A rule of thumb for the economic capital of a large credit portfolio

By

Rafael Weissbach

Institute of Business and Social Statistics

Department of Statistics

University of Dortmund *

November 2, 2004

*address for correspondence: Rafael Weissbach, Institute of Business and Social Statistics, Faculty of Statistics, University of Dortmund, 44221 Dortmund, Germany, email: Rafael.Weissbach@uni-dortmund.de, Fon: +49/231/7555419, Fax: +49/231/7555284.
Supported by DFG, SFB 475 “Reduction of Complexity in Multivariate Structures”, project B1.
Keywords. portfolio credit risk, counting process, economic capital, operational risk, martingale, mixture distribution, compounding, frailty.

Abstract

We derive approximate formulae for the credit value-at-risk and the economic capital of a large credit portfolio. The representation allows to change the risk horizon quickly and avoids simulation or numerical procedures. The Poisson mixture model is equivalent to CreditRisk$^+$ and uses the same parameters.

1 Introduction

The probability distribution of the future losses generated by a credit portfolio is of major interest in banking. The *credit value-at-risk* (CreditVaR) denotes a high quantile of the loss distribution and is a key parameter for the management of portfolio credit risk. The CreditVaR must be decomposed into a sum of the *expected loss* and the *unexpected loss*. The expected loss is a regular cost (see Weißbach and Sibbertsen (2004)) whereas the unexpected loss has to be covered by capital. For the latter reason it is denoted *economic capital* (EC). However, the latter is often named CreditVaR itself (see e.g. Grundke (2004)). Rules for the magnitude of the capital as required from the regulators have been release recently in the new basel capital accord by the Basel Commitee on Banking Supervision (Basel Commitee on Banking Supervision (2004)). As the rules in the regulations take portfolio effects only globally into account large banks maintain internal models. Internal credit risk portfolio models have the aim to account for the specific portfolio to be managed. Additionally, internal models may be used to with securitization. i.e. to price collateralized loan obligations (CLO). Strategically, large banks hope to use them to calculate their capital requirements as is already common practice portfolio market risk (Basel Commitee on Banking Supervision (1995)). But even more applications of the EC have an immense impact on banking. The EC is a risk management high level performance measure for a financial mother to control its children. The EC contribution to business units is an important steering tool. The return on risk capital (RoR) can be assigned to compare their performances. On the transaction level risk contributions can be calculated (e.g. by leave-one-out comparisons or by variance decomposition (see CSFB (1997)). The capital is associated with opportunity costs, as is the risk capital per trade. The costs must be incorporated into the price.

Portfolio models can be categorized into structural models and intensity based models. In structural models a mechanism for the default is assumed whereas in the intensity model the default is modeled directly as random event. We will restrict ourselves the intensity based modeling (see e.g. Jarrow and Turnbull (1995); Artzner and Delbaen (1995); Duffie and Singleton (1999)). However, for some structural and intensity model an equivalence is well documented (see Gordy (2000); Crouhy et al. (2000); Hamerle and Rösch (2004)). From a practical point of view to most common of the intensity based models is CreditRisk^{+TM} (Credit Suisse First Boston (CSFB) (1997)). The model is a Poisson mixture model and builds on the negative binomial distribution for the loss count as established already by Greenwood and Yule (1920). The default of a counterpart here is a Bernoulli random variable. One explicit point in time in the future is of interest. The question "Will a counterpart be defaulted by then?" suffices to be answered with the dichotomous states "Yes/No". In practice two problems arise from the specific set-up. In the first place, is it sufficient to consider one point in time to judge the bank's ability the take the load of risk? Especially, the time horizon for which the capital requirements are calculated is arbitrarily fixed. E.g. German banks often use a one-year horizon because the balance sheet for large companies (under German law) is calculated (and published) yearly. (Accounting schemes for the US (US-GAAP) and international rules (IAS) oblige a more frequent accounting of 3-4 times a year.) The link between the balance sheet and the risk management duty to warrant sufficient capital is the profit and loss account (P&L). Costs for the expected credit losses are provisioned in that account. The unexpected loss is calculated for convenience in the same cycles.

However, for the existential question "Has the bank enough capital to survive awkward events?" a time profile is needed. In order to overcome the drawback of a fixed time horizon we derive in the paper a model with variable time horizon. The calculation can be easily and quickly realized e.g. in spreadsheet software. The derived formulae build on well established asymptotic martingale approximations (see e.g. Rebolledo (1980); Andersen et al. (1993)). In order to overcome unrealistic independence assumption in a base formulation we incorporate the general economic activity analogous of the procedure in CreditRisk+. The derived frailty - or latent risk - model is typical in modern statistical failure analysis (Hougaard (2001)). As means of conditional modelling we use *counting processes* in the notion of Aalen (1978). Poisson- or Cox processes, usually used to model credit risk for the purpose of single (derivative) trade pricing, do not prove to be useful in the portfolio risk problem.

The paper is structured as follows: In Section 2 a simplified portfolio is analyzed, the counterparts are assumed to belong to one rating class only, their exposure is assumed to be identical and their defaults are assumed to be stochastically independent. The sample EC is derived in detail. In Section 3 the portfolio is assumed to be more realistic, namely we introduce rating classes and individual exposure levels. The lengthy calculation for that case are mostly deferred to the Appendix. In Section 4 the assumption of independence of defaults is replaced by a dependency model of the defaults related to economic activity. The loss distribution in that case is technically intractable. By neglecting cumulants of order higher than two, we fit a normal distribution.

