, g(i)) + \begin{cases} \frac{\alpha 2^{|\tilde{G}|}}{2^{|g(i)|}} \left(|\tilde{K}| + 1\right) & \text{, if } ij \in g \\ \frac{\alpha 2^{|\tilde{G}|}}{2^{|g(i)|}} \left(|\tilde{K}|\right) & \text{, if } ij \notin g \end{cases}$$

which is always greater than $\pi_j^{Ba}(N, v, g(i))$. We get

$$\begin{aligned} \kappa_i^{Ba}(N, v_{\tilde{K},i}, g) = \kappa_i^{Ba}(N, v, g) + \underbrace{\frac{\alpha 2^{|\tilde{G}|}}{2^{|g(i)|}}}_{:=\beta} \Big(|g_i| + |N| - |\mathcal{C}_i| - 2|\tilde{K}|\Big) \\ - \underbrace{\frac{\alpha 2^{|\tilde{G}|}}{2^{|g(i)|}}}_{=\beta} \frac{1}{|\mathcal{C}_i|} \left(2|g_i| + |N| - |\mathcal{C}_i| - 2|\tilde{K}| + (|\mathcal{C}_i| - 1)|\tilde{K}|\right) \end{aligned} \tag{18}$$

To prove **WOOS** we have to find a network on N under which outside option $\tilde{K}$ occurs and for which the part behind $\kappa_i^{Ba}(N, v, g)$ will be different from zero. Consider network $\bar{g}$ such that $\mathcal{C}(N, \bar{g}) = \{\{\tilde{K}\}, \{N \setminus \tilde{K}\}\}$ and $|\bar{g}_i| = 1$ (i.e., i is connected to only one other player and all players either are in i's connected component or in the outside option $\tilde{K}$. Obviously, if $\tilde{K}$ is an outside option for i in (N, v, g), it is an outside option in $(N, v, \bar{g})$. Using $|\tilde{K}| = |N| - |\mathcal{C}_i(\bar{g})|$ we obtain

$$\begin{aligned} \kappa_i^{Ba}(N, v_{\tilde{K},i}, \bar{g}) &= \kappa_i^{Ba}(N, v, \bar{g}) + \beta\Big(1 + |N| - |\mathcal{C}_i| - 2(|N| - \mathcal{C}_i)\Big) \\ &\quad - \frac{\beta}{|\mathcal{C}_i|}\left(2 + |N| - |\mathcal{C}_i| - 2(|N| - \mathcal{C}_i) + (|\mathcal{C}_i| - 1)(|N| - \mathcal{C}_i)\right) \\ &= \kappa_i^{Ba}(N, v, \bar{g}) + \frac{\beta}{|\mathcal{C}_i|}\Big(|\mathcal{C}_i| - |\mathcal{C}_i|(|N| - |\mathcal{C}_i|) - 2 - (|\mathcal{C}_i| - 2)(|N| - \mathcal{C}_i)\Big) \\ &= \kappa_i^{Ba}(N, v, \bar{g}) + \underbrace{\frac{\beta}{|\mathcal{C}_i|}}_{>0}\Big(\underbrace{|\mathcal{C}_i| - 2\big((|\mathcal{C}_i| - 1)(|N| - \mathcal{C}_i) + 1\big)}_{:=F(c,n) \text{ with } c:=|\mathcal{C}_i|, n:=|N|}\Big) \end{aligned}$$

Analyzing roots of the function $F(n, c)$ we obtain that any solution is of the form

$$c - 1 \neq 0 \text{ and } n = \frac{2c^2 - c - 2}{2(c-1)}$$

where the only integer solutions (remember that c and n are cardinalities) are

$$c = 0, n = 1 \text{ and } c = 2, n = 2$$

which both will not occur since $|\mathcal{C}_i| > 1$ (first solution impossible) and $|N| - |\mathcal{C}_i| = |\tilde{K}| > 0$ (second solution impossible), hence we have

$$\kappa_i^{Ba}(N, v_{\tilde{K},i}, \bar{g}) \neq \kappa_i^{Ba}(N, v, \bar{g})$$

which provides **WOOS**. □

Lemma 6.11. *The Banzhaf-kappa-value is not (strongly) outside-option-sensitive.*

Proof. From Equ. 18 in the previous proof we know that

$$\kappa_i^{Ba}(N, v_{\tilde{K},i}, g) = \kappa_i^{Ba}(N, v, g) + \frac{\beta}{c} F(n, c, g, k)$$
$$\text{with } F(n, c, g, k) := c(g + n - c - 2k) - (2g + n - c - 2k + (c-1)k)$$
$$\text{and } n := |N|, c := |\mathcal{C}_i|, g := |g_i|, k := |\tilde{K}|$$

and we have $F(n, c, g, k) = 0$ for example for

$$n = 5, c = 2, g = 1 \text{ and } k = 1$$

which contradicts **OOS** as in this case we have

$$\kappa_i^{Ba}(N, v_{\tilde{K},i}, g) = \kappa_i^{Ba}(N, v, g)$$

that is, neutralizing $\tilde{K}$ does not change payoffs as the negative effect player i experiences by neutralizing $\tilde{K}$ (power of links into $\tilde{K}$decrease) equalizes the positive effect player i experiences by neutralizing $\tilde{K}$ (power of other links increase). □

Remark 6.3 (Strength of **OOS**). *Note that the Banzhaf-kappa-value, even though it does not satisfy **OOS**, can still be seen as being affected by outside options in a very general sense. Neutralizing any outside option always negatively affects a player's Banzhaf-kappa-value as ev-*

ery player's link (of the LOOG) into the outside option decrease in its importance by $\frac{\alpha 2^{|\tilde{G}|}}{2^{|g(i)|-1}} > 0$*. However, due to the link-based-approach, the player's other links increase in their power by the same factor. This fact causes a positive effect. Clearly, the overall effect can be positive, negative or even zero, depending on the number of links into the outside option (which is the size of the outside option itself) and the number of other links the player has in the LOOG. However, there will always be an inner effect. This is why one could argue that* ***OOS*** *actually is a strong axiom as it ignores these inner effects.*

6.6 CONCLUSION

Motivated by the analysis of weighted voting games, we analyzed the Position value, which takes into account the position of a player within the network, and the graph-χ-value, which takes into account outside options. We found that, for this class of games, the graph-χ-value does not differ within networks referring to the same coalititon, hence does not take into account the position an agent has within the network. This motivated the use of the Position value in order to capture the fact that the position within the network/the path of information flow matters. But the Position value does generally not take into account outside options. We defined and characterized a new allocation rule for networks which combines outside option sensitivity and sensitivity to the position of an agent within the network: the kappa-value. The kappa-value provides an elegant use of the intuitive concept of the Position value, lacking its drawbacks by using the outside option graph used for the concept of the graph-χ-value. There is no need for new characterizing axioms, only known and approved ones (or weakened versions of them) are used. As a variant of the kappa-value, we further introduced the Banzhaf-kappa-value which might be more applicable especially for political applications and/or incompatibilities.

This chapter provided an extension of the approaches presented in Chapter 5, now applicable if outside options are of interest. Hence, for forecasting issues, the approaches from Chapter 5 can be used if outside options are absent while the extensions of this chapter solve the problem whenever outside options are relevant. Furthermore, during the analysis of outside-option-sensitivity of the new link-based values we found that outside options actually have a more complex effect than discussed in Chapter 4. This led to the finding that the suggested "weak" axiom indeed is not that weak while the "normal" axiom indeed is sort of strong. By this we extended of the discussion of outside-option-sensitivity in Chapter 4 and can now understand the effect of outside options in a highly detailed sense.

