A study on pattern formation in reaction-diffusion equations with spatial inhomogeneity

空間非一様性を伴う反応拡散方程式におけるパターン形成に関する研究

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Michio URANO

浦野 道雄
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早稲田大学大学院 理工学研究科

Michio URANO

浦野 道雄
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Introduction

Reaction-diffusion equations are mathematical models formulated on the basis of partial differential equations. Typically, they arise in the fields of chemical reactions, where they are observed in various aspects in which some chemical substances spread into spaces. In general, they are given in the form of

$$\frac{\partial u}{\partial t} = d\Delta u + f(u).$$

Here $u = u(x,t)$ is an unknown function which denotes a kind of state or density of some chemical substances at position $x = (x_1, x_2, \ldots, x_N)$ and time $t$. $\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $d$ is a positive constant. $\Delta u$ expresses a phenomenon that chemical substances diffuse into spaces; therefore $d\Delta u$ is called a diffusion term and $d$ is called a diffusion coefficient which denotes the rate of diffusion. Moreover the effect brought about by this term is called a diffusion effect. $f(u)$ is a nonlinearity describing some property of the chemical reaction. This term is called a reaction term. In the study of reaction-diffusion equations, there are many important subjects of mathematics. Among them, this thesis is devoted to the study of steady-state solutions. Here a steady-state solution means a time-independent solution $u = u(x)$ of the reaction-diffusion equation. Therefore steady-state solutions satisfy the corresponding stationary problems:

$$d\Delta u + f(u) = 0.$$ 

These problems are regarded as models describing the states that come about after enough time has passed. Hence their analysis helps us to know dynamics of time-dependent solutions. In particular, information on multiplicity, stability or profiles of steady-state solutions is significant to understand the mechanism of phenomena described by reaction-diffusion equations.

Concerning reaction-diffusion equations, it is striking that the dynamics of solutions is controlled by the interaction of the diffusion and the reaction terms. 

...
terms. When \( f(u) \equiv 0 \), the solution \( u(x,t) \) of the diffusion equation

\[
\frac{\partial u}{\partial t} = d\Delta u
\]

converges to a spatial homogeneous solution (See for instance the monograph of Smoller [23]). Moreover, Conway, Hoff and Smoller [7] proved that, if \( f(u) \) satisfies some appropriate conditions, then a similar fact to the diffusion equation occurs, provided that the diffusion coefficient \( d \) is sufficiently large. These facts imply that the diffusion term tends to smooth out irregularities of solutions as time goes by and that, if the diffusion coefficient is large enough to control the dynamics, then there is no spatially inhomogeneous steady-state solution. From this point of view, our interest is attracted to the case that the diffusion effect is not so large because the appearance of spatially inhomogeneous steady-state solutions is expected. Indeed, since Turing [25] suggested this assertion, a great number of mathematicians have attacked this kind of reaction-diffusion equations; and many kinds of such solutions have been observed. A series of studies as above is called a pattern formation problem. For these problems, in view of the role of the diffusion effect, it becomes important to take notice of reaction terms. If a function \( u = u^*(t) \) satisfies the ordinary differential equation

\[
\frac{du}{dt} = f(u),
\]

which corresponds to the case that \( d = 0 \), then \( u^* \) also satisfies the reaction-diffusion equation for all \( d > 0 \) as a spatially homogeneous solution. Hence it is expected that patterns are influenced by their properties. This conjecture is indeed correct and there appear varieties of patterns. This fact also implies that the interaction of the diffusion and reaction terms has a strong influence on the pattern formation.

Among various types of reaction-diffusion equations, our target is one dimensional reaction-diffusion equations with spatial inhomogeneity and bistability. In the real world, diffusion and reaction processes evolve in nonuniform media. Therefore, from the standpoint of application, it is natural to consider the spatial inhomogeneity. Bistability means a property of reaction terms which describes phenomena with two stable states. Mathematically, a reaction term \( f(u) \) defined in terms of a double-well potential \( W(u) \) as \( f(u) = W_u(u) \) is called a bistable nonlinearity. Here a \( C^2 \)-function \( W \) is called a double-well potential, if \( W \) has exactly three critical points \( a, b \) and \( c \) with
$a < b < c$ satisfying $W_{uu}(a) > 0$, $W_{uu}(b) < 0$ and $W_{uu}(c) > 0$. This feature implies that $W$ takes its local minima at $u = a$ and $c$; and both of them correspond to the stable states. These kinds of reaction-diffusion equations appear as models which describe phase transition phenomena in various fields. See the monograph of Fife [11] and the references therein. For them, it is well known that there appear solutions with transition layers and spikes when the diffusion coefficient is sufficiently small. Here a transition layer means a spatial part of a solution where its value drastically changes, and a spike is also a spatial part of a solution where the solution shapes something like a prickle. They do not necessary appear standing aloof from others. Indeed, it is observed that several transition layers and spikes appear in a neighborhood of a certain point as a cluster. Such a cluster of transition layers and that of spikes called a multi-layer and multi-spike, respectively. While, a transition layer which is away from other transition layers is called a single-layer. Similarly, the term a single-spike is determined.

The main purpose of this thesis is to make a detailed study on patterns of steady-state solutions with oscillating profiles as above. In particular, the influence of the spatial inhomogeneity on the patterns will be found out. Also the stability of each pattern will be discussed.

In the present thesis, two kinds of reaction-diffusion equations are studied, so that this thesis is divided into two main parts.

**Part I**

In this part, we consider the following reaction-diffusion problem:

\[
\begin{align*}
  u_t &= \varepsilon^2 u_{xx} + f(x, u), \quad 0 < x < 1, \quad t > 0, \\
  u_x(0, t) &= u_x(1, t) = 0, \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad 0 < x < 1.
\end{align*}
\]

(P1)

Here $\varepsilon$ is a positive parameter and $f(x, u)$ is given by

\[f(x, u) = u(1 - u)(u - a(x)),\]

where $a$ is a $C^2$-function with the following properties:

(A1) $0 < a(x) < 1$ in $[0, 1]$.

(A2) If $\Sigma := \{x \in [0, 1] : a(x) = 1/2\}$, then $\Sigma$ is a non-empty finite set, $0, 1 \not\in \Sigma$ and $a_x(x) \neq 0$ at any $x \in \Sigma$. 

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If $\Lambda := \{ x \in [0, 1] ; a_x(x) = 0 \}$, then $\Lambda$ is a non-empty finite set.

Furthermore, we introduce the following notation:

$$
\Sigma^+ := \{ x \in \Sigma ; a_x(x) > 0 \},
\Sigma^- := \{ x \in \Sigma ; a_x(x) < 0 \},
\Lambda^+ := \{ x \in \Lambda ; a(x) < 1/2 \text{ and } a_{xx}(x) < 0 \},
\Lambda^- := \{ x \in \Lambda ; a(x) > 1/2 \text{ and } a_{xx}(x) > 0 \},
\Lambda^0 := \Lambda \setminus (\Lambda^+ \cup \Lambda^-).
$$

Concerning (P1), we will mainly discuss the stationary problem associated with (P1):

$$
\begin{cases}
\varepsilon^2 u_{xx} + f(x,u) = 0, & 0 < x < 1, \\
u_x(0) = u_x(1) = 0.
\end{cases}
$$

Our interest lies in the study of solutions of (SP1) with transition layers and spikes. For (SP1), Angenent, Mallet-Paret and Peletier [3] classified all stable solutions. Among them, we can observe a stable solution $u^\varepsilon$ which possesses a single-layer near a point $x_0 \in \Sigma$ with $u_x^\varepsilon(x_0)a_x(x_0) < 0$ when $\varepsilon$ is sufficiently small. We also refer to the work of Hale and Sakamoto [12], who proved that (SP1) admits an unstable solution $u^\varepsilon$ with a single-layer near $x_0 \in \Sigma$ satisfying $u_x^\varepsilon(x_0)a_x(x_0) > 0$. Moreover, Dancer and Yan [8] have shown the existence of a solution $u^\varepsilon$ of (SP1) with multi-layers. More precisely, it is proved that there exists a solution which possesses any prescribed number of transition layers near any designated point $x_0 \in \Sigma$. They have discussed such solutions in a ball of $\mathbb{R}^N$.

An important reference is a work of Ai, Chen and Hastings [1], who have studied solutions of (SP1) with transition layers and spikes almost in the same time as we have done. And the author had a chance to see their preprint. They have proved remarkable results on the structure of solutions of (SP1) with transition layers and spikes. They also prove the existence of solutions with such profiles and establish their stability properties.

We are highly stimulated by their works. However, our main purpose is to derive more precise results on the profiles of solutions with transition layers and spikes. We will develop more general results on the asymptotic behavior of $u^\varepsilon(x)$ as $\varepsilon$ goes to 0 (Theorems 1.3.3, 1.3.6, 1.3.7 and 1.3.8). Furthermore, we will discuss patterns by using these results, so that our approach is different from that of Ai, Chen and Hastings [1].
In order to analyze a solution $u^\varepsilon$ of (SP1) with oscillatory profiles such as transition layers and spikes, it is useful to take account of the number of intersecting points of the graphs of $u^\varepsilon$ and $a$ in $(0, 1)$. We introduce the notion of **n-mode solutions**:

**Definition.** For any fixed integer $n \geq 1$, a solution $u^\varepsilon$ of (SP1) is called an $n$-mode solution, if $u^\varepsilon - a$ has exactly $n$ zero points in the interval $(0, 1)$.

Let $u^\varepsilon$ be any $n$-mode solution and define

$$\Xi := \{ x \in [0, 1] ; u^\varepsilon(x) = a(x) \}.$$

Roughly speaking, for any $n$-mode solution $u^\varepsilon$ of (SP1), it will be shown in Lemmas 1.2.3 and 1.2.5 that its graph is classified into the following three portions:

(i) $u^\varepsilon(x)$ is close to 0 or 1,

(ii) $u^\varepsilon(x)$ forms a transition layer connecting 0 and 1,

(iii) $u^\varepsilon(x)$ forms a spike based on 0 or 1.

Moreover, it will be seen that $u^\varepsilon$ forms a transition layer or a spike near a point in $\Xi$. Indeed, one can show that each transition layer or spike is characterized by a suitable $C^2$-function, which corresponds to a heteroclinic orbit or a homoclinic orbit of $U_{zz} + f(\xi^*, U) = 0$ with some $\xi^* \in [0, 1]$ (see Lemma 1.2.5).

In order to discuss patterns, we intend to focus on getting precise information on profiles of $n$-mode solutions of (SP1). We will give our main result on the asymptotic rates of $u^\varepsilon(x)$ or $1 - u^\varepsilon(x)$ as $\varepsilon \to 0$:

**Theorem 0.0.1.** Let $u^\varepsilon$ be any $n$-mode solution of (SP1). Let $\xi_1, \xi_2 \in \Xi$ be successive points satisfying $u^\varepsilon(x) - a(x) > 0$ in $(\xi_1, \xi_2)$ and let $\zeta$ be the unique local maximum point in $(\xi_1, \xi_2)$. Then the following assertions hold true:

(i) If $(\zeta - \xi_1)/\varepsilon \to \infty$ as $\varepsilon \to 0$, then there exist positive constants $C_1, C_2, r$ and $R$ with $C_1 < C_2$ and $r < R$ satisfying

$$C_1 \exp \left( -\frac{R(\zeta - \xi_1)}{\varepsilon} \right) < 1 - u^\varepsilon(x) < C_2 \exp \left( -\frac{r(x - \xi_1)}{\varepsilon} \right) \quad \text{in } [\xi_1, \zeta],$$

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provided that \( \varepsilon \) is sufficiently small.

(ii) If \( (\xi_2 - \zeta) / \varepsilon \to \infty \) as \( \varepsilon \to 0 \), then there exist positive constants \( C_1, C_2, r \) and \( R \) with \( C_1 < C_2 \) and \( r < R \) satisfying

\[
C_1 \exp \left( -\frac{R(\xi_2 - \zeta)}{\varepsilon} \right) < 1 - u^\varepsilon(x) < C_2 \exp \left( -\frac{r(\xi_2 - x)}{\varepsilon} \right)
\]

in \([\zeta, \xi_2]\), provided that \( \varepsilon \) is sufficiently small.

We should note that analogous theorems also hold true when \( u^\varepsilon(x) \) is very close to zero (see Theorems 1.3.7 and 1.3.8). These theorems play a very important role in the study of the location of transition layers and spikes as well as they give precise information on the profiles of solutions with transition layers and spikes.

Our basic results on the location of transition layers and spikes are given by the following theorems:

**Theorem 0.0.2** (Location of transition layers and their multiplicity). Let \( u^\varepsilon \) be any \( n \)-mode solution of (SP1). If \( \varepsilon \) is sufficiently small, then the following assertions hold true:

(i) If \( u^\varepsilon \) has a single-layer, then it appears only in an \( O(\varepsilon) \)-neighborhood of a point in \( \Sigma \).

(ii) If \( u^\varepsilon \) has a multi-layer, then it consists of odd number of transition layers. Thus the multi-layer is shaped like a cluster of transition layers connecting from 0 to 1 or 1 to 0. Moreover, if \( u^\varepsilon \) has a multi-layer connecting from 0 to 1 (resp. 1 to 0), it appears only in an \( O(\varepsilon |\log \varepsilon|) \)-neighborhood of a point in \( \Sigma^+ \) (resp. \( \Sigma^- \)).

**Theorem 0.0.3** (Location of spikes and their multiplicity). Let \( u^\varepsilon \) be any \( n \)-mode solution of (SP1) with spikes. Then any spike appears only in a neighborhood of a point in \( \Lambda \cup \{0, 1\} \), provided that \( \varepsilon \) is sufficiently small. Moreover the following assertions hold true:

(i) Any spike based on 0 (resp. 1) appears only in a neighborhood of a point \( \lambda \in \Lambda \cup \{0, 1\} \) with \( a(\lambda) < 1/2 \) (resp. \( a(\lambda) > 1/2 \)).

(ii) If \( u^\varepsilon \) has a multi-spike based on 0 (resp. 1), then it appears only in an \( O(\varepsilon |\log \varepsilon|) \)-neighborhood of a point in \( \Lambda^+ \) (resp. \( \Lambda^- \)), a boundary point 0 with \( a(0) < 1/2 \) and \( a_x(0) < 0 \) (resp. \( a(0) > 1/2 \) and \( a_x(0) > 0 \)) or a boundary point 1 with \( a(1) < 1/2 \) and \( a_x(1) > 0 \) (resp. \( a(1) > 1/2 \) and \( a_x(1) < 0 \)).

The assertions of Theorems 0.0.2 and 0.0.3 are the same as the corresponding results of Ai, Chen and Hastings [1]. Our strategy to prove these theorems

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is to use Theorem 0.0.1 and determine the location of transition layers and 
spikes by contradiction argument. However, Ai, Chen and Hastings have de-
related asymptotic results as $\varepsilon \to 0$ for $u^\varepsilon(\zeta)$, where $\zeta$ denotes a local maximum 
or minimum point of $u^\varepsilon$. Using these asymptotic results for $u^\varepsilon(\zeta)$, they reduce 
the pattern determination problem to a certain kind of an algebraic system. 
Then patterns of transition layers and spikes are determined by solving this 
algebraic system.

We will now give some information on profiles of multi-layers and multi-
spikes:

**Theorem 0.0.4** (Profile of a multi-layer). Let $\sigma \in \Sigma^+$ (resp. $\Sigma^-$) and let 
$\delta$ be a small positive number. For an n-mode solution $u^\varepsilon$, set $\{\xi_k\}_{k=1}^{2m-1} = 
\Xi \cap (\sigma - \delta, \sigma + \delta)$ with $m \in \mathbb{N}$ and $2 \leq m \leq (n+1)/2$. Moreover, let 
$\{\zeta_k\}_{k=0}^{2m-1}$ be a unique set of critical points of $u^\varepsilon$ satisfying $\zeta_0 < \xi_1 < \zeta_1 < 
\xi_2 < \cdots < \xi_{2m-1} < \zeta_{2m-1}$, where $\zeta_0 := \sup\{x; u_x^\varepsilon(x) = 0 \text{ and } x < \xi_1\}$ and 
$\zeta_{2m-1} := \inf\{x; u_x^\varepsilon(x) = 0 \text{ and } x > \xi_{2m-1}\}$. Then it holds that 
$$u^\varepsilon(\zeta_{k-2}) < u^\varepsilon(\zeta_k) \quad (\text{resp. } u^\varepsilon(\zeta_{k-2}) > u^\varepsilon(\zeta_k)) \quad \text{for } k = 2, 3, \ldots, 2m-1.$$ 

**Theorem 0.0.5** (Profile of a multi-spike). Let $\lambda \in \Lambda^+$ (resp. $\Lambda^-$) and let $\delta$ be 
a small positive number. For an $n$-mode solution $u^\varepsilon$, set $\{\xi_k\}_{k=1}^{2m} = 
\Xi \cap (\lambda - \delta, \lambda + \delta)$ with $m \in \mathbb{N}$ and $2 \leq m \leq n/2$. Moreover, let $\{\zeta_k\}_{k=0}^{2m}$ be a unique set 
of critical points of $u^\varepsilon$ satisfying $\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \cdots < \xi_{2m} < \zeta_{2m}$, where 
$\zeta_0 := \sup\{x; u_x^\varepsilon(x) = 0 \text{ and } x < \xi_1\}$ and $\zeta_{2m} := \inf\{x; u_x^\varepsilon(x) = 0 \text{ and } x > \xi_{2m}\}$. Then it holds that 
$$\begin{cases} 
    u^\varepsilon(\zeta_{k-2}) < u^\varepsilon(\zeta_k) & \text{if } \zeta_0 \leq \zeta_{k-2} < \zeta_k < \lambda, \\
    u^\varepsilon(\zeta_{k-2}) > u^\varepsilon(\zeta_k) & \text{if } \lambda \leq \zeta_{k-2} < \zeta_k \leq \zeta_{2m}. 
\end{cases}$$ 

(resp. 
$$\begin{cases} 
    u^\varepsilon(\zeta_{k-2}) > u^\varepsilon(\zeta_k) & \text{if } \zeta_0 \leq \zeta_{k-2} < \zeta_k < \lambda, \\
    u^\varepsilon(\zeta_{k-2}) < u^\varepsilon(\zeta_k) & \text{if } \lambda \leq \zeta_{k-2} < \zeta_k \leq \zeta_{2m}. 
\end{cases}$$)

**Remark 0.0.6.** In Theorem 0.0.5, we do not refer to a multi-spike near the 
boundary 0 with $a_x(0) \neq 0$ or the boundary 1 with $a_x(1) \neq 0$. Concerning 
such multi-spikes, see Remark 1.5.13.

Here we should refer to the existence results of Ai, Chen and Hastings [1]. 
They have shown that (SP1) has solutions with arbitrary prescribed number
of transition layers and spikes at places allowed by Theorems 0.0.2 and 0.0.3
with the use of the shooting argument developed in Ai and Hastings [2].

Furthermore, Ei and Matsuzawa [10] and Matsuzawa [16] have discussed
the existence of a solution with transition layers and their locations for similar
problems corresponding to the case where \( a_x \) vanishes in a certain interval
\( I \subset [0,1] \).

In this part, we will also discuss the stability of a solution with transition
layers and spikes. For this purpose, consider the following linearized eigenvalue
problem:

\[
\begin{cases}
-\varepsilon^2 \phi_{xx} - f_u(x,u^\varepsilon)\phi = \mu \phi, & 0 < x < 1, \\
\phi_x(0) = \phi_x(1) = 0,
\end{cases}
\]  
(EVP)

where \( u^\varepsilon \) is a solution of (SP1) with transition layers or spikes. By virtue of
the Sturm-Liouville theory, it is well known that (EVP) has infinitely many
eigenvalues \( \{\mu_j\}_{j=1}^\infty \) satisfying

\[-\infty < \mu_1 < \mu_2 < \ldots < \mu_j < \ldots \to \infty \quad \text{as } j \to \infty.\]

It is also well known that, if \( \mu_1 > 0 \), then \( u^\varepsilon \) is stable, while, if there exists
an eigenvalue \( \mu_j \) with \( \mu_j < 0 \), then \( u^\varepsilon \) is unstable (see the monograph of
Henry [13]).

In general, the stability property of a solution has a close relationship with
its profile. For example, the results of Angenent, Mallet-Paret and Peletier [3],
and Hale and Sakamoto [12] tell us that the stability of solutions with single-
layers is decided by the direction of each transition layer. Therefore, our results
concerning the pattern of \( u^\varepsilon \) will play an important role in the stability analysis.

Using the information on the profile of \( u^\varepsilon \) and the Sturm-Liouville theory, we
will derive stability results. In particular, we will discuss the relationship
between the profile of \( u^\varepsilon \) and its Morse index. Here the notion of Morse
index is defined as follows:

**Definition** (Morse index). Let \( u^\varepsilon \) be a solution of (SP1). Then, Morse index
\( \text{Ind}(u^\varepsilon) \) of \( u^\varepsilon \) is defined by the number of negative eigenvalues of (EVP).

In order to state stability properties of solutions with transition layers and
spikes, we will introduce some notation. Let \( n_{l+} \) be the number of transition
layers satisfying \( u^\varepsilon_x(\xi)a_x(\xi) > 0 \), where \( \xi \) is a point in \( \Xi \) included in the corre-
sponding transition layer. Similarly, \( n_{l-} \) is defined by the number of transition
layers satisfying \( u^\varepsilon_x(\xi)a_x(\xi) < 0 \). Furthermore, let \( n_{sp} \) denote the number of
spikes.
Our stability analysis is based on the variational structure of (SP1). So we will use a method based on Courant’s min-max principle and the spectral theory for Schrödinger operators. In our analysis, the information on the precise profile of $u^\varepsilon$ such as our asymptotic results (Theorems 1.3.3, 1.3.6, 1.3.7 and 1.3.8) or the characterization of each transition layer and each spike (Lemma 1.2.5) are also useful. Stability results are given by the following theorem:

**Theorem 0.0.7** (Stability of solutions with transition layers and spikes). Let $u^\varepsilon$ be an $n$-mode solution of (SP1). Then the following assertions hold true:

(i) If $u^\varepsilon$ has only single-layers and $n_{t^+} = 0$ (i.e., $n_{t} = n$), then $u^\varepsilon$ is stable.

(ii) If $u^\varepsilon$ has only single-layers and $n_{t^+} > 0$, then $u^\varepsilon$ is unstable and $\text{Ind}(u^\varepsilon) = n_{t^+}$.

(iii) If $n_{t^+} > 0$ or $n_{sp} > 0$, then $u^\varepsilon$ is unstable and $\text{Ind}(u^\varepsilon) \geq n_{t^+} + n_{sp}$.

For the stability of solutions with transition layers and spikes, we should refer to the results of Ai, Chen and Hastings [1]. In [1], they have succeeded in deciding Morse index of any solution with transition layers and spikes exactly. More precisely, it is shown that

$$\text{Ind}(u^\varepsilon) = n_{t^+} + 2n_{sp^\pm} + n_{sp^0},$$

where $n_{sp^\pm}$ and $n_{sp^0}$ denote the number of spikes near points in $\Lambda^+ \cup \Lambda^-$ and $\Lambda^0$, respectively. In order to obtain the stability results, they introduced an auxiliary function $w$ satisfying the first equation of (EVP) with $\mu = 0$ and estimated the number of zero points of $w$; and this information played an important role to determine Morse index.

Although we do not obtain exact Morse index, our approach is very simple and extremely depending on the information on the profile of $u^\varepsilon$. We will divide the interval $(0, 1)$ into suitable subintervals and consider a kind of localized eigenvalue problem. In each problem, since the interval is small, we can recognize the feature of the profile of $u^\varepsilon$ and derive information on the relationship between $u^\varepsilon$ and its stability property. Though Ai, Chen and Hastings [1] have already established a result on exact Morse index, the author will intend to obtain the same one in a different way from theirs.
Part II

In this part, we will consider the following reaction-diffusion problem:

\[
\begin{cases}
    u_t = \varepsilon^2 (d(x)^2 u_x)_x + h(x)^2 f(u), & 0 < x < 1, \ t > 0, \\
    u_x(0, t) = u_x(1, t) = 0, & t > 0, \\
    u(x, 0) = u_0(x), & 0 < x < 1.
\end{cases}
\]  

(P2)

Here \( \varepsilon \) denotes a positive parameter and \( f(u) \) is a nonlinearity given by

\[ f(u) = u(1 - u)(u - 1/2). \]

Moreover \( d \) and \( h \) are \( C^2 \)-functions with the following properties:

(\( \Phi 1 \)) \( d(x) > 0 \) and \( h(x) > 0 \) in \([0, 1]\).

(\( \Phi 2 \)) Define \( \varphi(x) := d(x)h(x) \) and \( \Sigma := \{ x \in [0, 1]; \varphi_x(x) = 0 \} \). Then \( \Sigma \) is a non-empty finite set and \( \varphi_{xx}(x) \neq 0 \) at any \( x \in \Sigma \).

(\( \Phi 3 \)) \( d_x(0) = d_x(1) = h_x(0) = h_x(1) = 0. \)

For (P2), we will mainly discuss steady-state solutions with transition layers by assuming the existence of such solutions. The stationary problem associated with (P2) is given as follows:

\[
\begin{cases}
    \varepsilon^2 (d(x)^2 u_x)_x + h(x)^2 f(u) = 0, & 0 < x < 1, \\
    u_x(0) = u_x(1) = 0.
\end{cases}
\]  

(SP2)

In (SP2), we should pay attention to the functions \( d \) and \( h \). They cause spatial inhomogeneity to our problem and their interaction yields many kinds of solutions of (SP2). In addition, they have much effect on patterns of such solutions.

We now present some related results. When both \( d \) and \( h \) are constant functions, Chafee and Infant [4] proved that, for any \( n \in \mathbb{N} \), if \( \varepsilon \) is sufficiently small, then (SP2) admits a solution with \( n \) transition layers, and that every non-constant solution is unstable. See Chipot and Hale [5] who also proved that all non-constant solutions are unstable when \( d \) satisfies \( d_{xx}(x) < 0 \) in \([0, 1]\) and \( h \) is a constant function.

Concerning stable non-constant solutions, Yanagida [26] gave a remarkable result. He pointed out the possibility of the existence of a stable non-constant
solution under the condition that \( d \) possesses a local minimum point and \( h \) is a constant function. Moreover, under the same assumptions as above, Miyata and Yanagida [17] showed the existence of a stable solution with single-layers and that each of them appears in a neighborhood of a certain local minimum point of \( d \).

In the case that \( d \) is a constant function, Nakashima [19] has proved that there exists a solution with transition layers and that each transition layer must be located in a neighborhood of a critical point of \( h \). Furthermore, she has also shown that there exists a solution with multi-layers and that any multi-layer appears only in a neighborhood of a local maximum point of \( h \). As to the stability of solutions with transition layers, it is shown that, if every transition layer is a single-layer located in a neighborhood of a local minimum point of \( h \), then the solution is stable. It is also proved that any solution with a multi-layer is unstable by using Morse index.

Also we refer to Nakashima [18] in which she discussed a similar problem and proved the existence of a solution with single-layers. See also Nakashima and Tanaka [20]. They have studied a similar problem and shown the existence of solutions with single-layers and multi-layers by using variational method. Moreover, Ei and Matsuzawa [10] and Matsuzawa [15] have discussed the existence of a solution with transition layers and their locations for similar problems whose spatial inhomogeneity degenerates in an interval.

We are motivated by many works as above. Among them, Miyata and Yanagida [17] and Nakashima [19] provide great incentives to our research. In this part, we will study solutions of (SP2) with transition layers. As is stated previously, our main purpose is not to show the existence of such solutions but to investigate their patterns. In particular, taking account of the interaction of \( d \) and \( h \), we will characterize all patterns of solutions with transition layers and determine the location of transition layers including multi-layers completely.

When we concentrate ourselves on an oscillatory behavior of a solution \( u^\varepsilon \) of (SP2), it is convenient to take account of intersecting points of \( u = u^\varepsilon(x) \) and \( u = 1/2 \). We will introduce the notion of \textbf{\( n \)-mode solutions} as in Part I:

**Definition.** For any fixed integer \( n \geq 1 \), a solution \( u^\varepsilon \) of (SP2) is called an \( n \)-mode solution, if \( u^\varepsilon - 1/2 \) has exactly \( n \) zero points in the interval \((0, 1)\).

For an \( n \)-mode solution \( u^\varepsilon \), we define

\[
\Xi := \{ x \in [0, 1] ; \ u^\varepsilon(x) = 1/2 \}.
\]
Clearly, the number of elements in $\Xi$ is $n$.

In the study of $n$-mode solutions, it will turn out in Section 3.2 that an $n$-mode solution $u^\varepsilon$ forms a transition layer in a neighborhood of a point in $\Xi$ and that the graph of $u^\varepsilon$ is classified into the following two portions when $\varepsilon$ is small:

(i) $u^\varepsilon(x)$ is very close to either 0 or 1.

(ii) $u^\varepsilon(x)$ forms a transition layer connecting 0 and 1.

Therefore, the study of solutions with transition layers is essentially the same as that of $n$-mode solutions. Investigating $n$-mode solutions, we will obtain the following theorems concerning the location of transition layers:

**Theorem 0.0.8** (Location of transition layers for solutions of (SP2)). Let $u^\varepsilon$ be any $n$-mode solution of (SP2) with transition layers. Then any transition layer appears only in an $O(\varepsilon |\log \varepsilon|)$-neighborhood of a point in $\Sigma$. Moreover, the following assertions hold true:

(i) If $u^\varepsilon$ has a multi-layer, then it appears only in a neighborhood of a local maximum point of $\varphi$.

(ii) If $u^\varepsilon$ has a transition layer in a neighborhood of a local minimum point of $\varphi$, then it must be a single-layer.

(iii) If $\varphi_{xx}(0) > 0$ (resp. $\varphi_{xx}(1) > 0$), then $u^\varepsilon$ has no transition layer in a neighborhood of 0 (resp. 1).

This theorem implies that any multi-layer appears in a neighborhood of a local maximum point of $\varphi(x)$.

**Remark 0.0.9.** We now give a comment on the assumption ($\Phi_3$). Under this assumption, we have obtained Theorem 0.0.8. However, we can show similar results without this assumption. Indeed, if ($\Phi_3$) is not supposed, then we can show that the boundary point 0 (resp. 1) becomes a candidate for the location where transition layers appear when $\phi_x(0) < 0$ (resp. $\phi_x(1) > 0$) in addition to the locations determined by Theorem 0.0.8. If we address such cases, then the argument becomes somewhat complicated, so that we assume ($\Phi_3$) in this thesis for the sake of simplicity.

We can also study a profile of each multi-layer.