2 An expository model for the loss

2.1 The loss process

Our model for the default of a counterpart in a credit portfolio is the jump process $D_t := I_{\{\tau \leq t\}}$ where τ denotes the default time, i.e. D_t is 1 if the default occurs prior to t and 0 otherwise. D_t is assumed to be adapted to a filtration $\{\mathcal{F}_t\}$ where τ is a $\mathcal{F}_t$-stopping time. D_t generalizes the Bernoulli model $B(PD)$ $(= D_1)$ being the basic model in the major credit risk portfolio model CreditRisk$^+$.

Consider a homogeneous portfolio where all independent counterparts $i = 1, \ldots, n$ owe the same amount of money, w.l.o.g. 1 unit, to the creditor. The portfolio is equivalent to the basic model of CreditRisk$^+$ where at first stage the distribution of the loss *count* is calculated.

The elementary parameter of the model is the notion of the "instantaneous probability of default". The notion is that each counterpart moves though time, i.e. "lives", with the permanent risk to default. The probability of instantaneous default is quantified as $P(\tau \in [t + dt] \mid \tau \geq t) \approx h(t)dt$. The probability is dependent on the time, t, via the function $h(t)$. We assume w.l.o.g. that the $t \geq 0$, i.e. we contemplate the future and define our present time as 0. The function *hazard rate* $h(t)$ is long known, consider Kalbfleisch and Prentice (1980) as a reference.

In practice the one-year probability of default (PD) is given, e.g. by the rating of the counterpart, expert guesses, etc. For the moment we assume that the tendency to default does not change with time, that means for the default time τ: $P(\tau \in [t, t + dt] | \tau \geq t) \equiv h$ dt $\forall$ t. The constant hazard rate h and the PD are linked by the relationship $PD = 1 - e^{-h}$, so that $h = -\log(1 - PD)$ and $F(t) = 1 - (1 - PD)^t$.

As further simplification to the unit exposure, we assume a homogeneous portfolio where all counterparts belong to the same rating class with a common PD and hence hazard rate. Denoted by $\tau_1, \ldots, \tau_n$ the independent identically distributed default times.

Define the *loss process*

$$L_t := \sharp\{i : \tau_i \leq t\}.$$

In order to analyzing the distribution we distinguish between trend and noise, i.e we need a decomposition of L_t so that $L_t = \Lambda_t + M_t$. Λ_t denotes the compensator of the process whereas M_t is the residual martingale.

For the calculation of Λ_t consider its increments. Clearly the expected increase in count is for independent failures

$$E(dL_t \mid \mathcal{F}_{t-}) = \sharp\{i : \tau_i \geq t\}hdt. \tag{1}$$

Define $Y_t := \sharp\{i : \tau_i \geq t\}$, the number of counterparts "at risk" and the *intensity* $\lambda(t) := Y_t h$.

The compensator is $\Lambda_t := \int_0^t \lambda(s)ds$.

2.2 The economic capital

Credit portfolio models are common only in large international financial institutes. The portfolios contain usually more than 5000 counterparts enabling asymptotic approximations, as we will see later.

For large portfolios, i.e. for $n \to \infty$, the proportion of the counterparts at risk is approximately given by the survival function: $\frac{Y_s}{n} \xrightarrow{a.s.} 1 - F(s)$ (because $1 - \frac{Y_s}{n} = F_n(s) \xrightarrow{a.s.} F(s)$), so that $\frac{\Lambda_t}{n} \xrightarrow{a.s.} 1 - (1 - PD)^t$, owing to $h(s) = \frac{f(s)}{1-F(s)}$.

For large n the path of $\frac{L_t}{\sqrt{n}}$ is almost smooth. The compensator is

$$\frac{\Lambda_t}{\sqrt{n}} \approx \sqrt{n} - \sqrt{n}(1 - PD)^t \tag{2}$$

The predictable variation process is $\frac{\langle M_t \rangle}{n} = \frac{\Lambda_t}{n}$ and approximately $v(t) = 1 - (1 - PD)^t$ (see Appendix). The latter is a deterministic smooth function.

There exits only one stochastic process with smooth path and deterministic variance function, namely the gaussian process. In fact, we have that $\frac{M_t}{\sqrt{n}}$ is asymptotically a Gaussian martingale with variance function

$F(t) = 1-(1-PD)^t$ (see Rebolledo (1980) or Andersen et al. (1993)). For a central Gaussian process with variance function $v(t)$ the increments are normally distributed with expectation 0 and variance $v(t)-v(0)$, so that

$$P\left(\frac{\frac{1}{\sqrt{n}}((L_t-\Lambda_t)-(L_0-\Lambda_0))}{\sqrt{F(t)-F(0)}} \geq u_{1-\alpha}\right) \approx \alpha.$$

The denominator reduces to $F(t)$ because $F(0)=0$. With $L_0 \overset{f.s.}{=} 0$, $\Lambda_0 \overset{f.s.}{=} 0$ and $\frac{\Lambda_t}{n} \approx F(t)$ we have the credit value-at-risk, $CreditVaR_{t,1-\alpha} = u_{1-\alpha}\sqrt{F(t)}\sqrt{n} + nF(t)$. The expected loss is clearly $E(L_t) = nF(t)$ so that we have the following result:

Theorem 2.1 *Consider a credit portfolio of n counterparts with debts* 1. *Their default times are assumed to be identical and independently distributed with constant hazard rate. The annual probability of default is denoted by PD and assumed to be known. The economic capital for the risk horizon t at level $1-\alpha$ is approximately for large n*

$$EC_{t,1-\alpha} = u_{1-\alpha}\sqrt{1-(1-PD)^t}\sqrt{n},$$

where $u_{1-\alpha}$ denotes the upper α-quantile of the standard normal distribution.