7 CONCLUSION AND OUTLOOK

7.1 CLOSING THE CIRCLE: SUMMARY

In this thesis we have analyzed outside options from various perspectives to understand their effects and formal construction, especially in the framework of networks. We started the analysis with a pilot experiment where we investigated the effect of outside options on negotiation in double auction markets. We found evidence that outside options do affect negotiation which shed some first light on the effect and importance of outside options. Analyzing a probabilistic forecasting model theoretically supported the importance of outside options, therefore, we explicitly formalized outside options and outside-option-sensitivity axioms to understand how to formally differentiate between outside-option-sensitive and -insensitive allocation rules. Applying these axioms shed some further light on the structural effects of outside options on allocation. During this analysis, we found that there is only one outside-option-sensitive allocation rule in the framework of network structures and also further issues ocurred that motivated a deeper analysis of network structures in general. First focussing on situations where outside options are absent, we provided a new (axiomatically characterized) allocation rule based on centrality analysis. We discussed its performance in application on political networks as a power index and in application as centrality measure for indetifying top key nodes. Beside discussions on moderate relative proportions ("Banzhaf vs. Shapley") we found that accounting for consequences of link failure rather than node failure leads to more convincing and appropriate performances in applications for measuring political power or centrality which emphazised the drawbacks of player-based values and the advantages of link-based values. Finally, we derived and characterized link-based allocation rules that account for centrality within networks *and* are outside-option-sensitive. By the analysis of the effect of outside options on these allocation rules we found that this effect is actually more complex in the link-based setting as outside options actually affect structurally different links in a different way. This finally provided an understanding of the effect of outside options in a highly detailed sense. This basically not only completes our "golden thread", it indeed even closes a circle by the discussion in the end of Chapter 6 leading back to a deeper understanding of the axioms suggested in Chapter 4.

7.2 FURTHER RESEARCH: OUTLOOK

We have raised issues that could be of interest for further research in the conlusion parts of the previous chapters as for example a deeper analysis of learning effects and the relation to the ultimatum game for Chapter 3, a further analysis of the explicit structural effects of outside options on allocation for Chapter 4, a deeper analysis of the eigenvector centrality approach or applications for weighted networks for Chapter 5 or applications of the kappa-values from Chapter 6 for unconnected political networks.

Moreover, there is another issue that is connected to the analysis of the probabilistic forecasting model in Chapter 4 which fits in the "golden thread" of this thesis and seems appealing: Recall the difference between the cooperative and the non-cooperative approach pointed out by Robert Aumann:

> Perhaps a better name for cooperative would be "outcome oriented" [..] and for noncooperative "strategically oriented". [In: Damme, 1998, p. 196]

In this thesis, we analyzed networks and outside options from the "outcome oriented" perspective only. We discussed the need of allocation rules to be outside-option-sensitive and provided such rules. However, the existence of outside options of an agent might bear a certain *risk for the other agents involved*. Recall von Neumann and Morgenstern [1944] (p. 36): "Even if [...] one particular alliance is actually formed, the others are present in virtual existence [...]." While we considered and analyzed the presence of this virtual existence and suggested to provide higher payoffs to agents that obtain bargaining power due to virtually existing alternatives, we did not consider that these agents might *actually deviate* from the materialized situation to one of the situations that virtually exist. We did not analyze how the other agents should account for that from a *strategic point of view*, that is, taking into account the *risk* that arises due to outside options.

When networks are of interest, sooner or later the question of stability arises: stable networks can be seen as an equilibrium of a strategic game, can be used as a proxy for forecasting issues and it might be reasonable that an evolutionary process converges to a stable network. In contrast to the generalized models discussed in Chapter 4 that took into account the likelihood of possible structures, one alternatively could analyze a formation process and arising equilibria. In most stability concepts, a network is said to be stable if there are no *beneficial deviations*, where the sort of deviation differs corresponding to the stability concept. However, existing stability concepts analyze outcomes of networks as if these networks have already materialized ("outcome oriented"). Doing so, the existence of alternatives that are present before the actual formation of a network (more precisely, indifference of agents due to outside options), is ignored. This obviously becomes problematic if the analysis of stability is for example used for forecasting issues: the ignorance of existing alternatives can lead to an overestimation of the evaluation of a network and hence, a network is classified as being stable while it is not likely to actually occur from a strategic point of view.

For further research we will take an allocation rule as given (used as a payoff punction) and analyze the impact of outside options on stability. Hence, we will shift our point of view a bit more to the "strategically oriented" perspective.

For motivating issues, consider the following example: There are 3 parties in a government with the following seat shares after an election:

party	1	2	3
seat share	50	25	25

To pass a bill, a threshold of 75% is needed. Now, parties have to simultaneously build bilateral binding agreements with each other and only after these agreements have been build, the resulting coalitions are formed. Let (relative) political power (i. e., payoffs) of parties in the winning coalition strictly decrease in the number of parties in the winning coalition[1]. Existing stability concepts as pairwise stability (Jackson and Wolinsky, 1996) suggest agreements between parties 1 and 2 or parties 1 and 3 or between *all* parties as being stable, while the latter is not stable under coalitional deviations (strong stability, cf. Jackson and van den Nouweland, 2005). Considering the refinement of strongly stable agreements (party 1 with *either* party 2 *or* 3) suggests that an agreement between parties 2 and 3 is unstable.

In these network stability concepts, parties 2 and 3 evaluate their outcome of building agreements comparing payoffs of materialized coalitions. In fact, party 1 is indifferent between building an agreement with 2 or 3 (but not with both of them due to strictly decreasing power) and will do one or the other while her final choice is unknown by 2 and 3. Following this, one could argue that parties 2 and 3 should evaluate their outcomes as an expected payoff of realizations of the indifferent actions of party 1 according to their *beliefs* of what party 1 will do. In the end, it is quite likely that parties 2 and 3 build an agreement in order to lower their risk. Hence, the structures suggested to be strongly stable might not be likely to actually occur.

Interpreting proposals of building agreements as strategies, we can model the situation by a coordination game: For simplicity, assume party 3 would build all agreements proposed to her (which is a priori fixed). For party i ($i = 1, 2$) let x_i and y_i with $0 < y_i < x_i$ denote party i's relative political power in a coalition with two parties and three parties, respectively. Consider the (simplified) payoff matrix of parties 1 and 2 according to their strategies:

		Party 1	
		choose 2	choose 3
Party 2	choose only 1	x_1, x_2	$x_1, 0$
	choose 1& 3	y_1, y_2	y_1, y_2

We see that both {choose 2, choose only 1} and {choose 3, choose 1 & 3} are Nash equilibria of the simplified game (if we take party 3's strategy as a priori fixed). While the first one (no agreement between 2

1 If a party joins the winning coalition, it obtains a payoff strictly greater than zero.

and 3) provides a higher payoff for party 2 and the latter (where 2 and 3 build an agreement) is less risky for party 2.

This difference is related to the idea behind the two equilibrium selection possibilities of Harsanyi and Selten [1988]: selection by *payoff dominance*, that is, providing the highest payoff for all agents, and *risk dominance*, that is, minimizing deviation losses of all agents (risk averse selection). If we only consider party 2's strategies, the strategy ″choose only 1″ payoff dominates ″choose 1 & 3″ while the strategy ″choose 1 & 3″ risk dominates ″choose only 1″. While the equilibrium in which there is no agreement between 2 and 3 is indeed payoff dominant (also player 1 obtains a higher payoff), the one with the agreement is not risk dominant as player 1's deviation losses of both equilibria are equal due to indifference of player 1.

Schmidt et al. [2003] find in their experimental study on payoff and risk dominance, that changes in the level of risk dominance significantly affects behavior while changes in the level of payoff dominance does not. Hence, accounting for risk of agents seems favorable. Note that the concept of risk dominance is not clearly determined for games with more than two players and also, indifference as in the example above cannot be valued properly.