**Theorem 0.0.10** (Profile of a multi-layer). Let $\sigma \in \Sigma$ satisfy $\varphi_{xx}(\sigma) < 0$ and let $\delta$ be a small positive number. For an $n$-mode solution $u^\varepsilon$, set $\{\xi_k\}_{k=1}^m =$
\( \Xi \cap (\sigma - \delta, \sigma + \delta) \) with \( m \in \mathbb{N} \) and \( 2 \leq m \leq n \). Moreover, let \( \{\zeta_k\}_{k=0}^m \) be a unique set of critical points of \( u^\varepsilon \) satisfying \( \zeta_0 < \xi_1 < \xi_2 < \cdots < \xi_m < \zeta_m \), where

\[
\zeta_0 := \sup \{ x; \ u^\varepsilon_x(x) = 0 \text{ and } x < \xi_1 \} \quad \text{and} \quad \zeta_m := \inf \{ x; \ u^\varepsilon_x(x) = 0 \text{ and } x > \xi_m \}.
\]

If \( u^\varepsilon_x(\xi_1) < 0 \) (resp. \( u^\varepsilon_x(\xi_1) > 0 \)), then it holds that

\[
\begin{aligned}
&u^\varepsilon(\zeta_{2k-2}) > u^\varepsilon(\zeta_{2k}) \quad \text{if } \zeta_0 \leq \zeta_{2k-2} < \zeta_{2k} < \sigma, \\
u^\varepsilon(\zeta_{2k-1}) < u^\varepsilon(\zeta_{2k+1}) \quad \text{if } \zeta_{1} \leq \zeta_{2k-1} < \zeta_{2k+1} < \sigma,
\end{aligned}
\]

and

\[
\begin{aligned}
u^\varepsilon(\zeta_{2k-2}) < u^\varepsilon(\zeta_{2k}) \quad \text{if } \sigma \leq \zeta_{2k-2} < \zeta_{2k} < \zeta_m, \\
u^\varepsilon(\zeta_{2k-1}) > u^\varepsilon(\zeta_{2k+1}) \quad \text{if } \sigma \leq \zeta_{2k-1} < \zeta_{2k+1} \leq \zeta_m.
\end{aligned}
\]

(resp.,

\[
\begin{aligned}
u^\varepsilon(\zeta_{2k-2}) < u^\varepsilon(\zeta_{2k}) \quad \text{if } \zeta_0 \leq \zeta_{2k-2} < \zeta_{2k} < \sigma, \\
u^\varepsilon(\zeta_{2k-1}) > u^\varepsilon(\zeta_{2k+1}) \quad \text{if } \zeta_{1} \leq \zeta_{2k-1} < \zeta_{2k+1} < \sigma,
\end{aligned}
\]

and

\[
\begin{aligned}
u^\varepsilon(\zeta_{2k-2}) > u^\varepsilon(\zeta_{2k}) \quad \text{if } \sigma \leq \zeta_{2k-2} < \zeta_{2k} \leq \zeta_m, \\
u^\varepsilon(\zeta_{2k-1}) < u^\varepsilon(\zeta_{2k+1}) \quad \text{if } \sigma \leq \zeta_{2k-1} < \zeta_{2k+1} \leq \zeta_m.
\end{aligned}
\]

After this thesis has been almost completed, Professor Tanaka has pointed out to the author that a suitable change of variable enables us to reduce (SP2) to the balanced bistable equation studied by Nakashima \[19\]. Indeed, define a new variable \( y \) by

\[
y = \int_0^x \frac{ds}{d(s)^2} \quad (0.0.1)
\]

and set \( v(y) = u(x) \) with use of (0.0.1). Then

\[
v_y = d(x)^2 u_x, \quad v_{yy} = d(x)^2 (d(x)^2 u_x)_x,
\]

so that one can reduce the first equation of (SP2) to

\[
\varepsilon^2 v_{yy} + \varphi(x)^2 f(v) = 0, \quad y \in [0, L] \quad \text{with} \quad L = \int_0^1 \frac{ds}{d(s)^2},
\]

where \( \varphi(x) = d(x) h(x) \) and \( x \) should be regarded as a function of \( y \). Hence it directly follows from her results in \[19\] that any transition layer appears in a neighborhood of a point in the set of critical points of \( \varphi(x) \) and that any multi-layer appears only in a neighborhood of a point in the set of local maximum points of \( \varphi(x) \) (Theorem 0.0.8). Moreover, it is also possible to
show the existence of a solution having multi-layers with prescribed number of transition layers near an arbitrary chosen set of local maximum points and a single-layer near an arbitrary chosen set of local minimum points of $\varphi(x)$.

In this sense, it now turns out that our results are covered by the results of Nakashima [19]. However, the author believes that the ideas and arguments developed in Part II will be useful to other subjects. For example, the study on stability properties of solutions of (SP2) attracts our interest. Concerning stationary problems, a problem is essentially the same as the corresponding problem transformed by the change of variables such as (0.0.1). However, for the linearized eigenvalue problem, the situation changes and it will require the different argument than [19], so that we expect that our method will help us to establish the stability results. Also we consider that it is challenging to apply our method to the other problems with different nonlinearity such as $u(1 - u)(u - a(x))$ or $u(1 - a(x))u^2$. 
Part I

Transition Layers and Spikes for a Bistable Reaction-Diffusion Equation
Chapter 1

Patterns of Solutions with Transition Layers and Spikes

1.1 Introductory Section of Chapter 1

In this chapter we consider the following reaction-diffusion equation:

\[
\begin{cases}
  u_t = \varepsilon^2 u_{xx} + f(x, u), & 0 < x < 1, \ t > 0, \\
  u_x(0, t) = u_x(1, t) = 0, & t > 0, \\
  u(x, 0) = u_0(x), & 0 < x < 1.
\end{cases}
\]

(P1)

Here \( \varepsilon \) is a positive parameter and \( f(x, u) \) is given by

\[
f(x, u) = u(1-u)(u-a(x)),
\]

where \( a \) is a \( C^2 \)-function with the following properties:

(A1) \( 0 < a(x) < 1 \) in \( [0, 1] \).

(A2) If \( \Sigma := \{x \in [0, 1]; a(x) = 1/2\} \), then \( \Sigma \) is a non-empty finite set, 0, 1 \( \notin \Sigma \) and \( a_x(x) \neq 0 \) at any \( x \in \Sigma \).

(A3) If \( \Lambda := \{x \in [0, 1]; a_x(x) = 0\} \), then \( \Lambda \) is a non-empty finite set.

Furthermore, we introduce the following notation:

\[
\begin{align*}
  \Sigma^+ &:= \{x \in \Sigma; a_x(x) > 0\}, \\
  \Sigma^- &:= \{x \in \Sigma; a_x(x) < 0\}, \\
  \Lambda^+ &:= \{x \in \Lambda; a(x) < 1/2 \text{ and } a_{xx}(x) < 0\}, \\
  \Lambda^- &:= \{x \in \Lambda; a(x) > 1/2 \text{ and } a_{xx}(x) > 0\}, \\
  \Lambda^0 &:= \Lambda \setminus (\Lambda^+ \cup \Lambda^-).
\end{align*}
\]
In this chapter, we will mainly discuss patterns of steady-state solutions with transition layers. The stationary problem associated with (P1) is given by

\[
\begin{align*}
\varepsilon^2 u'' + f(x,u) &= 0, \quad 0 < x < 1, \\
u_x(0) &= u_x(1) = 0.
\end{align*}
\] (SP1)

The appearance of transition layers is closely related to the bistability of the reaction term \( f(x,u) \). As an energy functional of (P1), one can find

\[
E(u) = \int_0^1 \left[ \frac{1}{2} \varepsilon^2 (u_x(x))^2 + W(x,u(x)) \right] dx,
\]

where

\[
W(x,u) = - \int_{\phi_0(x)}^{u} f(x,s) ds
\] (1.1.1)

with

\[
\phi_0(x) = \begin{cases} 
0 & \text{if } a(x) \leq 1/2, \\
1 & \text{if } a(x) > 1/2.
\end{cases}
\]

Note that \( W \) is a double-well potential which takes its local minima both at \( u = 0 \) and \( u = 1 \). It is well known that every solution of (P1) converges to a solution of (SP1) as \( t \to \infty \) and that \( E(u(\cdot,t)) \) is monotone decreasing with respect to \( t \). Therefore, a minimizer of \( E \) will be a stable solution of (SP1).

Proofs of these facts will be found in Matano [14].

We should note that the minimum of \( W(x,\cdot) \) is attained at \( u = 1 \) (resp. \( u = 0 \)) when \( a(x) < 1/2 \) (resp. \( a(x) > 1/2 \)). Intuitively, this fact assures that \( E \) has a minimizer \( u^\varepsilon \) with a transition layer near an \( x_0 \in \Sigma \) with \( u^\varepsilon(x_0) a_x(x_0) < 0 \). Also this fact implies that the spatial inhomogeneity has much effect on patterns of solutions of (SP1).

In order to investigate solutions of (SP1) with oscillatory profiles such as transition layers and spikes, we introduce the notion of \( n \)-mode solutions as follows:

**Definition 1.1.1.** For any fixed integer \( n \geq 1 \), a solution \( u^\varepsilon \) of (SP1) is called an \( n \)-mode solution, if \( u^\varepsilon - a \) has exactly \( n \) zero points in the interval \((0,1)\).

Hereafter, we denotes the set of all \( n \)-mode solutions by \( S_{n,\varepsilon} \). Moreover, for \( u^\varepsilon \in S_{n,\varepsilon} \), we define

\[
\Xi = \{ x \in [0,1] ; u^\varepsilon(x) = a(x) \}.
\]

It should be noted that the number of the elements in \( \Xi \) is \( n \).
In what follows, we sometimes extend \( u^\varepsilon \) to a function over \( \mathbb{R} \) by the standard reflection. This is possible because \( u^\varepsilon \) satisfies \( u^\varepsilon_1(0) = u^\varepsilon_1(1) = 0 \). Therefore \( u^\varepsilon \) is regarded as a periodic function with period 2. Similarly, \( f(x,u) \) can be extended for \( (x,u) \in \mathbb{R} \times \mathbb{R} \) by the reflection with respect to \( x \)-variable and \( f(x,u) \) is a piecewise smooth function. Hence we may consider that \( u^\varepsilon \) satisfies (SP1) for all \( x \in \mathbb{R} \).

Roughly speaking, for any \( u^\varepsilon \in S_{n,\varepsilon} \), its graph is classified into the following three groups:

(i) \( u^\varepsilon(x) \) is close to 0 or 1,

(ii) \( u^\varepsilon(x) \) forms a transition layer connecting 0 and 1,

(iii) \( u^\varepsilon(x) \) forms a spike based on 0 or 1.

Moreover, \( u^\varepsilon \) forms a transition layer or a spike near a point in \( \Xi \). These facts will be shown in Section 1.2 (Lemmas 1.2.3 and 1.2.5).

By the assertions above, it is fair to say that \( u^\varepsilon \) comes close to 0 or 1 almost everywhere in \([0,1]\). In particular, \( u^\varepsilon \) approaches 0 or 1 in a neighborhood of its extremal point except for a peak of a spike. From this viewpoint, it will be important to study the asymptotic rate of \( 1 - u^\varepsilon(x) \) (resp. \( u^\varepsilon(x) \)) as \( \varepsilon \) goes to 0 in a certain interval containing one local maximum point (resp. local minimum point) of \( u^\varepsilon \). The analysis to get the asymptotic rates will be carried out by a kind of barrier method. The following theorem is one of our main results describing the asymptotic rates of \( 1 - u^\varepsilon(x) \):

**Theorem 1.1.1.** For \( u^\varepsilon \in S_{n,\varepsilon} \), let \( \xi_1, \xi_2 \in \Xi \) be successive points satisfying \( u^\varepsilon(x) - a(x) > 0 \) in \((\xi_1, \xi_2)\) and let \( \zeta \) be the unique local maximum point in \((\xi_1, \xi_2)\). Then the following assertions hold true:

(i) If \( (\zeta - \xi_1)/\varepsilon \to \infty \) as \( \varepsilon \to 0 \), then there exist positive constants \( C_1, C_2, r \) and \( R \) with \( C_1 < C_2 \) and \( r < R \) satisfying

\[
C_1 \exp \left( -\frac{R(\zeta - \xi_1)}{\varepsilon} \right) < 1 - u^\varepsilon(x) < C_2 \exp \left( -\frac{r(x - \xi_1)}{\varepsilon} \right)
\]

in \([\xi_1, \zeta]\), provided that \( \varepsilon \) is sufficiently small.

(ii) If \( (\xi_2 - \zeta)/\varepsilon \to \infty \) as \( \varepsilon \to 0 \), then there exist positive constants \( C_1, C_2, r \) and \( R \) with \( C_1 < C_2 \) and \( r < R \) satisfying

\[
C_1 \exp \left( -\frac{R(\xi_2 - \zeta)}{\varepsilon} \right) < 1 - u^\varepsilon(x) < C_2 \exp \left( -\frac{r(\xi_2 - x)}{\varepsilon} \right)
\]

in \([\zeta, \xi_2]\), provided that \( \varepsilon \) is sufficiently small.
It should be noted that analogous theorems describing the asymptotic rates of $u^\varepsilon(x)$ also hold true (see Theorems 1.3.7 and 1.3.8).

Using these theorems, we have the following theorems describing patterns of solution with transition layers and spikes:

**Theorem 1.1.2** (Location of transition layers and their multiplicity). For $u^\varepsilon \in S_{n,\varepsilon}$, if $\varepsilon$ is sufficiently small, the following assertions hold true:

(i) If $u^\varepsilon$ has a single-layer, then it appears only in an $O(\varepsilon)$-neighborhood of a point in $\Sigma$.

(ii) If $u^\varepsilon$ has a multi-layer, then it consists of odd number of transition layers. Thus the multi-layer is shaped like a cluster of transition layers connecting from 0 to 1 or 1 to 0. Moreover, if $u^\varepsilon$ has a multi-layer connecting from 0 to 1 (resp. 1 to 0), it appears only in an $O(\varepsilon \log \varepsilon)$-neighborhood of a point in $\Sigma^+$ (resp. $\Sigma^-$).

**Theorem 1.1.3** (Location of spikes and their multiplicity). For $u^\varepsilon \in S_{n,\varepsilon}$ with spikes, any spike appears only in a neighborhood of a point in $\Lambda \cup \{0, 1\}$, provided that $\varepsilon$ is sufficiently small. Moreover the following assertions hold true:

(i) Any spike based on 0 (resp. 1) appears only in a neighborhood of a point $\lambda \in \Lambda \cup \{0, 1\}$ with $a(\lambda) < 1/2$ (resp. $a(\lambda) > 1/2$).

(ii) If $u^\varepsilon$ has a multi-spike based on 0 (resp. 1), then it appears only in an $O(\varepsilon \log \varepsilon)$-neighborhood of a point in $\Lambda^+$ (resp. $\Lambda^-$), a boundary point 0 with $a(0) < 1/2$ and $a_x(0) < 0$ (resp. $a(0) > 1/2$ and $a_x(0) > 0$) or a boundary point 1 with $a(1) < 1/2$ and $a_x(1) > 0$ (resp. $a(1) > 1/2$ and $a_x(1) < 0$).

The profile of each multi-layer or multi-spike is described as follows:

**Theorem 1.1.4** (Profile of a multi-layer). Let $\sigma \in \Sigma^+$ (resp. $\Sigma^-$) and $\delta$ be a small positive number. For $u^\varepsilon \in S_{n,\varepsilon}$, set $\{\xi_k\}_{k=1}^{2m-1} = \Xi \cap (\sigma - \delta, \sigma + \delta)$ with $m \in \mathbb{N}$ and $2 \leq m \leq (n + 1)/2$. Moreover, let $\{\xi_k\}_{k=0}^{2m-1}$ be a unique set of critical points of $u^\varepsilon$ satisfying $\zeta_0 < \xi_1 < \xi_2 < \cdots < \xi_{2m-1} < \zeta_{2m-1}$, where $\zeta_0 := \sup \{x; u^\varepsilon_x(x) = 0 \text{ and } x < \xi_1\}$ and $\zeta_{2m-1} := \inf \{x; u^\varepsilon_x(x) = 0 \text{ and } x > \xi_{2m-1}\}$. Then it holds that

\[ u^\varepsilon(\xi_{k-2}) < u^\varepsilon(\xi_k) \quad (\text{resp. } u^\varepsilon(\xi_{k-2}) > u^\varepsilon(\xi_k)) \quad \text{for } k = 2, 3, \ldots, 2m - 1. \]

**Theorem 1.1.5** (Profile of a multi-spike). Let $\lambda \in \Lambda^+$ (resp. $\Lambda^-$) and $\delta$ be a small positive number. For $u^\varepsilon \in S_{n,\varepsilon}$, set $\{\xi_k\}_{k=1}^{2m} = \Xi \cap (\sigma - \delta, \sigma + \delta)$ with
\(m \in \mathbb{N}\) and \(2 \leq m \leq n/2\). Moreover, let \(\{\zeta_k\}_{k=0}^{2m}\) be a unique set of critical points of \(u^\varepsilon\) satisfying \(\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \cdots < \xi_{2m} < \zeta_{2m}\), where \(\zeta_0 := \sup\{x; u^\varepsilon_x(x) = 0\text{ and } x < \xi_1\}\) and \(\zeta_{2m} := \inf\{x; u^\varepsilon_x(x) = 0\text{ and } x > \xi_{2m}\}\). Then it holds that
\[
\begin{align*}
\{\ u^\varepsilon(\zeta_{k-2}) < u^\varepsilon(\zeta_k) & \quad \text{if } \zeta_0 \leq \zeta_{k-2} < \zeta_k < \lambda, \\
\ u^\varepsilon(\zeta_{k-2}) > u^\varepsilon(\zeta_k) & \quad \text{if } \lambda \leq \zeta_{k-2} < \zeta_k \leq \zeta_{2m}.
\end{align*}
\]

Remark 1.1.6. In Theorem 1.1.5, we do not refer to a multi-spike near the boundary 0 with \(a_x(0) \neq 0\) or the boundary 1 with \(a_x(1) \neq 0\). Concerning such multi-spikes, see Remark 1.5.13.

The content of this chapter is as follows. In Section 1.2, we will give some fundamental properties of \(u^\varepsilon \in S_{n,\varepsilon}\). In Section 1.3, the asymptotic rates of \(1 - u^\varepsilon(x)\) and \(u^\varepsilon(x)\) will be discussed. The asymptotic results are given by Theorems 1.3.3, 1.3.6, 1.3.7 and 1.3.8. In Section 1.4, using these theorems, we will show that any transition layer (resp. spike) appears only in a neighborhood of a point of \(\Sigma\) (resp. \(\Lambda \cup \{0, 1\}\)). Finally, Section 1.5 is devoted to the study of multi-layers and multi-spikes. As a consequence, summarizing the assertions in Sections 1.4 and 1.5, one will obtain Theorems 1.1.2, 1.1.3, 1.1.4 and 1.1.5.

### 1.2 Transition Layers and Spikes of \(n\)-mode Solutions

In this section we will give some basic properties of solutions of (SP1).

The first one will be shown by the use of the maximum principle developed in Protter and Weinberger [21].

**Lemma 1.2.1.** Let \(u^\varepsilon\) be a solution of (SP1). Then
\[
0 \leq u^\varepsilon(x) \leq 1 \quad \text{in } [0, 1].
\]

Furthermore, if \(u^\varepsilon \neq 0\) or 1, then it holds that
\[
0 < u^\varepsilon(x) < 1 \quad \text{in } [0, 1]. \tag{1.2.1}
\]
Proof. Assume that
\[
u^\varepsilon(x_0) = \max\{u^\varepsilon(x) ; x \in [0, 1]\} > 1 \quad (1.2.2)
\]
for some \(x_0 \in [0, 1]\). It follows from \(u^\varepsilon_{xx}(x_0) \leq 0\) that \(f(x_0, u^\varepsilon(x_0)) \geq 0\). However (1.2.2) implies \(f(x_0, u^\varepsilon(x_0)) < 0\), which is a contradiction. Hence \(u^\varepsilon(x) \leq 1\).

Similarly, it is easy to show that \(u^\varepsilon(x) \geq 0\).

In order to give a proof of (1.2.1), suppose that \(u^\varepsilon(x_1) = \max\{u^\varepsilon(x) ; x \in [0, 1]\} = 1\) at some \(x_1 \in [0, 1]\). In this case, it holds that \(u^\varepsilon_{xx}(x_1) = 0\). Therefore, by virtue of the uniqueness of solutions for the initial value problem of the second order differential equation, we immediately get \(u^\varepsilon \equiv 1\). Hence \(u^\varepsilon \equiv 1\) in \([0, 1]\) unless \(u^\varepsilon \equiv 0\). Similarly, one can see that, if \(u \neq 0\), then \(u(x) > 0\) in \([0, 1]\).

This completes the proof.

Remark 1.2.2. If \(u^\varepsilon \in S_{n,\varepsilon}\), then \(u^\varepsilon \not\equiv 0\) and 1. Therefore (1.2.1) is valid for any \(u^\varepsilon \in S_{n,\varepsilon}\).

Lemma 1.2.3. For \(n \in \mathbb{N}\), it holds that
\[
\lim_{\varepsilon \to 0} \sup_{u^\varepsilon \in S_{n,\varepsilon}} \max_{x \in [0, 1]} u^\varepsilon(x)(1 - u^\varepsilon(x)) \left[ \frac{1}{2} \varepsilon^2 (u^\varepsilon_x(x))^2 - W(x, u^\varepsilon(x)) \right] = 0, \quad (1.2.3)
\]
where \(W(x, u)\) is defined by (1.1.1).

Proof. Suppose that (1.2.3) is not true, then there exists a set \(\{(\varepsilon_k, u^k, x_k)\}\) such that \(u^k \in S_{n,\varepsilon_k}\), \(x_k \in [0, 1]\) and
\[
\left| u^k(x_k)(1 - u^k(x_k)) \left[ \frac{1}{2} \varepsilon_k^2 (u^k_x(x_k))^2 - W(x_k, u^k(x_k)) \right] \right| \geq \delta \quad (1.2.4)
\]
with some \(\delta > 0\).

We use the change of variable \(x = x_k + \varepsilon_k z\) and introduce a new function \(U^k\) by \(U^k(z) = u^k(x_k + \varepsilon_k z)\). Clearly, \(U^k\) satisfies
\[
U^k_{zz} + f(x_k + \varepsilon_k z, U^k) = 0 \quad \text{in } \mathbb{R}. \quad (1.2.5)
\]

We first prove the uniform boundedness of \(\{U^k\}\), \(\{U^k_z\}\) and \(\{U^k_{zz}\}\). By Lemma 1.2.1, it holds that \(\sup\{|U^k(z)| ; z \in \mathbb{R}\} < 1\), so that (1.2.5) enables us to see that \(K := \sup\{|U^k_{zz}(z)| ; z \in \mathbb{R}\} < \infty\). In order to study \(U^k_z\), we
take any \( z \in \mathbb{R} \). The mean value theorem assures that there exists a number \( z_0 \in (z, z + 1) \) such that

\[
U^k_z(z_0) = U^k(z + 1) - U^k(z).
\]

Therefore it follows from Lemma 1.2.1 that \( -1 < U^k_z(z_0) < 1 \). Hence we have

\[
|U^k_z(z)| = \left| U^k_z(z_0) + \int_{z_0}^z U^k_z(s)ds \right| < 1 + K.
\]

These estimates imply that \( \{U^k\}, \{U^k_z\} \) and \( \{U^k_{zz}\} \) are uniformly bounded in \( \mathbb{R} \). Therefore, it is easy to see that \( \{U^k\} \) and \( \{U^k_z\} \) are equi-continuous. Moreover, it follows from (1.2.5) that \( \{U^k_{zz}\} \) is also equi-continuous.

On account of the results above, applying Ascoli-Arzelà’s theorem and using a diagonal argument, one can show that \( \{U^k\} \) has a subsequence, which is still denoted by \( \{U^k\} \), such that

\[
\lim_{k \to \infty} U^k = U \quad \text{in } C^2_{\text{loc}}(\mathbb{R})
\]

with a suitable function \( U \in C^2(\mathbb{R}) \). Here we recall that \( \{x_k\} \) is bounded. Hence one can choose a convergent subsequence from \( \{x_k\} \) satisfying

\[
\lim_{k \to \infty} x_k = x^* \quad \text{with some } x^* \in [0, 1].
\]

Then it is easy to see that \( U \) satisfies

\[
U_{zz} + f(x^*, U) = 0 \quad \text{in } \mathbb{R}.
\]  \hspace{1cm} (1.2.6)

Multiplying (1.2.6) by \( U_z \) and integrating the resulting expression with respect to \( t \), we get

\[
\frac{1}{2} (U_z(z))^2 - W(x^*, U(z)) = C \quad \text{in } \mathbb{R}
\]  \hspace{1cm} (1.2.7)

with some constant \( C \). If \( U \equiv 0 \) or \( U \equiv 1 \), then it is easy to derive a contradiction to (1.2.4) from (1.2.7).

We will show \( C = 0 \) in (1.2.7) in the case that \( U \not\equiv 0 \) and \( U \not\equiv 1 \). If \( C > 0 \), then we see from the phase plane analysis that \( U \) is unbounded. This is impossible because \( \{U^k\} \) is bounded. If \( C < 0 \), then the phase plane analysis tells us that \( U \) is a periodic function. Hence the graph of \( U(z) \) has infinitely many intersecting points with that of \( a(x^*) \). Therefore the same holds true.
for the graph of $U^k(z)$ when $k$ is sufficiently large. This fact implies that, if $k$ is sufficiently large, then $u^k(x) - a(x)$ has infinitely many zero points near $x = x^*$. This result contradicts the definition of $n$-mode solutions. Thus we have proved $C = 0$ in (1.2.7).

Hence

$$\lim_{k \to \infty} \left| \frac{1}{2} \varepsilon_k^2 (u^k_z(x_k))^2 - W(x_k, u^k(x_k)) \right| = \lim_{k \to \infty} \left| \frac{1}{2} (U^k_z(0))^2 - W(x_k, U_k(0)) \right| = \left| \frac{1}{2} (U_z(0))^2 - W(x^*, U(0)) \right| = 0,$$

which contradicts (1.2.4). Thus the proof is complete. \hfill \Box

**Lemma 1.2.4.** For $u^\varepsilon \in S_{n, \varepsilon}$, set $\Xi = \{ \xi_k \}_{k=1}^n$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. If $\varepsilon$ is sufficiently small, then there exists a unique set of critical points $\{ \zeta_k \}_{k=0}^n$ of $u^\varepsilon$ satisfying

$$0 = \zeta_0 < \zeta_1 < \zeta_2 < \zeta_3 < \cdots < \zeta_{n-1} < \zeta_n = 1.$$

**Proof.** Let $\xi \in \Xi$ and take any small $\eta > 0$. Then Lemma 1.2.3 implies that, if $\varepsilon$ is sufficiently small, then

$$\left| u^\varepsilon(\xi)(1 - u^\varepsilon(\xi)) \left[ \frac{1}{2} \varepsilon^2 (u^\varepsilon_z(\xi))^2 - W(\xi, u^\varepsilon(\xi)) \right] \right| = \left| a(\xi)(1 - a(\xi)) \left[ \frac{1}{2} \varepsilon^2 (u^\varepsilon_z(\xi))^2 - W(\xi, a(\xi)) \right] \right| < \eta.$$

Since $a(\xi)(1 - a(\xi)) > M$ with some $\varepsilon$-independent $M > 0$, we get

$$-\frac{\eta}{M} < \frac{1}{2} \varepsilon^2 (u^\varepsilon_z(\xi))^2 - W(\xi, a(\xi)) < \frac{\eta}{M}.$$

Observe that $W(\xi, a(\xi)) \geq K_1$, where $K_1$ is a positive constant independent of $\varepsilon$. Hence, taking a sufficiently small $\varepsilon > 0$, one can conclude that

$$\varepsilon^2 (u^\varepsilon_z(\xi))^2 \geq K^2_2$$

with some positive constant $K_2$. Thus

$$\left| u^\varepsilon_z(\xi) \right| > \frac{K_2}{\varepsilon} \quad \text{(1.2.8)}$$

for sufficiently small $\varepsilon$. 23
We study the case that \( u^\varepsilon(x) > a(x) \) in \((\xi_k, \xi_{k+1})\). By (1.2.8) and the boundedness of \( a_x(x) \), it is easy to see \( u^\varepsilon(x) > a(x) \) and \( \partial_x u^\varepsilon(x) < 0 \). Moreover, since \( u_{xx}(x) < 0 \) in \((\xi_k, \xi_{k+1})\), \( u^\varepsilon \) has a unique zero point in \((\xi_k, \xi_{k+1})\), which is denoted by \( \zeta_k \). Clearly \( u^\varepsilon \) attains its local maximum at \( x = \zeta_k \).

Since the proof is analogous for the case \( u^\varepsilon(x) < a(x) \) in \((\xi_k, \xi_{k+1})\), it remains to show the nonexistence of zero point of \( u^\varepsilon \) in \((0, \xi_1) \cup (\xi_n, 1)\). Assume \( u^\varepsilon(x) > a(x) \) in \((0, \xi_1)\). Since \( u^\varepsilon(0) = 0 \) and \( u^\varepsilon(x) < 0 \) in \((0, \xi_1)\), it is clear that \( u^\varepsilon(x) = 0 \) in \((0, \xi_1)\). The other cases can be discussed in the same way. 

\[ \square \]

**Lemma 1.2.5.** For \( u^\varepsilon \in S_{n, \varepsilon} \), let \( \xi \) be any point in \( \Xi \) and define \( U^\varepsilon \) by \( U^\varepsilon(z) = u^\varepsilon(\xi + \varepsilon z) \). Then there exists a subsequence \( \varepsilon_k \downarrow 0 \) such that

\[
\lim_{k \to \infty} \xi^\varepsilon_k = \xi^* \quad \text{and} \quad \lim_{k \to \infty} U^\varepsilon_k = U \quad \text{in } C^2_{loc}(\mathbb{R}),
\]

where \( \xi^* \in [0, 1] \) and \( U \in C^2(\mathbb{R}) \) is a function satisfying one of the following properties:

(i) In the case that \( a(\xi^*) = 1/2 \), \( U \) is the unique solution of

\[
\begin{cases}
  U_{zz} + f(\xi^*, U) = 0 & \text{in } \mathbb{R}, \\
  U(-\infty) = 0, \quad U(+\infty) = 1 & (\text{resp. } U(-\infty) = 1, \quad U(+\infty) = 0), \\
  U(0) = 1/2,
\end{cases}
\]

if \( U_z(0) > 0 \) (resp. \( U_z(0) < 0 \)). Moreover, \( U_z(z) > 0 \) for \( z \in \mathbb{R} \) if \( U_z(0) > 0 \), while \( U_z(z) < 0 \) for \( z \in \mathbb{R} \) if \( U_z(0) < 0 \).

(ii) In the case that \( a(\xi^*) < 1/2 \), \( U \) is a solution of

\[
\begin{cases}
  U_{zz} + f(\xi^*, U) = 0 & \text{in } \mathbb{R}, \\
  U(0) = a(\xi^*), \quad U(\pm\infty) = 0.
\end{cases}
\]

Moreover, \( U \) satisfies \( \sup\{U(z) : z \in \mathbb{R}\} > a(\xi^*) \).

(iii) In the case \( a(\xi^*) > 1/2 \), \( U \) is a solution of

\[
\begin{cases}
  U_{zz} + f(\xi^*, U) = 0 & \text{in } \mathbb{R}, \\
  U(0) = a(\xi^*), \quad U(\pm\infty) = 1.
\end{cases}
\]

Moreover, \( U \) satisfies \( \inf\{U(z) : z \in \mathbb{R}\} < a(\xi^*) \).

**Proof.** Clearly, \( u^\varepsilon \) satisfies

\[
u^\varepsilon_{zz} + f(\xi + \varepsilon z, U^\varepsilon) = 0 \quad \text{and} \quad U^\varepsilon(0) = a(\xi^*).
\]
As in the proof of Lemma 1.2.3, one can prove that there exists a subsequence \( \{\varepsilon_k\} \downarrow 0 \) such that
\[
\lim_{k \to \infty} U^\varepsilon_k = U \quad \text{in } C^2_{\text{loc}}(\mathbb{R})
\]
(1.2.9) with some \( U \in C^2(\mathbb{R}) \). Moreover, since \( \{\xi^\varepsilon_k\} \) is bounded, we may assume \( \lim_{k \to \infty} \xi^\varepsilon_k = \xi^* \) with some \( \xi^* \in [0, 1] \). Hence the limiting procedure yields that
\[
\begin{cases}
U_{zz}(z) + f(\xi^*, U(z)) = 0 \quad \text{in } \mathbb{R}, \\
U(0) = a(\xi^*).
\end{cases}
\]
The same argument as in the proof of (1.2.7) with \( C = 0 \) also shows
\[
\frac{1}{2}(U_z(z))^2 - W(\xi^*, U(z)) = 0 \quad \text{in } \mathbb{R}.
\]
Therefore the phase plane analysis enables us to conclude that \( U \) satisfies one of (i)-(iii).

Lemma 1.2.6. For \( u^\varepsilon \in S_{n, \varepsilon} \), let \( \xi_1^\varepsilon, \xi_2^\varepsilon \) be two successive points in \( \Xi \). Then one of the following properties hold true:
(i) \((\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon \) is unbounded as \( \varepsilon \to 0 \),
(ii) For sufficiently small \( \varepsilon > 0 \), it holds that
\[
M_1 < \frac{\xi_2^\varepsilon - \xi_1^\varepsilon}{\varepsilon} < M_2,
\]
where \( M_1 \) and \( M_2 \) are positive constants independent of \( \varepsilon \).