Note: The result is not surprising because the theorem of De Moivre-Laplace for the mean of the Bernoulli variables $I_{\{\tau_i \leq t\}} \sim B(F(t))$ yields (note $L_t = \sum_{i=1}^n I_{\{\tau_i \leq t\}}$)

$$\frac{L_t - nF(t)}{\sqrt{n}\sqrt{F(t)}} \overset{F(t)^2 \approx 0}{\approx} \frac{L_t - nF(t)}{\sqrt{n}\sqrt{F(t)(1-F(t))}} \xrightarrow{\mathcal{D}} N(0,1)$$
$$\Leftrightarrow P(L_t \geq u_{1-\alpha}\sqrt{F(t)}\sqrt{n} + nF(t)) \approx \alpha.$$

However, in the terminology of the counting process the loss count can be generalized to the full loss.

Large sample approximations seem applicable in banking although simulations must clarify when the asymptotic behavior "kicks in". It is a well known implication of the Berry-Esséen theorem that small probabilities of default obstruct the asymptotics (see e.g. Shiryaev (1996)). In a sample portfolio of 5000 counterparts with the common annual PD of 1% the 99% one-year EC is exactly given by 67 (with cumulative probability of 99.13%) compared to a normal approximation of 66.4. A detailed comparison is given in Table 1. It can be seen that the approximation is excellent for practical proposes. And even if the annual PD attains the smallest possible value allowed for the regulatory capital (Basel Commitee on Banking Supervision (2004)), 0.03% for at portfolio of 10000 counterparts and level of 99% the difference between the exact value of 8 and the approximation of 7.5 is negligible.

2.3 Calibration of the model

The relation between the hazard rate and the density $f(\cdot)$ of the default time is well-known to be $h(t) = \frac{f(t)}{1-F(t)}$ and enables the expression of the cumulative distribution function $F(\cdot)$ as argument of the cumulative hazard rate function $H(t) := \int_0^t h(s)ds$, namely $F(t) = 1 - e^{-H(t)}$. The cumulative distribution gives the annual PD's. For the one-year PD the relation $PD = P(\tau \leq 1) = F(1)$ has already been used to calibrate a constant, i.e. one parameter, hazard rate. However, the shape of the hazard rate is not defined, we must

Table 1: Comparison of the exact economic capital and the approximate economic capital. The exact value is given as integer from the binomial distribution closest to the level. The approximate value is given by the normal asymptotics.

	Portfolio size							
	5000				10000			
	PD				PD			
Level	1%		0.5%		1%		0.5%	
	exact	asym.	exact	asym.	exact	asym.	exact	asym.
90%	59	59.1	31	31.4	112	112.8	59	59.1
95%	61	61.6	33	33.2	116	116.4	61	61.6
99%	67	66.4	37	36.6	123	123.3	67	66.4
99.5%	69	68.2	38	37.9	126	125.8	69	68.2

only insure that

$$\int_0^1 h(s)ds = H(1) = -\log(1 - PD) \tag{3}$$

and that the number of parameters matches the number of input variables.

We can e.g. assume the hazard rate to be piece-wise constant (see e.g. Hougaard (2001)), a assumption which often used when pricing credit derivatives. For the first year we derive

$$h(t) = -\log(1 - PD) \quad \text{for} \quad 0 < t \leq 1.$$

The procedure can easily be extended for $t > 1$ given PD's exist for the two-year, three-year, etc PD's. Usually, the development of the PD over time is defined using migration matrices for the change of counterparts between rating classes. Each rating class can be attributed a one-year PD which implicitly defines two-year, three-year, etc PD's. If no assumption is made on the longer development of the PD the hazard rate can be assumed to be constant. Note that a constant hazard rate $\equiv h$ implies an exponentially distributed failure time with parameter h. It has to be mentioned that doubts exist that the hazard rates are constant (see e.g. Hakenes and Altrock (2001), Jones and Mingo (1998)) and Nickell et al. (2000), Weißbach and Sibbertsen (2004)) which is accounted for in the next Section.

3 Introducing rating classes and exposure

3.1 Inhomogeneous hazard rate

In a realistic situation a portfolio consists of counterparts with different PD's, i.e. from different rating classes. Assume our portfolio contains counterparts with default times

$$\tau_{ij} \sim F_j$$

for $i = 1, \ldots, n_j$ independent counterparts with hazard rates $h_j(\cdot)$ from $j = 1, \ldots, J$ rating classes. Note that we do not assume a constant hazard rate now.