For further research, we will follow Dutta et al. [1998] and Myerson [1991] who model the process of network formation via a strategic form game in which payoffs are determined by a cooperative allocation rule (which is exogenously given). Taking this game as a basis, we will define a transformed game which uses beliefs over classes of payoff equivalent strategies in terms of materialized outcomes (i. e., outside options with the same impact/value). This will be used to define *indifference-proof* stable networks following the idea behind the concepts of Nash equilibria (Nash, 1951), pairwise stability (Jackson and Wolinsky, 1996) and strong stability (Jackson and van den Nouweland, 2005). One could interpret this kind of stability concepts as concepts where agents aim to lower their risk, that is, as risk averse stability concepts.

Part IV

APPENDIX

A APPENDIX CHAPTER 3

A.1 COMPUTATIONAL EFFORT ANALYSIS

Corollary A.1. *Computational effort of*

$$Sh_i(N, v_{gg}) = \begin{cases} \frac{1}{2} + \frac{W-S}{2\cdot S} \sum_{k=1}^{S} \frac{W!S!}{(W+k)!(S-k)!} & \text{, if } i \text{ is strong player} \\ \frac{1}{2} - \frac{W-S}{2\cdot W} \sum_{k=0}^{S} \frac{W!S!}{(W+k)!(S-k)!} & \text{, if } i \text{ is weak player} \end{cases}$$

is of polynomial order.

Proof. First of all, one only needs to compute 2 expressions and not an expression for every $i \in N$ individually. The summation can be approximated from above by $S-1$ times computing the expressions $(|N|-S)!$ and $S!$ in the nominator and again $(|N|-S)!$ and $S!$ in the denominator. Hence, computational effort can be approximated by $O((|N|-S)^4 \cdot S^4 \cdot (S-1) \cdot \log^2(|N|-S) \cdot \log^2(S))$ (and computing the fraction in front of the sum, but this will not change the polynomial order). □

Remark A.1. *Computational effort of the original* AD*-value as well as the modified version for glove games and minimal winning coalitions is negligible. However, imbalancedness of the market is underestimated/ignored.*

Corollary A.2. *Computational complexity of the general formula of the* χ *value is at least of order* $O\left(2^{n \log n}\right)$ *while computational complexity of the special case of glove games for efficient coalition structures is of polynomial order.*

Proof. Follows from the findings for the Shapley formula. □

Lemma A.1. *Consider a glove game* $(N, v_{gg}, \mathcal{P})$ *and let* $\mathcal{P}$ *be efficient with* $S > 0$*. While computational complexity of the general Owen formula is higher than polynomial order, computational complexity of the special case of glove games for efficient coalition structures is of polynomial order.*

Proof. Consider the general formula of the Owen value:.

$$Ow_i(N, v, \mathcal{P}) := \frac{1}{|\Sigma(N, \mathcal{P})|} \sum_{\sigma \in \Sigma(N, \mathcal{P})} [v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})]$$

Approximating computational effort of the marginal contribution by 1 for each $\sigma \in \Sigma(N, \mathcal{P})$, computational effort of the sum can be approximated from below by $|\Sigma(N, \mathcal{P})|$ computations. This has to be multiplied by $|N|$ (calculation has to be done for each agent $i \in N$). Additionally, $\Sigma(N, \mathcal{P})$ has to be computed. In case of efficient coalition structures and $S > 0$, $|\Sigma(N, \mathcal{P})|$ can be approximated from below by $\left\lfloor \frac{|N|}{2} \right\rfloor !$: there are W components in $\mathcal{P}$ and, hence, $W!$ permutations of components and for every of the S pairs, there are two inner permutations. Now neglect multiplicity due to inner permutations and use that $W \leqslant \frac{|N|}{2}$.

To compute $\Sigma(N, \mathcal{P})$, one has to check for each possible order σ over N whether $\sigma \in \Sigma(N, \mathcal{P})$. As $\mathcal{P}$ is efficient, this is checking whether $|\sigma(i) - \sigma(j)| = 1$ for each pair (i, j) in $\mathcal{P}$. There are $|N|!$ possible orders over N and S pairs.

Hence, computational complexity is, approximated from below, at least of order

$$O\left(\left\lfloor \frac{|N|}{2} \right\rfloor ! \cdot |N| + |N|! \cdot S\right) \geqslant O\left(\left\lfloor \frac{|N|}{2} \right\rfloor ! \cdot |N| + |N|!\right)$$

Now consider the new formula given in Equation (4). Following Corollary A.1, the expression is of polynomial order. □

A.2 SUPPLEMENTARY: INSTRUCTIONS AND SCREENS

The following instructions are translated from the original German instructions.

INSTRUCTIONS

PRELIMINARY REMARK. You are participating at a study of decision making behavior in the context of experimental economics. During the study you and the other participants will be asked to make decisions. You can earn money with this study. How much money you earn is depending on your decisions. Directly after the experiment you are paid in cash. The experiment lasts approximately 60 minutes. All participants receive exactly the same instructions and orders. No participant will receive any information about the identity of the other participants during the experiment. This experiment consists of two parts. After part 1 you will receive the instructions for part 2.

INSTRUCTIONS PART 1

NOTES ON READING. Please read the following instructions. Approximately five minutes after you received the instructions, we will come to you and answer open questions. Please note that during the first part of the experiment, no further questions can be answered anymore.

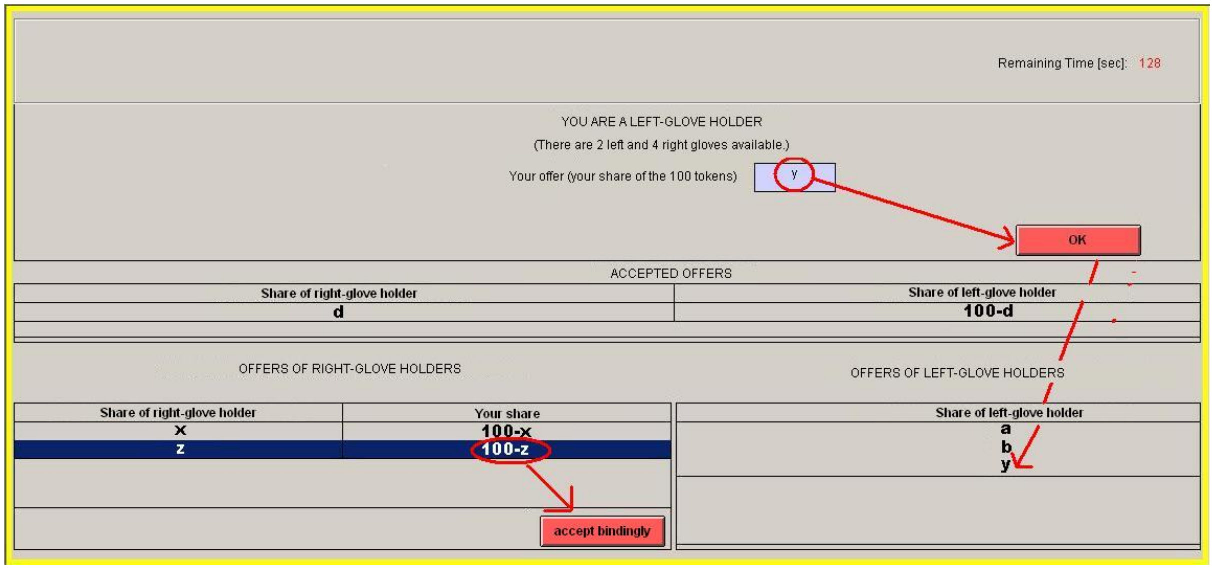
THE EXPERIMENT. You play in a group of in total 6 participants, the composition of the group stays unchanged over the whole experiment. There will be two (imaginary) left gloves and four (imaginary) right gloves randomly distributed in your group, this means you are either a left-glove holder or a right-glove holder. A matching pair consists of exactely one right and one left glove. You can try to find a partner in order to build such a matching pair. Please note that there are more right gloves than left gloves, hence, **not every participant in your group will be able to find a partner**. In total **two matching pairs** can be built in your group.