Proof. Putting \( U^\varepsilon(z) = u^\varepsilon(\xi_1^\varepsilon + \varepsilon z) \), we have
\[
\begin{cases}
U^\varepsilon_z(0) = \varepsilon u^\varepsilon_x(\xi_1^\varepsilon), \\
U^\varepsilon_z((\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon) = \varepsilon u^\varepsilon_x(\xi_2^\varepsilon).
\end{cases}
\]
In view of (1.2.8) we see
\[
U^\varepsilon_z(0) U^\varepsilon_z((\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon) = \varepsilon^2 u^\varepsilon_x(\xi_1^\varepsilon) u^\varepsilon_x(\xi_2^\varepsilon) < -c_2^2 < 0.
\]
(1.2.10)
Suppose that (i) does not holds true; i.e., \( \{(\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon\} \) is bounded. Then one can choose a subsequence \( \{\varepsilon_k\} \) such that
\[
0 \leq M := \lim_{k \to \infty} \frac{\xi_2^{\varepsilon_k} - \xi_1^{\varepsilon_k}}{\varepsilon_k} < +\infty.
\]
(1.2.11)
Recalling the proof of Lemma 1.2.5, we may regard \( \{U^\varepsilon_k\} \) as a convergent sequence satisfying (1.2.9). Setting \( \varepsilon = \varepsilon_k \) in (1.2.10) and letting \( k \to \infty \) we get
\[
U_z(0)U_z(M) \leq -K,
\]
where \( K \) is a positive constant. This together with 1.2.11 enables us to obtain that \( M > 0 \). Thus we have shown (ii) when (i) does not hold.

\[\square\]

### 1.3 Asymptotic profiles of \( n \)-mode solutions

In this section we will derive some asymptotic behavior of \( u^\varepsilon \) or \( 1 - u^\varepsilon \) as \( \varepsilon \) goes to 0 in a certain interval containing an extremal point of \( u^\varepsilon \). For this purpose, we first prepare the following lemma:

**Lemma 1.3.1.** Let \( g(v) = v(1-v)(v-a_0) \) with \( a_0 \in (0,1) \). Then for any \( \delta \in (0,1) \) satisfying \( \delta > \max\{a_0, (a_0 + 1)/3\} \) and \( M > 0 \), there exists a unique solution \( v = v(z) \) of
\[
\begin{align*}
\begin{cases}
v_{zz} + g(v) &= 0 \quad \text{in } (-M, 0), \\
v(-M) &= \delta, \ v_z(0) = 0, \\
v > \delta, v_z > 0 & \quad \text{in } (-M, 0).
\end{cases}
\end{align*}
\]
Moreover, there exists a constant \( \delta_0 \in ((a_0 + 1 + \sqrt{a_0^2 - a_0 + 1})/3, 1) \) such that, if \( \delta > \delta_0 \), then
\[
c_1 \exp(-RM) < 1 - v(0) < c_2 \exp(-rM),
\]
where \( r = \sqrt{-g_v(\delta)}, \ R = \sqrt{-g_v(1)} \) and \( c_1, c_2 (0 < c_1 < c_2) \) are positive constants depending only on \( \delta \).

**Proof.** In order to solve (1.3.1), we employ the time-map method developed in Smoller and Wasserman [24]. Take \( \delta \in (0,1) \) with \( \delta > \max\{a_0, (a_0 + 1)/3\} \) and consider the following initial value problem:
\[
\begin{align*}
\begin{cases}
v_{zz} + g(v) &= 0 \quad \text{for } z > -M, \\
v(-M) &= \delta, \ v_z(-M) = p,
\end{cases}
\end{align*}
\]
where \( p \) is a positive parameter. Let \( v(z; p) \) the solution of (1.3.3). Multiplying (1.3.3) by \( v_z(z; p) \) and integrating the resulting expression over \((-M, z)\), we get
\[
\frac{1}{2}(v_z(z; p))^2 - G(v(z; p)) = \frac{1}{2}p^2,
\]
26
where

\[ G(v) = - \int_{\delta}^{v} g(s) ds. \]

Since we look for \( p \) satisfying \( v_z(0;p) = 0 \) and \( v_z(z;p) > 0 \) for \( z \in (-M, 0) \), we have to restrict the range of \( p \). By virtue of the phase plane analysis, we can see that \( p \) must satisfy \( 0 < p < \sqrt{-2G(1)} =: p^* \). Here we should note that \( G(1) < 0 \) because \( \delta > a_0 \).

For such \( p \), define \( \alpha(p) \in (\delta, 1) \) by \( \frac{p^2}{2} = -G(\alpha(p)) \), and let \( T(p) \) be a time-map given by

\[ T(p) = \inf \{ z > -M : v(z) = \alpha(p) \} + M. \]

Then \( \alpha(p) = \max\{v(z;p) : z > -M\} \) and \( T(p) \) denotes the distance from \( z = -M \) to the first critical point of \( v \). If we can find a number \( p_M \) satisfying \( T(p_M) = M \), then \( v(z;p_M) \) gives a solution of (1.3.1). Hence the study of \( T(p) \) is essential to solve (1.3.1).

As a first step, we will show that \( T(p) \) is strictly monotone increasing for \( 0 < p < p^* \). It follows from (1.3.4) that

\[ \frac{1}{\sqrt{G(v) - G(\alpha(p))}} \frac{dv}{dz} = \sqrt{2} \]

Integrating this equation over \( (-M, -M + T(p)) \) with respect to \( z \), we obtain

\[ \sqrt{2}T(p) = \int_{\delta}^{\alpha(p)} \frac{dv}{\sqrt{G(v) - G(\alpha(p))}}. \] (1.3.5)

From the definition, \( \alpha(p) \) is a strictly increasing function of \( p \) satisfying \( \alpha(p) \to \delta \) as \( p \to 0 \) and \( \alpha(p) \to 1 \) as \( p \to p^* \). Hence it is convenient to treat \( T(p) \) in (1.3.5) as a function of \( \alpha \) in place of \( p \). Set

\[ S(\alpha) = \int_{\delta}^{\alpha} \frac{dv}{\sqrt{G(v) - G(\alpha)}} = \int_{0}^{1} \frac{\alpha - \delta}{\sqrt{(s(\alpha - \delta) + \delta) - G(\alpha)}} ds. \]

We will prove that \( S(\alpha) \) is strictly monotone increasing for \( \alpha \in (\delta, 1) \). Differentiation of \( S(\alpha) \) with respect to \( \alpha \) gives

\[ \frac{\partial S}{\partial \alpha}(\alpha) = \int_{0}^{1} \frac{2(G(v) - G(\alpha)) + (\alpha - \delta)sg(s(\alpha - \delta) + \delta) - (\alpha - \delta)g(\alpha)}{2(G(v) - G(\alpha))^{3/2}} ds \]

\[ = \frac{1}{\alpha - \delta} \int_{\delta}^{\alpha} \frac{\theta(v) - \theta(\alpha)}{2(G(v) - G(\alpha))^{3/2}} dv, \] (1.3.6)
\[ \theta(v) = 2G(v) + (v - \delta)g(v). \]

Remark that \( G(v) - G(\alpha) > 0 \) when \( v \) lies in \((\delta, \alpha)\). We will investigate \( \theta \) in order to show that \( \partial S / \partial \alpha \) is positive for \( \alpha \in (\delta, 1) \). It is easy to see that

\[ \theta_v(v) = -g(v) + (v - \delta)g_v(v) \quad \text{and} \quad \theta_{vv}(v) = (v - \delta)g_{vv}(v). \]

Observe \( \theta_v(\delta) = -g(\delta) < 0 \) for \( a_0 < \delta < 1 \). Moreover, \( \theta_{vv}(v) < 0 \) in \((\delta, \alpha)\) by the concavity of \( g(v) \). Therefore, \( \theta_v(v) < 0 \) in \((\delta, \alpha)\). Since \( \theta \) is monotone decreasing in \((\delta, \alpha)\), we see from (1.3.6) that \( \partial S / \partial \alpha \) is positive for \( \alpha \in (\delta, 1) \). Therefore, \( S(\alpha) \) is monotone increasing in \((\delta, 1)\), so that \( T(p) \) is monotone increasing in \((0, p^*)\).

Furthermore, we will show

\[ \lim_{p \to 0^+} T(p) = 0 \] (1.3.7)

and

\[ \lim_{p \to p^*} T(p) = \infty. \] (1.3.8)

Since, for \( v \in (\delta, \alpha) \), it holds that

\[ G(v) - G(\alpha) = \int_\alpha^v g(s)ds \geq \min\{g(\alpha), g(\delta)\}(\alpha - v), \]

one can see that \( \lim_{\alpha \to \delta} S(\alpha) = 0 \), which implies (1.3.7). In order to prove (1.3.8), we note that \( \alpha(p) \to 1 \) when \( p \to p^* \). For \( \alpha \to 1 \), we see that

\[ G(v) - G(\alpha) \to -\frac{1}{2}g_v(1)(v - 1)^2 + o((v - 1)^2) \quad \text{as} \quad v \to 1. \]

Therefore, \( \lim_{\alpha \to 1} S(\alpha) = \infty \), which implies (1.3.8).

We have shown that \( T(p) \) is a strictly increasing function satisfying (1.3.7) and (1.3.8). Hence it is easy to see that, for each \( M > 0 \), there exists a unique \( p_M \in (0, p^*) \) such that \( T(p_M) = M \). Clearly, \( p_M \) is strictly increasing and continuous with respect to \( M \) and \( \lim_{M \to \infty} p_M = p^* \). Setting \( v_M = v(0; p_M) \), we see that \( v_M \) is also strictly increasing and continuous with respect to \( M \). Furthermore it holds that \( \lim_{M \to \infty} v_M = 1 \).

We will prove that \( v_M \) satisfies (1.3.2). Recall

\[ \sqrt{2M} = \int_\delta^{v_M} \frac{dv}{\sqrt{G(v) - G(v_M)}}, \] (1.3.9)
from (1.3.5). By the mean value theorem, there exists a constant $\theta_1 \in (\delta, v_M)$ satisfying

$$\frac{G(v) - G(v_M)}{(1 - v)^2 - (1 - v_M)^2} = -\frac{g(\theta_1)}{2(\theta_1 - 1)} = -\frac{g(\theta_1) - g(1)}{2(\theta_1 - 1)}. \quad (1.3.10)$$

Using the mean value theorem again, we see that the right-hand side of (1.3.10) is equal to $-g_v(\theta_2)/2$ with some $\theta_2 \in (\theta_1, 1)$. Now we take a positive constant $\delta^*$ lying in $((a_0 + 1 + \sqrt{a_0^2 - a_0 + 1})/3, 1)$. It should be noted that $g_v(v)$ is decreasing and negative for $v \in (\delta^*, 1)$. Then for $\delta \in (\delta^*, 1)$

$$\frac{r^2}{2} < -\frac{g_v(\theta_2)}{2} < \frac{R^2}{2} \quad (1.3.11)$$

with $r = \sqrt{-g_v(\delta)}$ and $R = \sqrt{-g_v(1)}$. With use of (1.3.10) and (1.3.11), it follows from (1.3.9) that

$$\frac{1}{R} B_M < M < \frac{1}{r} B_M, \quad (1.3.12)$$

where

$$B_M = \int_\delta^{v_M} \frac{dv}{\sqrt{(1 - v)^2 - (1 - v_M)^2}} = \log \left( b_M + \sqrt{b_M^2 - 1} \right),$$

with $b_M = (1 - \delta)/(1 - v_M)$. Since $B_M \in [\log b_M, \log 2b_M]$, (1.3.12) yields

$$(1 - \delta) \exp(-R M) < 1 - v_M < 2(1 - \delta) \exp(-r M).$$

Thus the proof is complete. \qed

Replacing $z$ by $-z$ in the proof of Lemma 1.3.1, we can show the following lemma.

**Lemma 1.3.2.** Let $g$ be the same function as in Lemma 1.3.1. Then for any $\delta \in (0, 1)$ satisfying $\delta > \max\{a_0, (a_0 + 1)/3\}$ and $M > 0$, there exists a unique solution $v = v(z)$ of

$$\begin{cases}
  v_{zz} + g(v) = 0 & \text{in } (0, M), \\
  v(M) = \delta, v_z(0) = 0 \\
  v > \delta & \text{in } (0, M).
\end{cases}$$

Furthermore, there exists a constant $\delta^* \in ((a_0 + 1 + \sqrt{a_0^2 - a_0 + 1})/3, 1)$ such that, if $\delta > \delta^*$, then $v$ satisfies (1.3.2).
Now, we will study asymptotic behavior of \( u^\varepsilon \) in \( (\xi_1, \xi_2) \) as \( \varepsilon \downarrow 0 \).

**Theorem 1.3.3.** For \( u^\varepsilon \in S_{n,x} \), let \( \xi_1 \) and \( \xi_2 \) be successive points in \( \Xi \) satisfying
\[
 u^\varepsilon(x) - a(x) > 0 \quad \text{in} \quad (\xi_1, \xi_2),
\]
and let \( \zeta \) be the unique local maximum point in \( (\xi_1, \xi_2) \). If \( (\zeta - \xi_1)/\varepsilon \to \infty \) as \( \varepsilon \to 0 \), then there exist positive constants \( C_1, C_2, r \) and \( R \) with \( C_1 < C_2 \) and \( r < R \) satisfying
\[
 C_1 \exp \left( -\frac{R(\zeta - \xi_1)}{\varepsilon} \right) < 1 - u^\varepsilon(x) < C_2 \exp \left( -\frac{r(x - \xi_1)}{\varepsilon} \right) \quad \text{in} \quad [\xi_1, \xi_2],
\]
provided that \( \varepsilon \) is sufficiently small.

**Remark 1.3.4.** Lemma 1.2.4 assures the existence of \( \zeta \) as in Theorem 1.3.3.

**Proof of Theorem 1.3.3.** We only consider the case that \( x \) lies in \([\xi_1, \xi_2]\).

We begin with the proof of the right-hand side inequality of (1.3.13). Let \( a^* \) be a constant which satisfies \( a^* > \max\{a(x); x \in [\xi_1, \xi_2]\} \) and take \( \delta^* \in (a^*, 1) \) which is close to 1. By Lemma 1.2.5, one can find a point \( \tilde{\xi}_1 \in (\xi_1, \zeta) \) such that \( u^\varepsilon(\tilde{\xi}_1) = \delta^* \) and \( u^\varepsilon(x) > \delta^* \) in \( (\tilde{\xi}_1, \zeta) \) provided that \( \varepsilon \) is sufficiently small.

Clearly, \( \tilde{\xi}_1 - \xi_1 = O(\varepsilon) \) as \( \varepsilon \to 0 \), so that \( \zeta - \tilde{\xi}_1 > \varepsilon \) when \( \varepsilon \) is small.

We now take any \( x^* \in (\tilde{\xi}_1 + \varepsilon, \zeta) \) and apply Lemma 1.3.1. Let \( v(z) \) be a solution of (1.3.1) with \( a_0 = a^* \), \( \delta = \delta^* \) and \( M = (x^* - \tilde{\xi}_1 - \varepsilon)/\varepsilon \). We use the change of variable \( z = (x - x^*)/\varepsilon \) and define \( V_1 \) by \( V_1(x) = v((x - x^*)/\varepsilon) \). Then
\[
 \begin{cases}
 \varepsilon^2 V_1'' + V_1(1 - V_1)(V_1 - a^*) = 0 & \text{in} \quad (\tilde{\xi}_1 + \varepsilon, x^*), \\
 V_1(\tilde{\xi}_1 + \varepsilon) = \delta^*, \; V_1'(x^*) = 0, \\
 V_1 > \delta^* & \text{in} \quad (\tilde{\xi}_1 + \varepsilon, x^*),
\end{cases}
\]
where \( ' ' \) denotes the derivative with respect to the autonomous variable. By virtue of Lemma 1.3.1, \( V_1 \) satisfies
\[
 c_1 e^R \exp \left( -\frac{R(x^* - \tilde{\xi}_1)}{\varepsilon} \right) < 1 - V_1(x^*) < c_2 e^r \exp \left( -\frac{r(x^* - \tilde{\xi}_1)}{\varepsilon} \right),
\]
where \( c_1, c_2, r \) and \( R \) are positive constants depending only on \( a^* \) and \( \delta^* \).

We will show
\[
 V_1(x) \leq u^\varepsilon(x) \quad \text{in} \quad (\tilde{\xi}_1 + \varepsilon, x^*). \tag{1.3.16}
\]
For this purpose, introduce the following auxiliary function:
\[
 h_1(x) = \frac{V_1(x) - a^*}{u^\varepsilon(x) - a^*} \quad \text{in} \quad [\tilde{\xi}_1 + \varepsilon, x^*].
\]
In order to get (1.3.16), we will prove that \( h_1(x) \leq 1 \) in \( [\tilde{\xi}_1 + \varepsilon, x^*] \). Suppose that there exists an \( x_1 \in [\tilde{\xi}_1 + \varepsilon, x^*] \) such that

\[
h_1(x_1) = \max \{ h_1(x) ; x \in [\tilde{\xi}_1 + \varepsilon, x^*] \} = \frac{1}{\eta} > 1.
\]

Then it holds that

\[
\begin{align*}
V_\eta(x) &\leq u^\varepsilon(x) \quad \text{in } [\tilde{\xi}_1 + \varepsilon, x^*], \\
V_\eta(x_1) &= u^\varepsilon(x_1),
\end{align*}
\]

where

\[
V_\eta(x) = \eta(V_1(x) - a^*) + a^* \quad (< V_1(x)).
\]

We will prove

\[
V''_\eta(x_1) \leq u^\varepsilon_{xx}(x_1). \quad (1.3.17)
\]

Clearly, \( h_1(\tilde{\xi}_1 + \varepsilon) < 1 \). Moreover, since \( u^\varepsilon_{xx}(x^*) > 0 \) and \( V_1'(x^*) = 0 \), it is easy to see \( h'_1(x^*) < 0 \). Therefore, \( x_1 \) must be an interior point in \( (\tilde{\xi}_1 + \varepsilon, x^*) \), so that

\[
h'_1(x_1) = 0 \quad \text{and} \quad h''_1(x_1) \leq 0. \quad (1.3.18)
\]

From the definition of \( h_1 \), we have

\[
h_1(x)(u^\varepsilon(x) - a^*) = V_1(x) - a^*.
\]

Differentiating the above identity two times with respect to \( x \) and setting \( x = x_1 \) we get

\[
u^\varepsilon_{xx}(x_1) + 2\eta u^\varepsilon_{x}(x_1)h'_1(x_1) + \eta(u^\varepsilon(x_1) - a^*)h''_1(x_1) = \eta V''_1(x_1) = V''_\eta(x_1). \quad (1.3.19)
\]

Then (1.3.18) and (1.3.19) imply (1.3.17).

We next use \( f(x, V_\eta) > \eta V_1(1 - V_1)(V_1 - a^*) \). Indeed, since \( V_1(x) > a^* > a(x) \) in \( (\xi_1 + \varepsilon, x^*) \), a simple calculation yields that

\[
f(x, V_\eta) = V_\eta(1 - V_\eta)(V_\eta - a(x)) \\
= \eta(V_1 - a^*)V_\eta(1 - V_\eta) + (a^* - a(x))V_\eta(1 - V_\eta) \\
> \eta(V_1 - a^*)V_\eta(1 - V_\eta) \\
> \eta V_1(1 - V_1)(V_1 - a^*),
\]

provided that \( \delta^* \) is sufficiently close to 1. Hence it follows from (1.3.14) that

\[
\epsilon^2 V''_\eta + f(x, V_\eta) = \eta \epsilon^2 V''_1 + f(x, V_\eta) > \eta \{ \epsilon^2 V''_1 + V_1(1 - V_1)(V_1 - a^*) \} = 0.
\]
Therefore, using (1.3.17), we have
\[ 0 = \varepsilon^2 u_{xx}(x_1) + f(x_1, u^\varepsilon(x_1)) \geq \varepsilon^2 V''(x_1) + f(x_1, V_\eta(x_1)) > 0, \]
which is a contradiction. Thus we have shown (1.3.16).

From (1.3.15) and (1.3.16), we obtain that
\[ 1 - u^\varepsilon(x^*) \leq 1 - V_1(x^*) < c_2 e^r \exp \left( -\frac{r(x^* - \tilde{\xi}_1)}{\varepsilon} \right). \]  
Here we should note that \( c_2 \) and \( r \) can be chosen independently of \( x^* \). Recalling that \( x^* \) is an arbitrary point in \((\tilde{\xi}_1 + \varepsilon, \zeta)\), one can conclude that
\[ 1 - u^\varepsilon(x) < c_2 e^r \exp \left( -\frac{r(x - \tilde{\xi}_1)}{\varepsilon} \right) \text{ in } (\tilde{\xi}_1 + \varepsilon, \zeta). \]  
(1.3.20)  
Moreover, since \( \tilde{\xi}_1 - \xi_1 < K_1 \varepsilon \) with some \( K_1 > 0 \), it follows from (1.3.20) that
\[ 1 - u^\varepsilon(x) < c_2 e^r \exp \left( -\frac{r(x - \xi_1)}{\varepsilon} \right) \exp \left( \frac{r(\tilde{\xi}_1 - \xi_1)}{\varepsilon} \right) \]  
\[ < c_2 e^{r(K_1+1)} \exp \left( -\frac{r(x - \xi_1)}{\varepsilon} \right) \]  
(1.3.21)  
for \( x \in (\tilde{\xi}_1 + \varepsilon, \zeta) \). We should remark that
\[ \exp(-r(K_1 + 1)) < \exp \left( -\frac{r(x - \xi_1)}{\varepsilon} \right) \text{ in } (\xi_1, \tilde{\xi}_1 + \varepsilon). \]
Hence, we can choose a sufficiently large constant \( K_2 > 0 \) such that
\[ 1 - u^\varepsilon(x) \leq 1 - u^\varepsilon(\xi_1) = 1 - a(\xi_1) \]
\[ < K_2 \exp(-r(K_1 + 1)) < K_2 \exp \left( -\frac{r(x - \xi_1)}{\varepsilon} \right) \]  
(1.3.22)  
for \( x \in (\xi_1, \tilde{\xi}_1 + \varepsilon) \). Thus (1.3.21) and (1.3.22) tell us to see that (1.3.20) is valid for all \( x \in [\xi_1, \zeta] \). For \( x = \zeta \) of this extension, it is sufficient to use the continuity of \( u^\varepsilon \) with respect to \( x \).

We will prove the left-hand side inequality of (1.3.13). Let \( a_* \) be a constant satisfying \( a_* < \min\{a(x) : x \in [\xi_1, \zeta]\} \) and let \( \delta_* \in (a_*, 1) \) satisfy \( \delta_* > \max\{1/2, \max\{a(x) : x \in [\xi_1, \zeta]\}\} \). Then there exists a point \( \xi \in (\xi_1, \zeta) \) such
that \( u^\varepsilon(\tilde{\xi}_1) = \delta_* \) and \( \tilde{\xi}_1 - \xi_1 = O(\varepsilon) \). If \( \varepsilon \) is sufficiently small, then \( \zeta - \tilde{\xi}_1 > \varepsilon \).

We apply Lemma 1.3.1. Let \( v \) be the solution of (1.3.1) with putting \( \delta = \delta_* \), \( a_0 = a_* \) and \( M = (\zeta - \xi_1 + \varepsilon)/\varepsilon \). By using the change of variable \( z = (x - \zeta)/\varepsilon \), we see that \( V_2(x) = v((x - \zeta)/\varepsilon) \) satisfies

\[
\begin{align*}
\varepsilon^2 V_2'' + V_2(1 - V_2)(V_2 - a_*) &= 0 & \text{in } (\bar{\xi}_1 - \varepsilon, \zeta), \\
V_2(\bar{\xi}_1 - \varepsilon) &= \delta_* \quad \text{and} \quad V_2'(\zeta) = 0, \\
V_2 > \delta_* & \quad \text{in } (\bar{\xi}_1 - \varepsilon, \zeta).
\end{align*}
\]

Furthermore, Lemma 1.3.1 gives

\[
c_{1} e^R \exp \left( -\frac{R(\zeta - \bar{\xi}_1)}{\varepsilon} \right) < 1 - V_2(\zeta). \tag{1.3.23}
\]

We will prove

\[
V_2(x) \geq u^\varepsilon(x) \quad \text{in } [\bar{\xi}_1 - \varepsilon, \zeta]. \tag{1.3.24}
\]

This together with (1.3.23) yields our desired inequality because \( u^\varepsilon(\zeta) \) is the maximum of \( u^\varepsilon \) in \([\xi_1, \zeta]\) and \( \xi_1 < \bar{\xi}_1 < \zeta \). In order to prove (1.3.24), introduce the following function

\[
h_2(x) = \frac{u^\varepsilon(x) - a_*}{V_2(x) - a_*} \quad \text{in } [\bar{\xi}_1, \zeta].
\]

We will show \( h_2(x) \leq 1 \). Assume that there exists \( x_2 \in [\bar{\xi}_1, \zeta] \) such that

\[
h_2(x_2) = \max\{h_2(x) : x \in [\bar{\xi}_1, \zeta]\} = \eta > 1.
\]

This implies that

\[
\begin{align*}
u^\varepsilon(x) &\leq W_\eta(x) \quad \text{in } [\bar{\xi}_1, \zeta], \\
u^\varepsilon(x_2) &= W_\eta(x_2),
\end{align*}
\]

where \( W_\eta(x) = \eta(V_2(x) - a_*) + a_* \). Since \( h_2(\bar{\xi}_1) < 1 \), \( x_2 \) must satisfy \( \bar{\xi}_1 < x_2 \leq \zeta \). If \( x_2 \) lies in \((\bar{\xi}_1, \zeta)\), then it is easy to see that

\[
u^\varepsilon(x_2) \leq W_\eta''(x_2). \tag{1.3.25}
\]

For the case \( x_2 = \zeta \), note \( h_2'(x_2) = h_2'(\zeta) = 0 \). Therefore, (1.3.25) is also valid for \( x_2 = \zeta \).

Next we will prove that

\[
f(x, W_\eta) < \eta V_2(1 - V_2)(V_2 - a_*). \tag{1.3.26}
\]
As a function of \( \eta \), set \( P(\eta) = \eta V_2(1 - V_2)(V_2 - a_\ast) - f(x, W_\eta) \). Then
\[
P'(\eta) = V_2(1 - V_2)(V_2 - a_\ast) - (V_2 - a_\ast)f_u(x, W_\eta) = (V_2 - a_\ast)Q(\eta),
\]
where
\[Q(\eta) = V_2(1 - V_2) - f_u(x, W_\eta).\]

Observe that
\[Q'(\eta) = -f_{uu}(x, W_\eta)(V_2 - a_\ast) = 2(V_2 - a_\ast)\{(W_\eta - a(x)) + (2W_\eta - 1)\}.
\]

Recalling the definition of \( \delta_\ast \) and \( \eta > 1 \), we can see that \( W_\eta(x) \geq V_2(x) > \delta_\ast > \max\{1/2, \max\{a(x); x \in [\xi_1, \zeta]\}\} \) in \( (\xi_1, \zeta) \), so that \( Q'(\eta) > 0 \). Therefore,
\[Q(\eta) \geq Q(1) = (V_2 - a(x))(2V_2 - 1) > 0,
\]
which leads to \( P'(\eta) > 0 \) for \( \eta \geq 1 \). Hence we get
\[P(\eta) \geq P(1) = V_2(1 - V_2)(a(x) - a_\ast) > 0
\]
and (1.3.26) is proved.

We finally combine (1.3.25) and (1.3.26) to obtain that
\[
0 = \varepsilon^2 u_{xx}(x_2) + f(x_2, u(x_2)) \\
\leq \varepsilon^2 W''_\eta(x_2) + f(x_2, W_\eta(x_2)) \\
< \eta\{(\varepsilon^2 V''_2(x_2) + V_2(x_2)(1 - V(x_2))(V(x_2) - a_\ast)\} = 0.
\]

Since this is a contradiction, we have shown (1.3.24). Thus the proof is complete. \( \square \)

**Remark 1.3.5.** In (1.3.13), we can choose \( r = \sqrt{1 - A^*} + o(1) \) and \( R = \sqrt{1 - A_\ast} + o(1) \) where \( A_\ast = \min\{a(x); x \in [\xi_1, \zeta]\} \) and \( A^* = \max\{a(x); x \in [\xi_1, \zeta]\} \). These facts can be shown from the proof of Theorem 1.3.3 by taking account of the definition of \( r \) and \( R \) in Lemma 1.3.1.

Using the same method as the proof of Theorem 1.3.3 one can prove the following result from Lemma 1.3.2:

**Theorem 1.3.6.** For \( u \in S_{n, \varepsilon} \), let \( \xi_1, \xi_2 \) and \( \zeta \) satisfy the conditions as in Theorem 1.3.3. If \( (\xi_2 - \zeta)/\varepsilon \to \infty \), then there exist positive constants \( C_1, C_2, r \) and \( C_1 < C_2 \) and \( r < R \) satisfying
\[
C_1 \exp \left( -\frac{R(\xi_2 - \zeta)}{\varepsilon} \right) < 1 - u(x) < C_2 \exp \left( -\frac{r(\xi_2 - \zeta)}{\varepsilon} \right)
\]
in \( [\zeta, \xi_2] \),
\[
(1.3.27)
\]
provided that \( \varepsilon \) is sufficiently small.
Theorem 1.3.3 deals with the case that $u^\epsilon$ attains its local maximum at $x = \zeta$. In other words, Theorem 1.3.3 gives us information on the asymptotic property of $u^\epsilon$ when $u^\epsilon$ approaches 1. If $\zeta$ is a local minimum point of $u^\epsilon$ and $(\zeta - \xi_1)/\epsilon \to \infty$ or $(\xi_2 - \zeta)/\epsilon \to \infty$ as $\epsilon \to 0$, then we can derive similar estimates as (1.3.13) and (1.3.27) with $1 - u^\epsilon(x)$ replaced by $u^\epsilon(x)$ as follows:

**Theorem 1.3.7.** For $u^\epsilon \in S_n,\epsilon$, let $\xi_1$ and $\xi_2$ be successive points in $\Xi$ satisfying $u^\epsilon(x) - a(x) < 0$ in $(\xi_1, \xi_2)$, and let $\zeta$ be the unique local minimum point in $(\xi_1, \xi_2)$. If $(\xi_2 - \zeta)/\epsilon \to \infty$, then there exist positive constants $C_1$, $C_2$, $r$ and $R$ with $C_1 < C_2$ and $r < R$ satisfying

$$C_1 \exp \left( -\frac{R(\zeta - \xi_1)}{\epsilon} \right) < u^\epsilon(x) < C_2 \exp \left( -\frac{r(x - \xi_1)}{\epsilon} \right) \text{ in } [\xi_1, \zeta],$$

provided that $\epsilon$ is sufficiently small.

**Theorem 1.3.8.** For $u \in S_n,\epsilon$, let $\xi_1$, $\xi_2$ and $\zeta$ satisfy the conditions as in Theorem 1.3.7. If $(\xi_2 - \zeta)/\epsilon \to \infty$, then there exist positive constants $C_1$, $C_2$, $r$ and $R$ with $C_1 < C_2$ and $r < R$ satisfying

$$C_1 \exp \left( -\frac{R(\xi_2 - \zeta)}{\epsilon} \right) < u^\epsilon(x) < C_2 \exp \left( -\frac{r(\xi_2 - x)}{\epsilon} \right) \text{ in } [\zeta, \xi_2],$$

provided that $\epsilon$ is sufficiently small.

### 1.4 Location of Transition Layers and Spikes

We will study the location of transition layers and spikes of $u^\epsilon \in S_n,\epsilon$.

**Theorem 1.4.1.** For $u^\epsilon \in S_n,\epsilon$, let $\xi$ be any point in $\Xi$. Then $\xi$ lies in a neighborhood of a point in $\Sigma \cup \Lambda \cup \{0,1\}$ when $\epsilon$ is sufficiently small. Moreover, if $u^\epsilon$ has a transition layer near a point $x_0 \in \Sigma \cup \Lambda \cup \{0,1\}$, then $x_0$ belongs to $\Sigma$, and if $u^\epsilon$ has a spike near a point $x_0 \in \Sigma \cup \Lambda \cup \{0,1\}$, then $x_0$ belongs to $\Lambda \cup \{0,1\}$.

We will show this theorem, by means of asymptotic properties developed in Section 1.3.

**Proof of Theorem 1.4.1.** Define $\{\xi_k\}_{k=1}^n$ and $\{\zeta_k\}_{k=0}^n$ as in Lemma 1.2.4. By Lemma 1.2.5 it can be shown that, if $u^\epsilon \in S_n,\epsilon$ has a transition layer in a neighborhood of $\xi_k \in \Xi$, then $\xi_k$ must be very close to one of the elements of
Σ when ε is sufficiently small. Therefore, it is sufficient to show that if \( u^\varepsilon \) has a spike near \( \xi_k \), then \( \xi_k \) lies in a vicinity of a point in \( \Lambda \cup \{0, 1\} \).