We simply need to redefine the *loss count process*

$$L_t := \sharp\{(i,j), \tau_{ij} \leq t\}$$

with compensator and predictable variation process

$$\Lambda_t = \sum_{j=1}^{J} \int_0^t Y_{js} h_j(s) ds.$$

Here the counterparts "at risk" are counted per rating class $Y_{jt} := \#\{i, \tau_{ij} \geq t\}$ for $j = 1, \ldots, J$.

3.2 Inhomogeneous portfolio

As in CreditRisk$^+$ we assume the exposure ν_{ij}, or more correctly the "loss given default" to be deterministic. The portfolio loss is the sum of individual loss of the counterparts, or default entities

$$L_t = \sum_{j=1}^{J} \sum_{i=1}^{n_j} \nu_{ij} I_{\{\tau_{ij} \leq t\}}. \tag{4}$$

Theorem 3.1 *Consider a portfolio containing exposure in J rating classes with n_j counterparts in rating class j and loss given default of ν_{ij}, $i = 1, \ldots, n_j$ in each. The default times for the n_j counterparts in rating class j are assumed to be identical and independently distributed with hazard rates $h_j(\cdot)$. For a risk horizon t, a risk level $1 - \alpha$ and approximately for large $n = \sum_{j=1}^{J} n_j$ with not negligible proportion in each rating class $\frac{n_j}{n} \to c_j > 0$ the economic capital is given by*

$$EC_{t,1-\alpha} = u_{1-\alpha} \sqrt{\sum_{j=1}^{J} (1 - e^{-\int_0^t h_j(s) ds}) \sum_{i=1}^{n_j} \nu_{ij}^2},$$

where $u_{1-\alpha}$ denotes the upper α-quantile of the standard normal distribution.

The proof is deferred to the Appendix.

Corollary 3.1 *The approximate credit value-at-risk is now given by*

$$CreditVaR_{t,1-\alpha} = u_{1-\alpha}\sqrt{\sum_{j=1}^{J}(1-e^{-\int_0^t h_j(s)ds})\sum_{i=1}^{n_j}\nu_{ij}^2} + \sum_{j=1}^{J}(1-e^{-\int_0^t h_j(s)ds})\sum_{i=1}^{n_j}\nu_{ij}$$

Proof. The expected loss is clearly $E(L_t) = \sum_{j=1}^{J}\sum_{i=1}^{n_j}(1-e^{-\int_0^t h_j(s)ds})\nu_{ij}$. □

Corollary 3.2 *For the assumption of constant hazard rates h_j for each rating class defined by the annual PD_j the economic capital simplifies to*

$$EC_{t,1-\alpha} = u_{1-\alpha}\sqrt{\sum_{j=1}^{J}(1-(1-PD_j)^t)\sum_{i=1}^{n_j}\nu_{ij}^2}.$$

Proof. For a constant hazard rate $e^{-\int_0^t h(s)ds} = (1-PD)^t$. □

Note: The same result is achieved using the central limit theorem and the Lindeberg-Feller condition (see Ferguson (1996)) for the Bernoulli formulation of the problem.

4 Introducing economic activity

In the derived model so far the economic capital can be reduced to any aimed level by diversifying the portfolio, i.e. by splitting the exposure. The reason is that the economic capital converges to zero for the maximal exposure $\max\{\nu_{ij} : i = 1, \ldots, n_j, j = 1, \ldots, J\}$ going to infinity for fixed total exposure $\sum_{j=1}^{J} \sum_{i=1}^{n_j} \nu_{ij}$. The basis for that is the assumed mutual independence for the default of any pair of counterparts. From an economic view this is not reasonable. It is well understood that the - potentially stratified - economic activity influences the probability of default of all counterparts - in a sector. Credit Suisse First Boston (CSFB) (1997) models the dependence in the Bernoulli formulation for the default by assuming a latent economic activity variable increasing or reducing the probability of default PD for counterpart ij compared to the mean PD_j. In detail, if the counterpart is solely influenced by the economic activity in sector k ($k = 1, \ldots, K$) the default behaves as $I_{ij} \sim B(PD_j X_k)$ with $E(X_k) = 1$ for any k and $Var(X_k) = \sigma_k^2$.

The resulting dependence of the defaults between two counterparts $A = (i, j)$ and $B = (\tilde{i}, \tilde{j})$ can easily be seen to be

$$Corr(I_A, I_B) = \frac{\sqrt{PD_A PD_B}\rho_{k_A k_B}\sigma_{k_A}\sigma_{k_B}}{\sqrt{(1 - PD_A)(1 - PD_B)}}.$$

Here, k_A and k_B are index the sectors the counterparts A and B are active in. $\rho_{k_A k_B}$ denotes the correlation between the economic activity variables X_{k_A} and X_{k_B}. If the latter correlation is 0, the inter-sectorial *default* correlation is 0. If it is 1, the inter-sectorial default correlation equals $\frac{\sqrt{PD_A PD_B}\sigma_{k_A}\sigma_{k_B}}{\sqrt{(1-PD_A)(1-PD_B)}}$, which is the intra-sectorial default correlation is $k_A = k_B$.

We want to make use of the established model and integrate the idea in our hazard rate based model.