PROCESS OF BUILDING A MATCHING PAIR. A matching pair has a value of 100 tokens (where 1 token is 12.5 Euro-Cents, that is, 100 tokens are EUR 12.50). A matching pair is built if a left-glove holder and a right-glove holder agree on a distribution of these 100 tokens. You can make offers by inserting your share of the 100 tokens in the window "Your offer". Your offer will, visible for all participants in your group, appear on the screen an can be taken by any holder of a matching glove. By this, a matching pair is built. The other participants can also make offers and you can take offers from holders of a matching glove. To take an offer, select with your mouse the desired offer from the list and confirm with "Accept bindingly". Taken offers will disappear from the list of offers and will appear in the overlying list "Accepted offers". As soon as a player enters into any agreement, all not-accepted offers of this player will disappear from the screen.
Please note that, as soon as you made an offer, you cannot take it back. If another participant takes this offer, it will be binding and the matching pair is built. You can make as many offers as you like, however, all of your offers will remain in the offers-list and all of your offers can be taken. If you accept an offer of another participant, it will also be binding.
Part 1 is finished as soon as **two matching pairs have been built** or if the **playing time of 2 minutes (120 seconds)** has expired, even if no two matching pairs have been built until then.

Please find examplary screens on the next pages.

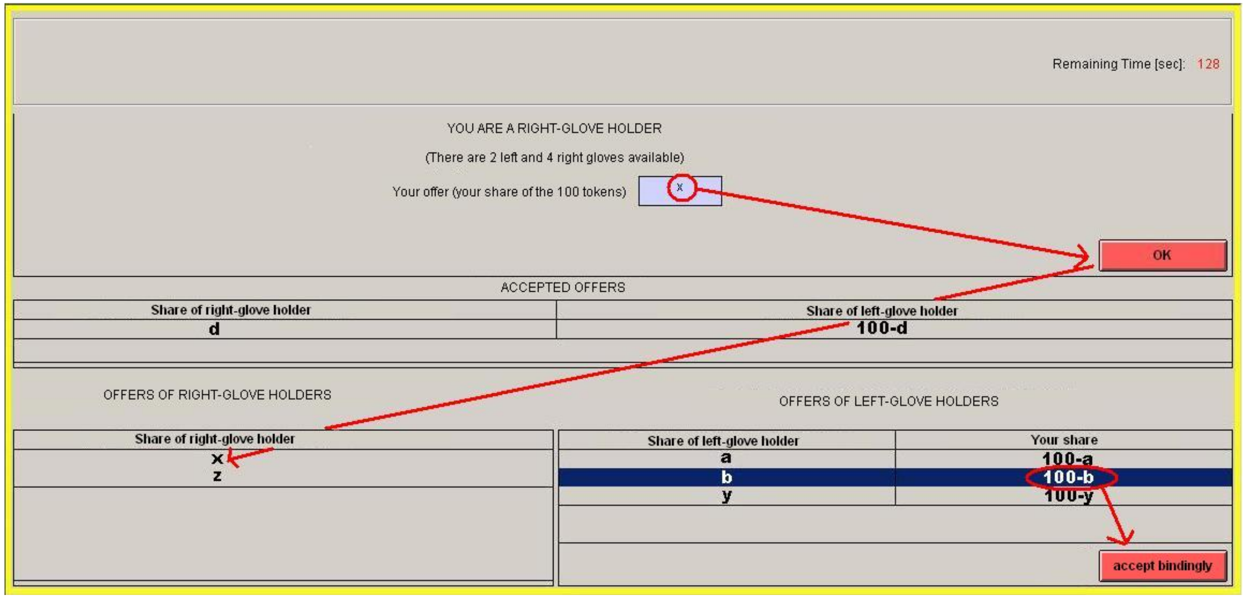
YOUR PAYOFF FROM PART 1. If you were able to find a partner, you earned your share of the distribution of tokens. Your payoff from part 1 are the tokens, converted to EUR, you earned.

If you have got a left glove



Here, the offer „y tokens“ has been submitted (after confirmation by „OK“, the offer will appear in the list). Furthermore, one can accept an offer of a compatible glove. In this example, a right-glove holder submitted the offer „z tokens“ (Hence, your share would be 100-z tokens). To accept this, select the offer with your mouse and confirm with “accept bindingly”. At „Accepted offers“ you see in this example that two of your takers already entered into an agreement (with a distribution of d tokens for the right-glove holder and 100-d tokens for the left-glove holder).
At the top on the right you see the remaining time. In this example you have got 128 seconds left.

If you have got a right glove



Here, the offer „x tokens“ has been submitted (after confirmation by „OK“, the offer will appear in the list). Furthermore, one can accept an offer of a compatible glove. In this example, a left-glove holder submitted the offer „b tokens“ (Hence, your share would be 100-b tokens). To accept this, select the offer with your mouse and confirm with "accept bindingly". At „Accepted offers“ you see in this example that two of your takers already entered into an agreement (with a distribution of d tokens for the right-glove holder and 100-d tokens for the left-glove holder).
At the top on the right you see the remaining time. In this example you have got 128 seconds left.

COMPREHENSION QUESTIONS PART 1

Please evaluate the below-mentioned statement for understanding of the experiment. Your answers will have no consequences for your payoff or the sequel of the experiment.

1. There can be built 3 pairs.

   Dialogue-box for wrong answer: Your answer is not correct. There can be built at most 2 matching pairs, since there are exactely 2 left gloves available. Please read again the corresponding part of the instructions if necessary.

2. There are less left gloves than right gloves.

   Dialogue-box for wrong answer: Your answer is not correct. Please read again the corresponding part of the instructions if necessary.

3. If I took an offer, I can change my mind and take another one.

   Dialogue-box for wrong answer: Your answer is not correct. If you take an offer, it will be binding. Please read again the corresponding part of the intructions if necessary.

4. I can make as many offers as I wish.

   Dialogue-box for wrong answer: Your answer is not correct. Please read again the corresponding part of the instructions if necessary.

5. If I make a new offer, all my previous offers expire and cannot be accepted by the other participants anymore.

   Dialogue-box for wrong answer: Your answer is not correct. Each offer you made can be accepted. Please read again the corresponding part of the instructions if necessary.

6. If I did not find a partner after expiration of the 120 seconds, I earned 0 tokens.

   Dialogue-box for wrong answer: Your answer is not correct. Please read again the corresponding part of the instructions if necessary.

INSTRUCTIONS PART 2

NOTES ON READING. Please read the following instructions. Approximately five minutes after you received the instructions, we will come to you and answer open questions. Please note that after the part 2 of the experiment started, no further questions can be answered anymore.

THE EXPERIMENT. This part of the experiment is identical to the first part, just that you play multiple rounds now. In each round, the two (imaginary) left and four (imaginary) right gloves will be randomly distributed. In total, **two matching pairs** can be build per round in your group and the round is finished as soon as **two matching pairs have been built** or if the **round time of 2 minutes** has expired, even if no two matching pairs have been built until then. The process of the single rounds is the same as in the first part of the experiment.

YOUR OVERALL PAYOFF. If you were able to find a partner, you earned your share of the distribution of tokens. There will be played multiple rounds. After each round, you can see how many tokens you earned in the previous rounds. Please note that part 2 starts with round 2, since you **played round 1 already in the part 1**. After the last round, there will be **randomly drawn one payment-relevant round additionally** to round 1. Your overall payoff is the sum of tokens, converted to EUR, you earned in this round and round 1 plus a fixed show-up fee of EUR 4.