We will prove by contradiction that every spike lies near a point in \( \Lambda \cup \{0, 1\} \).

Take a positive constant \( \delta \) independent of \( \varepsilon \) and suppose that there exists an interval \( I \) such that \( \text{dist}(I, \Sigma \cup \Lambda \cup \{0, 1\}) > \delta \) and a spike lies in \( I \).

We should note that there are many candidates for the interval \( I \), so that we only consider the typical interval as \( I \) for the sake of simplicity. In order to choose such \( I \), put \( \Sigma = \{\sigma_j\}_{j=1}^m \) with \( m \in \mathbb{N} \) and \( 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_m < 1 \). Moreover, for definiteness, assume \( a(x) - 1/2 > 0 \) in \( (\sigma_j, \sigma_{j+1}) \) with some \( j \in \{1, 2, \ldots, m\} \) and denote all points of \( \Lambda \cap (\sigma_j, \sigma_{j+1}) \) by \( \{\lambda_i\}_{i=1}^l \) with \( \sigma_j < \lambda_1 < \lambda_2 < \cdots < \lambda_l < \sigma_{j+1} \). Then we put \( I = (\sigma_j + \delta, \lambda_1 - \delta) \) and assume that \( u^\varepsilon \) has a spike in \( I \). Note \( a_x(x) > 0 \) in this interval.

By (iii) of Lemma 1.2.5, there exist \( \xi_k \) and \( \xi_{k+1} \) such that
\[
\sigma_j + \delta < \xi_k < \xi_{k+1} < \lambda_1 - \delta, \quad u^\varepsilon_x(\xi_k) < 0 \quad \text{and} \quad u^\varepsilon_x(\xi_{k+1}) > 0,
\]
if \( \varepsilon \) is sufficiently small. By Lemma 1.2.4 there exist a unique set of critical points \( \{\zeta_{k-1}, \zeta_k, \zeta_{k+1}\} \) of \( u^\varepsilon \) satisfying \( \zeta_{k-1} < \xi_k < \zeta_k < \xi_{k+1} < \zeta_{k+1} \) with \( \xi_k := \sup\{x; u^\varepsilon_x(x) = 0 \text{ and } x < \xi_k\} \) and \( \zeta_{k+1} := \inf\{x; u^\varepsilon_x(x) = 0 \text{ and } x > \xi_k\} \).

We will show
\[
1 - u^\varepsilon(\zeta_{k-1}) > \kappa\sqrt{\varepsilon}
\]
with some \( \kappa > 0 \), in the case that neither \( \zeta_{k-1} \) nor \( \zeta_{k+1} \) belongs to \( (\sigma_j, \lambda_1) \). The other cases can be discussed in the same way and the proof is easier.

We rewrite (SP1) as
\[
\varepsilon^2 u^\varepsilon_{xx} + f(\zeta_k, u^\varepsilon) = u^\varepsilon(1 - u^\varepsilon)(a(x) - a(\zeta_k)).
\]

Multiply (1.4.2) by \( u^\varepsilon_x \) and integrate the resulting expression over \( (\zeta_{k-1}, \zeta_{k+1}) \) with respect to \( x \). Then we have
\[
W(\zeta_k, u^\varepsilon(\zeta_{k-1})) - W(\zeta_k, u^\varepsilon(\zeta_{k+1})) = \int_{\xi_{k-1}}^{\xi_{k+1}} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_k))u^\varepsilon_x(x)dx
\]
\[
= I + II + III.
\]
where

\[
I = \int_{\zeta_{k-1}}^{\sigma_j} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_k))u^\varepsilon_x(x)dx,
\]

\[
II = \int_{\sigma_j}^{\lambda_1} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_k))u^\varepsilon_x(x)dx,
\]

\[
III = \int_{\lambda_1}^{\zeta_{k+1}} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_k))u^\varepsilon_x(x)dx.
\]

We will estimate I, II and III. We begin with the study of II. Since \(a\) is monotone increasing in \((\sigma_j, \lambda_1)\), Taylor’s expansion enables us to see that

\[
II > \int_{\zeta_k+\varepsilon}^{\lambda_1} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_k))u^\varepsilon_x(x)dx
\]

\[
> (a(\zeta_k + \varepsilon) - a(\zeta_k)) \int_{\zeta_k+\varepsilon}^{\lambda_1} u^\varepsilon(x)(1 - u^\varepsilon(x))u^\varepsilon_x(x)dx
\]

\[
= (a(\zeta_k + \varepsilon) - a(\zeta_k)) \int_{u^\varepsilon(\zeta_k+\varepsilon)}^{u^\varepsilon(\lambda_1)} s(1 - s)ds
\]

\[
> K_1\varepsilon \int_{u^\varepsilon(\zeta_k+\varepsilon)}^{u^\varepsilon(\lambda_1)} s(1 - s)ds
\]

with a positive constant \(K_1\). Moreover Theorem 1.3.3 assures that there exist some positive constants \(C_1\) and \(r_1\) satisfying

\[
1 - u^\varepsilon(\lambda_1) < C_1 \exp\left(-\frac{r_1(\lambda_1 - \zeta_k)}{\varepsilon}\right) < C_1 \exp\left(-\frac{r_1\delta}{\varepsilon}\right).
\]

We should note that, if \(\varepsilon\) is sufficiently small, then Lemma 1.2.5 implies \(u^\varepsilon(\zeta_k + \varepsilon)\) is close to 0. Hence there exists a positive constant \(K_2\), which is independent of \(\varepsilon\), satisfying

\[
\int_{u^\varepsilon(\zeta_k+\varepsilon)}^{u^\varepsilon(\lambda_1)} s(1 - s)ds > K_2,
\]

so that \(II > K_1K_2\varepsilon\).

We next estimate I such as

\[
|I| \leq \int_{\zeta_{k-1}}^{\sigma_j} |u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_k))u^\varepsilon_x(x)|dx
\]

\[
\leq \int_{\zeta_{k-1}}^{\sigma_j} u^\varepsilon(x)(1 - u^\varepsilon(x))|u^\varepsilon_x(x)|dx
\]

\[
= \int_{u^\varepsilon(\sigma_j)}^{u^\varepsilon(\zeta_{k-1})} s(1 - s)ds
\]

\[
\leq 1 - u^\varepsilon(\sigma_j),
\]

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Theorem 1.3.6 implies the existence of positive constants $C_2$ and $r_2$ satisfying

$$1 - u^\varepsilon(\sigma_j) \leq C_2 \exp\left(-\frac{r_2(\xi_k - \sigma_j)}{\varepsilon}\right) \leq C_2 \exp\left(-\frac{r_2\delta}{\varepsilon}\right).$$

Therefore, we get $|I| = O(\exp(-r_2\delta/\varepsilon))$. We should note that a similar estimate is also valid for III. Thus we get

$$W(\zeta_k, u^\varepsilon(\zeta_{k-1})) - W(\zeta_k, u^\varepsilon(\zeta_{k+1})) = I + II + III > K_3\varepsilon \quad (1.4.5)$$

with some $K_3 > 0$.

On the other hand, we will estimate the left-hand side of (1.4.3). In the same way as in the proof of (1.3.10), one can see

$$W(\zeta_k, u^\varepsilon(\zeta_{k-1})) - W(\zeta_k, u^\varepsilon(\zeta_{k+1})) = -\frac{1}{2} f_u(\zeta_k, \theta) \{(1 - u^\varepsilon(\zeta_{k-1}))^2 - (1 - u^\varepsilon(\zeta_{k+1}))^2\}$$

with some $\theta \in (u^\varepsilon(\zeta_{k-1}), 1)$. Since $\theta$ is very close to 1, there exists a positive constant $K_4$, which is independent of $\varepsilon$, such that

$$W(\zeta_k, u^\varepsilon(\zeta_{k-1})) - W(\zeta_k, u^\varepsilon(\zeta_{k+1})) < K_4(1 - u^\varepsilon(\zeta_{k-1}))^2. \quad (1.4.6)$$

Hence (1.4.1) follows from (1.4.5) and (1.4.6).

We use (1.4.1) and Theorem 1.3.6 with $x = \zeta_{k-1}$ and $\xi_2 = \xi_k$ to get

$$\kappa \sqrt{\varepsilon} < C_3 \exp\left(-\frac{r_3(\xi_k - \xi_{k-1})}{\varepsilon}\right) \quad (1.4.7)$$

with some positive constants $C_3$ and $r_3$. Here recall that $u^\varepsilon$ can be regarded as a periodic function of $\mathbb{R}$ with period 2. Thus there exists a point $\xi_{k-1} \in \Xi$ such that $u^\varepsilon(x) > a(x)$ for $x \in (\xi_{k-1}, \xi_k)$. Therefore, Theorem 1.3.3 together with (1.4.1) implies

$$\kappa \sqrt{\varepsilon} < 1 - u^\varepsilon(\zeta_{k-1}) < C_4 \exp\left(-\frac{r_4(\xi_{k-1} - \xi_{k-1})}{\varepsilon}\right) \quad (1.4.8)$$

with some positive constants $C_4$ and $r_4$. Hence (1.4.7) and (1.4.8) imply

$$\xi_k - \xi_{k-1} < K_5\varepsilon|\log \varepsilon| \quad (1.4.9)$$

with some positive constant $K_5$. This fact implies that $\xi_{k-1}$ belongs to the interval $I$ if $\varepsilon$ is sufficiently small.
When $\xi_{k-1}$ lies in $(\sigma_j + \delta, \lambda_1 - \delta)$, Lemma 1.2.5 tells us that there must be another spike near $\xi_{k-1}$, so that there exists another element $\xi_{k-2}$ of $\Xi$ satisfying $\sigma_j + \delta < \xi_{k-2} < \xi_{k-1} < \lambda_1 - \delta$ and $u^\varepsilon_x(\xi_{k-2}) < 0$. Here we should note that $u^\varepsilon_x(\xi_{k-1}) > 0$. Moreover, we also remark that $u^\varepsilon$ has a peak at $x = \zeta_{k-1}$.

Repeating this procedure, we see that the number of points of $\Xi \cap I$ increases in each process. This contradicts the definition of $n$-mode solutions, so that $u^\varepsilon$ has no spikes in $I$.

Thus the proof is complete. \qed

We will discuss the location of each single-layer more carefully.

**Theorem 1.4.2.** Let $u^\varepsilon \in S_{n,\varepsilon}$ possess a single-layer near $\sigma \in \Sigma$ for sufficiently small $\varepsilon > 0$. If $\Xi \cap (\sigma - \delta, \sigma + \delta) = \{\xi\}$ with some $\delta > 0$, then $\xi - \sigma = O(\varepsilon)$.

**Proof.** We only consider the case that $a_x(\sigma) > 0$, $\sigma < \xi$ and $u^\varepsilon_x(\xi) > 0$ for the sake of simplicity. The other case can be shown in the same way as follows.

Choose critical points $\zeta_0$ and $\zeta_1$ of $u^\varepsilon$ such that $u^\varepsilon_x(x) > 0$ in $(\zeta_0, \zeta_1)$, and define $\xi^* := \inf\{x; x > \xi \text{ and } x \in \Xi\}$. The existence of such $\xi^*$ is assured by the notion that $u^\varepsilon$ is a function defined for all $x \in \mathbb{R}$ by reflection. In this case, it is clear that $u^\varepsilon(x) > a(x)$ for $x \in (\xi, \xi^*)$. Moreover, it is also easy to see that $\xi^* > \xi + \delta$ because $\Xi \cap (\sigma - \delta, \sigma + \delta) = \{\xi\}$. Therefore $\zeta_1$ is distant from either $\xi$ or $\xi^*$ independently of $\varepsilon$, so that Theorem 1.3.3 or 1.3.6 enables us to get

$$1 - u^\varepsilon(\zeta_1) = O\left(\exp\left(-\frac{K_1}{\varepsilon}\right)\right)$$

(1.4.10)

with some positive constant $K_1$. Similarly, we can also show that there exists a positive constant $K_2$ such that

$$u^\varepsilon(\zeta_0) = O\left(\exp\left(-\frac{K_2}{\varepsilon}\right)\right).$$

(1.4.11)

We introduce

$$\tilde{W}(x, u) := -\int_{\tilde{\phi}_0(x)}^u f(x, s)ds$$

with

$$\tilde{\phi}_0(x) := \begin{cases} 0 & \text{in } (\zeta_0, \xi), \\ 1 & \text{in } (\xi, \zeta_1). \end{cases}$$
From (1.4.10) and (1.4.11), it is easy to see that
\[ \tilde{W}(\zeta_0, u^\varepsilon(\zeta_0)) = O\left(\exp\left(-\frac{K_3}{\varepsilon}\right)\right) \text{ and } \tilde{W}(\zeta_1, u^\varepsilon(\zeta_1)) = O\left(\exp\left(-\frac{K_3}{\varepsilon}\right)\right) \]
with some positive constant $K_3$.

We use the following identity for $x \in (\zeta_0, \xi) \cup (\xi, \zeta_1)$:
\[
\frac{d}{dx}\left\{ \frac{1}{2}\varepsilon^2 u_x^\varepsilon(x)^2 - \tilde{W}(x, u^\varepsilon(x)) \right\} = \{\varepsilon^2 u_{xx}^\varepsilon(x) + f(x, u^\varepsilon(x))\}u_x^\varepsilon(x) - \tilde{W}_x(x, u^\varepsilon(x)) = a_x(x)G(u(x)), \tag{1.4.13}
\]
where
\[
G(u(x)) := \begin{cases} 
-(u(x))^2/2 + (u(x))^3/3 & \text{in } (\zeta_0, \xi), \\
(1 - (u(x))^2)/2 - (1 - (u(x))^3)/3 & \text{in } (\xi, \zeta_1).
\end{cases}
\]
It follows from Theorem 1.3.8 that there exist some positive constants $C$ and $r$ satisfying
\[
|a_x(x)G(u(x))| < C \exp\left(-\frac{r(\xi - x)}{\varepsilon}\right) \text{ in } (\zeta_0, \xi).
\]

Therefore, there exists a positive constant $K_4$ such that
\[
\left| \int_{\zeta_0}^\xi a_x(x)G(u(x))dx \right| < \int_{\zeta_0}^\xi C \exp\left(-\frac{r(\xi - x)}{\varepsilon}\right) dx < K_4\varepsilon. \tag{1.4.14}
\]
On the other hand, integrating the left-hand side of (1.4.13) over $(\zeta_0, \xi)$ yields that
\[
\int_{\zeta_0}^\xi \frac{d}{dx}\left\{ \frac{1}{2}\varepsilon^2 u_x^\varepsilon(x)^2 - \tilde{W}(x, u^\varepsilon(x)) \right\} dx = \frac{1}{2}\varepsilon^2 u_x^\varepsilon(\xi)^2 + \int_0^{u(\xi)} f(\xi, s)ds + \tilde{W}(\zeta_0, u^\varepsilon(\zeta_0)).
\]
Hence it follows from (1.4.13) and (1.4.14) that
\[
\frac{1}{2}\varepsilon^2 u_x^\varepsilon(\xi)^2 + \int_0^{u(\xi)} f(\xi, s)ds + \tilde{W}(\zeta_0, u^\varepsilon(\zeta_0)) \leq K_4\varepsilon. \tag{1.4.15}
\]
Using the same argument as above with $(\zeta_0, \xi)$ replaced by $(\xi, \zeta_1)$, one can also obtain that
\[
-\frac{1}{2}\varepsilon^2 u_x^\varepsilon(\xi)^2 - \tilde{W}(\zeta_1, u^\varepsilon(\zeta_1)) + \int_{u(\xi)}^1 f(\xi, s)ds \leq K_5\varepsilon \tag{1.4.16}
\]
with some positive constant $K$. Therefore (1.4.15) and (1.4.16) imply that

$$
\tilde{W}(\zeta_0, u^\varepsilon(\zeta_0)) - \tilde{W}(\zeta_1, u^\varepsilon(\zeta_1)) + \int_0^1 f(\xi,s) ds = O(\varepsilon).
$$

This together with (1.4.12) enables us to see that

$$
\int_0^1 f(\xi,s) ds = O(\varepsilon).
$$

Taking account of

$$
\int_0^1 f(\xi,s) ds = \frac{1}{6} a(\xi) + \frac{1}{12}
$$

and

$$
a(\xi) = \frac{1}{2} + a_x(\sigma)(\xi - \sigma) + O((\xi - \sigma)^2),
$$

we can conclude that $\xi - \sigma = O(\varepsilon)$. \qed

1.5 Location of Multi-Layers and Multi-Spikes

In this section, we will discuss multi-layers and multi-spikes. By Theorem 1.4.1, a multi-layer appears only in a neighborhood of a point in $\Sigma$ if it exists. While if there is a multi-spike, then it must lie in a neighborhood a point in $\Lambda$.

We recall the following notation in order to study multi-layers and multi-spikes:

$$
\Sigma^+ := \{ x \in \Sigma ; a_x(x) > 0 \}, \\
\Sigma^- := \{ x \in \Sigma ; a_x(x) < 0 \}, \\
\Lambda^+ := \{ x \in \Lambda ; a(x) < 1/2 \text{ and } a_{xx}(x) < 0 \}, \\
\Lambda^- := \{ x \in \Lambda ; a(x) > 1/2 \text{ and } a_{xx}(x) > 0 \}, \\
\Lambda^0 := \Lambda \setminus (\Lambda^+ \cup \Lambda^-).
$$

We begin with the study of multi-layers. We only discuss the case that $u^\varepsilon$ has a multi-layer in a neighborhood of a point $\sigma \in \Sigma^+$ because the analysis for the case that $\sigma \in \Sigma^-$ is almost the same.

By virtue of Lemma 1.2.5, there exists a one-to-one correspondence between a transition layer and a point in $\Xi$. 

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Lemma 1.5.1. Take $\sigma \in \Sigma^+$ and a small positive constant $\delta$. For $u^\varepsilon \in S_{n,\varepsilon}$, let $\xi_1, \xi_2 \in (\sigma - \delta, \sigma + \delta)$ be successive points in $\Xi$. Then the following assertions hold true, provided $\varepsilon$ is sufficiently small:

(i) If $u^\varepsilon_x(\xi_1) < 0$ and $u^\varepsilon_x(\xi_2) > 0$, then there exists another $\xi \in \Xi$ such that $\sigma - \delta < \xi < \xi_1$. Moreover it holds that

\begin{equation}
\xi_1 - \xi = O(\varepsilon|\log \varepsilon|). \tag{1.5.1}
\end{equation}

(ii) If $u^\varepsilon_x(\xi_1) > 0$ and $u^\varepsilon_x(\xi_2) < 0$, then there exists another $\xi \in \Xi$ such that $\xi_2 < \xi < \sigma + \delta$. Moreover it holds that

\begin{equation}
\xi - \xi_2 = O(\varepsilon|\log \varepsilon|). \tag{1.5.2}
\end{equation}

Proof. We give a proof of (i). By Lemma 1.2.4, there exist critical points $\zeta_0, \zeta_1$ and $\zeta_2$ of $u^\varepsilon$ with $\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \zeta_2$. Since $a_x(x) > 0$ in $(\sigma - \delta, \sigma + \delta)$, the argument used in the proof of (1.4.1) works well, so that we obtain that

\begin{equation}
1 - u^\varepsilon(\zeta_0) > \kappa \sqrt{\varepsilon} \tag{1.5.3}
\end{equation}

with some $\kappa > 0$ independent of $\varepsilon$. Theorem 1.3.3 implies the existence of the other successive point $\xi \in \Xi$ to $\xi_1$ with $\xi < \xi_1$ satisfying

\begin{equation}
1 - u^\varepsilon(\zeta_0) < C \exp\left(-\frac{r(\zeta_0 - \xi)}{\varepsilon}\right) \tag{1.5.4}
\end{equation}

with some positive constants $C$ and $r$. As in the proof of (1.4.9), it follows from (1.5.3) and (1.5.4) that $\xi \in \Xi$ satisfies $\xi < \xi_1$, $\xi_1 - \xi < K\varepsilon|\log \varepsilon|$ and $u^\varepsilon_x(\xi) > 0$. Hence $\xi$ lies in $(\sigma - \delta, \sigma + \delta)$ if $\varepsilon$ is sufficiently small. Furthermore, (1.5.1) is also proved.

\[\square\]

Lemma 1.5.2. Let $\sigma \in \Sigma^+$ and assume that $u^\varepsilon \in S_{n,\varepsilon}$ has a multi-layer in $(\sigma - \delta, \sigma + \delta)$ with some $\delta > 0$. If

\begin{equation}
\Xi \cap (\sigma - \delta, \sigma + \delta) = \{\xi_k\}_{k=1}^m \tag{1.5.5}
\end{equation}

with $\xi_1 < \xi_2 < \ldots < \xi_m$, then $m$ is odd. Moreover it holds that $u^\varepsilon_x(\xi_1) > 0$ and $u^\varepsilon_x(\xi_m) > 0$.

Proof. Suppose that $m$ is even. Then either one of the following two properties holds true:

(i) $'u^\varepsilon_x(\xi_1) < 0$ and $u^\varepsilon_x(\xi_2) > 0'$ and $'u^\varepsilon_x(\xi_m-1) < 0$ and $u^\varepsilon_x(\xi_m) > 0.'$

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(ii) '\( u_x^ε(ξ_1) > 0 \) and \( u_x^ε(ξ_2) < 0 \)' and '\( u_x^ε(ξ_{m-1}) > 0 \) and \( u_x^ε(ξ_m) < 0 \).'

Lemma 1.5.1 implies that, if (i) (resp. (ii)) is true, then there exists a point \( ξ_0 \in Ξ \) (resp. \( ξ_{m+1} \in Ξ \)) such that \( σ - δ < ξ_0 < ξ_1 \) (resp. \( ξ_m < ξ_{m+1} < σ + δ \)). This contradicts (1.5.5). Hence \( m \) is odd.

When \( m \) is odd, it is clear that the signs of \( u_x^ε(ξ_1) \) and \( u_x^ε(ξ_m) \) are the same. Therefore there are two possible cases that both of them are positive or negative. However, the latter case cannot occur because Lemma 1.5.1 assures the existence of another element \( ξ_0 \) of \( Ξ \cap (σ - δ, σ + δ) \). This fact contradicts (1.5.5), so that the proof is complete.

Let \( u^ε \) possess a multi-layer in a neighborhood of \( σ \in Σ^+ \). Set \( Ξ \cap (σ - δ, σ + δ) = \{ξ_k\}_{k=1}^{2m-1} \) with some \( δ > 0 \). By Lemma 1.2.4, \( u^ε \) has critical points \( ζ_0, ζ_1, \ldots, ζ_{2m-1} \) satisfying \( ζ_0 < ζ_1 < ζ_1 < \cdots < ζ_{2m-1} < ζ_{2m-1} \) with \( ζ_0 := \sup \{x ; x < ξ_1 \} \) and \( ζ_{2m-1} := \inf \{x ; x > ξ_{2m-1} \} \) and \( u_x^ε(ξ_0) = 0 \). Here we should note that \( u^ε(ζ_0) \) is close to 0, while \( u^ε(ζ_{2m-1}) \) is close to 1. Such a multi-layer is called a multi-layer from 0 to 1. A multi-layer from 1 to 0 is defined in a similar manner.

We can also show that, if there exists a multi-layer in a neighborhood of a point in \( Σ^- \), it must be a multi-layer from 1 to 0.

Summarizing these facts, we have the following theorem:

**Theorem 1.5.3.** For \( u^ε \in S_{n,ε} \), a multi-layer from 0 to 1 (resp. from 1 to 0) appears only in a neighborhood of a point in \( Σ^+ \) (resp. \( Σ^- \)).

Next we will study multi-spikes. Note that each spike corresponds to exactly two points in \( Ξ \). Hence, if \( u^ε \) has a multi-spike in a neighborhood of a point \( λ \in Λ \), then we can denote \( Ξ \cap (λ - δ, λ + δ) = \{ξ_1, ξ_2, \ldots, ξ_{2m}\} \) with some \( δ > 0 \) and some \( m \in N \). Moreover, by Lemmas 1.2.4 and 1.2.5, there exists a set of critical points \( \{ξ_k\}_{k=0}^{2m} \) of \( u^ε \) with \( ζ_0 < ζ_1 < ζ_1 < \cdots < ζ_{2m} < ζ_{2m} \). Here \( ζ_0 := \sup \{x ; x < ξ_1 \} \) and \( u_x^ε(ζ_0) = 0 \) and \( ζ_{2m} := \inf \{x ; x > ξ_{2m-1} \} \) and \( u_x^ε(ζ_{2m}) = 0 \).

We also remark that both \( u^ε(ζ_0) \) and \( u^ε(ζ_{2m}) \) are sufficiently close to 0 or 1. If \( u^ε(ζ_0) \) and \( u^ε(ζ_{m}) \) are close to 0 (resp. 1), then such a multi-spike is called a multi-spike based on 0 (resp. 1).

**Theorem 1.5.4.** For \( u^ε \in S_{n,ε} \), a multi-spike based on 0 (resp. 1) appears only in a neighborhood of a point in \( Λ^+ \) (resp. \( Λ^- \)), a boundary point 0 with \( a(0) < 1/2 \) and \( a_x(0) < 0 \) (resp. \( a(0) > 1/2 \) and \( a_x(0) > 0 \)) or a boundary point 1 with \( a(1) < 1/2 \) and \( a_x(1) > 0 \) (resp. \( a(1) > 1/2 \) and \( a_x(1) < 0 \)).
Proof. We only discuss a multi-spike based on 1. Recall that any spike based on 1 appears only in a neighborhood of a point $\lambda \in \Lambda$ with $a(\lambda) > 1/2$. Hence we assume that $a(\lambda) > 1/2$ throughout this proof.

For multi-spikes lying away from boundary points, it suffices to show that any multi-spike based on 1 cannot appear in a neighborhood of a local maximum point $\lambda$ of $a$. We will show this assertion by contradiction. For this purpose, let $\lambda \in \Lambda$ be an interior local maximum point of $a$ and assume that $u^{\varepsilon}$ has a multi-spike based on 1 in $(\lambda - \delta, \lambda + \delta)$ with some $\delta > 0$. In this case, one can denote that

$$\Xi \cap (\lambda - \delta, \lambda + \delta) = \{\xi_1, \xi_2, \ldots, \xi_{2m}\}$$

(1.5.6)

with some $m \in \mathbb{N}$. Furthermore, it follows from Lemma 1.2.4 that we can choose a set of critical points $\{\zeta_k\}_{k=0}^{2m}$ of $u^{\varepsilon}$ with $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m} < \zeta_{2m}$. Here $\zeta_0 := \sup\{x ; x < \xi_1 \text{ and } u^{\varepsilon}_x(x) = 0\}$ and $\zeta_{2m} := \inf\{x ; x > \xi_{2m} \text{ and } u^{\varepsilon}_x(x) = 0\}$. Moreover, Lemma 1.2.5 implies that $\xi_{k+1} - \xi_k = O(\varepsilon)$ for $k = 1, 3, 5, \ldots, 2m - 1$, so that at least two points in $\Xi$ belong to either $(\lambda - \delta, \lambda)$ or $(\lambda, \lambda + \delta)$.

We will consider the case that $\xi_1$ and $\xi_2$ lie in $(\lambda - \delta, \lambda)$. Note that $a_x(x) > 0$ in $(\lambda - \delta, \lambda)$. For the sake of simplicity, we assume that $\zeta_0$ also lies in $(\lambda - \delta, \lambda)$. If not, see the argument developed in the proof of Theorem 1.4.1. Similarly to the proof of (1.4.2) and (1.4.3) we have

$$W(\zeta_1, u^{\varepsilon}(\zeta_0)) - W(\zeta_1, u^{\varepsilon}(\zeta_2)) = \int_{\zeta_0}^{\zeta_1} u^{\varepsilon}_x(x)(1 - u^{\varepsilon}_x(x))(a(x) - a(\zeta_1))u^{\varepsilon}_x(x)dx.$$  

(1.5.7)

For the left-hand side of (1.5.7), observe that (1.4.6) is valid with $k$ replaced by 1. Thus we have

$$W(\zeta_1, u^{\varepsilon}(\zeta_0)) - W(\zeta_1, u^{\varepsilon}(\zeta_2)) < K_1(1 - u^{\varepsilon}(\zeta_0))^2$$

(1.5.8)

with some positive constant $K_1$. We next consider the right-hand side of (1.5.7). Since (A3) implies that $a_{xx}(x) < 0$ in $(\lambda - \delta, \lambda)$, the right-hand side of
(1.5.7) is bounded from below as
\[
\int_{\zeta_0}^{\zeta_1} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_1))u^\varepsilon_x(x)dx \\
> \int_{\zeta_0}^{\zeta_1 - \varepsilon} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(\zeta_1) - a(x))(-u^\varepsilon_x(x))dx \\
> \int_{\zeta_0}^{\zeta_1 - \varepsilon} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(\zeta_1) - a(\zeta_1 - \varepsilon))(-u^\varepsilon_x(x))dx \\
= (a(\zeta_1) - a(\zeta_1 - \varepsilon)) \int_{u^\varepsilon(\zeta_1 - \varepsilon)}^{u^\varepsilon(\zeta_0)} s(1 - s)ds.
\]
when \(\varepsilon\) is sufficiently small. By Taylor’s expansion, we see that
\[
a(\zeta_1) - a(\zeta_1 - \varepsilon) = -\frac{a_{xx}(\lambda)}{2}\varepsilon\{(\lambda - \zeta_1) + (\lambda - \zeta_1 + \varepsilon)\} + h.o.t.
\]
We should note that Lemma 1.2.5 implies \(\lambda - \zeta_1 > \xi_2 - \zeta_1 > K_2\varepsilon\) with some positive constant \(K_2\) independent of \(\varepsilon\). Thus there exists a positive constant \(K_3\) such that
\[
a(\zeta_1) - a(\zeta_1 - \varepsilon) > K_3\varepsilon^2. \quad (1.5.10)
\]
Moreover, the same argument as in the proof of (1.4.4) leads to
\[
\int_{u^\varepsilon(\zeta_1 - \varepsilon)}^{u^\varepsilon(\zeta_0)} s(1 - s)ds > K_4 \quad (1.5.11)
\]
with some positive constant \(K_4\). Thus, summarizing (1.5.7), (1.5.8), (1.5.9), (1.5.10) and (1.5.11), one can obtain that
\[
1 - u^\varepsilon(\zeta_0) > \kappa\varepsilon
\]
with some \(\kappa > 0\). Recalling the argument developed in the proof of Theorem 1.4.1, we see that there appears another spike in \((\lambda - \delta, \lambda)\) when \(\varepsilon\) is sufficiently small. This fact contradicts (1.5.6).

Now we will consider the appearance of multi-spikes in a neighborhood of boundary points. We only discuss multi-spikes based on \(1\) near the left-hand side boundary point \(0\). Thus we should assume that \(a(0) > 1/2\).

When \(0 \in \Lambda\), by the standard reflection at \(x = 0\), \(0\) is regarded as an interior critical point of \(a\) in the interval \((-1, 1)\). Therefore, similar arguments as above enable us to see that a multi-spike based on \(1\) cannot appear near
0 when $0 \not\in \Lambda$. We next consider the case that $0 \not\in \Lambda$. In this case, we assume that there exists a multi-spike based on 1 near the boundary point 0 with $a_x(0) < 0$. Then the same arguments as above also works well and we can derive a contradiction.

Thus we complete the proof.

Now we will discuss $\varepsilon$-dependence of the distance from a multi-layer and a multi-spike to the corresponding points in $\Sigma$ and $\Lambda$, respectively.

For this purpose, we will collect important properties of multi-layers and multi-spikes.