4.1 Global economic activity

At first stage we assume the default of counterparts is driven by *one* economic activity variable as is implied by the famous Merton model, i.e. $K = 1$.

The easiest way one can think of is to use a random factor Y to the hazard rate, often denoted as *frailty* to achieve the - now stochastic - hazard rate

$$g_j(t) = Zh_j(t) \quad j = 1, \ldots, J.$$

Here, $h_j(t)$ is the deterministic baseline hazard rate in rating class j as used before. The compounding of the two distributions is a common approach in modern survival analysis (see Hougaard (2001)). For our simplified model of a constant hazard rate the approach is equivalent to the model of a random factor to default time rather than the hazard rate. Theorem 3.1 implies that $\frac{1}{\sqrt{n}} L_t$ converges in distribution and conditional on Z to

$$N\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{J}(1 - e^{-\int_0^t Zh_j(s)ds})\sum_{i=1}^{n_j}\nu_{ij}, \frac{1}{n}\sum_{j=1}^{J}(1 - e^{-\int_0^t Zh_j(s)ds})\sum_{i=1}^{n_j}\nu_{ij}^2\right)$$

As small PD's imply small hazard rates we may use the Taylor approximation $e^x \approx 1+x$ to simplify the arguments to $\frac{1}{\sqrt{n}} L_t \mid_Z \xrightarrow{\mathcal{D}} N(Za, Zb)$ where $a := \frac{1}{\sqrt{n}} \sum_{j=1}^{J} H_j(t) \sum_{i=1}^{n_j} \nu_{ij}$ and $b = \frac{1}{n} \sum_{j=1}^{J} H_j(t) \sum_{i=1}^{n_j} \nu_{ij}^2$.

To calculate the marginal distribution we need a parametric assumption for the distribution of the frailty Z. As one of our goals to stick to the assumptions (including notation) of CreditRisk$^+$, we need to take the relation between Z and X into consideration. In Credit Suisse First Boston (CSFB) (1997) X is assumed to be Γ-distributed. This assumption erroneously enables annual PD's larger than 1. However, the probability of that error is negligible when the variance of the distribution is selected in practice (see e.g. Rosenow et al. (2004)). As technical advantage of the Gamma assumption over, say a Beta-distribution, the probability generating function (PGF) of the loss can be given in closed form. As we are not interested in the PGF, we may change the distributional assumption with little harm and decide to use a log-normal distribution. Again, we enable PD's larger than 1 but argue as above. Formula (3) clarifies the relation between the conditional annul PD and the conditional hazard to be $ZH(1) = -\log(1 - XPD)$. As a consequence, we derive a normal assumption $Z \sim N(\mu, \vartheta^2)$ as plausible. The deficiency of PD's larger than 1 translates into the possibility of hazard rates smaller than 0. The expectation μ needs to be 1 because of $E(X) = 1$. The volatility parameter derives from the assumption as in Credit Suisse First Boston (CSFB) (1997) of $Var(X) = \sigma^2$ and denoted for that reason as $\vartheta^2(\sigma)$.

Let us consider the marginal distribution of L_t. The density of $\frac{1}{\sqrt{n}} L_t$ is $\int_{\mathbb{R}} \phi^{za,zb}(t) \phi^{\mu,\vartheta(\sigma)}(z) dz = \int_{\mathbb{R}} \frac{1}{zb\sqrt{2\pi}} \exp\left\{-\frac{(t-za)^2}{2zb}\right\} \frac{1}{\vartheta(\sigma)\sqrt{2\pi}} \exp\left\{-\frac{(z-\mu)^2}{2\vartheta^2(\sigma)}\right\} dz$ where $\phi^{l,m}(\cdot)$ denotes the density of the normal distribution with expectation l and variance m. The integral is not feasible. We like to fit a distribution in two moments. The symmetry of the conditional asymptotic distribution of $\frac{1}{\sqrt{n}} L_t$ together with the symmetry of the distribution of Z suggests to use a normal distribution. The expectation is clearly $E(E(\frac{1}{\sqrt{n}} L_t|Z)) = \frac{1}{\sqrt{n}} \sum_{j=1}^{J} H_j(t) \sum_{i=1}^{n_j} \nu_{ij}$. The variance is $E(Var(\frac{1}{\sqrt{n}} L_t|Z)) + Var(E(\frac{1}{\sqrt{n}} L_t|Z)) =$

$E(Zb) + Var(Za) = \frac{1}{n}\sum_{j=1}^{J} H_j(t)\sum_{i=1}^{n_j}\nu_{ij}^2 + \vartheta^2(\sigma)\frac{1}{\sqrt{n}}\sum_{j=1}^{J} H_j(t)\sum_{i=1}^{n_j}\nu_{ij}.$

Theorem 4.1 *Consider a portfolio containing exposure in J rating classes with n_j counterparts in rating class j and loss given default of ν_{ij}, $i = 1, \ldots, n_j$ in each. The default times for the n_j counterparts in rating class j are assumed to be identical and independently distributed with hazard rates $Zh_j(\cdot)$ conditional on Z. The frailty Z is assumd to be normally distributed with $E(Z) = 1$ and $Var(Z) = \vartheta^2(\sigma)$. For a risk horizon t, a risk level $1 - \alpha$ and large $n = \sum_{j=1}^{J} n_j$ with not negligible proportion in each rating class $\frac{n_j}{n} \to c_j > 0$ the economic capital is approximately given by*

$$EC_{1-\alpha} = u_{1-\alpha}\sqrt{\frac{1}{n}\sum_{j=1}^{J} H_j(t)\sum_{i=1}^{n_j}\nu_{ij}^2 + \vartheta^2(\sigma)\frac{1}{\sqrt{n}}\sum_{j=1}^{J} H_j(t)\sum_{i=1}^{n_j}\nu_{ij}},$$

where $u_{1-\alpha}$ denotes the upper α-quantile of the standard normal distribution.