END OF THE EXPERIMENT. After all rounds are finished, the payment-relevant part of the experiment is over. It follows a short not-payment-relevant part. You will find the instructions for that on screen. Afterwards you will be asked a few questions about personal details, they will of course be treated confidentially and anonymously.

THANK YOU FOR PARTICIPATING AT THE EXPERIMENT!

COMPREHENSION QUESTIONS PART 2

Please evaluate the below-mentioned statement for understanding of the experiment. Your answers will have no consequences for your payoff or the sequel of the experiment.

1. My position (right- or left-glove holder) was fixed in round 1 and will not change anymore.

Dialogue-box for wrong answer: Your answer is not correct. In each round the gloves will newly be assigned. Please read again the corresponding part of the instructions if necessary.

2. 2 matching pairs can be built in each round.

Dialogue-box for wrong answer: Your answer is not correct. 2 matching pairs can be built in each round since there are exactely two left gloves available. Please read again the corresponding part of the instructions if necessary.

SCREENS

The following screens are from the translation of the original German program.

Figure 9: Examples Comprehension Questions

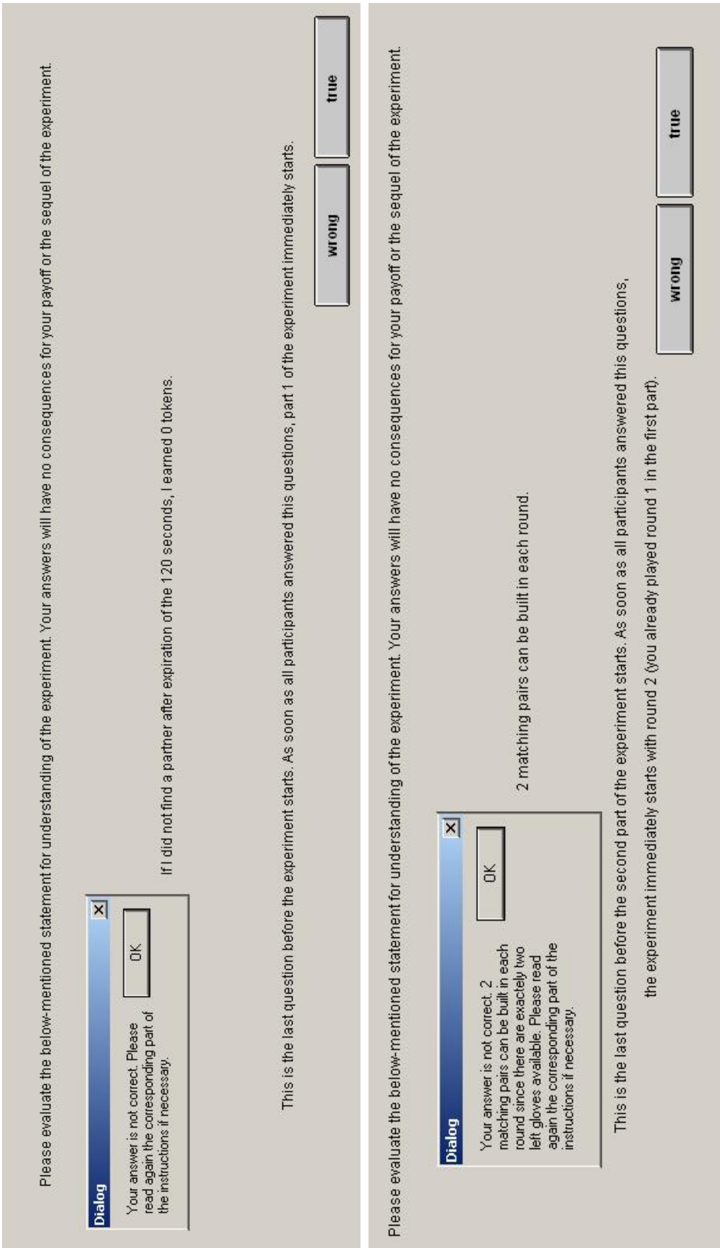
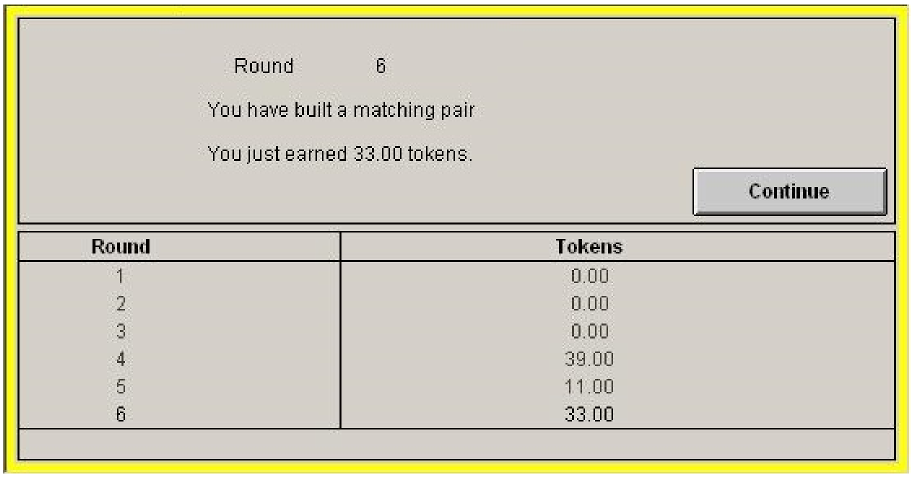
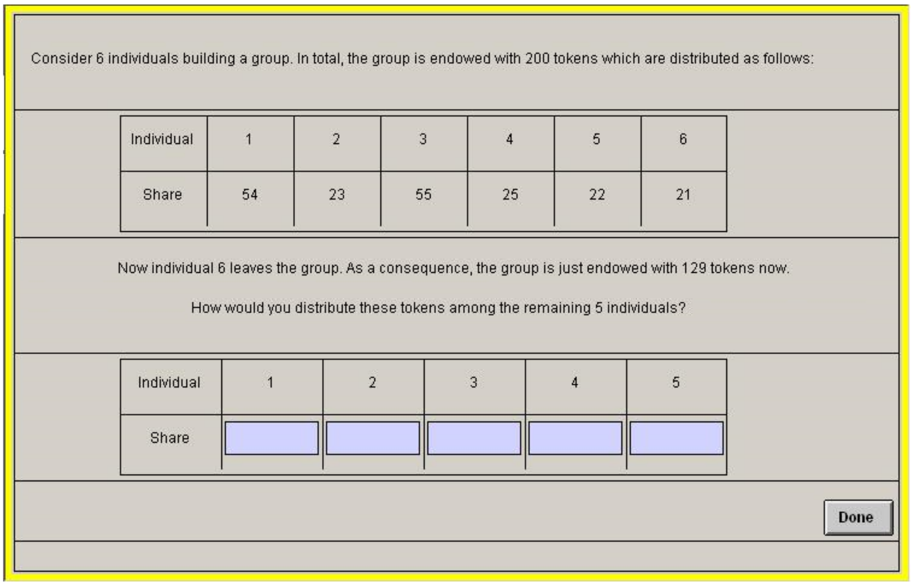