By Lemma 1.5.2 any multi-layer consists of an odd number of transition layers. If $u^\varepsilon \in S_{n,\varepsilon}$ has a multi-layer in $\delta$-neighborhood of a point $\sigma \in \Sigma = \Sigma^+ \cup \Sigma^-$ with small $\delta > 0$, then there exist $m \in \mathbb{N} \setminus \{1\}$ and $\{\xi_k\}_{k=1}^{2m-1} \subset \Xi$ satisfying

$$\Xi \cap (\sigma - \delta, \sigma + \delta) = \{\xi_k\}_{k=1}^{2m-1} \quad (1.5.12)$$

with $\xi_1 < \xi_2 < \cdots < \xi_{2m-1}$ when $\varepsilon$ is sufficiently small. In this case, we can show the following lemma:

**Lemma 1.5.5.** Under the assumption $(1.5.12)$, it holds that

$$\xi_{k+1} - \xi_k = O(|\varepsilon| \log |\varepsilon|) \quad (1.5.13)$$

for all $k = 1, 2, \ldots, 2m - 2$. Moreover, let $\zeta_0 := \sup\{x; x < \xi_1 \text{ and } u_x^\varepsilon(x) = 0\}$ and $\zeta_{2m-1} := \inf\{x; x > \xi_{2m-1} \text{ and } u_x^\varepsilon(x) = 0\}$. Then the following assertions hold true:

(i) If $\sigma \in \Sigma^+$, then

$$u(\zeta_0) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right) \quad \text{and} \quad 1 - u(\zeta_{2m-1}) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right)$$

with some positive constant $C$.

(ii) If $\sigma \in \Sigma^-$, then

$$1 - u(\zeta_0) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right) \quad \text{and} \quad u(\zeta_{2m-1}) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right)$$

with some positive constant $C$.

**Proof.** First we will show $(1.5.13)$. We only consider the case that $\sigma \in \Sigma^+$. For $l = 1, 2, \ldots, m - 1$, take a set of three components $\{\xi_{2l-1}, \xi_{2l}, \xi_{2l+1}\}$ of
$\Xi \cap (\sigma - \delta, \sigma + \delta)$ satisfying $\xi_{2l-1} < \xi_{2l} < \xi_{2l+1}$ and $\Xi \cap (\xi_{2l-1}, \xi_{2l+1}) = \{\xi_{2l}\}$. In this case, it follows from Theorem 1.5.3 that $u^\varepsilon_\Lambda(\xi_{2l-1}) > 0$, $u^\varepsilon_\Lambda(\xi_{2l}) < 0$ and $u^\varepsilon_\Lambda(\xi_{2l+1}) > 0$. Therefore, using the assertion (i) in Lemma 1.5.1 with replacing $\xi_1$ and $\xi_2$ by $\xi_{2l}$ and $\xi_{2l+1}$, respectively, one can see that $\xi_{2l} - \xi_{2l-1} = O(\varepsilon \log \varepsilon)$. Also the assertion (ii) in Lemma 1.5.1 with replacing $\xi_1$ and $\xi_2$ by $\xi_{2l-1}$ and $\xi_{2l}$, respectively, yields that $\xi_{2l+1} - \xi_{2l} = O(\varepsilon \log \varepsilon)$. These facts show (1.5.13).

For (i) and (ii), the technique developed in the proofs of (1.4.10) and (1.4.11) are valid. Therefore we can easily obtain our desired results. \hfill \Box

For a multi-spike, similar arguments as above work well. However, we should pay attention to treat a multi-spike near the boundary point. If the boundary point is an element of $\Lambda^+ \cup \Lambda^-$, then the standard reflection enables us to see that the boundary point is regarded as an interior critical point of $a$, while, if the derivative of $a$ does not vanish at the boundary point, then we need another argument. Then, for a moment, we do not consider the case that $a_x(0) \neq 0$ and $a_x(1) \neq 0$.

If $u^\varepsilon \in S_{n,\varepsilon}$ has a multi-spike in a neighborhood of $\lambda \in \Lambda^+ \cup \Lambda^- \subset \Lambda$, then there exist a number $m \in \mathbb{N} \setminus \{1\}$, a subset $\{\xi_k\}_{k=1}^{2m} \subset \Xi$ and critical points $\{\zeta_k\}_{k=1}^{2m}$ of $u^\varepsilon$ satisfying

$$\Xi \cap (z - \delta, z + \delta) = \{\xi_k\}_{k=1}^{2m}$$

(1.5.14)

with $\zeta_0 < \xi_1 < \xi_2 < \cdots < \xi_{2m} < \xi_2m$. Here $\xi_0 := \sup \{x; x < \xi_1 \text{ and } u^\varepsilon_\Lambda(x) = 0\}$ and $\xi_{2m} := \inf \{x; x > \xi_{2m} \text{ and } u^\varepsilon_\Lambda(x) = 0\}$. Observe that Lemma 1.2.5 implies that $\xi_{2k} - \xi_{2k-1} = O(\varepsilon)$ for any $k = 1, 2, \ldots, m$. Furthermore, by the same argument as in the proof of Lemma 1.5.5, we obtain that $\xi_{2k+1} - \xi_{2k} = \frac{\varepsilon}{\log \varepsilon}$ for any $k = 1, 2, \ldots, m-1$. It should be noted that we can also show that, if $y \in \Lambda^+$, then $u^\varepsilon(y_0) = O(\varepsilon \exp(-C/\varepsilon))$ and $u^\varepsilon(y_{2m}) = O(\varepsilon \exp(-C/\varepsilon))$ with some positive constant $C$. While if $y \in \Lambda^-$, then there exists a positive constant $C$ such that $1 - u^\varepsilon(y_0) = O(\varepsilon \exp(-C/\varepsilon))$ and $1 - u^\varepsilon(y_{2m}) = O(\varepsilon \exp(-C/\varepsilon))$. Summarizing these facts above, we have the following lemma:

**Lemma 1.5.6.** Under the assumption (1.5.14), it holds that

$$\xi_{2k} - \xi_{2k-1} = O(\varepsilon)$$

for all $k = 1, 2, \ldots, m$, and

$$\xi_{2k+1} - \xi_{2k} = O(\varepsilon \log \varepsilon)$$

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for $k = 1, 2, \ldots, m - 1$. Moreover, let $\zeta_0 := \sup\{x ; x < \xi_1 \text{ and } u_\varepsilon(x) = 0\}$ and $\zeta_{2m} := \inf\{x ; x > \xi_{2m} \text{ and } u_\varepsilon(x) = 0\}$. Then the following assertions hold true:

(i) If $\lambda \in \Lambda^+$, then

$$u(\zeta_0) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right) \quad \text{and} \quad u(\zeta_{2m}) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right)$$

with some positive constant $C$.

(ii) If $\lambda \in \Lambda^-$, then

$$1 - u(\zeta_0) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right) \quad \text{and} \quad 1 - u(\zeta_{2m}) = O\left(\exp\left(-\frac{C}{\varepsilon}\right)\right)$$

with some positive constant $C$.

We now ready to show the following theorems describing the $\varepsilon$-dependence of the distances between a multi-layer (resp. a multi-spike) and the corresponding point in $\Sigma$ (resp. $\Lambda$):

**Theorem 1.5.7.** Let $u^\varepsilon \in S_{n,\varepsilon}$ possess a multi-layer satisfying (1.5.12) for sufficiently small $\varepsilon > 0$. Then $\xi_k - \sigma = O(\varepsilon \log \varepsilon)$ for $k = 1, 2, \ldots, 2m - 1$.

**Proof.** We only consider the case that $\sigma \in \Sigma^+$. In this case, there exists a set of critical points $\{\zeta_k\}_{k=0}^{2m-1}$ of $u^\varepsilon$ with $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m-1} < \zeta_{2m-1}$. Here $\zeta_0 := \sup\{x ; x < \xi_1 \text{ and } u_\varepsilon(x) = 0\}$ and $\zeta_{2m} := \inf\{x ; x > \xi_{2m} \text{ and } u_\varepsilon(x) = 0\}$. We remark that a multi-layer located near a point in $\Sigma^+$ is a multi-layer from 0 to 1. Moreover it follows from Lemma 1.5.5 that

$$\xi_{k+1} - \xi_k = O(\varepsilon \log \varepsilon) \quad (1.5.15)$$

for $k = 1, 2, \ldots, 2m - 2$. Hence, if $\sigma \in (\xi_1, \xi_{2m-1})$, then we can immediately obtain our desired result. Therefore, it suffices to consider the case that $\xi_1 > \sigma$ or $\xi_{2m-1} < \sigma$. We will give a proof for the latter case.

In this case, we first show that

$$\zeta_{2m-1} \geq \sigma. \quad (1.5.16)$$

For this purpose, we assume that (1.5.16) does not hold true. Rewrite (SP1) as

$$\varepsilon^2 u_{xx}^\varepsilon + f(\sigma, u^\varepsilon) = u^\varepsilon(1-u^\varepsilon)(a(x) - 1/2). \quad (1.5.17)$$
Then, multiplying (1.5.17) by $u^\varepsilon_x$ and integrating the resulting expression over $(\zeta_{2m-2}, \zeta_{2m-1})$, we get

$$W(\sigma, u^\varepsilon(\zeta_{2m-2})) - W(\sigma, u^\varepsilon(\zeta_{2m-1})) = \int_{\zeta_{2m-2}}^{\zeta_{2m-1}} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - 1/2)u^\varepsilon_x(x)dx. \quad (1.5.18)$$

We should note that $a(x) - 1/2 < 0$ in $(\zeta_{2m-2}, \zeta_{2m-1})$ when $\zeta_{2m-1} < \sigma$ and that $u^\varepsilon(x) > 0$ in $(\zeta_{2m-2}, \zeta_{2m-1})$. These facts together with (1.5.18) yield that

$$W(\sigma, u^\varepsilon(\zeta_{2m-2})) < W(\sigma, u^\varepsilon(\zeta_{2m-1})). \quad (1.5.19)$$

Here we should note that the graph of $W(\sigma, u)$ is axisymmetric with respect to the line $u = 1/2$. Thus (1.5.19) enables us to see that

$$u^\varepsilon(\zeta_{2m-2}) < 1 - u^\varepsilon(\zeta_{2m-1}). \quad (1.5.20)$$

For the left-hand side of (1.5.20), using Theorem 1.3.8, one can see that there exist positive constants $C_1$ and $R$ satisfying

$$u^\varepsilon(\zeta_{2m-2}) > C_1 \exp\left(-\frac{R(\xi_{2m-1} - \zeta_{2m-2})}{\varepsilon}\right).$$

Moreover, this together with (1.5.15) enables us to see that

$$u^\varepsilon(\zeta_{2m-2}) > K_1 \varepsilon \quad (1.5.21)$$

with some positive constant $K_1$. On the other hand, apply Lemma 1.5.5 to the right-hand side of (1.5.20). Then there exists a positive constant $K_2$ such that

$$1 - u(\zeta_{2m-1}) = O\left(\exp\left(-\frac{K_2}{\varepsilon}\right)\right). \quad (1.5.22)$$

Combining (1.5.20), (1.5.21) and (1.5.22), we derive a contradiction, so that (1.5.16) holds true.

Under the condition (1.5.16), multiplying (1.5.17) by $u^\varepsilon_x$ and integrating the resulting expression over $(\zeta_{2m-2}, \sigma)$, we get

$$\frac{1}{2} \varepsilon^2 u^\varepsilon_x(\sigma)^2 - W(\sigma, u^\varepsilon(\sigma)) + W(\sigma, u^\varepsilon(\zeta_{2m-2})) = \int_{\zeta_{2m-2}}^{\sigma} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - 1/2)u^\varepsilon_x(x)dx. \quad (1.5.23)$$
We should note that $a(x) - 1/2 < 0$ and $u^\varepsilon(x) > 0$ in $(\zeta_{2m-2}, \sigma)$. Therefore the right-hand side of (1.5.23) is negative. This implies that $W(\sigma, u^\varepsilon(\sigma)) > W(\sigma, u^\varepsilon(\zeta_{2m-2}))$. Taking account of the profile of the graph of $W(\sigma, u)$, we obtain that

$$u^\varepsilon(\zeta_{2m-2}) < 1 - u^\varepsilon(\sigma). \quad (1.5.24)$$

For the left-hand side of the inequality above, (1.5.21) is also valid. Moreover, applying Theorem 1.3.3 to the right-hand side of (1.5.24), we can obtain that

$$K_1 \varepsilon < u^\varepsilon(\zeta_{2m-2}) < 1 - u^\varepsilon(\sigma) < C_2 \exp \left(-\frac{r(\sigma - \xi_{2m-1})}{\varepsilon}\right)$$

with some positive constants $C_2$ and $r$. This implies that there is a positive constant $K_3$ such that

$$0 < \sigma - \xi_{2m-1} < K_3 \varepsilon |\log \varepsilon|$$

when $\varepsilon$ is sufficiently small. Thus the proof is complete.

**Lemma 1.5.8.** Let $u^\varepsilon \in S_{n, \varepsilon}$ possess a multi-spike satisfying (1.5.14) for sufficiently small $\varepsilon > 0$. Then $\xi_k - \lambda = O(\varepsilon |\log \varepsilon|)$ for $k = 1, 2, \ldots, 2l$.

**Proof.** We only consider the case that $u^\varepsilon$ has a multi-spike based on 1 near a point $\lambda \in \Lambda^-$. Then there exists a set of critical points $\{\zeta_k\}_{k=0}^{2m}$ of $u^\varepsilon$ satisfying $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m} < \zeta_{2m}$ with $\zeta_0 := \sup \{x; x < \xi_1 \text{ and } u^\varepsilon_x(x) = 0\}$ and $\zeta_{2m} := \inf \{x; x > \xi_{2m} \text{ and } u^\varepsilon_x(x) = 0\}$. By Lemma 1.5.6, it is sufficient to discuss the case that $\xi_1 > \lambda$ or $\xi_{2m} < \lambda$. We only consider the latter case. In this case, observe that a similar method as in the proof of (1.5.16) works well, so that we can show that $\zeta_{2m} \geq \lambda$.

We rewrite (SP1) as

$$\varepsilon^2 u^\varepsilon_{xx} + f(\zeta_{2m-1}, u^\varepsilon) = u^\varepsilon(1 - u^\varepsilon)(a(x) - a(\zeta_{2m-1})). \quad (1.5.25)$$

Then, multiplying (1.5.25) by $u^\varepsilon_x$ and integrating the resulting expression over $(\zeta_{2m-2}, \lambda)$ with respect to $x$, we obtain

$$\frac{1}{2} \varepsilon^2 u^\varepsilon_x(\lambda)^2 - W(\zeta_{2m-1}, u^\varepsilon(\lambda)) + W(\zeta_{2m-1}, u^\varepsilon(\zeta_{2m-2}))$$

$$= \int_{\zeta_{2m-2}}^{\lambda} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_{2m-1})) u^\varepsilon_x(x) dx. \quad (1.5.26)$$
Since \( a_x(x) < 0 \) in \((\zeta_{2m-2}, \lambda)\), \( u^x(x) > 0 \) in \((\zeta_{2m-1}, \lambda)\) and \( u^x(x) < 0 \) in \((\zeta_{2m-2}, \zeta_{2m-1})\), the right-hand side of (1.5.26) is negative. This fact implies
\[
W(\zeta_{2m-1}, u^\xi(\zeta_{2m-2})) < W(\zeta_{2m-1}, u^\xi(\lambda)),
\]
so that
\[
1 - u^\xi(\zeta_{2m-2}) < 1 - u^\xi(\lambda).
\]
Applying Theorems 1.3.3 and 1.3.6 to the inequality above, we can obtain that
\[
C_1 \exp \left( - \frac{R(\zeta_{2m-1} - \zeta_{2m-2})}{\varepsilon} \right) < C_2 \exp \left( - \frac{r(\lambda - \zeta_{2m})}{\varepsilon} \right)
\]
with some positive constants \( C_1, C_2, r \) and \( R \). By using Lemma 1.5.6, it holds that \( \xi_{2m-1} - \xi_{2m-2} = O(\varepsilon |\log \varepsilon|) \). This together with (1.5.27) enables us to conclude that \( \lambda - \xi_{2m} = O(\varepsilon |\log \varepsilon|) \). Thus the proof is complete.

Now we will consider a multi-layer near the boundary point in the case that \( a_x \) does not vanish at the boundary. We only consider a multi-spike based on 1 near the boundary 0 with \( a(0) > 1/2 \) and \( a_x(0) > 0 \). We assume \( \Xi \cap (0, \delta) = \{ \xi_k \}^n_{k=1} \) with some \( \delta > 0 \) and \( m \in \mathbb{N} \). Moreover, we can choose a set of critical points \( \{ \xi_k \}^n_{k=0} \) of \( u^\varepsilon \) with \( 0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-1} < \xi_m := \inf \{ x; u^\varepsilon(x) = 0 \text{ and } x > \xi_m \} \). We should note that \( m \) is not necessarily an even number.

If \( \xi_0 \) is a peak of a spike, then \( \xi_1 \) lies in an \( O(\varepsilon) \)-neighborhood of 0. Moreover, using similar arguments as in the proof of Lemma 1.5.8, we can show that, for all \( k = 2, 3, \ldots, m \), \( \xi_k \) lies in an \( O(\varepsilon |\log \varepsilon|) \)-neighborhood of a point of \( \xi_1 \). It also follows from the arguments as in the proof of Lemma 1.5.8 that, if \( \xi_0 \) is not a peak of a spike, then \( \xi_1 \) must lie in an \( O(\varepsilon |\log \varepsilon|) \)-neighborhood of 0.

Summarizing these facts and Lemma 1.5.8, we have the following theorem:

**Theorem 1.5.9.** Let \( u^\varepsilon \in S_{n,\varepsilon} \) possess a multi-spike in a neighborhood of a point \( \lambda \in \Lambda^+ \cup \Lambda^- \cup \{0, 1\} \) and let \( \delta \) satisfy \( \Xi \cap (\lambda - \delta, \lambda + \delta) = \{ \xi_k \}^m_{k=1} \) with some \( m \in \mathbb{N} \). Then \( \xi_k - \lambda = O(\varepsilon |\log \varepsilon|) \) for \( k = 1, 2, \ldots, m \).

At the end of this chapter, we will discuss more precise profiles of a multi-layer and a multi-spike.

**Lemma 1.5.10.** For \( u^\varepsilon \in S_{n,\varepsilon} \), let \( \{ \xi_k \}^n_{k=0} \) be a set of all critical points of \( u^\varepsilon \) with \( 0 = \xi_0 < \xi_1 < \cdots < \xi_n = 1 \). Moreover, let \( I \) be an interval which satisfies
\[
\zeta_k, \zeta_{k+2} \in I \subset \{ x; a_x(x) > 0 \} \quad \text{(resp. } I \subset \{ x; a_x(x) < 0 \} \text{)} \quad (1.5.28)
\]
with some \( k = 1, 2, \ldots, n \). Then it holds that
\[
\varepsilon \xi (\zeta_k) \leq \xi (\zeta_{k+2}) \quad (\text{resp. } \xi (\zeta_k) > \xi (\zeta_{k+2})).
\] (1.5.29)

**Proof.** Rewrite (SP1) as
\[
\varepsilon^2 u_{xx} + f(\zeta_{k+1}, u^\varepsilon) = u^\varepsilon(1 - u^\varepsilon)(a(x) - a(\zeta_{k+1})).
\] (1.5.30)

Then, multiplying (1.5.30) by \( u^\varepsilon_x \) and integrating the resulting expression over \((\zeta_{k}, \zeta_{k+2})\) with respect to \( x \), we obtain that
\[
W(\zeta_{k+1}, u^\varepsilon(\zeta_k)) - W(\zeta_{k+1}, u^\varepsilon(\zeta_{k+2}))
= \int_{\zeta_k}^{\zeta_{k+2}} u^\varepsilon(x)(1 - u^\varepsilon(x))(a(x) - a(\zeta_{k+1}))u^\varepsilon_x(x)dx.
\] (1.5.31)

We first consider the case that \( u^\varepsilon \) attains its local maxima both at \( x = \zeta_k \) and \( \zeta_{k+2} \). Remark that both \( u^\varepsilon(\zeta_k) \) and \( u^\varepsilon(\zeta_{k+2}) \) are close to 1. In this case, it is now standard to show that the right-hand side of (1.5.31) is positive. This yields that
\[
W(\zeta_{k+1}, u^\varepsilon(\zeta_k)) > W(\zeta_{k+1}, u^\varepsilon(\zeta_{k+2})).
\]

Therefore, taking account of the profile of the graph of \( W \), we can immediately conclude (1.5.29).

On the other hand, if \( u^\varepsilon \) attains its local minima both at \( x = \zeta_k \) and \( \zeta_{k+2} \), then one can also show that the right-hand side of (1.5.31) is negative. Hence we have
\[
W(\zeta_{k+1}, u^\varepsilon(\zeta_k)) < W(\zeta_{k+1}, u^\varepsilon(\zeta_{k+2})).
\]
This implies that (1.5.29) holds true because both \( u^\varepsilon(\zeta_k) \) and \( u^\varepsilon(\zeta_{k+2}) \) are close to 0. Thus the proof is complete. \( \square \)

As a corollary of Lemma 1.5.10, we can show the following theorems which describe precise profiles of multi-layers and multi-spikes:

**Theorem 1.5.11** (Profile of a multi-layer). Let \( \sigma \in \Sigma^+ \) (resp. \( \Sigma^- \)) and \( \delta \) be a small positive number. For \( u^\varepsilon \in S_{n,\varepsilon} \) and \( m \in \mathbb{N} \) with \( 2 \leq m \leq \frac{n+1}{2} \), set \( \{\zeta_k\}_{k=1}^{2m} = \Xi \cap (\sigma - \delta, \sigma + \delta) \). Moreover, let \( \{\zeta_k\}_{k=0}^{2m-1} \) be a unique set of critical points of \( u^\varepsilon \) satisfying \( \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{2m-1} < \zeta_{2m-1} \), where \( \zeta_0 := \sup \{x ; u^\varepsilon_x(x) = 0 \text{ and } x < \zeta_1\} \) and \( \zeta_{2m-1} := \inf \{x ; u^\varepsilon_x(x) = 0 \text{ and } x > \zeta_{2m-1}\} \). Then it holds that
\[
u^\varepsilon(\zeta_{k-2}) < \nu^\varepsilon(\zeta_k) \quad (\text{resp. } \nu^\varepsilon(\zeta_{k-2}) > \nu^\varepsilon(\zeta_k))
\] (1.5.32)
for \( k = 2, 3, \ldots, 2m - 1 \).
Proof. We will use Lemma 1.5.10. As an interval $I$, we define $I := (\sigma - \delta, \sigma + \delta)$. Then it is clear that $\{\xi_k\}_{k=1}^{2m}$ satisfies (1.5.28), so that (1.5.32) is valid for $k = 3, 4, \ldots, 2m - 2$. For $\zeta_0$ and $\zeta_{2m-1}$, it is not necessary the case that they belong to $I$. However, Lemma 1.5.5 enables us to see that (1.5.32) is also valid for $k = 2$ and $2m - 1$. This completes the proof.

Theorem 1.5.12 (Profile of a multi-spike). Let $\lambda \in \Lambda^+$ (resp. $\Lambda^-$) and $\delta$ be a small positive number. For $u^x \in S_{n,e}$ and $m \in \mathbb{N}$ with $2 \leq m \leq n/2$, set $\{\xi_k\}_{k=1}^{2m} = \Xi \cap (\lambda - \delta, \lambda + \delta)$. Moreover, let $\{\zeta_k\}_{k=0}^{2m}$ be a unique set of critical points of $u^x$ satisfying $\zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{2m} < \zeta_2m$, where $\zeta_0 := \sup\{x; u^x_\lambda(x) = 0 \text{ and } x < \xi_1\}$ and $\zeta_{2m} := \inf\{x; u^x_\lambda(x) = 0 \text{ and } x > \xi_{2m}\}$. Then it holds that

\[
\begin{align*}
&\begin{cases}
    u^x(\zeta_{k-2}) < u^x(\zeta_k) & \text{if } \zeta_0 \leq \zeta_{k-2} < \zeta_k < \lambda,
    \\
    u^x(\zeta_{k-2}) > u^x(\zeta_k) & \text{if } \lambda \leq \zeta_{k-2} < \zeta_k \leq \zeta_{2m}.
\end{cases}
\end{align*}
\]

(resp.

\[
\begin{align*}
&\begin{cases}
    u^x(\zeta_{k-2}) > u^x(\zeta_k) & \text{if } \zeta_0 \leq \zeta_{k-2} < \zeta_k < \lambda,
    \\
    u^x(\zeta_{k-2}) < u^x(\zeta_k) & \text{if } \lambda \leq \zeta_{k-2} < \zeta_k \leq \zeta_{2m}.
\end{cases}
\end{align*}
\]

Proof. We only give a proof of the first assertion for the case of $\lambda \in \Lambda^+$. As an interval $I$ of Lemma 1.5.10, put $I := (\lambda - \delta, \lambda)$. Then $\{\xi_k\}_{k=1}^{2m}$ with $\xi_1 < \lambda$ satisfies (1.5.28). Hence, taking notice of Lemma 1.5.6, we can derive our desired result.

Remark 1.5.13. In Theorem 1.5.12, we do not refer to a multi-spike near the boundary 0 with $a_x(0) \neq 0$ or the boundary 1 with $a_x(1) \neq 0$. However, essentially the same arguments as in the proof of Theorems 1.5.11 and 1.5.12 can work well. The reason is that the monotonicity property of extremal points of $u^x$ developed in Theorems 1.5.11 and 1.5.12 depends only on that of $a$. For example, we consider a multi-layer based on 0 near the boundary 0 with $a_x(0) < 0$. In this case, we can show that local maxima (the peaks of the multi-spike) of $u^x$ are placed in a monotone decreasing order. Also local minima of $u^x$ are placed in a monotone decreasing order.
Chapter 2

Stability of Solutions with Transition Layers and Spikes

2.1 Introductory Section of Chapter 2

In this chapter, we will discuss the stability property of an $n$-mode solution $u^\varepsilon$ for (SP1) with transition layers and spikes. For this purpose, we will consider the following linearized eigenvalue problem:

$$\begin{cases}
-\varepsilon^2 \phi_{xx} - f_u(x, u^\varepsilon) \phi = \mu \phi, & 0 < x < 1, \\
\phi_x(0) = \phi_x(1) = 0.
\end{cases}$$

(EVP)

In general, the profile of a solution and its stability property have a close relation. In this chapter, we will discuss such relations with the use of Morse index defined as follows:

**Definition 2.1.1 (Morse index).** Let $u^\varepsilon$ be a solution of (SP1). Then, Morse index $\text{Ind}(u^\varepsilon)$ of $u^\varepsilon$ is defined by the number of negative eigenvalues of (EVP).

In order to state results on stability properties, we will introduce some notation. Remark that, it follows from Lemma 1.2.3 that any $u^\varepsilon \in S_{n,\varepsilon}$ forms a transition layer or a spike near a point in $\Xi$. Then we define a number $n_{t^+}$ by the number of transition layers with $u^\varepsilon_x(\xi) a_x(\xi) > 0$, where $\xi$ is a point in $\Xi$ included in this transition layer. Similarly, $n_{t^-}$ is defined by the number of transition layers with $u^\varepsilon_x(\xi) a_x(\xi) < 0$. Furthermore we define a number $n_{sp}$ by the number of spikes. Then we obtain the following theorem:
Theorem 2.1.1 (Stability of solutions with transition layers and spikes). Let $u^\varepsilon$ be an $n$-mode solution of (SP1). Then the following assertions hold true:

(i) If $u^\varepsilon$ has only single-layers and $n_+ = 0$ (i.e., $n_- = n$), then $u^\varepsilon$ is stable.

(ii) If $u^\varepsilon$ has only single-layers and $n_+ > 0$, then $u^\varepsilon$ is unstable and $\text{Ind}(u^\varepsilon) = n_+$.

(iii) If $n_+ > 0$ or $n_{sp} > 0$, then $u^\varepsilon$ is unstable and $\text{Ind}(u^\varepsilon) \geq n_+ + n_{sp}$.

For $u^\varepsilon \in S_{n,\varepsilon}$, let $\Xi = \{\xi_k\}_{k=1}^n$ with $\xi_1 < \xi_2 < \cdots < \xi_n$ and let $\{\zeta_k\}_{k=0}^n$ be the set of all critical points of $u^\varepsilon$ satisfying $0 = \zeta_0 < \zeta_1 < \xi_1 < \zeta_2 < \cdots < \xi_{n-1} < \xi_n < \zeta_n = 1$. We should note that Lemma 1.2.4 assures the existence of such $\{\zeta_k\}_{k=0}^n$. In order to study (EVP), taking a subset $\{\zeta_{k_i}\}_{i=0}^l \subset \{\zeta_k\}_{k=0}^n$ with $0 = \zeta_0 = \zeta_{k_0} < \zeta_{k_1} < \cdots < \zeta_{k_l} = \zeta_n = 1$, we introduce the following eigenvalue problems:

\[
\begin{align*}
-\varepsilon^2 \phi_{xx} - f_u(x, u^\varepsilon) \phi &= \mu \phi, & \xi_i < x < \xi_{i+1}, \\
\phi_x(\zeta_{k_i}) &= 0, & \zeta_{k_i} < x < \zeta_{k_{i+1}},
\end{align*}
\]

for $i = 0, 1, 2, \ldots, l - 1$. Remark that $u^\varepsilon|_{x \in (\zeta_{k_i}, \zeta_{k_{i+1}})}$ is a solution of

\[
\begin{align*}
\varepsilon^2 u_{xx} + f(x, u) &= 0, & \xi_{k_i} < x < \xi_{k_{i+1}}, \\
u_x(\zeta_{k_i}) &= 0, & \zeta_{k_{i+1}},
\end{align*}
\]

for each $i = 0, 1, \ldots, l - 1$. In what follows, for (2.1.2), we denotes Morse index of $u^\varepsilon|_{x \in (\zeta_{k_i}, \zeta_{k_{i+1}})}$ by $\text{Ind}(u^\varepsilon; \zeta_{k_i}, \zeta_{k_{i+1}})$.

The content of this chapter is as follows. In Section 2.2, we will recall the Sturm-Liouville theory. Also, as an application of this theory, we will show that $\text{Ind}(u^\varepsilon)$ is given by the summation of $\text{Ind}(u^\varepsilon; \zeta_{k_i}, \zeta_{k_{i+1}})$. In Section 2.3, we will derive a result on the stability property of a solution with transition layers. Section 2.4 is devoted to the study of a solution with spikes. Finally, summa-
izing the results developed in Sections 2.2, 2.3 and 2.4, one will immediately derive Theorem 2.1.1.

2.2 Basic Theory for Sturm-Liouville Eigenvalue Problem

In this section, we recall the Sturm-Liouville theory for (EVP). For the proofs, see Coddington and Levinson [6] or Egorov and Kondratiev [9].
Proposition 2.2.1. Let $\mu$ be an eigenvalue of (EVP). Then $\mu$ is real and simple. Furthermore, there exist infinitely number of eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ of (EVP) such that

$$-\infty < \mu_1 < \mu_2 < \cdots < \mu_j < \cdots \to \infty \quad \text{as} \quad j \to \infty,$$

and the eigenfunction corresponding to $\mu_j$ has exactly $j-1$ zeros in $(0,1)$.

Proposition 2.2.2. Let $\mu_j$ be the $j$th eigenvalue of (EVP) and let $\phi_j$ be the corresponding eigenfunction of $\mu_j$. Then it holds that $(\phi_j, \phi_k)_{L^2(0,1)} = 0$ when $j \neq k$.

The following result is well-known as Courant’s min-max principle.

Proposition 2.2.3. Let $\mu_j$ be the $j$th eigenvalue of (EVP). Then $\mu_j$ is characterized by

$$\mu_1 = \inf_{\phi \in H^1(0,1) \setminus \{0\}} \frac{\mathcal{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}},$$

$$\mu_j = \sup_{\psi_1, \ldots, \psi_{j-1} \in L^2(0,1)} \inf_{\phi \in X[\psi_1, \ldots, \psi_{j-1}]} \frac{\mathcal{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}}, \quad \text{for} \quad j = 2, 3, \ldots,$$

where

$$\mathcal{H}(\phi) = \int_0^1 \left\{ \varepsilon^2 (\phi_x(x))^2 - f_u(x, u'(x))(\phi(x))^2 \right\} \, dx \quad (2.2.1)$$

and

$$X[\psi_1, \ldots, \psi_j] = \{ \phi \in H^1(0,1) \setminus \{0\} \mid (\phi, \psi_i)_{L^2(0,1)} = 0 \quad (i = 1, 2, \cdots, j) \}.$$

Remark 2.2.4. In Proposition 2.2.3, if $\psi_j$ denotes the eigenfunction corresponding to the $j$th eigenvalue $\mu_j$, then $\mu_j$ is characterized by

$$\mu_j = \inf_{\phi \in X[\psi_1, \ldots, \psi_{j-1}]} \frac{\mathcal{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}}.$$

As a corollary of Proposition 2.2.3, we can see that the following assertion holds true:

Corollary 2.2.5. Let $\{w_j\}_{j=1}^{m}$ be a family of functions in $H^1(0,1)$ satisfying $(w_j, w_k)_{L^2(0,1)} = 0$ when $j \neq k$ and $\mathcal{H}(w_j) < 0$ for $j = 1, 2, \ldots, m$ where $\mathcal{H}$ is defined by (2.2.1). Then, the $m$th eigenvalue of (EVP) is negative.
Remark 2.2.6. All propositions cited above in Section 2.2 can work well on (2.1.1), if $(0, 1)$ is replaced by $(\zeta_k, \zeta_{k+1})$ for $i = 1, 2, \ldots, l - 1$.