Note. For a stable economy reflected by the a small variance $\vartheta^2(\sigma)$ the EC is close (and in the limit equal) to the economic capital with independent defaults as in Theorem 3.1.

4.2 Sectorial economic activity

To account for several sectorial economic activities as is the case in CreditRisk$^+$ we suggest to use the idea of Bürgisser et al. (1999). The variance σ^2 of the latent variable X is calibrated so that the variance of the loss in the one-factor model is equal to the variance in the multi-factor model with $X_1, \ldots, X_K$, where $Var(X_k) = \sigma_k^2$ $k = 1, \ldots, K$ and $Corr(X_{k_A}, X_{k_B})) = \rho_{k_A k_B}$. To that end, we assume as in a slight simplification of Credit Suisse First Boston

(CSFB) (1997) that each counterpart ij belongs to *one* economic sector. A straight forward calculation proves our last

Theorem 4.2 *Under the conditions of Theorem 4.1 and the model for the default of* $I_{\{\tau_{ij}\le t\}} \sim B(X_k F(t))$ *with variances* $Var(X_k) = \sigma_k^2$ *and correlation* $Corr(X_{k_A}, X_{k_B})) = \rho_{k_A k_B}$ *the economic capital is approximately given by* $EC_{1-\alpha} = u_{1-\alpha}\sqrt{\frac{1}{n}\sum_{j=1}^{J} H_j(t)\sum_{i=1}^{n_j}\nu_{ij}^2 + \vartheta^2(\tilde{\sigma})\frac{1}{\sqrt{n}}\sum_{j=1}^{J} H_j(t)\sum_{i=1}^{n_j}\nu_{ij}}$ *with* $\tilde{\sigma}^2 = \frac{1}{\varepsilon(t)^2}\ \sum_{k,l=1}^{K}\rho_{kl}\ \sigma_k\sigma_l\ \varepsilon_k(t)\varepsilon_l(t)$*, where* $\varepsilon(t) = \sum_{j=1}^{J}(1-e^{-H_j(t)ds})\sum_{i=1}^{n_j}\nu_{ij}$ *denotes the total expected loss and* $\varepsilon_k(t)$ *denotes the expected loss restricted to counterpart in sector* k*.*

5 Summary

Based on the model CreditRisk$^+$ we give an approximate closed form expression of the economic capital for a diversified portfolio. The formula has the risk horizon as covariate which enables either to change the view quickly or to print an economic capital profile. The approximation is based on conditional asymptotic arguments. The calculation of the marginal loss distribution by means of compounding is avoided by approximation with a feasible distribution. The default behavior is described by the hazard rate of their default time enabling to use the one-year PD as parameter, forward PD's, or any assumed default behavior. The economic activity is taken into account and restricts the diversification potential by introducing dependency between the counterparts' credit risk. Replacement of CreditRisk$^+$ calculations are applicable for large credit portfolios and yield reduced operational risk due to the lack of numerical procedures.

Acknowledgement: The financial support of the Deutsche Forschungsgemeinschaft (SFB 475 "Reduction of Complexity for Multivariate Data Structures") is gratefully acknowledged.

A Appendix

A.1 Derivation of compensator and variation for Theorem 2.1

The process $\Lambda_t := \int_0^t \lambda(s)ds$ is the compensator of the *count process* L_t. One can show that the residual $M_t := L_t - \Lambda_t$ is a martingale (see Andersen et al. (1993)).

The predictable variation process $\langle M_t \rangle$ is the compensator of the squared martingale M_t.

Because $dM_t^2 = M_{(t+dt)-}^2 - M_t^2 = (M_{t-} + dM_t)^2 - M_{t-}^2 = (dM_t)^2 + 2dM_tM_{t-}$, we have

$$E(dM_t^2 \mid \mathcal{F}_{t-}) = E((dM_t)^2 \mid \mathcal{F}_{t-}),$$

i.e. the increments of the compensator of M_t^2 are the conditional variances of the increments of M, since the conditional expectation is 0:

$$Var(dM_t \mid \mathcal{F}_{t-}) = d\langle M_t \rangle.$$

The increments of L_t are in $\{0, 1\}$, therefore the increments of M_t are in $\{-d\Lambda_t, 1 - d\Lambda_t\}$. The variance is invariant altering the location. Adding $d\Lambda_t$ implies values 0 or 1, i.e. the shifted variable is Bernoulli distributed. The expectation is $d\Lambda_t$ because before the shift is was 0. Hence:

$$Var(dM_t \mid \mathcal{F}_{t-}) = d\Lambda_t(1 - d\Lambda_t) \approx d\Lambda_t,$$

because $d\Lambda_t$ is small and $d\Lambda_t^2$ negligible.