Figure 10: Results Screen and Splitting Screen





A.3 SUPPLEMENTARY: RESULTS

Table 20: Results for Left-glove Holders of all Rounds

Round	Group	1st Pair	2nd Pair	Round	Group	1st Pair	2nd Pair
1	1	65	66	1	7	75	75
2	1	66	66	2	7	80	30
3	1	66	10	3	7	80	90
4	1	65	2	4	7	90	90
5	1	65	65	5	7	90	95
6	1	90	99	6	7	95	95
1	2	80	79	1	8	80	80
2	2	90	80	2	8	80	80
3	2	88	10	3	8	80	90
4	2	90	85	4	8	80	98
5	2	90	95	5	8	90	98
6	2	95	90	6	8	98	98
1	3	50	55	1	9	70	70
2	3	65	80	2	9	75	80
3	3	50	80	3	9	80	75
4	3	70	60	4	9	85	80
5	3	65	70	5	9	90	70
6	3	70	70	6	9	75	10
1	4	75	73	1	10	80	51
2	4	82	78	2	10	80	90
3	4	78	80	3	10	55	90
4	4	80	80	4	10	66	80
5	4	20	81	5	10	57	90
6	4	84	20	6	10	90	99
1	5	50	80	1	11	60	70
2	5	50	10	2	11	70	60
3	5	55	90	3	11	60	75
4	5	10	70	4	11	30	70
5	5	90	80	5	11	70	85
6	5	90	87	6	11	90	95
1	6	60	90	1	12	70	90
2	6	90	92	2	12	40	90
3	6	90	92	3	12	70	80
4	6	80	95	4	12	70	80
5	6	95	1	5	12	60	70
6	6	50	99	6	12	60	70

Mean 1st Pair: 72.50; Mean 2nd Pair: 73.60

B

APPENDIX CHAPTER 4

B.1 THE PROBABILISTIC FORECASTING MODEL - AN EXAMPLE[1]

As an example of how the probabilistic forecasting model can be used, consider the glove game from Chapter 2.4 with four right-glove holders $r_1, \ldots, r_4$ and two left-glove holders l_1, l_2. Instead of assuming that coalitions have already been built, we now consider the situation *before* any coalitions have been established. Assume that the glove holders have the following (common) knowledge: r_1 and r_3 are known to be more attractive to the left-glove holder l_1 while l_2 is known to prefer r_2 and r_4. This (certain) *knowledge* leads to the following set of possible coalition structures that might occur:

$$\begin{aligned} \mathcal{P}_1 &= \{\{l_1, r_1\}, \{l_2, r_2\}, \{r_3\}, \{r_4\}\}, \; \mathcal{P}_2 = \{\{l_1, r_1\}, \{l_2, r_4\}, \{r_3\}, \{r_2\}\} \\ \mathcal{P}_3 &= \{\{l_1, r_3\}, \{l_2, r_2\}, \{r_1\}, \{r_4\}\}, \; \mathcal{P}_4 = \{\{l_1, r_3\}, \{l_2, r_4\}, \{r_1\}, \{r_2\}\} \end{aligned}$$

Beside this knowledge, the right-glove holders have some further *beliefs*: l_1 is supposed to prefer r_1 over r_3 while l_2 is supposed not to distinguish between r_2 and r_4. Let $p(\mathcal{P}_i)$ be the probability that the coalition structure $\mathcal{P}_i$ occurs. The beliefs lead to $p(\mathcal{P}_1) = p(\mathcal{P}_3) \wedge p(\mathcal{P}_2) = p(\mathcal{P}_4)$ (preference/indifference of l_1) and $p(\mathcal{P}_1) = p(\mathcal{P}_2) \wedge p(\mathcal{P}_3) = p(\mathcal{P}_4)$ (preference/indifference of l_2). Assume that we have

$$p(\mathcal{P}_1) = p(\mathcal{P}_2) = 0.3, \; p(\mathcal{P}_3) = p(\mathcal{P}_4) = 0.2.$$

For this situation, the values for the probabilistic AD-value, the probabilistic χ-value and the probabilistic Owen value are given in Table 21.

Table 21: Expected ex ante Payoffs for the Glove Game

glove holder	prob. AD-value	prob χ-value	prob. Owen value
l_1, l_2	0.5	0.8	0.833
r_1	0.3	0.12	0.1
r_2	0.25	0.1	0.083
r_3	0.2	0.08	0.067
r_4	0.25	0.1	0.083

Source: Belau [2010] and own calculations

The values can be interpreted as the expected ex ante payoffs of the right-glove holders (the left-glove holders will obtain their share of the worth for sure).

1 based on Belau [2010]

B.2 NETWORKS VS. PARTITIONS

For $m = 2$ and $m = 1$, the inner grouping corresponds to a finer grouping with respect to the robustness of the network, that is, the number of links within a coalition