At the end of this section, we will show another proposition which will play an important role in the study of Morse index of $u^\varepsilon$.

Proposition 2.2.7. For a solution $u^\varepsilon$ of (SP1), it holds that

$$\text{Ind}(u^\varepsilon) = \sum_{i=1}^{l} \text{Ind}(u^\varepsilon; \zeta_k, \zeta_{k+1}).$$

(2.2.2)

Proof. For the sake of simplicity, we will show that

$$\text{Ind}(u^\varepsilon) = \text{Ind}(u^\varepsilon; 0, \zeta) + \text{Ind}(u^\varepsilon; \zeta, 1)$$

with some critical point $\zeta$ of $u^\varepsilon$. If the $j$th eigenvalue of (EVP) is denoted by $\mu_j$, then it suffices to show that

$$\mu_{m_1+m_2} < 0 \quad \text{and} \quad \mu_{m_1+m_2+1} > 0$$

(2.2.3)

where $m_1 = \text{Ind}(u^\varepsilon; 0, \zeta)$ and $m_2 = \text{Ind}(u^\varepsilon; \zeta, 1)$.

Let $\mu_j^{(1)}$ be the $j$th eigenvalue of (2.1.1) with replacing $(\zeta_k, \zeta_{k+1})$ by $(0, \zeta)$ and let $\phi_j^{(1)}$ be the corresponding eigenfunction of $\mu_j^{(1)}$. Moreover, we define

$$\mathcal{H}^{(1)}(\phi) = \int_{0}^{\zeta} \left\{ \varepsilon^2 (\phi_x(x))^2 - f_u(x, u^\varepsilon(x))(\phi(x))^2 \right\} dx$$

and

$$X^{(1)}[\psi_1, \ldots, \psi_j] = \{ \phi \in H^1(0, \zeta) \setminus \{0\}; (\phi, \psi_i)_{L^2(0, \zeta)} = 0 (i = 1, 2, \cdots, j) \}.$$

Similarly we denotes the $j$th eigenvalue of (2.1.1) with replacing $(\zeta_k, \zeta_{k+1})$ by $(\zeta, 1)$ by $\mu_j^{(2)}$ and its corresponding eigenfunction by $\phi_j^{(2)}$. We also define

$$\mathcal{H}^{(2)}(\phi) = \int_{\zeta}^{1} \left\{ \varepsilon^2 (\phi_x(x))^2 - f_u(x, u^\varepsilon(x))(\phi(x))^2 \right\} dx$$

and

$$X^{(2)}[\psi_1, \ldots, \psi_j] = \{ \phi \in H^1(\zeta, 1) \setminus \{0\}; (\phi, \psi_i)_{L^2(\zeta, 1)} = 0 (i = 1, 2, \cdots, j) \}.$$
We first prove the first assertion of (2.2.3). Since \( \text{Ind}(u^\varepsilon; 0, \zeta) = m_1 \), it is easy to see that
\[
\mathcal{H}(1)(\phi_j^{(1)}) = \mu_j^{(1)}\|\phi_j^{(1)}\|^2_{L^2(0,\zeta)} < 0 \quad \text{for } j = 1, 2, \ldots, m_1. \tag{2.2.4}
\]
In a similar way, one can obtain that
\[
\mathcal{H}(2)(\phi_j^{(2)}) < 0 \quad \text{for } j = 1, 2, \ldots, m_2. \tag{2.2.5}
\]
Now we define a family of \( L^2(0,1) \)-functions \( \{w_j\}_{j=1}^{m_1+m_2} \) by
\[
w_j = \begin{cases} 
\phi_j^{(1)} & \text{in } (0, \zeta), \\
0 & \text{in } (\zeta, 1),
\end{cases} \quad \text{for } j = 1, 2, \ldots, m_1,
\]
\[
w_j = \begin{cases} 
0 & \text{in } (0, \zeta), \\
\phi_j^{(2)} & \text{in } (\zeta, 1),
\end{cases} \quad \text{for } j = m_1 + 1, m_1 + 2, \ldots, m_1 + m_2. \tag{2.2.6}
\]
By Proposition 2.2.2, this family is linearly independent in \( L^2(0,1) \). Furthermore, (2.2.4) and (2.2.5) imply that
\[
\mathcal{H}(w_j) = \mathcal{H}(1)(w_j) + \mathcal{H}(2)(w_j) < 0
\]
for \( j = 1, 2, \ldots, m_1 + m_2 \). Therefore Corollary 2.2.5 enables us to conclude that \( \mu_{m_1+m_2} < 0 \).

Next we will show \( \mu_{m_1+m_2+1} > 0 \). It suffices to show that there exists a set of linearly independent functions \( \{\psi_j\}_{j=1}^{m_1+m_2} \subset L^2(0,1) \) satisfying
\[
\inf_{\phi \in X[\psi_1, \ldots, \psi_{m_1+m_2}]} \frac{\mathcal{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}} \geq \mu^* \tag{2.2.7}
\]
with some \( \mu^* > 0 \). Indeed, if (2.2.7) holds true, then Proposition 2.2.3 implies that \( \mu_{m_1+m_2+1} \geq \mu^* > 0 \).

By Remark 2.2.4, for any \( w^{(1)} \in X^{(1)}[\phi_1^{(1)}, \ldots, \phi_{m_1}^{(1)}] \), it holds that
\[
\mu_{m_1+1} = \inf_{\phi \in X^{(1)}[\phi_1^{(1)}, \ldots, \phi_{m_1}^{(1)}]} \frac{\mathcal{H}(\phi)}{\|\phi\|^2_{L^2(0,\zeta)}} \leq \frac{\mathcal{H}(1)(w^{(1)})}{\|w^{(1)}\|^2_{L^2(0,\zeta)}}.
\]
Thus we get
\[
\mathcal{H}(1)(w^{(1)}) \geq \mu_{m_1+1}^{(1)}\|w^{(1)}\|^2_{L^2(0,\zeta)} > 0 \tag{2.2.8}
\]
because Morse index of \( u^\varepsilon \) for (2.1.1) is \( m_1 \). By using the same technique as above, for any \( w^{(2)} \in X^{(2)}[\phi_1^{(2)}, \ldots, \phi_{m_2}^{(2)}] \), we can also obtain that
\[
\mathcal{H}(2)(w^{(2)}) \geq \mu_{m_2+1}^{(2)}\|w^{(2)}\|^2_{L^2(\zeta,1)} > 0. \tag{2.2.9}
\]
We now put $\psi_j = w_j$ for $j = 1, 2, \ldots, m_1 + m_2$, where $w_j$ is defined by (2.2.6) and prove that this family satisfies (2.2.7). Recalling the property of $\{w_j\}_{j=1}^{m_1+m_2}$, we see that $\{\psi_j\}_{j=1}^{m_1+m_2}$ is a set of linearly independent functions in $L^2(0, 1)$. Moreover it is easy to show that, for any $w \in X[\psi_1, \psi_2, \ldots, \psi_{m_1+m_2}]$, it holds that

$$
(w, \psi_j)_{L^2(0,1)} = \begin{cases}
(w, \phi_j^{(1)})_{L^2(0,\zeta)} = 0 & \text{for } j = 1, 2, \ldots, m_1, \\
(w, \phi_j^{(2)})_{L^2(\zeta,1)} = 0 & \text{for } j = m_1 + 1, m_1 + 2, \ldots, m_1 + m_2.
\end{cases}
$$

This implies that $w\big|_{(0,\zeta)}$ and $w\big|_{(\zeta,1)}$ also belong to $X^{(1)}[\phi_1^{(1)}, \phi_2^{(1)}, \ldots, \phi_{m_1}^{(1)}]$ and $X^{(2)}[\phi_1^{(2)}, \phi_2^{(2)}, \ldots, \phi_{m_2}^{(2)}]$, respectively. Hence (2.2.8) and (2.2.9) yield that

$$
\mathcal{H}(w) = \mathcal{H}^{(1)}(w) + \mathcal{H}^{(2)}(w) > \mu^{(1)}_{m_1+1}\|w\|_{L^2(0,\zeta)}^2 + \mu^{(2)}_{m_2+1}\|w\|_{L^2(\zeta,1)}^2
\geq \mu^*\|w\|_{L^2(0,1)}^2,
$$

where

$$
\mu^* = \min \left\{ \mu^{(1)}_{m_1+1}, \mu^{(2)}_{m_2+1} \right\} > 0.
$$

This implies (2.2.7), so that we obtain (2.2.3).

Finally, one will find out that the argument as above is valid for the proof of (2.2.2). Thus the proof is complete.

\textbf{Remark 2.2.8.} For $u^\varepsilon \in S_{n,\varepsilon}$, each transition layer or spike is included in an interval $(\zeta_k, \zeta_l)$. Here $\zeta_k$ and $\zeta_l$ are critical points of $u^\varepsilon$ which satisfies $\min\{u^\varepsilon(\zeta_k), 1 - u^\varepsilon(\zeta_k)\} = O(\exp(K/\varepsilon))$ and $\min\{u^\varepsilon(\zeta_l), 1 - u^\varepsilon(\zeta_l)\} = O(\exp(K/\varepsilon))$ with some positive constant $K$. We should remark that Proposition 2.2.7 implies that, summing up $\text{Ind}(u^\varepsilon; \zeta_k, \zeta_l)$ for each interval, we can obtain $\text{Ind}(u^\varepsilon)$. Therefore, in order to study Morse index of $u^\varepsilon$ which possesses transition layers and spikes, it suffices to consider the case that $u^\varepsilon$ has any one of a single-layer, a multi-layer, a single-spike or a multi-spike in $(0, 1)$.

\section*{2.3 Stability of Solutions with Transition Layers}

In this section, we will discuss the stability of solutions with transition layers. By virtue of Proposition 2.2.7 and Remark 2.2.8, it suffices to concentrate ourselves on the study of a solution with an oscillating profile in a neighborhood
of a point in $\sigma \in \Sigma$ in order to show Theorem 2.1.1. Then, for $u^\varepsilon \in S_{n,\varepsilon}$, take a small positive constant $\delta$ and assume that

$$\Xi = \Xi \cap (\sigma - \delta, \sigma + \delta) = \{\xi_k\}_{k=1}^n$$

(2.3.1)

with $\xi_1 < \xi_2 < \cdots < \xi_n$. We should note that (2.3.1) together with Theorems 1.1.2 and 1.1.3 imply that $u^\varepsilon$ has no oscillation except for transition layers. Moreover, considering the profiles of transition layers, we can see that

$$n = 2m - 1$$

(2.3.2)

with some $m \in \mathbb{N}$. Also remark that, if $m = 1$, then $u^\varepsilon$ forms a single-layer near $\sigma$, while, if $m \geq 2$, then $u^\varepsilon$ forms a multi-layer near $\sigma$.

The stability property of solutions with transition layers is described as follows:

**Theorem 2.3.1.** For $u^\varepsilon \in S_{n,\varepsilon}$, assume (2.3.1) and (2.3.2). Then the following assertions hold true:

(i) If $m = 1$ and $u^\varepsilon_x(x) a_x(x) < 0$, then $u^\varepsilon$ is stable.

(ii) If $m = 1$ and $u^\varepsilon_x(x) a_x(x) > 0$, then $u^\varepsilon$ is unstable. Moreover it holds that $\text{Ind}(u^\varepsilon) = 1$.

(iii) If $m \geq 2$, then $u^\varepsilon$ is unstable. Moreover it holds that $\text{Ind}(u^\varepsilon) \geq m$.

For Theorem 2.3.1, we only give a proof of (iii) because the stability or instability of a solution with single-layers has already been obtained by Angenent, Mallet-Paret and Peletier [3] or Hale and Sakamoto [12]. For this purpose, letting $\mu_j$ be the $j^{th}$ eigenvalue of (EVP), we will show the following lemma:

**Lemma 2.3.2.** Under the assumptions of (iii) in Theorem 2.3.1, it holds that $\mu_m < 0$.

Proof. For definiteness, we assume $a_x(\sigma) > 0$. In this case, we may consider that $a_x(x) > 0$ in $(\sigma - \delta, \sigma + \delta)$. Moreover it follows from Lemma 1.2.4 that one can choose the unique set of critical points $\{\zeta_k\}_{k=0}^{2m-1}$ of $u^\varepsilon$ satisfying $0 = \zeta_0 < \zeta_1 < \zeta_1 < \cdots < \zeta_{2m-1} < \zeta_{2m-1} = 1$. In this case, Theorem 1.1.2 implies that, if $m \geq 2$, then this is a multi-layer from 0 to 1. Thus it holds that $u^\varepsilon_x(x) > 0$ in $(\zeta_{2k}, \zeta_{2k+1})$ for $k = 0, 1, \ldots, m - 1$, while, $u^\varepsilon_x(x) < 0$ in $(\zeta_{2k-1}, \zeta_{2k})$ for $k = 1, 2, \ldots, m - 1$. 

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Define \( \{w_k\}_{k=1}^m \) by
\[
  w_k(x) = \begin{cases} 
    u_x^\varepsilon(x) & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \\
    0 & \text{in } (0,1) \setminus (\zeta_{2k-2}, \zeta_{2k-1}).
  \end{cases}
\]

Then \( \{w_k\}_{k=1}^m \) is a family of linearly independent functions in \( H^1(0,1) \) satisfying \( (w_k, w_l)_{L^2(0,1)} = 0 \) when \( k \neq l \). Furthermore, \( w_k \) satisfies
\[
  \begin{cases} 
    \varepsilon^2 (w_k)_{xx} + f_s(x, u^\varepsilon) w_k + f_x(x, u^\varepsilon) = 0 & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \\
    w(\zeta_{2k-2}) = w(\zeta_{2k-1}) = 0,
  \end{cases} \tag{2.3.3}
\]
for \( k = 1, 2, \ldots, m \). We will show
\[
  \mathcal{H}(w_k) < 0 \tag{2.3.4}
\]
for all \( k = 1, 2, \ldots, m \). If (2.3.4) holds true, then we can immediately obtain \( \mu_m < 0 \) by virtue of Corollary 2.2.5.

It follows from (2.3.3) that
\[
  \mathcal{H}(w_k) = -\int_{\zeta_{2k-2}}^{\zeta_{2k-1}} a_x(x) u^\varepsilon(x)(1 - u^\varepsilon(x)) u_x^\varepsilon(x) dx.
\]
Since \( a \) is monotone increasing in \( (\sigma - \delta, \sigma + \delta) \), it is easy to see that (2.3.4) is valid for \( k = 2, \ldots, m-1 \). However \( a_x \) is not necessarily positive in \( (\zeta_0, \zeta_1) \) and \( (\zeta_{2m-2}, \zeta_{2m-1}) \), so that we have to prove \( \mathcal{H}(w_1) < 0 \) and \( \mathcal{H}(w_m) < 0 \) without the monotonicity condition of \( a \).

We only prove \( \mathcal{H}(w_1) < 0 \). Define
\[
  \lambda := \max \{ x ; x \in \Lambda \text{ and } x < \sigma \}.
\]
If \( \lambda = 0 \) (\( = \zeta_0 \)), then \( a \) is monotone increasing in \( (\zeta_0, \zeta_1) \) and (2.3.4) is valid for \( k = 1 \) by the same reasoning as the case of \( k = 2, 3, \ldots, m-1 \). Hence it is sufficient to consider the case \( \lambda > 0 \).

Set
\[
  \mathcal{H}(w_1) = -\int_{\zeta_0}^{\lambda} a_x(x) u^\varepsilon(x)(1 - u^\varepsilon(x)) u_x^\varepsilon(x) dx \\
  - \int_{\lambda}^{\zeta_1} a_x(x) u^\varepsilon(x)(1 - u^\varepsilon(x)) u_x^\varepsilon(x) dx \tag{2.3.5}
\]
\[
  =: I + II.
\]
Then I is estimated as follows:

\[
|I| \leq a^* \int_{\zeta_0}^{\lambda} u^\varepsilon(x)(1 - u^\varepsilon(x))u_x^\varepsilon(x)dx \\
< a^* \int_{\zeta_0}^{\lambda} u^\varepsilon(x)u_x^\varepsilon(x)dx \\
< \frac{1}{2}a^*(u^\varepsilon(\lambda))^2,
\]

where \(a^* = \max\{|a_x(x)|; x \in [\zeta_0, \lambda]\} > 0\). Note that \(\xi_1\) lies in \((\sigma - \delta, \sigma + \delta)\) and \(\xi_1 - \lambda > \sigma - \delta - \lambda > K_1\) with some positive constant \(K_1\). By Theorem 1.3.8, there exist a positive constant \(C\) and \(r\) such that

\[
u^\varepsilon(\lambda) < C \exp \left(-\frac{r(\xi_1 - \lambda)}{\varepsilon}\right) < C \exp \left(-\frac{rK_1}{\varepsilon}\right).
\]

This estimate together with (2.3.6) implies

\[
I = O \left(\exp \left(-\frac{2rK_1}{\varepsilon}\right)\right).
\]

For II, we should note that \(a_x(x)u_x^\varepsilon(x) > 0\) in \((\lambda, \zeta_1)\) and \(\zeta_1 - \lambda > \xi_1 - \lambda > K_1\). Therefore, it holds that

\[
\Pi \leq -a_* \int_{\lambda+K_1}^{\zeta_1} u^\varepsilon(x)(1 - u^\varepsilon(x))u_x^\varepsilon(x)dx = -a_* \int_{u^\varepsilon(\lambda+K_1)}^{u^\varepsilon(\zeta_1)} s(1 - s)ds,
\]

where \(a_* = \min\{|a_x(x)|; x \in [\lambda + K_1, \zeta_1]\} > 0\). Remark that \(u^\varepsilon(\zeta_1)\) is very close to 1. It also should be noted that Theorem 1.1.2 implies that \(u^\varepsilon(\lambda + K_1) < u^\varepsilon(\xi_1) = a(\xi_1) = 1/2 + O(\varepsilon|\log \varepsilon|)\). Hence there exists a positive number \(K_2\) satisfying

\[
\int_{u^\varepsilon(\lambda+K_1)}^{u^\varepsilon(\zeta_1)} s(1 - s)ds > K_2.
\]

Thus we get

\[
\Pi < -a_*K_2.
\]

This together with (2.3.5) and (2.3.7) yields that \(\mathcal{H}(w_1) < 0\) when \(\varepsilon\) is sufficiently small. We can also get \(\mathcal{H}(w_m) < 0\) in a similar way. Thus the proof is complete. \(\square\)
2.4 Stability of Solutions with Spikes

In this section, we will derive stability results on $u^\varepsilon \in S_{n,\varepsilon}$ with spikes. One will find that the spectral theory of Schrödinger operators developed in Egorov and Kondratiev [9] and Reed and Simon [22] will play major roles here.

In order to focus such solutions, take a point $\lambda \in \Lambda$ and suppose that there is a positive constant $\delta$ satisfying

$$\Xi = \Xi \cap (\lambda - \delta, \lambda + \delta) = \{\xi_k\}_{k=1}^n$$

(2.4.1)

with $\xi_1 < \xi_2 < \cdots < \xi_n$. Then, it follows from Theorems 1.1.2 and 1.1.3 that $u^\varepsilon$ has no oscillation except for spikes, so that

$$n = 2m$$

(2.4.2)

with some $m \in \mathbb{N}$. We should note that Proposition 2.2.7 and Remark 2.2.8 imply that it is sufficient to consider a solution satisfying (2.4.1) and (2.4.2) in order to prove Theorem 2.1.1. Then we will show the following theorem:

**Theorem 2.4.1.** For $u^\varepsilon \in S_{n,\varepsilon}$, assume (2.4.1) and (2.4.2). Then $u^\varepsilon$ is unstable and $\text{Ind}(u^\varepsilon) \geq m$.

**Proof.** For definiteness, we will assume that $a(\lambda) < 1/2$. Then, Theorem 1.1.3 implies that spikes around $\lambda$ is based on 0.

We fist consider the case that $m = 1$. In this case, there is a single-spike near $\lambda$. For $u^\varepsilon \in S_{n,\varepsilon}$, let $\xi^\varepsilon$ be an element of $\{\xi_1, \xi_2\}$ and define $U^\varepsilon$ by $U^\varepsilon(z) = u^\varepsilon(\xi^\varepsilon + \varepsilon z)$. Then, Lemma 1.2.5 yields that $\{\varepsilon\}$ has a subsequence $\{\varepsilon_k\} \downarrow 0$ satisfying $\lim_{k \to \infty} \xi^\varepsilon_k = \xi^\varepsilon$ and $\lim_{k \to \infty} U^\varepsilon_k = U$ in $C^2_{\text{loc}}(\mathbb{R})$ with some $\xi^* \in (0, 1)$ and $U \in C^2(\mathbb{R})$. Moreover, $U$ satisfies

$$\begin{cases}
U_{zz} + f(\xi^*, U) = 0 & \text{in } \mathbb{R}, \\
U(\pm \infty) = 0.
\end{cases}$$

(2.4.3)

Define an operator $\mathcal{F} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$-\frac{d^2}{dz^2} - f_u(\xi^*, U(z)).$$

We should note that $\mathcal{F}$ is self-adjoint, so that its spectrum $\sigma(\mathcal{F})$ lies on the real axis. Also remark that $\lim_{|z| \to \infty} (-f_u(\xi^*, U(z))) = a(\xi^*) \in (0, 1/2)$. Therefore,
it follows from the spectral property of Schrödinger operators that there exists a positive constant $K$ such that $\sigma(\mathcal{I}) \cap (-\infty, K)$ consists of finite number of eigenvalues with finite multiplicity. Differentiating the first equation of (2.4.3) with respect to $z$, we can easily see that $U_z$ is an eigenfunction corresponding to 0. Taking notice that the principal eigenvalue of $\mathcal{I}$ is simple and that its corresponding eigenfunction does not change its sign. Therefore, we obtain that

$$\inf\sigma(\mathcal{I}) < 0. \quad (2.4.4)$$

For any $\psi \in C_0^\infty(\mathbb{R}) \setminus \{0\}$, put $\psi_k(x) = \psi(z)$ with $x = \xi^{\epsilon_k} + \epsilon_k z$. It follows from $\lim_{k \to \infty} \xi^{\epsilon_k} = \xi^*$ that $\psi_k \in C_0^\infty(0, 1)$ when $k$ is sufficiently large. Hence it follows from Proposition 2.2.3 that the principal eigenvalue $\mu_{1}^{(k)}$ of (EVP) with replacing $\epsilon$ by $\epsilon_k$ satisfies

$$\mu_{1}^{(k)} \leq \frac{\mathcal{H}(\psi_k)}{\|\psi_k\|_{L^2(0,1)}}.$$

Therefore, changing the variable as $x = \xi^{\epsilon_k} + \epsilon_k z$ in the above integral and using the limiting procedure, we obtain that

$$\limsup_{i \to \infty} \mu_{1}^{(k)} \leq \inf_{\psi \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \left\{ (\psi_z(z))^2 - f_u(\xi^*, U(z))(\psi(z))^2 \right\} dz}{\|\psi\|_{L^2(\mathbb{R})}^2} = \inf\sigma(\mathcal{I}).$$

Hence one can see that

$$\limsup_{k \to \infty} \mu_{1}^{(k)} \leq \inf_{\psi \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \left\{ (\psi_z(z))^2 - f_u(\xi^*, U(z))(\psi(z))^2 \right\} dz}{\|\psi\|_{L^2(\mathbb{R})}^2} = \inf\sigma(\mathcal{I}).$$

This together with (2.4.4) yields that

$$\limsup_{k \to \infty} \mu_{1}^{(k)} \leq \inf\sigma(\mathcal{I}) < 0.$$

This implies that $u^\varepsilon$ is unstable.

Next we consider the case that $m \geq 2$. Remark that the arguments as above are valid for each spike included in a multi-spike. Therefore we can see that $\text{Ind}(u^\varepsilon; \zeta_{2k-2}, \zeta_{2k}) \geq 1$ for $k = 1, 2, \ldots, m$, so that it follows from Proposition 2.2.7 that $\text{Ind}(u^\varepsilon) \geq m$. Thus the proof is complete. \qed
Remark 2.4.2. Under the assumptions (2.4.1) and (2.4.2), if $\lambda \in \Lambda^+ \cup \Lambda^-$, then we can easily see that there exists a subset $\{\xi_{k_i}\}_{i=1}^m \subset \{\xi_k\}_{k=1}^{2m}$ satisfying $u_x^\epsilon(\xi_{k_i})a_x(\xi_{k_i}) > 0$ for all $i = 1, 2, \ldots, m$. Therefore, the same technique as in the proof of Lemma 2.3.2 is valid for this case. This is another proof of Theorem 2.4.1 in the case that $\lambda \in \Lambda^+ \cup \Lambda^-$. 
Part II

Transition Layers for a Bistable Reaction-Diffusion Equation with Spatial Inhomogeneity Both in Its Diffusion and Reaction Terms
Chapter 3

Patterns of Solutions with Transition Layers

3.1 Introductory Section of Chapter 3

In this chapter, we will consider the following reaction-diffusion problem:

\[
\begin{aligned}
  \frac{\partial u}{\partial t} &= \varepsilon^2 (d(x)^2 u_x)_x + h(x)^2 f(u), & 0 < x < 1, \ t > 0, \\
  u_x(0, t) &= u_x(1, t) = 0, & t > 0, \\
  u(x, 0) &= u_0(x), & 0 < x < 1.
\end{aligned}
\]  

(P2)

Here \( \varepsilon \) denotes a positive parameter and \( f(u) \) is a nonlinearity given by

\[ f(u) = u(1 - u)(u - 1/2). \]

Moreover \( d \) and \( h \) are \( C^2 \)-functions satisfying the following properties:

(\( \Phi_1 \)) \( d(x) > 0 \) and \( h(x) > 0 \) in \([0, 1]\).

(\( \Phi_2 \)) Define \( \varphi(x) := d(x)h(x) \) and \( \Sigma := \{x \in [0, 1]: \varphi_x(x) = 0\} \). Then \( \Sigma \) is a non-empty finite set and \( \varphi_{xx}(x) \neq 0 \) at any \( x \in \Sigma \).

(\( \Phi_3 \)) \( d_x(0) = d_x(1) = h_x(0) = h_x(1) = 0. \)

This problem appears in various fields such as physics, chemistry and mathematical biology. It is well known that (P2) describes a phase transition phenomenon and that this kind of problem admits a solution with transition layers when \( \varepsilon \) is sufficiently small.
For (P2), we will mainly discuss patterns of steady-state solutions of (P2) when the existence of such solutions is assumed. The stationary problem associated with (P2) is given as follows:

\[
\begin{align*}
\varepsilon^2 (d(x)^2 u_x)_x + h(x)^2 f(u) &= 0, \quad 0 < x < 1, \\
u_x(0) &= u_x(1) = 0.
\end{align*}
\] (SP2)

The thing which attracts our interest is that both the diffusion and reaction terms include \(x\)-dependent functions \(d\) and \(h\), respectively. They cause spatial inhomogeneity to our problem and their interaction yields many kinds of solutions of (SP2). In addition, they have much effect on patterns of such solutions.

We now consider the relation of (P2) and (SP2). As an energy functional, one can take

\[
E(u) := \int_0^1 \left[ \frac{1}{2} \varepsilon^2 d(x)^2 (u_x(x))^2 + h(x)^2 W(u(x)) \right] dx
\]
with

\[
W(u) := -\int_0^u f(s) ds.
\] (3.1.1)

Here \(W\) is a bistable potential which attains its local minima both at \(u = 0\) and \(1\). Moreover we should remark that the depths of two potential wells are equal. This case is called a balanced case. It is known that every solution of (P2) converges to a solution of (SP2) as \(t \to \infty\) and that \(E(u(\cdot, t))\) is monotone decreasing with respect to \(t\). Hence the minimizer of \(E\) will be a stable steady-states. For proofs of these facts, see Matano [14].

In this chapter, we will study solutions of (SP2) with transition layers. As previously indicated, our main purpose is not to show the existence of such solutions but to investigate their patterns. In particular, taking account of the interaction of \(d\) and \(h\), we will characterize all patterns of solutions with transition layers and determine the location of transition layers including multi-layers completely.

In order to concentrate ourselves on a solution \(u^\varepsilon\) of (SP2) with oscillatory profiles, we will introduce the notion of \(n\)-mode solutions as follows:

**Definition 3.1.1.** For any fixed integer \(n \geq 1\), a solution \(u^\varepsilon\) of (SP2) is called an \(n\)-mode solution, if \(u^\varepsilon - 1/2\) has exactly \(n\) zero points in the interval \((0, 1)\).
Hereafter we will denote the set of all \( n \)-mode solutions of (SP2) by \( S_{n,\varepsilon} \). Furthermore, for \( u^\varepsilon \in S_{n,\varepsilon} \), set

\[ \Xi := \{ x \in (0, 1) ; u^\varepsilon(x) = 1/2 \}. \]

In the study of \( n \)-mode solutions, we can show that, if \( \varepsilon \) is sufficiently small, then the graph of any \( u^\varepsilon \in S_{n,\varepsilon} \) is classified into the following two portions (see Lemma 3.2.5):

(i) \( u^\varepsilon(x) \) is very close to either 0 or 1.

(ii) \( u^\varepsilon(x) \) forms a transition layer connecting 0 and 1.

Therefore, the study of solutions with transition layers and that of \( n \)-mode solutions are essentially the same. Investigating \( n \)-mode solutions, we obtain the following theorems concerning the location of transition layers and their precise profiles:

**Theorem 3.1.1** (Location of transition layers for solutions of (SP2)). For \( u^\varepsilon \in S_{n,\varepsilon} \) with transition layers, then any transition layer appears only in an \( O(\varepsilon |\log \varepsilon|) \)-neighborhood of a point in \( \Sigma \). Moreover, the following assertions hold true:

(i) If \( u^\varepsilon \) has a multi-layer, then it appears only in a neighborhood of a local maximum point of \( \varphi \).

(ii) If \( u^\varepsilon \) has a transition layer in a neighborhood of a local minimum point of \( \varphi \), then it must be a single-layer.

(iii) If \( \varphi_{xx}(0) > 0 \) (resp. \( \varphi_{xx}(1) > 0 \)), then \( u^\varepsilon \) has no transition layer in a neighborhood of 0 (resp. 1).

**Theorem 3.1.2.** Let \( \sigma \in \Sigma \) satisfy \( \varphi_{xx}(\sigma) < 0 \) and \( \delta \) be a small positive number. For \( u^\varepsilon \in S_{n,\varepsilon} \), set \( \{ \zeta_k \}_{k=1}^m = \Xi \cap (\sigma - \delta, \sigma + \delta) \) with \( m \in \mathbb{N} \) and \( 2 \leq m \leq n \). Moreover, let \( \{ \zeta_k \}_{k=0}^m \) be a unique set of critical points of \( u^\varepsilon \) satisfying \( \zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \cdots < \xi_m < \zeta_m \), where \( \zeta_0 := \sup \{ y ; u_{\varepsilon}^y(y) = 0 \text{ and } y < \xi_1 \} \) and \( \zeta_m := \inf \{ y ; u_{\varepsilon}^y(y) = 0 \text{ and } y > \xi_m \} \). If \( u_{\varepsilon}^\sigma(\xi_1) < 0 \) (resp. \( u_{\varepsilon}^\sigma(\xi_1) > 0 \)), then it holds that

\[
\begin{align*}
\{ u^\varepsilon(\zeta_{2k-2}) > u^\varepsilon(\zeta_{2k}) \} & \quad \text{if } \zeta_0 \leq \zeta_{2k-2} < \zeta_{2k} < \sigma, \\
\{ u^\varepsilon(\zeta_{2k-1}) < u^\varepsilon(\zeta_{2k+1}) \} & \quad \text{if } \zeta_1 \leq \zeta_{2k-1} < \zeta_{2k+1} < \sigma,
\end{align*}
\]

and

\[
\begin{align*}
\{ u^\varepsilon(\zeta_{2k-2}) < u^\varepsilon(\zeta_{2k}) \} & \quad \text{if } \sigma \leq \zeta_{2k-2} < \zeta_{2k} \leq \zeta_m, \\
\{ u^\varepsilon(\zeta_{2k-1}) > u^\varepsilon(\zeta_{2k+1}) \} & \quad \text{if } \sigma \leq \zeta_{2k-1} < \zeta_{2k+1} \leq \zeta_m.
\end{align*}
\]
In the study of (SP2), it is also suitable to make the change of variables

\[ x \mapsto y = \int_0^x \frac{h(s)}{d(s)} ds. \]  

Then (SP2) is transformed into

\[
\begin{cases}
\varepsilon^2 u_{yy} + \varepsilon^2 \gamma(y) u_y + f(u) = 0, & 0 < y < L, \\
u_y(0) = u_y(L) = 0,
\end{cases}
\]

(SP2')

where

\[
\gamma(y) := \frac{\varphi_x(x)}{h(x)^2} \quad \text{and} \quad L := \int_0^1 \frac{h(s)}{d(s)} ds.
\]

From (Φ1)-(Φ3), γ satisfies the following conditions:

(Γ1) \( \gamma(0) = \gamma(L) = 0. \)

(Γ2) Let \( \hat{\Sigma} := \{ y \in [0, L] ; \gamma(y) = 0 \} \). Then \( \hat{\Sigma} \) is a non-empty finite set and \( \gamma_y(y) \neq 0 \) at any \( y \in \hat{\Sigma} \).