Note, that the same reasoning was applied when deriving the Poisson distribution as the distribution of a sum of Bernoulli distributed defaults Credit Suisse First Boston (CSFB) (1997).

A.2 Proof of Theorem 3.1

The compensator of the loss (4) is

$$\Lambda_t = \sum_{j=1}^{J} \int_0^t \sum_{i=1}^{n_j} \nu_{ij} I_{\{\tau_{ij} \geq s\}} h_j(s) ds. \tag{5}$$

Now $\sum_{i=1}^{n_j} \nu_{ij} I_{\{X_{ij} \geq s\}} \stackrel{n_j \to \infty}{\longrightarrow} \sum_{i=1}^{n_j} \nu_{ij}(1 - F_j(s))$ where $1 - F_j(s) = e^{-\int_0^t h_j(s)ds}$ because for $X_i \stackrel{iid}{\sim} B(p)$, $i = 1, \ldots, n$ and $a_i \in \mathbb{R}^+$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} a_i = n$ follows $Y := \frac{1}{n} \sum_{i=1}^{n} a_i X_i \stackrel{P}{\longrightarrow} p$. (Define $a_i := \frac{n_j \nu_{ij}}{\sum_{i=1}^{n_j} \nu_{ij}}$ and $X_i := I_{\{X_{ij} \geq s\}}$.)

It follows that $\int_0^t \sum_{i=1}^{n_j} \nu_{ij} I_{\{\tau_{ij} \geq s\}} h_j(s) ds \stackrel{n_j \to \infty}{\longrightarrow} \sum_{i=1}^{n_j} \nu_{ij}(1 - e^{-\int_0^t h_j(s)ds})$ and finally $\Lambda_t \to \sum_{j=1}^{J}(1 - e^{-\int_0^t h_j(s)ds}) \sum_{i=1}^{n_j} \nu_{ij}$.

The predictable variation process $\langle M \rangle_t$ is the compensator of the squared compensated loss process $M_t^2 = (L_t - \Lambda_t)^2$. Note that $E(dM_t^2 | \mathcal{F}_{t-}) = E((dM_t)^2 | \mathcal{F}_{t-})$ as in the case of Section 2.1 and proven in the Appendix. dM_t is a multinomial random variable with values in $\{\nu_{ij} I_{\{\tau_{ij} \geq t\}} - d\Lambda_t, i = 1, \ldots, n_j, j = 1, \ldots, J, -d\Lambda_t\}$. (Especially it is $-d\Lambda_t$ for $dL_t = 0$.) We assume that the ν_{ij}'s do not contain ties.

$$\begin{aligned} d\langle M\rangle_t = E((dM_t)^2|\mathcal{F}_{t-}) &= \sum_{j=1}^{J}\sum_{i=1}^{n_j}(\nu_{ij} - d\Lambda_t)^2 P(dL_t = \nu_{ij}|\mathcal{F}_{t-}) I_{\{\tau_{ij}\geq t\}} \\ &+ -(d\Lambda_t)^2 P(dL_t = 0|\mathcal{F}_{t-}) \\ &\approx \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(\nu_{ij} - d\Lambda_t\right)^2 h_j(t) dt I_{\{\tau_{ij}\geq t\}} \end{aligned}$$

because $P(dL_t = \nu_{ij}|\mathcal{F}_{t-}) = P(\tau_{ij} \in [t, t{+}dt[|\tau_{ij} \geq t) I_{\{\tau_{ij}\geq t\}} = h_j(t) dt I_{\{\tau_{ij}\geq t\}}$ and $(d\Lambda_t)^2 \approx 0$.

Again with $(d\Lambda_t)^2 \approx 0$ and the definition of Λ_t (5) we have

$$d\langle M\rangle_t \approx \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(\nu_{ij}^2 - 2\nu_{ij}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\nu_{ij} I_{\{\tau_{ij}\geq t\}} h_j(t) dt\right) h_j(t) dt I_{\{\tau_{ij}\geq t\}}$$

Ignoring terms of order $(dt)^2$ yields

$$d\langle M\rangle_t \approx \sum_{j=1}^{J} h_j(t) dt \sum_{i=1}^{n_j}\nu_{ij}^2 I_{\{\tau_{ij}\geq t\}}$$

Note, that the process $\langle M\rangle_t$ simplifies to (1) for the situation of one rating class and $\nu_{ij} = 1 \; \forall \, i, j$.

Let us consider again the dampened loss process $\frac{L_t}{\sqrt{n}}$ where $n := \sum_{j=i}^{J} n_j$ denotes the number of all counterparts. The increments of its variation process are

$$\frac{d\langle M\rangle_t}{n} \approx \sum_{j=1}^{J} h_j(t) dt \left(\frac{1}{n}\sum_{i=1}^{n_j}\nu_{ij}^2 I_{\{\tau_{ij}\geq t\}}\right)$$

