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The notion of regulated function in the calculus teaching by Professor Igor Kluvánek

Everybody familiar with basic calculus remembers properties of continuous functions defined on a bounded closed interval I. Some of the properties can be extended to suitable discontinuous function, namely to functions having right and left limits in each point of I, such functions are called regulated. We shall deal with a special class of regulated functions consisting of piece-wise functions.

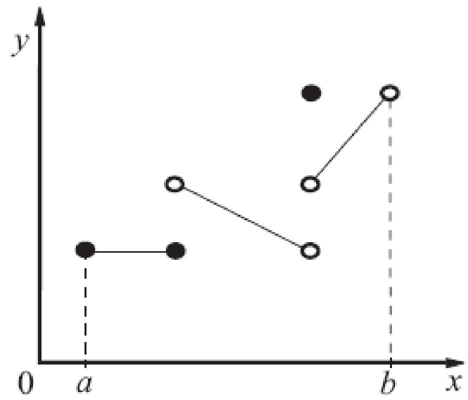
Introduction

Definition 1: A function f is said to be regulated on an interval I if

a) the function f has the left limit at every point of the interval I except the left end point and
b) the function f has the right limit at every point of the interval I except the right end point.

In this definition, we do not require that the right limit and the left limit of the function at a point are the same. The function is not required to be defined at every point of the interval I.

A graph of a regulated function an I = $\langle a; b\rangle$.



Properties of the regulated functions

Lemma 1: Each function continuous on an interval I is regulated on I.

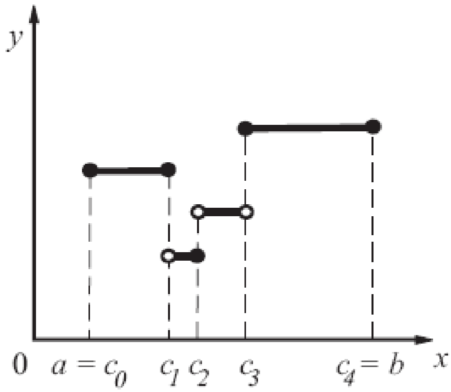
Lemma 2: Each monotonic and bounded function on an interval I is regulated on I.

Theorem 1: Let *f* be a regulated function on a bounded and closed interval I. Then *f* is bounded on it.

An important class of a regulated function consists of a piece-wise constant ones.

Definition 2: A function *f* is said to be piece-wise constant on an interval I with end points a and b, a < b, (we admit $a = -\infty$ and $b = \infty$), if there is natural number n and interior points c_j, $j = 1, 2, \ldots, n - 1$ of the interval I such that $c_{j-1} < c_j$, for $j = 2, 3, \ldots, n - 1$, and if $c_0 = a$ and $c_n = b$, then the function *f* is constant on each of the interval (c_{j-1}, c_j), for $j = 1, 2, \ldots, n$.

Because of its shape piece-wise constant functions are also called step functions. It follows from the definition that if *f* is regulated on a interval I, then it is also regulated in each interval J (J $\subseteq$ I).



The following theorem states that a function regulated on a bounded closed interval can be replaced with arbitrary accuracy by a step function.

Theorem 2 : For a and b be number such that $a < b$ let *f* be a function regulated on the interval $\langle a; b\rangle$ and let ε be a positive number. Then there exists a step function g such that

$$|f(x) - g(x)| < \varepsilon$$

for each point x of the interval $\langle a; b\rangle$ belonging to the domain of *f*. If the function *f* is continuous on the interval $\langle a; b\rangle$, then we can choose the function g to be right-continuous at each point of the interval $\langle a; b)$, or left-continuous at each point of the interval $(a; b\rangle$.

x	$\sqrt{x}$
20	4,4721
21	4,5826
22	4,6904
23	4,7958
24	4,8990
25	5,0000
26	5,0990
27	5,1962
28	5,2915
29	5,3852
30	5,4772

This theorem has numerous applications. It says, among other things, that for each regulated function on a compact interval (bounded and closed interval) an arbitrarily accurate table exists.

Example : In the Table, the values of the function $\sqrt{x}$ at the integers in the interval $\langle 20; 30\rangle$ are given. So, in this case $f(x)=\sqrt{x}$, $a = 20$, $b = 30$, n=10, $x_0 = a = 20$, $x_1 = 21$, $x_2 = 22$, and so on, $x_9 = 29$, $x_{10} = 30$. $y_0 = 4{,}4721$, $y_1 = 4{,}5826$, and so on, $y_9 = 5{,}3852$, $y_{10} = 5{,}4772$. For example, the approximate value of $\sqrt{x}$ at 27 is 5,1962, etc.

To determine approximately $\sqrt{26{,}4}$, we use the described procedure and find out from the Table that $\sqrt{26{,}4}$ is approximately 5,0990. Similarly, the approximate value of $\sqrt{x}$ at 27,8 is 5,2915, the approximate value of $\sqrt{x}$ at 21,5 is 4,5826.

It is now apparent that a table of a function f in an interval $\langle a; b\rangle$ together with the described procedure for determining approximate values of f represents a step function g defined on the interval $\langle a; b\rangle$. The number which this procedure produces as an approximate value of the function f at a point $x \in \langle a,b\rangle$ is equal to the value of the step function g at x. If the rule of taking y_{j-1} as an approximate value of f at $\frac{x_{j-1}+x_j}{2}$, for every $j = 1, 2, \ldots, n$ is adopted, then the corresponding step function g is left-continuous at each point of the interval $\langle a; b)$.

Conclusion

Generalization of the properties any continuous function on the interval $\langle a; b\rangle$ leads to the notion of the regulated function. First advance of this type of functions are application in different approximation regulated function by piece-wise function. Second advance is that the regulated function is integrable in the interval $\langle a; b\rangle$. These properties we can use the effective methods for actual calculation of integral.

References

[1] Gunčaga J.: *Grenzwertprozesse in der Schulmathematik.* In: *Journal für Mathematik-Didaktik.* Stuttgart – Leipzig – Wiesbaden, 2006, Heft 1, s. 77 – 78

[2] Gunčaga J., Tkačik Š.: *Derivative of a function at a point*. In: *XIIth Czech – Polish – Slovak Mathematical School*, Ústí nad Labem, UJEP, 2005, s. 120 – 124

[3] Eisenmann P.: *Propedeutika infinitezimálního počtu.* UJEP, Ústí nad Labem 2002

[4] Fulier J.: *Funkcie a funkčné myslenie vo vyučovaní matematickej analýzy.* UKF, Nitra 2001
[5] Kluvánek I.: *Manuscrips for Diferencial and integral calculus*
[6] Königsberger B.: *Analysis I.* Springer Verlag, Berlin–Heidelberg–New York 2001
[7] Pickert G.: *Aufbau der Analysis vom Stetigkeitsbegriff her*. In: *Der Mathematische und naturwissenschaftliche Unterricht*, 1968, 21. Band, Heft 11, s. 384 – 388
[8] Wachnicki E., Powązka Z.: *Problemy analizy matematycznej w zadaniach. Cęść I.* AP, Kraków 2002