Moreover, we set

\[
\gamma^* := \max_{y \in [0,L]} |\gamma(y)|
\]

and introduce the following notation:

\[
\hat{\Sigma}^+ := \{ y \in \hat{\Sigma} ; \gamma_y(y) > 0 \} \quad \text{and} \quad \hat{\Sigma}^- := \{ y \in \hat{\Sigma} ; \gamma_y(y) < 0 \}.
\]

Since \( d(x) > 0 \) and \( h(x) > 0 \) in \([0,1]\), every solution of (SP2) has a one-to-one correspondence to that of (SP2'). In particular, every \( n \)-mode solution of (SP2) corresponds to a solution of (SP2') which possesses exactly \( n \) intersecting points with \( u = 1/2 \) in \((0, L)\).

We now define an \( n \)-mode solution of (SP2').
Definition 3.1.2. For any fixed integer $n \geq 1$, a solution $\tilde{u}^\varepsilon$ of (SP2$'$) is called an $n$-mode solution, if $\tilde{u}^\varepsilon - 1/2$ has exactly $n$ zero points in the interval $(0, L)$.

We will study $n$-mode solutions of (SP2$'$) as substitute for that of (SP2). Hereafter we will denote the set of all $n$-mode solutions of (SP2$'$) by $\tilde{S}_{n,\varepsilon}$. Moreover, for $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$, we put

$$\tilde{\Xi} := \{ y \in [0, L] ; \tilde{u}^\varepsilon(y) = 1/2 \}.$$  

Since $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$ satisfies $\tilde{u}^\varepsilon_y(0) = \tilde{u}^\varepsilon_y(L) = 0$, one can extend $\tilde{u}^\varepsilon$ to a function over $\mathbb{R}$ by the standard reflection. Therefore, if necessary, we may regard $\tilde{u}^\varepsilon$ as a function in $\mathbb{R}$ satisfying (SP2$'$) for all $y \in \mathbb{R}$.

The content of this chapter is as follows. In Section 3.2, we will study some basic properties of $n$-mode solutions of (SP2$'$). Section 3.3 is devoted to the study of some asymptotic rates of $n$-mode solutions of (SP2$'$) as $\varepsilon \to 0$. In section 3.4, we will show some lemmas which play major roles in the study of location of transition layers. In Section 3.5, we will mainly discuss the location of transition layers for $n$-mode solutions of (SP2$'$). Moreover, we will study multi-layers; their location and profiles. Finally, at the end of this section, taking notice of the relation of (SP2) and (SP2$'$), we will show Theorems 3.1.1 and 3.1.2.

3.2 Transition Layers of $n$-mode Solutions

In this section, we will show some properties of $n$-mode solutions.

Lemma 3.2.1. Let $\tilde{u}^\varepsilon$ be a solution of (SP2$'$). Then

$$0 \leq \tilde{u}^\varepsilon(y) \leq 1 \quad \text{in} \ [0, L]. \quad (3.2.1)$$

Furthermore, it holds that

$$0 < \tilde{u}^\varepsilon(y) < 1 \quad \text{in} \ [0, L] \quad (3.2.2)$$

unless $\tilde{u}^\varepsilon \equiv 0$ or 1.

Proof. We first show (3.2.1). Suppose for contradiction that there exists a point $y_0 \in [0, L]$ such that

$$\tilde{u}^\varepsilon(y_0) = \max_{y \in [0, L]} \tilde{u}^\varepsilon(y) > 1.$$
Then taking notice of the boundary condition, we see $\hat{u}_y^\varepsilon(y_0) \leq 0$ and $\hat{u}_y^\varepsilon(y_0) = 0$. Therefore, it holds that

$$0 < -f(\hat{u}^\varepsilon(y_0)) = \varepsilon^2 \hat{u}_{yy}^\varepsilon(y_0) + \varepsilon^2 \gamma(y_0)\hat{u}_y^\varepsilon(y_0) \leq 0,$$

which is a contradiction. Hence we obtain $\hat{u}^\varepsilon(y) \leq 1$ in $[0, L]$. Similarly, we can show $\hat{u}^\varepsilon(y) \geq 0$ in $[0, L]$.

We next give a proof of (3.2.2). Assume that there exists a point $y_1 \in [0, L]$ satisfying

$$\hat{u}^\varepsilon(y_1) = \max_{y \in [0,L]} \hat{u}^\varepsilon(y) = 1.$$

Since $\hat{u}^\varepsilon(y_1) = 0$, we immediately get $u^\varepsilon \equiv 1$ by the uniqueness of solutions for initial value problems of second order ordinary differential equations. Therefore, $u^\varepsilon(y) < 1$ in $[0, L]$ unless $u^\varepsilon \equiv 1$. Similarly, one can see that, if $u \neq 0$, then $\hat{u}^\varepsilon(y) > 0$ in $[0, L]$. Thus the proof is complete. \hfill \Box

**Remark 3.2.2.** If $\hat{u}^\varepsilon \in \hat{S}_{n,\varepsilon}$, then $\hat{u}^\varepsilon \neq 0$ and 1. Therefore (3.2.2) is valid for any $\hat{u}^\varepsilon \in \hat{S}_{n,\varepsilon}$.

**Lemma 3.2.3.** For $\hat{u}^\varepsilon \in \hat{S}_{n,\varepsilon}$, let $\tilde{\xi} = \{\xi_k\}_{k=1}^n$ with $\xi_1 < \xi_2 < \cdots < \xi_n$. Then there exist a unique set of critical points $\{\xi_k\}_{k=0}^n$ of $\hat{u}^\varepsilon$ such that

$$0 = \tilde{\xi}_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1} < \xi_n < \tilde{\xi}_n = L$$

and $\hat{u}^\varepsilon$ takes either its local maximum or minimum at $\tilde{\xi}_k$ for $k = 0, 1, \ldots, n$.

**Proof.** We first prove $u_y^\varepsilon(\tilde{\xi}_k) \neq 0$ for $k = 1, 2, \ldots, n$. If $u_y^\varepsilon(\tilde{\xi}_k) = 0$ for some $k$, then $\hat{u}^\varepsilon$ satisfies

$$\begin{cases}
\varepsilon^2 \hat{u}_{yy} + \varepsilon^2 \gamma(y)\hat{u}_y + f(\hat{u}^\varepsilon) = 0 & \text{in } [0, L], \\
\hat{u}_y(\tilde{\xi}_k) = 1/2, \quad \hat{u}_y(\tilde{\xi}_k) = 0.
\end{cases}$$

By the uniqueness of solutions for initial value problems of second order ordinary differential equations, it is easy to see $\hat{u}^\varepsilon \equiv 1/2$. However, this fact contradicts $\hat{u}^\varepsilon \in \hat{S}_{n,\varepsilon}$.

We next prove that there exists a unique critical point in $(\tilde{\xi}_k, \tilde{\xi}_{k+1})$ for $k = 1, 2, \ldots, n - 1$. We only consider the case that $\hat{u}^\varepsilon(y) > 1/2$ in $(\tilde{\xi}_k, \tilde{\xi}_{k+1})$. Let $\tilde{\zeta} \in (\tilde{\xi}_k, \tilde{\xi}_{k+1})$ satisfy $u_y^\varepsilon(\tilde{\zeta}) = 0$. The existence of such a critical point is assured by Rolle’s theorem. It follows from $\hat{u}^\varepsilon(\tilde{\zeta}) > 1/2$ that

$$0 = \varepsilon^2 u_{yy}^\varepsilon(\tilde{\zeta}) + \varepsilon^2 \gamma(\tilde{\zeta})u_y^\varepsilon(\tilde{\zeta}) + f(\hat{u}^\varepsilon(\tilde{\zeta})) = \varepsilon^2 u_{yy}^\varepsilon(\tilde{\zeta}) + f(\hat{u}^\varepsilon(\tilde{\zeta})) > \varepsilon^2 \hat{u}_{yy}^\varepsilon(\tilde{\zeta}).$$
This implies that \( \tilde{\zeta} \) is a unique critical point in \((\tilde{\xi}_k, \tilde{\xi}_{k+1})\) and that \( \tilde{u}^\varepsilon \) takes its local maximum at \( y = \tilde{\zeta} \). Similarly, one can show that, if \( \tilde{u}^\varepsilon(y) < 1/2 \) in \((\tilde{\xi}_k, \tilde{\xi}_{k+1})\) and \( u_\varepsilon(y) = 0 \) with some \( \tilde{\zeta} \in (\tilde{\xi}_k, \tilde{\xi}_{k+1}) \), then \( \tilde{\zeta} \) is a unique critical point in \((\tilde{\xi}_k, \tilde{\xi}_{k+1})\) at which \( \tilde{u}^\varepsilon \) takes its local minimum.

Finally, we will show that there exists no critical point in \((0, \tilde{\xi}_1)\) and \((\tilde{\xi}_n, L)\). Consider the standard reflection at the boundary points \( y = 0 \) and \( y = L \). Using the same argument as in the previous paragraph, one can easily see that \( y = 0 \) is a unique critical point in \( (\tilde{\xi}_1, \tilde{\xi}_1) \). We can also show that \( y = L \) is a unique critical point in \( (\tilde{\xi}_n, 2L - \tilde{\xi}_n) \). Thus the proof is complete.

**Lemma 3.2.4.** For \( \tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon} \), if \( \varepsilon \) is sufficiently small, then there exists a positive constant \( \varepsilon^* \) satisfying

\[
\max_{y \in [0,L]} |\tilde{u}_y^\varepsilon(y)| \leq \frac{\varepsilon^*}{\varepsilon}.
\]

(3.2.3)

**Proof.** We will use the contradiction method. For this purpose, define a function \( \phi = \phi(\varepsilon) \) satisfying \( \phi(\varepsilon) = o(\varepsilon) \) as \( \varepsilon \to 0 \) and suppose

\[
|\tilde{u}_y^\varepsilon(y^*)| = \max_{y \in [0,L]} |\tilde{u}_y^\varepsilon(y)| = \frac{1}{\phi(\varepsilon)}
\]

(3.2.4)

with some \( y^* \in [0,L] \).

We only consider the case that \( \tilde{u}_y^\varepsilon(y^*) > 0 \). If we define \( U^\varepsilon \) by \( U^\varepsilon(z) = \tilde{u}^\varepsilon(y^* + \phi(\varepsilon)z) \), then \( U^\varepsilon \) satisfies

\[
\begin{cases}
\frac{\varepsilon^2}{\phi(\varepsilon)^2} U_{zz}^\varepsilon + \frac{\varepsilon^2}{\phi(\varepsilon)} \gamma(y^* + \phi(\varepsilon)z) U_z^\varepsilon + f(U^\varepsilon) = 0 & \text{in } \mathbb{R} \\
U_z^\varepsilon(0) = 1.
\end{cases}
\]

From Lemma 3.2.1, it is obvious that \( 0 < U^\varepsilon(z) < 1 \) in \( \mathbb{R} \). Furthermore, it follows from (3.2.4) that \( -1 \leq U_z^\varepsilon(z) \leq 1 \) in \( \mathbb{R} \). Hence both \( \{U^\varepsilon\} \) and \( \{U_z^\varepsilon\} \) are uniformly bounded in \( \mathbb{R} \). We also see that \( \{U_z^\varepsilon\} \) is uniformly bounded in \( \mathbb{R} \) because

\[
U_{zz}^\varepsilon = -\phi(\varepsilon)\gamma(y^* + \phi(\varepsilon)z) U_z^\varepsilon - \frac{\phi(\varepsilon)^2}{\varepsilon^2} f(U^\varepsilon) \quad \text{in } \mathbb{R}.
\]

Therefore we also obtain that \( \{U^\varepsilon\} \), \( \{U_z^\varepsilon\} \) and \( \{U_{zz}^\varepsilon\} \) are equi-continuous.

On account of the above results, using Ascoli-Arzelà’s theorem and a usual diagonal argument, one can see that \( \{U^\varepsilon\} \) has a subsequence \( \{U^{\varepsilon'}\} \) such that

\[
\lim_{\varepsilon' \to 0} U^{\varepsilon'} = U \quad \text{in } C^2_{\text{loc}}(\mathbb{R}),
\]
where $U$ is a $C^2(\mathbb{R})$-function satisfying $U_{zz} \equiv 0$ in $\mathbb{R}$. Moreover, $U$ satisfies

$$0 \leq U(z) \leq 1, \quad -1 \leq U_z(z) \leq 1, \quad U_{zz}(z) \equiv 0 \quad \text{in} \ \mathbb{R} \quad \text{and} \quad U_z(0) = 1.$$ 

Therefore, one can obtain that $U(z) = z + C$ with some $C \in \mathbb{R}$, so that $U$ is unbounded. However, this fact contradicts the boundedness of $U$. Thus we can prove (3.2.3).

\[ \tag{3.2.3} \]

**Lemma 3.2.5.** For $n \in \mathbb{N}$, it holds that

$$\lim_{\varepsilon \to 0} \sup_{u \in \tilde{S}_n, y \in [0, L]} \max \left| \tilde{u}^\varepsilon(y)(1 - \tilde{u}^\varepsilon(y)) \left[ \frac{1}{2} \varepsilon^2 (\tilde{u}^\varepsilon_y(y))^2 - W(\tilde{u}^\varepsilon(y)) \right] \right| = 0, \quad \text{(3.2.5)}$$

where $W$ is a function defined in (3.1.1).

**Proof.** Suppose that (3.2.5) is not true; then there exist a set $\{(\varepsilon_k, u^k, y_k)\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, $u^k \in \tilde{S}_n, \varepsilon_k$, $y_k \in [0, L]$ and

$$\left| u^k(y_k)(1 - u^k(y_k)) \left[ \frac{1}{2} \varepsilon_k^2 (u^k_y(y_k))^2 - W(u^k(y_k)) \right] \right| \geq \delta \quad \text{(3.2.6)}$$

with some $\delta > 0$.

We use the change of variables $y = y_k + \varepsilon_k z$ and put $U^k(z) = u^k(y_k + \varepsilon_k z)$. Then $U^k$ satisfies

$$U_{zz}^k + \varepsilon_k \gamma(y_k + \varepsilon_k z) U_z^k + f(U^k) = 0 \quad \text{in} \ \mathbb{R}. \quad \tag{3.2.7}$$

In view of (3.2.3), the same argument as in the proof of Lemma 3.2.4 works well, so that we can choose a subsequence $\{U^{k'}\}$ of $\{U^k\}$ satisfying

$$\lim_{k' \to \infty} U^{k'} = U \quad \text{in} \ C^2_{loc}(\mathbb{R})$$

with some $U \in C^2(\mathbb{R})$. Furthermore, one can easily see that $U$ satisfies

$$U_{zz} + f(U) = 0 \quad \text{in} \ \mathbb{R}. \quad \tag{3.2.7}$$

Multiplying (3.2.7) by $U_z$ and integrating the resulting expression with respect to $z$, we have

$$\frac{1}{2} (U_z(z))^2 - W(U(z)) = C \quad \tag{3.2.8}$$

with some constant $C$.

If $U \equiv 0$ or 1, then we can easily find a contradiction. We will show $C = 0$ when $U \not\equiv 0$ and 1. If $C > 0$, then the phase plain analysis tells us that $U$
is unbounded. This is impossible because \( \{U^{k'}\} \) is bounded in \( \mathbb{R} \). If \( C < 0 \), it follows from the phase plain analysis that \( U \) is a periodic function in \( \mathbb{R} \), so that the graph of \( U \) has infinitely many intersecting points with \( u = 1/2 \). Therefore, the graph of \( U^{k'} \) also has infinitely many intersecting points with \( u = 1/2 \) when \( k' \) is sufficiently large. This fact contradicts the definition of \( n \)-mode solutions. Thus we can conclude \( C = 0 \) in (3.2.8).

Therefore,
\[
\lim_{k' \to \infty} \left[ \frac{1}{2} \varepsilon^2 (u_y^{k'}(y_{k'}))^2 - W(u^{k'}(y_{k'})) \right] = \lim_{k' \to \infty} \left[ \frac{1}{2} (U_x^{k'}(0))^2 - W(U^{k'}(0)) \right] = \frac{1}{2} (U_z(0))^2 - W(U(0)) = 0,
\]
which contradicts (3.2.6). This completes the proof.

\[\square\]

**Remark 3.2.6.** For \( \tilde{u}^\varepsilon \in S_{n,\varepsilon} \), let \( Y^\delta := \{ y \in [0, L] ; \delta < \tilde{u}^\varepsilon(y) < 1 - \delta \} \) with some small positive constant \( \delta \). Then \( C\delta^2 < W(\tilde{u}^\varepsilon(y)) \leq 1/64 \) for \( y \in Y^\delta \) with some positive constant \( C \). Moreover, Lemma 3.2.5 implies that, for any \( \eta > 0 \), if \( \varepsilon \) is sufficiently small, then
\[
\left| \frac{1}{2} \varepsilon^2 (\tilde{u}_y^\varepsilon(y))^2 - W(\tilde{u}^\varepsilon(y)) \right| < \eta \quad \text{for} \ y \in Y^\delta.
\]
Therefore, it holds that
\[
\frac{C_1}{\varepsilon} \leq |\tilde{u}_y^\varepsilon(y)| \leq \frac{C_2}{\varepsilon} \quad \text{in} \ Y^\delta \tag{3.2.9}
\]
with some positive constants \( C_1 \) and \( C_2 \). Furthermore, taking notice of \( \tilde{\Xi} \subset Y^\delta \) and Lemma 3.2.4, we can see that (3.2.9) is valid for all \( y \) lying in an \( O(\varepsilon) \)-neighborhood of a point in \( \tilde{\Xi} \). In other words, \( \tilde{u}^\varepsilon \) forms a transition layer near a point in \( \tilde{\Xi} \).

Using a similar argument as in the proof of Lemma 3.2.5, we have the following lemma which gives information on the profile of a transition layer:

**Lemma 3.2.7.** For \( \tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon} \), let \( \tilde{\xi}^\varepsilon \) be a point in \( \tilde{\Xi} \) and define \( U^\varepsilon(z) = \tilde{u}^\varepsilon(\tilde{\xi}^\varepsilon + \varepsilon z) \). Then it holds that
\[
\lim_{\varepsilon \to 0} U^\varepsilon = U \quad \text{in} \ C_{loc}^2(\mathbb{R}),
\]

where \( U \in C^2(\mathbb{R}) \) is the unique solution of
\[
\begin{cases}
U_{zz} + f(U) = 0 & \text{in } \mathbb{R}, \\
U(-\infty) = 0, U(\infty) = 1 \quad \text{(resp. } U(-\infty) = 1, U(\infty) = 0) \\
U(0) = 1/2 & \text{in } \mathbb{R},
\end{cases}
\]
if \( U_z(0) > 0 \) (resp. \( U_z(0) < 0 \)).

**Proof.** It is easy to see that \( U^\varepsilon \) satisfies
\[
U_{zz}^\varepsilon + \varepsilon \gamma(\tilde{\xi}^\varepsilon + \varepsilon z)U_z^\varepsilon + f(U^\varepsilon) = 0 \quad \text{and } U^\varepsilon(0) = \frac{1}{2}.
\]
As in the proof of Lemma 3.2.5, there exists a subsequence \( \{\varepsilon_k\} \downarrow 0 \) such that \( \{U^\varepsilon_k\} \) converges to some \( U \in C^2(\mathbb{R}) \) in \( C^2_{\text{loc}}(\mathbb{R}) \) satisfying
\[
U_{zz} + f(U) = 0 \quad \text{and } U(0) = \frac{1}{2}.
\]
Furthermore, the same argument as in the proof of (3.2.8) with \( C = 0 \) enables us to obtain
\[
\frac{1}{2}U_z^2 - W(U) = 0 \quad \text{in } \mathbb{R}.
\]
Therefore, it follows from the phase plain analysis that \( U \) satisfies (3.2.10). Furthermore, taking account of the uniqueness of the solution for (3.2.10), we can see that the above argument does not depend on the choice of a subsequence \( \{\varepsilon_k\} \). Thus the proof is complete.

At the end of this section, putting
\[
J_\varepsilon(u; s, t) := \varepsilon^2 \int_s^t \gamma(y)(u_y(y))^2 dy,
\]
we will show the following lemma:

**Lemma 3.2.8.** For \( \tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon} \) and \( \tilde{\xi} \in \tilde{\Xi} \), put \( \tilde{\zeta}_0 := \sup\{y; y < \tilde{\xi} \text{ and } \tilde{u}^\varepsilon_y(y) = 0\} \) and \( \tilde{\zeta}_1 := \inf\{y; y > \tilde{\xi} \text{ and } \tilde{u}^\varepsilon_y(y) = 0\} \). If \( \tilde{u}^\varepsilon_y(\tilde{\xi}) > 0 \) (resp. \( \tilde{u}^\varepsilon_y(\tilde{\xi}) < 0 \)), then it holds that
\[
\begin{align*}
|J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\zeta}_0, y)| & \leq \varepsilon C^* \gamma^*(\tilde{u}^\varepsilon(y) - u^\varepsilon(\tilde{\zeta}_0)) \quad \text{in } [\tilde{\zeta}_0, \tilde{\xi}], \\
|J_\varepsilon(\tilde{u}^\varepsilon; y, \tilde{\zeta}_1)| & \leq \varepsilon C^* \gamma^*(\tilde{u}^\varepsilon(\tilde{\zeta}_1) - \tilde{u}^\varepsilon(y)) \quad \text{in } [\tilde{\xi}, \tilde{\zeta}_1].
\end{align*}
\]
Here, $\gamma^*$ and $C^*$ are positive constants defined in (3.1.3) and (3.2.3), respectively.

**Proof.** We will give a proof of the first inequality for the case of $\tilde{u}^\varepsilon(\tilde{z}) > 0$. Remark that $\tilde{u}^\varepsilon(y) > 0$ for $y \in (\tilde{z}_0, \tilde{z}_1)$. Then, Lemma 3.2.4 implies that

$$|J_\varepsilon(\tilde{u}^\varepsilon; \tilde{z}_0, y)| \leq \varepsilon C^* \gamma^* \int_{\tilde{z}_0}^y \tilde{u}^\varepsilon(z) dz = \varepsilon C^* \gamma^* (\tilde{u}^\varepsilon(y) - u^\varepsilon(\tilde{z}_0))$$

for any $y \in [\tilde{z}_0, \tilde{z}]$. Similarly, one can show that the other inequalities hold true, so that we omit their proofs. \hfill \Box

### 3.3 Asymptotic Profiles of $n$-mode Solutions

In this section, we will derive some estimates for $\tilde{u}^\varepsilon(y)$ and $1 - \tilde{u}^\varepsilon(y)$ as $\varepsilon \to 0$. They will be given in a certain interval which contains a local minimum or maximum point of $\tilde{u}^\varepsilon$.

**Lemma 3.3.1.** Let $g(v; a_0) = v(1-v)(v-a_0)$ with $a_0 \in (0, 1)$. Then for any $\delta \in (0, 1)$ satisfying $\delta > \max\{a_0, (a_0 + 1)/3\}$ and $M > 0$, there exists a unique solution $v = v(z)$ of

$$\begin{align*}
v_{zz} + g(v; a_0) &= 0 \quad \text{in } (-M, 0), \\
v(-M) &= \delta, \quad v_z(0) = 0, \\
v > \delta, v_z > 0 & \quad \text{in } (-M, 0).
\end{align*} \tag{3.3.1}
$$

Moreover, there exists a constant $\delta_0 \in ((a_0 + 1 + \sqrt{a_0^2 - a_0 + 1})/3, 1)$ such that, if $\delta > \delta_0$, then

$$c_1 \exp(-RM) < 1 - v(0) < c_2 \exp(-rM), \tag{3.3.2}
$$

where $r = \sqrt{-g_v(\delta; a_0)}$, $R = \sqrt{-g_v(1; a_0)}$ and $c_1, c_2 (0 < c_1 < c_2)$ are positive constants depending only on $\delta$.

Lemma 3.3.1 is the same lemma as Lemma 1.3.1. Thus we omit the proof.
Lemma 3.3.2. Let $\rho \in \mathbb{R}$ and $b_0 \in (0, 1)$. Then for any $\delta \in (0, 1)$ with $\delta > \max\{b_0, (b_0 + 1)/3\}$ and $M > 0$, there exists a solution $w = w(z)$ of
\[
\begin{cases}
w_{zz} + \rho w_z + g(w; b_0) = 0 & \text{in } (M, 0), \\
w(-M) = \delta, w_z(0) = 0, & \text{in } (M, 0),
\end{cases}
\]
(3.3.3)
provided that $\rho$ satisfies $\rho > -\rho_0$ with some positive constant $\rho_0$. Here, $g$ is a function defined in Lemma 3.3.1. Furthermore, if $\delta$ is sufficiently close to 1, then $w$ satisfies
\[1 - w(0) < C \exp(-rM)\]
with some positive constants $C$ and $r$.

Proof. We will construct a solution of (3.3.3) by using sub and supersolution method. Set
\[\mathcal{L}(w) = w_{zz} + \rho w_z + g(w; b_0).\]
Putting $\varpi \equiv 1$, we can easily see that $\varpi$ is a supersolution of (3.3.3) because $\mathcal{L}(\varpi) = 0$, $\varpi(-M) = 1 > \delta$ in $(-M, 0)$ and $\varpi_z(0) = 0$.

We will construct a subsolution of (3.3.3). Set $\underline{w} = v$ where $v$ is a solution of (3.3.1) with $a_0 > b_0$. Then, it holds that
\[
\mathcal{L}(\underline{w}) = v_{zz} + \rho v_z + g(v; b_0) \\
= v_{zz} + g(v; a_0) + \rho v_z + v(1 - v)(a_0 - b_0) \\
= \rho v_z + v(1 - v)(a_0 - b_0) \quad \text{in } (-M, 0).
\]
Recall that $v$ satisfies (3.3.2) if $\delta$ is sufficiently close to 1. Taking notice that $v_z(z) > 0$ in $(-M, 0)$, one can see that
\[\frac{v(z)(1 - v(z))}{v_z(z)} > 0 \quad \text{in } (-M, 0).\]
In this case, let $\rho_0$ be a positive constant satisfying
\[\rho_0 < \inf_{z \in (-M, 0)} \left( \frac{v(z)(1 - v(z))}{v_z(z)} \right) \cdot (a_0 - b_0).\]
If $\rho > -\rho_0$, then $\mathcal{L}(\underline{w}) > 0$ in $(-M, 0)$. Furthermore, $\underline{w}(-M) = \delta$ and $\underline{w}_z(0) = 0$. Hence $\underline{w}$ is a subsolution of (3.3.3).
It is obvious that \( w < \bar{w} \) in \((-M, 0)\), so that there exists a solution \( w \) of (3.3.3) such that
\[
v < w < 1 \quad \text{in } (-M, 0).
\]
Therefore, it holds that
\[
1 - w(0) \leq 1 - v(0) < C \exp(-rM)
\]
with some positive constants \( C \) and \( r \) when \( p \) is sufficiently close to 1. Thus the proof is complete. \( \square \)

We should note that the following lemma also holds true similarly to Lemma 3.3.2:

**Lemma 3.3.3.** Let \( \rho \in \mathbb{R} \) and \( b \in (0, 1) \). Then for any \( \delta \in (0, 1) \) with \( \delta > \max\{b_0, (b_0 + 1)/3\} \) and \( M > 0 \), there exists a solution \( w = w(z) \) of
\[
\begin{cases}
w_{zz} + \rho w_z + g(w; b_0) = 0 & \text{in } (0, M), \\
w(M) = \delta, w_z(0) = 0, \\
w > \delta & \text{in } (0, M),
\end{cases}
\]
provided that \( \rho \) satisfies \( \rho < \rho_0 \) with some positive constant \( \rho_0 \). Here \( g \) is a function defined in Lemma 3.3.1. Furthermore, if \( \delta \) is sufficiently close to 1, then \( w \) satisfies
\[
1 - w(0) < C \exp(-rM)
\]
with some positive constants \( C \) and \( r \).

By using Lemmas 3.3.2 and 3.3.3, we can obtain the following two theorems:

**Theorem 3.3.4.** For \( \tilde{u} \in \tilde{S}_{n, \varepsilon} \), let \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) be successive points in \( \tilde{\Xi} \) satisfying \( \tilde{u}(y) > 1/2 \) in \((\tilde{\xi}_1, \tilde{\xi}_2)\) and let \( \tilde{\zeta} \) be a unique critical point in \((\tilde{\xi}_1, \tilde{\xi}_2)\) where \( \tilde{u} \) takes its local maximum. Then it holds that
\[
1 - \tilde{u}(y) < C \exp \left( -\frac{r l(y)}{\varepsilon} \right) \quad \text{in } [\tilde{\xi}_1, \tilde{\xi}_2],
\]
with some positive constants \( C \) and \( r \). Here
\[
l(y) := \begin{cases}
y - \tilde{\xi}_1 & \text{in } [\tilde{\xi}_1, \tilde{\zeta}] \\
\tilde{\zeta} - y & \text{in } [\tilde{\zeta}, \tilde{\xi}_2].
\end{cases}
\]
Proof. We will only give a proof of (3.3.4) for \( y \in [\tilde{\xi}_1, \tilde{\xi}] \). It will be found that the other case can be treated in a similar way.

For \( b^* \in (1/2, 1) \), let \( \delta^* \in (0, 1) \) satisfy \( \delta^* > b^* (= \max\{b^*, (b^* + 1)/3\}) \) and define

\[
\tilde{\xi}_1^* := \inf\{ y \in [\tilde{\xi}_1, \tilde{\xi}] ; \tilde{u}^\varepsilon(y) = \delta^* \}.
\]

It follows from (3.2.9) that \( \tilde{\xi}_1^* - \tilde{\xi}_1 < K \varepsilon \) with some \( K > 0 \) when \( \varepsilon \) is sufficiently small.

We now take any \( y^* \in (\tilde{\xi}_1^*, \tilde{\xi}) \) and let \( w = w(z) \) be a solution of (3.3.3) with \( \delta = \delta^* \), \( \rho = -\varepsilon \gamma^* \), \( b = b^* \) and \( M = (y^* - \tilde{\xi}_1^*)/\varepsilon \), where \( \gamma^* \) is defined in (3.1.3).

We use the change of variables \( z = (y - y^*)/\varepsilon \) and define

\[
w^\varepsilon(y) := w \left( \frac{y - y^*}{\varepsilon} \right).
\]

Clearly, \( w^\varepsilon \) satisfies

\[
\begin{aligned}
\varepsilon^2 w^\varepsilon_{yy} - \varepsilon^2 \gamma^* w^\varepsilon_y + w^\varepsilon(1 - w^\varepsilon)(w^\varepsilon - b^*) &= 0 \quad \text{in } (\tilde{\xi}_1^*, y^*), \\
 w^\varepsilon(\tilde{\xi}_1^*) &= \delta^* \quad \text{in } (\tilde{\xi}_1^*, y^*), \\
 w^\varepsilon(y) &> \delta^* \quad \text{in } (\tilde{\xi}_1^*, y^*),
\end{aligned}
\]

and

\[
1 - w^\varepsilon(y^*) < C \exp \left( -\frac{r(y^* - \tilde{\xi}_1^*)}{\varepsilon} \right) < Ce^{-K} \exp \left( -\frac{r(y^* - \tilde{\xi}_1^*)}{\varepsilon} \right)
\]

with some \( C > 0 \) and \( r > 0 \).