Figure 11: Possible Networks for $m = 4$, $m = 3$ and $m = 2$

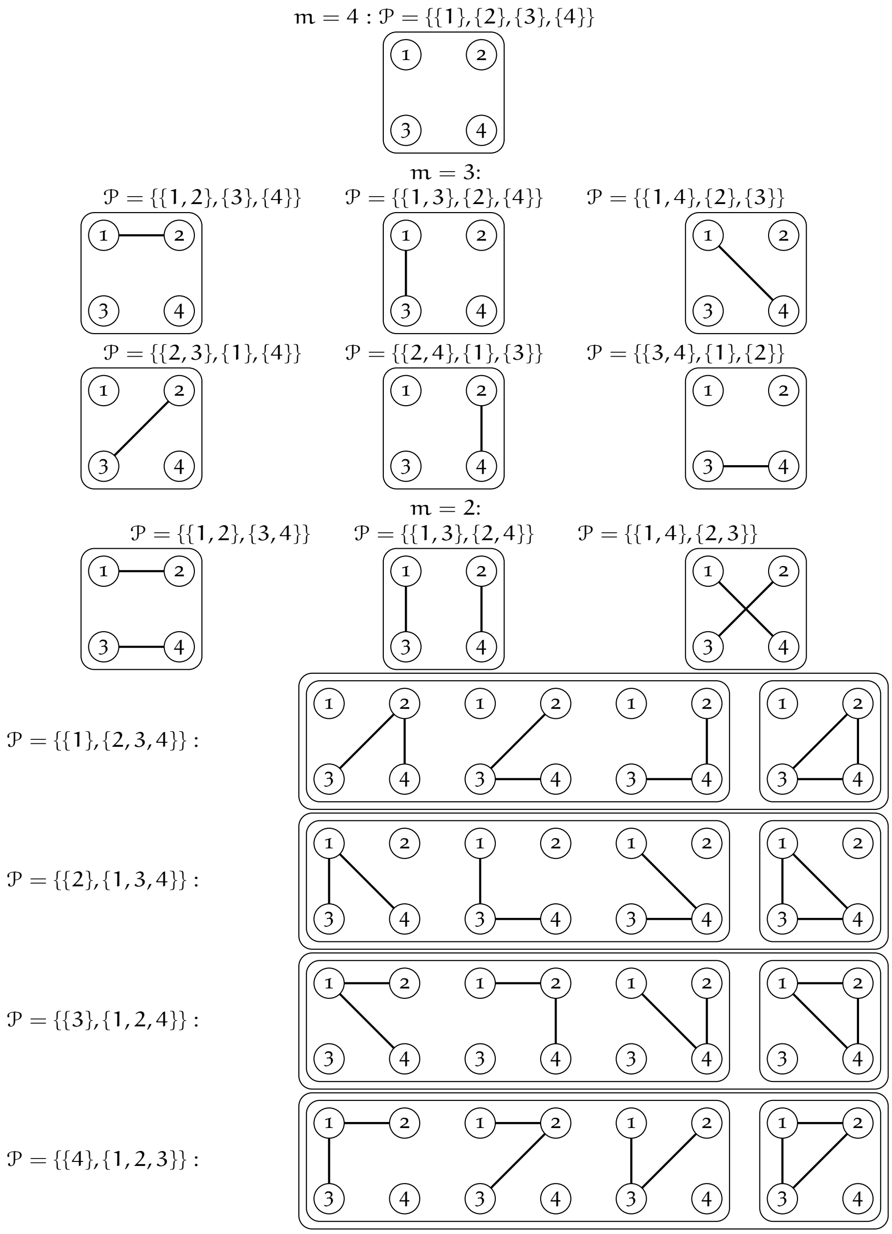
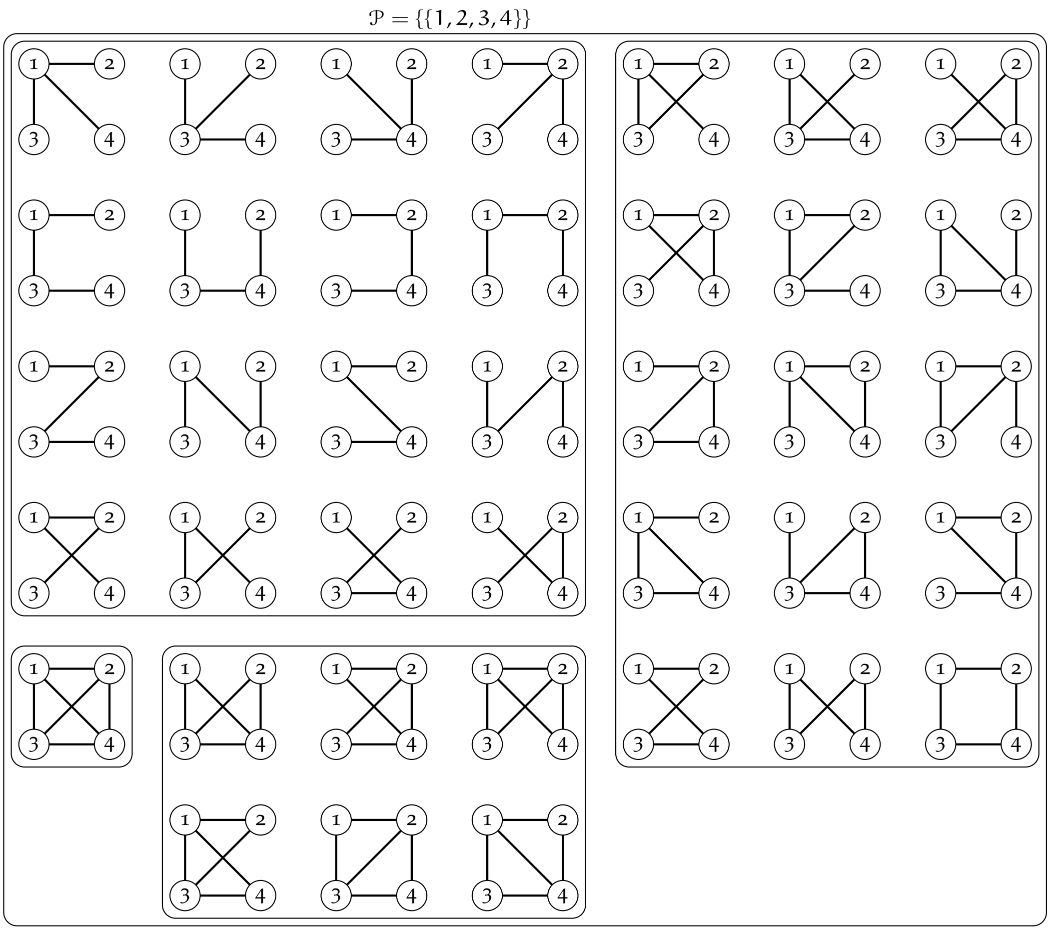


Figure 12: Possible Networks for $m = 1$



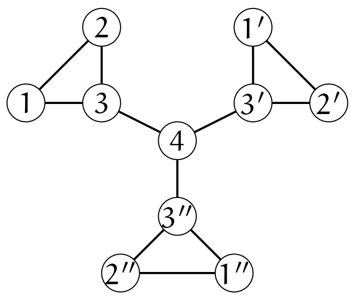
C

APPENDIX CHAPTER 5

C.1 A COMMUNICATION NETWORK - EXAMPLE

For a more detailed analysis of the classic centralities due to Freeman [1978]'s degree, closeness and betweenness measures or Bonacich [1972]'s eigenvector centrality consider the following communication network: Imagine there are three chairholders (3, $3'$ and $3''$), each having two phd students (1/2, $1'/2'$ and $1''/2''$). The three chairholders do not know each other personally, but have a common friend (4) through which they can get in contact to each other. The communication network is presented in Figure 13. Now, we are interested in the

Figure 13: Communication Network



relative power of each participant in the network. Centrality according to Bonacich's or Freeman's approaches (normalized for comparability issues) is presented in Table 22.

Table 22: Normalized Classic Centralities

Participant	1,2	3	4
Degree	8.33	12.50	12.50
Closeness	8.45	11.56	14.64
Betweenness	0	21.21	36.36
Eigenvector	8.10	12.26	14.63

Recall that, while having the advantage of very low computational complexity, the degree measure only considers the direct links of a node and not the whole network, that is, generally lacks to take into account importance for connectedness of the whole network. In the example we see that there is no difference between the centrality of the chairholders and the common friend who is connecting the whole network as they all have the same number of connections.

For the closeness, betweenness and eigenvector measure we obtain for the communication network the same problems as in the train example from the introduction in Chapter 5: For closeness and eigenvector centrality, the relative distance of centrality between the chairholders and the common friend is very small and the boundary nodes (phd students) still obtain a relatively high centrality. For betweenness, the boundary nodes obtain a betweenness-centrality of zero while their existence actually "creates" the power of the other nodes: if a boundary node fails, the betweenness measure of the corresponding chairholder decrases. Also for Freeman et al. [1991]'s betweenness measure based on network flow all boundary nodes would obtain a centrality of zero in the communication example.

Centrality according to the Position value and the new approaches from Chapter 5 are represented in Table 23. As we are interested in cohesion of the whole communication network, the unanimity game of the grand coalition is used as the underlying game.

Table 23: Normalized Position Centralities

Participant	1,2	3	4
(Shapley-) Position value	4.24	14.55	30.91
Banzhaf-Position value	6.67	13.33	20
Shapley-Eigenvector-value	2.41	18.19	30.99
Banzhaf-Eigenvector-value	5.285	14.96	23.41

In contrast to the classic centrality measures, relative distances are more plausible and also, boundary nodes do not obtain a centrality of zero. Note that the Shapley approaches yield much higher outcomes for the hub which is due to the "weighted voting".

C.2 CLOSENESS-, BETWEENNESS- AND EIGENVECTOR-VALUES

This section covers formal definitions of Freeman [1978]'s closeness and betweenness as well as Bonacich [1972]'s eigenvector measure and the transformation of these measures due to the procedure in Chapter 5.

Closeness and betweenness are designed by so-called *shortest paths*:

Definition C.1 ((Binary) Shortest Path)**.** *A* (binary) shortest path *between node i and j is defined by*

$$d(i,j) := \min(x_{ih_1} + ... + x_{h_k j})$$

where $h_1, ..., h_k$ are the intermediate nodes that have to passed between i and j.

Closeness is measured by the lengths of the shortest paths from a node to all other nodes (closeness to other nodes) while betweenness counts how often a node lies on a shortest paths between two other nodes (betweenness of nodes) relative to all shortest paths. Dijkstra [1959] suggests an algorithm to find shortest paths in weighted networks where weights are transmission costs. To implement this in a general weighted network where a "good" link usually has a high weight instead of low costs, Brandes [2001] and Newman [2001] invert weights to interpret them as costs:

Definition C.2 (Fastest Path or Weighted Shortest Path)**.** *A shortest path between two nodes in a weighted network is given by*

$$d^w(i,j) = \min\left(\frac{1}{w_{ih_1}} + \ldots + \frac{1}{w_{h_k j}}\right),$$

which can be interpreted as the fastest path between two nodes if weights represent the speed of for example information flow or passing speed in road networks.