Similar arguments as for the proof of (5) lead to $\sum_{i=1}^{n_j} \nu_{ij}^2 I_{\{\tau_{ij} \geq t\}} \stackrel{n_j \to \infty}{\longrightarrow} \sum_{i=1}^{n_j} \nu_{ij}^2 (1 - F_j(t))$ and hence

$$
\begin{aligned}
\frac{\langle M \rangle_t}{n} &\stackrel{n \to \infty}{\longrightarrow} \sum_{j=1}^{J} \int_0^t h_j(s) ds \left(\frac{1}{n} \sum_{i=1}^{n_j} \nu_{ij}^2 (1 - F_j(s)) \right) \\
&= \sum_{j=1}^{J} F_j(t) \frac{1}{n} \sum_{i=1}^{n_j} \nu_{ij}^2 \\
&= \sum_{j=1}^{J} (1 - e^{-\int_0^t h_j(s) ds}) \frac{1}{n} \sum_{i=1}^{n_j} \nu_{ij}^2
\end{aligned}
$$

In the case of an exposure of 1 for all counterparts, $\frac{1}{n} \sum_{i=1}^{n_j} \nu_{ij}^2$ reduces to 1. For the general case we will assume that the exposure sizes do not increase too much with growing portfolio, $\frac{1}{n} \sum_{i=1}^{n_j} \nu_{ij}^2 \longrightarrow a, 0 < a < \infty$.

The asymptotic variation process is deterministic and smooth function so that the limiting process is gaussian with variance function $v(t) := a \sum_{j=1}^{J} (1 - (1 - PD_j)^t)$. We have again

$$
P \left(\frac{\frac{1}{\sqrt{n}} ((L_t - \Lambda_t) - (L_0 - \Lambda_0))}{\sqrt{v(t) - v(0)}} \geq u_{1-\alpha} \right) \approx \alpha
$$

Slutzky's theorem enables us to replace $v(t)$ with its empirical analogue $\hat{v}(t) := \frac{1}{n} \sum_{j=1}^{J} (1 - e^{-\int_0^t h_j(s) ds}) \sum_{i=1}^{n_j} \nu_{ij}^2$. □

References

Aalen, O. (1978). Nonparametric estimation of partial transition probabilities in multiple decrement models. *Annals of Statistics*, 6:534–545.

Andersen, P., Borgan, Ø., Gill, R., and Keiding, N. (1993). *Statistical Models Based on Counting Processes*. Springer-Verlag New York.

Artzner, P. and Delbaen, F. (1995). Default risk insurance and incomplete markets. *Mathematical Finance*, 5:187–195.

Basel Commitee on Banking Supervision (April 1995). An internal model-based approach to market risk to market risk capital requirements. Technical report, Bank for International Settlements.

Basel Commitee on Banking Supervision (June 2004). International convergence of capital measurement and capital standards. Technical report, Bank for International Settlements.

Bürgisser, P., Kurth, A., Wagner, A., and Wolf, M. (1999). Integrating correlations. *Journal of Risk*, 07:57–60.

Credit Suisse First Boston (CSFB) (1997). CreditRisk+: A credit risk management framework. Technical report, Credit Suisse First Boston.

Crouhy, M., Galai, D., and Mark, R. (2000). Comparative analysis of current credit risk models. *Journal of Banking and Finance*, pages 59–117.

Duffie, D. and Singleton, K. (1999). Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12:687–720.

Ferguson, T. (1996). *A course in Large Sample Theory*. Chapmann & Hall, London.

Gordy, M. (2000). A comparative anatomy of credit risk models. *Journal of Banking and Finance*, 24:119–149.

Greenwood, M. and Yule, G. (1920). An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurence of multiple attacks of disease or of repeated accidents. *Journal of the Royal Statistical Society*, 83:255–279.

Grundke, P. (2004). How important is the modeling of interest rate and credit spread risk in standard and non-standard credit portfolio models? Technical report, University of Cologne.

Hakenes, H. and Altrock, F. (2001). Die Kalkulation ausfallrisikobedrohter Finanztitel mit Rating-übergangsmatrizen. *Finanzmarkt und Portfoliomanagement*, 15:187–200.

Hamerle, A. and Rösch, D. (2004). Parametrizing credit risk models. Technical report, University Regensburg.

Hougaard, P. (2001). *Analysis of Multivariate Survival Data*. Springer, New York.

Jarrow, R. and Turnbull, S. (1995). Pricing options on financial securities subject to credit risk. *Journal of Finance*, 50:53–86.

Jones, D. and Mingo, J. (1998). Industry practices in credit risk modeling and internal capital allocations: Implications for a models-based regulatory capital standard. *Economic Policy Review, Special Issue*, pages 53–60.

Kalbfleisch, J. and Prentice, R. (1980). *The Statistical Analysis of Failure Time Data*. John Wiley & Sons, New York.

Nickell, P., Perraudin, W., and Varotto, S. (2000). Stability of ratings transitions. *Journal of Banking and Finance*, 24:203.

Rebolledo, R. (1980). Central limit theorems for local martingales. *Zeitschrift für Wahrscheinlichkeit und verwandte Gebiete*, 51:269–286.

Rosenow, B., Weißbach, R., and Altrock, F. (2004). Modelling correlations in portfolio credit risk. Technical report, University Dortmund, Technical Report 5.

Shiryaev, A. (1996). *Probability.* Springer-Verlag, New York.

Weißbach, R. and Sibbertsen, P. (2004). The cost for the default of a loan - linking theory and practice -. Technical report, University Dortmund, Technical Report 33.