Brandes and Newman use this to extend the *Closeness* and *Betweenness* measures for weighted networks:

Definition C.3 ((generalized) Closeness and Betweenness Measure)**.** *The (generalized) closeness measure* C^c *and the (generalized) betweenness measure* C^b *for weighted network* $g(N,w)$ *are for every* $i \in N$ *given by*

$$C_i^c(g(N,w)) = \left[\sum_{j \in N \setminus \{i\}} d^w(i,j)\right]^{-1}$$

$$C_i^b(g(N,w)) = \sum_{(j,k) \in N \setminus \{i\} \times N \setminus \{i\}, j \neq k} \frac{|d^w(j,k)(i)|}{|d^w(j,k)|}$$

where $|d^w(j,k)|$ *is the number of (weighted) shortest paths between* j *and* k *and* $|d^w(j,k)(i)|$ *the number of those of them passing* i*. For notational reasons we will write*

$$\sum_{j \neq i \neq k}^{N} := \sum_{(j,k) \in N \setminus \{i\} \times N \setminus \{i\}, j \neq k}$$

Beside Freeman's centrality measures, Bonacich [1972] introduces the *Eigenvetor centrality*:

Definition C.4 ((generalized) Eigenvector Centrality)**.** *Consider the* adjacency matrix $A(g)_{ij} := (w_{ij})_{ij}$ *corresonding to network* g*. Then, the Eigenvector centrality* $C_i^{EV}(g)$ *of node* i *in network* g *is given by* *the i^{th} entry of the eigenvector corresonding to the largest eigenvalue of* A*, that is, the unique nonnegative solution*[1] *of*

$$A \cdot C^{EV} = \lambda \cdot C^{EV}.$$

The idea of this approach is that the centrality of a node should be proportional to the centralities of the node's neighbors.

Analogously to the definition of the Shapley/Banzhaf-Degree-value we define:

Definition C.5 (Shapley/Banzhaf-Closeness- and -Betweenness-value)**.** *For every network* g*, the Shapley-Closeness-value* CC^{Sh} *and the Banzhaf-Closeness-value* CC^{Ba} *are given by*

$$CC_i^Y(g, v) := C_i^c\left(g(N, \tilde{w}(v, Y))\right) = \left[\sum_{j \in N \setminus \{i\}} d^{\tilde{w}(v,Y)}(i, j)\right]^{-1}$$

where $Y = Sh$ *or* $Y = Ba$*, respectively.*

whereas the Shapley-Betweenness-value CB^{Sh} *and the Banzhaf-Betweenness-value* CB^{Ba} *are given by*

$$CB_i^Y(N, v, g) := C_i^b\left(g(N, \tilde{w}(v, Y))\right) = \sum_{j \neq i \neq k}^{N} \frac{|d^{\tilde{w}(v,Y)}(j,k)(i)|}{|d^{\tilde{w}(v,Y)}(j,k)|}$$

where $Y = Sh$ *or* $Y = Ba$*, respectively.*

Normalization can again be done by the multiplicative approach.

Definition C.6 (Shapley/Banzhaf-Eigenvector-value)**.** *For every network* g*, the Shapley-Eigenvector-value* CEV^{Sh} *and the Banzhaf-Eigenvector-value* CEV^{Ba} *are given by*

$$CEV_i^Y(N, v, g) := C_i^{EV}\left(g(N, \tilde{w}(v, Y))\right)$$

where $Y = Sh$ *or* $Y = Ba$*, respectively.*

That is, CEV^Y *is given by the unique nonnegative solution of*

$$(\tilde{w}(v, Y)_{ij})_{ij} \cdot CEV^Y = \lambda \cdot CEV^Y.$$

As before, normalization can be done by the multiplicative approach.

1 In general, a matrix has several eigenvalues but one can show that only the largest one yields to a corresponding eigenvector consisting of non-negative components only.

C.3 SUPPLEMENTARY MATERIAL POLITICAL EXAMPLE

Table 24: Swings & Banzhaf values for Parties & Connections
(less restrictive case in parantheses)

For parties (nodes)			
party i	swings for i	# of swings	$\overline{Ba}_i$ (to 100%)
1	$\{2\},\{3\},\{2,3\},\{2,5\},\{3,5\},\{4,5\}$	6	40.00
2	$\{1\},\{1,5\},\{3,5\}$	3	20.00
3	$\{1\},\{1,5\},\{2,5\}$	3	20.00
4	$\{1,5\}$	1	6.67
5	$\{1,4\},\{2,3\}$	2	13.33
For connections (links)			
link λ	swings for λ	# of swings	$\overline{Ba}_\lambda$ (to 100%)
12	$\emptyset,\{15\},\{23\},\{25\},\{35\},\{15,23\}$ [+{45},{23,45}]	6 [8]	20.00 [17.39]
13	$\emptyset,\{15\},\{23\},\{25\},\{35\},\{15,23\}$ [+{45},{23,45}]	6 [8]	20.00 [17.39]
14	$\{15\},\{45\}$ [+{15,23},{23,45}]	2 [4]	6.67 [8.70]
15	$\{14\},\{45\},\{25\},\{35\}$ [+{14,23},{23,45}]	4 [6]	13.33 [13.04]
23	$\{25\},\{35\}$ [+{14,25},{14,35}]	2 [4]	6.67 [8.70]
25	$\{15\},\{23\},\{35\},\{15,23\}$ [+{14,23},{14,35}]	4 [6]	13.33 [13.04]
35	$\{15\},\{23\},\{25\},\{15,23\}$ [+{14,23},{14,25}]	4 [6]	13.33 [13.04]
14	$\{14\},\{15\}$ [+{14,23},{15,23}]	2 [4]	6.67 [8.70]

Table 25: Classic Centrality Approaches

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
$\overline{C^d}_i$	**25.00**	18.75	18.75	12.50	**25.00**
$\overline{C^c}_i$	**23.44**	18.75	18.75	15.62	**23.44**
$\overline{C^b}_i$	**50.00**	0.00	0.00	0.00	**50.00**
$\overline{C^{EV}}_i$	**23.13**	19.91	19.91	13.92	**23.13**
Resulting Coalitional Power					
coalition	$\{1,2\}$	$\{1,3\}$	$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$
$\sum \overline{C^d}_i$	43.75	43.75	**62.50**	-	**62.50**
$\sum \overline{C^c}_i$	42.19	42.19	**62.50**	-	60.94
$\sum \overline{C^b}_i$	50.00	50.00	**100.00**	-	50.00
$\sum \overline{C^{EV}}_i$	43.04	43.04	60.18	-	**62.95**

(Highest individual and coalitional power are bolt face.)

Table 26: Banzhaf-Centrality Approaches, less restrictive case

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
$\overline{\pi^{Ba}}_i$	**28.26**	19.57	19.57	8.70	23.91
$\overline{CEV^{Ba}}_i$	**25.62**	21.21	21.21	9.60	22.35
Resulting Coalitional Power					
coalition	$\{1,2\}$	$\{1,3\}$	$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$
$\sum \overline{\pi^{Ba}}_i$	47.83	47.83	60.87	-	**63.05**
$\sum \overline{CEV^{Ba}}_i$	46.83	46.83	57.57	-	**64.77**

(Highest individual and coalitional power are bolt face.)

LIST OF AXIOM ACRONYMS

LIST OF SYMBOLS

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Dortmund, August 2014

Julia Belau