We will show

\[
w^\varepsilon(y) \leq \tilde{u}^\varepsilon(y) \quad \text{in } [\tilde{\xi}_1^*, y^*].
\]

For this purpose, we will introduce an auxiliary function

\[
h(y) := \frac{w^\varepsilon(y) - b^*}{\tilde{u}^\varepsilon(y) - b^*} \quad \text{in } [\tilde{\xi}_1^*, y^*],
\]

and prove \( h(y) \leq 1 \) in \([\tilde{\xi}_1^*, y^*]\). Suppose for contradiction that there exist a point \( y_1 \in [\tilde{\xi}_1^*, y^*] \) and a constant \( \eta \in (0, 1) \) satisfying

\[
\max_{y \in [\tilde{\xi}_1^*, y^*]} h(y) = h(y_1) = \frac{1}{\eta}.
\]
Then it holds that

$$\begin{cases}
    w^{\varepsilon,\eta}(y) \leq \tilde{u}^\varepsilon(y) & \text{in } [\tilde{\xi}_1^*, y^*], \\
    w^{\varepsilon,\eta}(y_1) = \tilde{u}^\varepsilon(y_1),
\end{cases} \tag{3.3.8}$$

where $w^{\varepsilon,\eta}(y) := \eta(w^\varepsilon - b^*) + b^*$. We should note that $y_1$ is an interior point in $[\tilde{\xi}_1^*, y^*]$ because $h(\tilde{\xi}_1^*) < 1$ and $h_\gamma(y^*) < 0$, so that we have

$$h_\gamma(y_1) = 0 \quad \text{and} \quad h_{y\gamma}(y_1) \leq 0. \tag{3.3.9}$$

By the definition of $h$, it is easy to show that

$$h_\gamma(y_1)(\tilde{u}^\varepsilon(y_1) - b^*) + h(y_1)\tilde{u}_y^\varepsilon(y_1) = w_y^\varepsilon(y_1).$$

Therefore, it follows from (3.3.7) and (3.3.9) that

$$\tilde{u}_y^\varepsilon(y_1) = \eta w_y^\varepsilon(y_1). \tag{3.3.10}$$

One can also easily see that

$$h_{y\gamma}(y_1)(\tilde{u}^\varepsilon(y_1) - b) + 2h_\gamma(y_1)\tilde{u}_y^\varepsilon(y_1) + h(y_1)\tilde{u}_{yy}^\varepsilon(y_1) = w_{yy}^\varepsilon(y_1).$$

This fact together with (3.3.7) and (3.3.9) implies that

$$w^{\varepsilon,\eta}(y_1) = \eta w_{yy}^\varepsilon(y_1) \leq \tilde{u}_{yy}^\varepsilon(y_1). \tag{3.3.11}$$

By direct calculation, we can estimate $f(w^{\varepsilon,\eta})$ from below such as

$$f(w^{\varepsilon,\eta}) = w^{\varepsilon,\eta}(1 - w^{\varepsilon,\eta})(w^{\varepsilon,\eta} - 1/2)$$
$$= w^{\varepsilon,\eta}(1 - w^{\varepsilon,\eta}\{\eta(w^\varepsilon - b^*) + (b^* - 1/2)\})$$
$$> \eta w^{\varepsilon,\eta}(w^{\varepsilon,\eta} - b^*) > \eta w^{\varepsilon}(1 - w^{\varepsilon})(w^{\varepsilon} - b^*). \tag{3.3.12}$$

Using (3.3.8), (3.3.10), (3.3.11) and (3.3.12), one can obtain that

$$0 = \varepsilon^2 \tilde{u}_{yy}^\varepsilon(y_1) + \varepsilon^2 \gamma(y_1)\tilde{u}_y^\varepsilon(y_1) + f(\tilde{u}^\varepsilon(y_1))$$
$$= \varepsilon^2 \tilde{u}_{yy}^\varepsilon(y_1) + \varepsilon^2 \gamma(y_1)\tilde{u}_y^\varepsilon(y_1) + f(w^{\varepsilon,\eta}(y_1))$$
$$\geq \varepsilon^2 \eta w_{yy}^\varepsilon(y_1) + \varepsilon^2 \gamma(y_1)\tilde{u}_y^\varepsilon(y_1) + f(w^{\varepsilon,\eta}(y_1))$$
$$> \varepsilon^2 \eta w_{yy}^\varepsilon(y_1) + \varepsilon^2 \gamma(y_1)\tilde{u}_y^\varepsilon(y_1) + \eta w^{\varepsilon}(1 - w^{\varepsilon}(y_1))(w^{\varepsilon}(y_1) - b^*)$$
$$= \varepsilon^2 \gamma(y_1)\tilde{u}_y^\varepsilon(y_1) + \eta e^2 w_{yy}^\varepsilon(y_1) + w^{\varepsilon}(y_1)(1 - w^{\varepsilon}(y_1))(w^{\varepsilon}(y_1) - b^*)$$
$$= \varepsilon^2 \gamma(y_1)\tilde{u}_y^\varepsilon(y_1) + \eta e^2 w_{yy}^\varepsilon(y_1) + e^2 \gamma y^* \tilde{u}_y^\varepsilon(y_1)$$
$$= \varepsilon^2 \tilde{u}_y^\varepsilon(y_1)(\gamma(y_1) + \gamma^*) \geq 0.$$
which is a contradiction. Thus we obtain (3.3.6).

Hence, (3.3.5) and (3.3.6) enable us to see that

\[ 1 - \tilde{u}^\varepsilon(y^*) \leq 1 - w^\varepsilon(y^*) < Ce^{rK} \exp \left( - \frac{r(y^* - \tilde{\xi}_1)}{\varepsilon} \right). \]

Recalling that \(y^*\) is an arbitrary point in \((\tilde{\xi}_1^*, \tilde{\zeta})\), we can conclude that (3.3.4) is valid for \(y \in (\tilde{\xi}_1^*, \tilde{\zeta})\) because \(C, r\) and \(K\) are independent of \(y^*\).

Finally, we will show that (3.3.4) also holds true for \(y \in [\tilde{\xi}_1, \tilde{\xi}_1^*]\). For \(y \in [\tilde{\xi}_1, \tilde{\xi}_1^*]\), it holds that

\[ e^{-rK} \leq \exp \left( - \frac{r(y - \tilde{\xi}_1)}{\varepsilon} \right). \]

Therefore, there exists a positive constant \(C'\) such that

\[ 1 - \tilde{u}^\varepsilon(y) < \frac{1}{2} < C' e^{-rK} \leq C' \exp \left( - \frac{r(y - \tilde{\xi}_1)}{\varepsilon} \right) \]

for \(y \in [\tilde{\xi}_1, \tilde{\xi}_1^*]\). Thus (3.3.4) is also valid for \(y \in [\tilde{\xi}_1, \tilde{\xi}_1^*]\).

In this section, we have discussed the asymptotic profile of an \(n\)-mode solution \(\tilde{u}^\varepsilon\) when \(\tilde{u}^\varepsilon\) approaches 1 as \(\varepsilon \to 0\). At the end of this section, we will state the following theorem which describes the asymptotic profile of an \(n\)-mode solution when it approaches 0.

**Theorem 3.3.5.** For \(\tilde{u}^\varepsilon \in \tilde{S}_{n, \varepsilon}\), let \(\tilde{\xi}_1\) and \(\tilde{\xi}_2\) be successive points in \(\tilde{\Xi}\) satisfying \(\tilde{u}^\varepsilon(y) < 1/2\) in \((\tilde{\xi}_1, \tilde{\xi}_2]\) and \(\tilde{\zeta}\) be a unique critical point in \((\tilde{\xi}_1, \tilde{\xi}_2]\) where \(\tilde{u}^\varepsilon\) takes its local minimum. Then it holds that

\[ \tilde{u}^\varepsilon(y) < C \exp \left( - \frac{r l(y)}{\varepsilon} \right) \quad \text{in} \quad [\tilde{\xi}_1, \tilde{\xi}_2], \]

where \(l(y)\) is defined as in Theorem 3.3.4.

Theorem 3.3.5 can be shown in a similar way to Theorem 3.3.4; so we omit the proof.
### 3.4 Key Lemmas for the Analysis of the Location of Transition Layers

In this section, we will show the following two lemmas. It will turn out in Section 3.5 that they will play important role in the study of the location and multiplicity of transition layers.

**Key Lemma 3.4.1.** For \( \tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon} \) and \( \tilde{\xi} \in \tilde{\Xi} \), define \( \tilde{\zeta}_0 := \sup \{ y; y < \tilde{\xi} \text{ and } \tilde{u}_y^\varepsilon(y) = 0 \} \) and \( \tilde{\zeta}_1 := \inf \{ y; y > \tilde{\xi} \text{ and } \tilde{u}_y^\varepsilon(y) = 0 \} \). Moreover, let \( \tilde{\sigma}_1, \tilde{\sigma}_2 \in \tilde{\Sigma} \) satisfy \( \gamma(y) > 0 \) (resp. \( \gamma(y) < 0 \)) in \( (\tilde{\sigma}_1, \tilde{\sigma}_2) \). If

\[
\tilde{\xi} \in (\tilde{\sigma}_1 + \delta, \tilde{\sigma}_2 - \delta) \tag{3.4.1}
\]

with some positive constant \( \delta \), which is independent of \( \varepsilon \), then there exists a positive constant \( K \) satisfying

\[
\begin{align*}
1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1) &> K\sqrt{\varepsilon} \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) > 0, \\
\tilde{u}^\varepsilon(\tilde{\zeta}_1) &> K\sqrt{\varepsilon} \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) < 0.
\end{align*}
\tag{3.4.2}
\]

(resp.

\[
\begin{align*}
\tilde{u}^\varepsilon(\tilde{\zeta}_0) &> K\sqrt{\varepsilon} \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) > 0, \\
1 - \tilde{u}^\varepsilon(\tilde{\zeta}_0) &> K\sqrt{\varepsilon} \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) < 0.
\end{align*}
\]

Furthermore, it holds that

\[
\tilde{\zeta}_1 - \tilde{\xi} = O(\varepsilon |\log \varepsilon|) \quad \text{(resp. } \tilde{\xi} - \tilde{\zeta}_0 = O(\varepsilon |\log \varepsilon|)). \tag{3.4.3}
\]

**Proof.** We only consider the case that \( \gamma(y) > 0 \) in \( (\tilde{\sigma}_1, \tilde{\sigma}_2) \) and \( \tilde{u}_y^\varepsilon(\tilde{\xi}) > 0 \).

In this case, we first show \( \tilde{\zeta}_1 \leq \tilde{\sigma}_2 \) by using a contradiction method. For this purpose, we will suppose \( \tilde{\zeta}_1 > \tilde{\sigma}_2 \).

Rewrite (SP2') as

\[
- \varepsilon^2 \tilde{u}_{yy}^\varepsilon - f(\tilde{u}^\varepsilon) = \varepsilon^2 \gamma(y)\tilde{u}_y^\varepsilon. \tag{3.4.4}
\]

Multiplying (3.4.4) by \( \tilde{u}_y^\varepsilon \) and integrating the resulting expression over \( [\tilde{\zeta}_0, \tilde{\zeta}_1] \) with respect to \( y \), we obtain

\[
W(\tilde{u}^\varepsilon(\tilde{\zeta}_1)) - W(\tilde{u}^\varepsilon(\tilde{\zeta}_0)) = J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_1), \tag{3.4.5}
\]

where \( W \) and \( J_\varepsilon \) are defined by (3.1.1) and (3.2.11), respectively.
In the left-hand side of (3.4.5), Taylor’s expansion enables us to get

\[
W(\tilde{u}^\varepsilon(\tilde{\zeta}_1)) = W(1) + W_u(1)(\tilde{u}^\varepsilon(\tilde{\zeta}_1) - 1) + \frac{W_{uu}(1)}{2}(\tilde{u}^\varepsilon(\tilde{\zeta}_1) - 1)^2 + o\left((1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1))^2\right)
\]

\[
= \frac{1}{4}(1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1))^2 + o\left((1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1))^2\right)
\]

(3.4.6)

because \(\tilde{u}^\varepsilon(\tilde{\zeta}_1)\) is close to 1. Hence, there is a positive constant \(K_1\) satisfying

\[
W(\tilde{u}^\varepsilon(\tilde{\zeta}_1)) - W(\tilde{u}^\varepsilon(\tilde{\zeta}_0)) \leq W(\tilde{u}^\varepsilon(\tilde{\zeta}_1)) \leq K_1(1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1))^2.\]

(3.4.7)

For the right-hand side of (3.4.5), we will establish an estimate such as

\[
J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_1) > K_2\varepsilon
\]

(3.4.8)

with some \(K_2 > 0\). Transform the right-hand side of (3.4.5) into

\[
J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_1) = \begin{cases} J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}_1) + J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\sigma}_1, \tilde{\sigma}_2) + J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\sigma}_2, \tilde{\zeta}_1) & \text{if } \tilde{\zeta}_0 < \tilde{\sigma}_1, \\ J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}_2) + J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\sigma}_2, \tilde{\zeta}_1) & \text{if } \tilde{\zeta}_0 \geq \tilde{\sigma}_1. \end{cases}
\]

Since (3.4.1) assures that

\[
\tilde{\sigma}_2 - (\tilde{\xi} + \varepsilon) > \delta - \varepsilon \geq \frac{\delta}{2}
\]

(3.4.9)

when \(\varepsilon\) is sufficiently small, one can easily see that

\[
J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_1) > \begin{cases} J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}_1) + J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\xi}, \tilde{\xi} + \varepsilon) + J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\sigma}_2, \tilde{\zeta}_1) & \text{if } \tilde{\zeta}_0 < \tilde{\sigma}_1, \\ J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\xi}, \tilde{\xi} + \varepsilon) + J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\sigma}_2, \tilde{\zeta}_1) & \text{if } \tilde{\zeta}_0 \geq \tilde{\sigma}_1. \end{cases}
\]

(3.4.10)

We only consider the case of \(\tilde{\zeta}_0 < \tilde{\sigma}_1\) because this case is more difficult than the other one.

For \(J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}_1)\), it follows from Lemma 3.2.8 and Theorem 3.3.5 that there exist positive constants \(C_1\) and \(r_1\) such that

\[
|J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}_1)| \leq \varepsilon C^*\gamma^*(\tilde{u}^\varepsilon(\tilde{\sigma}_1) - u^\varepsilon(\tilde{\zeta}_0)) < \varepsilon C^*\gamma^*\tilde{u}^\varepsilon(\tilde{\sigma}_1)
\]

\[
\leq C_1\varepsilon \exp\left(-\frac{r_1(\xi - \tilde{\sigma}_1)}{\varepsilon}\right) < C_1\varepsilon \exp\left(-\frac{r_1\delta}{\varepsilon}\right).
\]

(3.4.11)
Here, $\gamma^*$ and $C^*$ are positive constants defined in (3.1.3) and (3.2.3), respectively. Similarly, for $J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\sigma}_2, \tilde{\zeta}_1)$, we obtain an estimate such as

$$|J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\sigma}_2, \tilde{\zeta}_1)| < C_2 \varepsilon \exp \left(-\frac{r_2 \delta}{\varepsilon}\right)$$  \hspace{1cm} (3.4.12)

with some positive constants $C_2$ and $r_2$.

We next consider $J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\xi}, \tilde{\xi} + \varepsilon)$. By (3.4.9), there exists a positive constant $\gamma_\delta$, which is independent of $\varepsilon$, satisfying

$$\min_{y \in [\tilde{\xi}, \tilde{\xi} + \varepsilon]} \gamma(y) \geq \gamma_\delta$$

This fact together with (3.2.9) tells us to see that

$$J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\xi}, \tilde{\xi} + \varepsilon) > \varepsilon^2 \int_{\tilde{\xi}}^{\tilde{\xi} + \varepsilon} \gamma_\delta \left(\frac{K_3}{\varepsilon}\right)^2 dy = K_3^2 \gamma_\delta \varepsilon$$  \hspace{1cm} (3.4.13)

with some positive constant $K_3$.

Summarizing (3.4.10), (3.4.11), (3.4.12) and (3.4.13), one can see that (3.4.8) holds true.

By (3.4.5), (3.4.7) and (3.4.8), there exists a positive constant $K_4$ such that

$$1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1) > K_4 \sqrt{\varepsilon}.$$  \hspace{1cm} (3.4.14)

Applying Theorem 3.3.4 to (3.4.14), we get

$$C_3 \exp \left(-\frac{r_3 (\tilde{\zeta}_1 - \tilde{\zeta})}{\varepsilon}\right) \geq 1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1) > K_4 \sqrt{\varepsilon}$$

with some positive constants $C_3$ and $r_3$, so that

$$\tilde{\zeta}_1 - \tilde{\zeta} = O(\varepsilon |\log \varepsilon|).$$

However this is impossible because $\tilde{\zeta}_1 - \tilde{\zeta} > \tilde{\sigma}_2 - \tilde{\xi} > \delta$. Thus we can conclude that $\tilde{\zeta}_1 \leq \tilde{\sigma}_2$ holds true.

Finally, we will show (3.4.2) and (3.4.3). In view of the arguments cited above, it suffices to prove (3.4.8) in the case of $\tilde{\zeta}_1 \leq \tilde{\sigma}_2$. Remark that Lemma 3.2.7 implies that $\tilde{\zeta}_1 - \tilde{\xi} > \varepsilon$ when $\varepsilon$ is sufficiently small. Then, it holds that

$$J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_1) > \begin{cases} J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}_1) + J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\xi}, \tilde{\xi} + \varepsilon) & \text{if } \tilde{\zeta}_0 < \tilde{\sigma}_1, \\ J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\xi}, \tilde{\xi} + \varepsilon) & \text{if } \tilde{\zeta}_0 \geq \tilde{\sigma}_1. \end{cases}$$

Note that the arguments as in (3.4.11) and (3.4.13) are valid here. Hence, (3.4.8) is also valid for the case of $\tilde{\zeta}_1 \leq \tilde{\sigma}_2$. Thus the proof is complete.  \hspace{1cm} $\square$
Remark 3.4.2. We will give a comment on the proof of Key Lemma 3.4.1 for the case of \( \tilde{u}_y^\varepsilon(\tilde{\xi}) < 0 \). In this case, \( \tilde{u}_y^\varepsilon(\tilde{\xi}_1) \) is close to 0, so that, instead of (3.4.6), it holds that

\[
W(\tilde{u}_y^\varepsilon(\tilde{\xi}_1)) = W(0) + W_u(0)\tilde{u}_y^\varepsilon(\tilde{\xi}_1) + \frac{W_{uu}(0)}{2}(\tilde{u}_y^\varepsilon(\tilde{\xi}_1))^2 + o\left((\tilde{u}_y^\varepsilon(\tilde{\xi}_1))^2\right)
\]

\[
= \frac{1}{4}(\tilde{u}_y^\varepsilon(\tilde{\xi}_1))^2 + o\left((\tilde{u}_y^\varepsilon(\tilde{\xi}_1))^2\right).
\]

In the light of this fact, the same argument with replacing \( 1 - \tilde{u}_y^\varepsilon(\tilde{\xi}_1) \) by \( \tilde{u}_y^\varepsilon(\tilde{\xi}_1) \) works well for this case.

Key Lemma 3.4.3. For \( \tilde{u}_y^\varepsilon \in \tilde{S}_{n,\varepsilon} \) and \( \tilde{\xi} \in \tilde{\Xi} \), define \( \tilde{\xi}_0 := \sup\{y; y < \tilde{\xi} \text{ and } \tilde{u}_y^\varepsilon(y) = 0\} \) and \( \tilde{\xi}_1 := \inf\{y; y > \tilde{\xi} \text{ and } \tilde{u}_y^\varepsilon(y) = 0\} \). Let \( \tilde{\sigma} \in \tilde{\Sigma}^+ \) and a positive constant \( \delta \) independently of \( \varepsilon \) satisfy \( \gamma_y(y) > 0 \) in \( (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) \). If

\[
\tilde{\sigma} \leq \tilde{\xi}_0 < \tilde{\xi} < \tilde{\sigma} + \delta \quad \text{(resp. } \tilde{\sigma} - \delta < \tilde{\xi} < \tilde{\xi}_1 \leq \tilde{\sigma}),
\]

then

\[
\frac{1}{\varepsilon} \tilde{u}_y^\varepsilon(\tilde{\xi}_1) > K\varepsilon \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) > 0,
\]

\[
\tilde{u}_y^\varepsilon(\tilde{\xi}_1) > K\varepsilon \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) < 0.
\]

(resp.

\[
\frac{1}{\varepsilon} \tilde{u}_y^\varepsilon(\tilde{\xi}_0) > K\varepsilon \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) > 0,
\]

\[
1 - \tilde{u}_y^\varepsilon(\tilde{\xi}_0) > K\varepsilon \quad \text{if } \tilde{u}_y^\varepsilon(\tilde{\xi}) < 0.
\]

Moreover, it holds that

\[
\tilde{\xi}_1 - \tilde{\xi} = O(\varepsilon|\log \varepsilon|) \quad \text{(resp. } \tilde{\xi} - \tilde{\xi}_0 = O(\varepsilon|\log \varepsilon|)).
\]

Proof. We only consider the case that \( \tilde{\sigma} \leq \tilde{\xi}_0 < \tilde{\xi} < \tilde{\sigma} + \delta \) and \( \tilde{u}_y^\varepsilon(\tilde{\xi}) > 0 \). Recalling the argument as in the proof of Key Lemma 3.4.1, we can easily obtain that

\[
K_1(1 - \tilde{u}_y^\varepsilon(\tilde{\xi}_1))^2 \geq W(\tilde{u}_y^\varepsilon(\tilde{\xi}_1)) - W(\tilde{u}_y^\varepsilon(\tilde{\xi}_0)) = J_\varepsilon(\tilde{u}_y^\varepsilon; \tilde{\xi}_0, \tilde{\xi}_1)
\]

with some positive constant \( K_1 \).

We first prove

\[
\tilde{\xi}_1 \leq \tilde{\sigma}' := \inf\{y \in \tilde{\Sigma}; y > \tilde{\sigma}\}.
\]

For this purpose, we will assume \( \tilde{\xi}_1 > \tilde{\sigma}' \) and derive a contradiction. In this case, if \( \varepsilon \) is sufficiently small, then \( \tilde{\xi} + \varepsilon < \tilde{\sigma}' \) because \( \tilde{\xi} < \tilde{\sigma} + \delta < \tilde{\sigma}' \), so that it is clear that

\[
J_\varepsilon(\tilde{u}_y^\varepsilon; \tilde{\xi}_0, \tilde{\xi}_1) > J_\varepsilon\left(\tilde{u}_y^\varepsilon; \tilde{\xi} + \frac{\varepsilon}{2}, \tilde{\xi} + \varepsilon\right) + J_\varepsilon(\tilde{u}_y^\varepsilon; \tilde{\sigma}', \tilde{\xi}_1).
\]
Moreover, the same way to obtain (3.4.12) is valid for the second term of the right-hand side of (3.4.18). Hence there exist positive constants $K_2$ and $K_3$ satisfying
\[ |J_\varepsilon(\bar{u}^\varepsilon; \tilde{\sigma}', \tilde{\zeta}_1)| < K_2\varepsilon \exp \left( -\frac{K_3}{\varepsilon} \right). \] (3.4.19)

We next give an estimate for $J_\varepsilon(\bar{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_1)$. For $y \in [\tilde{\xi} + \varepsilon/2, \tilde{\xi} + \varepsilon]$, using Taylor’s expansion, one can see that
\[ \gamma(y) = \gamma(\tilde{\xi}) + \gamma_y(\tilde{\xi})(y - \tilde{\xi}) + o((y - \tilde{\xi})) = \gamma_y(\tilde{\xi})(y - \tilde{\xi}) + o((y - \tilde{\xi})). \]
Hence, there exists a positive constant $K_4$ such that
\[ \min_{y \in [\tilde{\xi} + \frac{\varepsilon}{2}, \tilde{\xi} + \varepsilon]} \gamma(y) > K_4 \cdot \min_{y \in [\tilde{\xi} + \frac{\varepsilon}{2}, \tilde{\xi} + \varepsilon]} (y - \tilde{\xi}) = K_4 \left( \frac{\varepsilon}{2} \right) > \frac{K_4}{2} \varepsilon. \]
This fact together with (3.2.9) implies that there is a positive constant $K_5$ satisfying
\[ J_\varepsilon \left( \bar{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_1 \right) \geq K_6 \varepsilon^2 \] (3.4.21)
with some positive constant $K_6$. Moreover, (3.4.17) and (3.4.21) imply
\[ 1 - \bar{u}'(\tilde{\zeta}_1) > K_7 \varepsilon \] (3.4.22)
with some positive constant $K_7$. Applying Theorem 3.3.4 to (3.4.22), we obtain that $\tilde{\zeta}_1 - \tilde{\xi} = O(\varepsilon |\log \varepsilon|)$. However, this is impossible because $\tilde{\zeta}_1 - \tilde{\xi} > \tilde{\sigma}' - (\tilde{\xi} + \delta)$. Thus we see that $\tilde{\zeta}_1 \leq \tilde{\sigma}'$.

We will show (3.4.15) and (3.4.16) in the case of $\tilde{\zeta}_1 \leq \tilde{\sigma}'$. Recalling the argument for the case of $\tilde{\zeta}_1 > \tilde{\sigma}'$, one can see that it suffices to show that (3.4.21) also holds true here.

We should remark that (3.4.17) is valid for the case of $\tilde{\zeta}_1 \leq \tilde{\sigma}'$. It also should be noted that Lemma 3.2.7 implies that $\min\{\tilde{\xi} - \tilde{\zeta}_0, \tilde{\zeta}_1 - \tilde{\xi}\} > \varepsilon$ when $\varepsilon$ is sufficiently small. Hence, it holds that
\[ J_\varepsilon(\bar{u}^\varepsilon; \tilde{\zeta}_0, \tilde{\zeta}_{1}) > J_\varepsilon \left( \bar{u}^\varepsilon; \tilde{\zeta}_0 + \frac{\varepsilon}{2}, \tilde{\zeta}_1 + \varepsilon \right). \] (3.4.23)
Furthermore, the same argument to get (3.4.20) works well for the right-hand side of (3.4.23), so that (3.4.21) also holds true. This completes the proof. □
3.5 Location of Transition Layers and Their Multiplicity

We will study the location of transition layers. By Lemma 3.2.5 and Remark 3.2.6, $\tilde{u}^\varepsilon \in \bar{S}_{n,\varepsilon}$ forms a transition layer near a point in $\bar{\Xi}$. Therefore, one of our goals is to determine the location of points in $\bar{\Xi}$.

Lemma 3.5.1. Let $\tilde{u}^\varepsilon \in \bar{S}_{n,\varepsilon}$ and take a positive constant $\delta$ independently of $\varepsilon$. If $\tilde{u}^\varepsilon$ has a transition layer, then it appears only in a $\delta$-neighborhood of a point in $\bar{\Sigma} \cup \{0, L\}$.

Proof. For $l \in \mathbb{N}$, set $\bar{\Sigma} \cup \{0, L\} = \{\bar{\sigma}_k\}_{k=0}^l$ with $0 = \bar{\sigma}_0 < \bar{\sigma}_1 < \cdots < \bar{\sigma}_l = L$. We will show this lemma by using a contradiction method. For this purpose, suppose

$$\hat{\Xi} \cap \bigcup_{k=0}^{l-1} (\bar{\sigma}_k + \delta, \bar{\sigma}_{k+1} - \delta) \neq \emptyset.$$  

In this case, we can choose a point $\hat{\xi}_1 \in \hat{\Xi}$ and a number $k \in \{0, 1, \ldots, l-1\}$ satisfying

$$\hat{\xi}_1 \in (\bar{\sigma}_k + \delta, \bar{\sigma}_{k+1} - \delta).$$

Moreover, we put $\hat{\xi}_1 := \inf\{y ; y > \hat{\xi}_1 \text{ and } \tilde{u}^\varepsilon(y) = 0\}$. We will consider the case that $\tilde{u}^\varepsilon(\hat{\xi}_1) > 0$ and $\gamma(y) > 0$ in $(\bar{\sigma}_k, \bar{\sigma}_{k+1})$.

Then, by virtue of Key Lemma 3.4.1, there exists a positive constant $K_1$ such that

$$1 - \tilde{u}^\varepsilon(\hat{\xi}_1) > K_1 \sqrt{\varepsilon} \quad \text{and} \quad \hat{\xi}_1 - \hat{\xi}_1 = O(\varepsilon \log \varepsilon). \quad (3.5.1)$$

This implies $\hat{\xi}_1 < L$ when $\varepsilon$ is sufficiently small. Therefore, (3.5.1) together with Lemma 3.2.3 and Theorem 3.3.4 enables us to see that there exists a point $\hat{\xi}_2 := \inf\{\xi \in \hat{\Xi} ; \hat{\xi} > \hat{\xi}_1\}$ satisfying $\tilde{u}^\varepsilon(\hat{\xi}_2) < 0$ and

$$K_1 \sqrt{\varepsilon} < 1 - \tilde{u}^\varepsilon(\hat{\xi}_1) < C_1 \exp \left( - \frac{r_1 (\hat{\xi}_2 - \hat{\xi}_1)}{\varepsilon} \right),$$

with some positive constants $C_1$ and $r_1$. Thus, we get

$$\hat{\xi}_2 - \hat{\xi}_1 = O(\varepsilon \log \varepsilon). \quad (3.5.2)$$

Furthermore, it follows from (3.5.1) and (3.5.2) that

$$\hat{\xi}_2 - \hat{\xi}_1 = O(\varepsilon \log \varepsilon). \quad (3.5.3)$$
Here, we should note that \( \tilde{\xi}_2 \in (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta) \) when \( \varepsilon \) is sufficiently small. This implies that there exists another point of \( \tilde{\Xi} \cap (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta) \) except for \( \tilde{\xi}_1 \).

Putting \( \tilde{\zeta}_2 := \inf\{y; y > \tilde{\xi}_2 \text{ and } \tilde{u}_y^\varepsilon(y) = 0\} \) and using Key Lemma 3.4.1 again, one can see that there is a positive constant \( K_2 \) satisfying

\[
K_2 \sqrt{\varepsilon} < \tilde{u}_y^\varepsilon(\tilde{\zeta}_2) \quad \text{and} \quad \tilde{\zeta}_2 - \tilde{\xi}_2 = O(\varepsilon|\log \varepsilon|).
\]

This implies \( \tilde{\zeta}_2 < L \) when \( \varepsilon \) is sufficiently small, so that Lemma 3.2.3 and Theorem 3.3.5 yield that there exists a point \( \tilde{\xi}_3 := \inf\{\tilde{\xi} \in \tilde{\Xi}; \tilde{\xi} > \tilde{\xi}_2\} \) satisfying

\[
\tilde{u}_y^\varepsilon(\tilde{\xi}_3) > 0 \quad \text{and} \quad K_2 \sqrt{\varepsilon} < \tilde{u}_y^\varepsilon(\tilde{\zeta}_2) < C_2 \exp\left(-r_2(\tilde{\zeta}_3 - \tilde{\xi}_3) \varepsilon\right).
\]

with some positive constants \( C_2 \) and \( r_2 \). Therefore, it holds that

\[
\tilde{\xi}_3 - \tilde{\xi}_2 = O(\varepsilon|\log \varepsilon|);
\]

i.e., we have shown the existence of another element of \( \tilde{\Xi} \cap (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta) \) except for \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \).

Repeating these procedures, one can see that the number of points in \( \tilde{\Xi} \cap (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta) \) increases in each process. However, this is impossible because \( \tilde{u}_y^\varepsilon \) belongs to \( \tilde{\mathcal{S}}_{n,\varepsilon} \).

Thus, the proof is complete. \( \square \)

**Remark 3.5.2.** Take \( \tilde{\sigma}, \tilde{\sigma}' \in \tilde{\Sigma} \) with \( \tilde{\sigma} < \tilde{\sigma}' \) and any small \( \delta > 0 \). For \( \tilde{u}_y^\varepsilon \in \tilde{\mathcal{S}}_{n,\varepsilon} \), we consider the case that \( \tilde{\Xi} \cap (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) \neq \emptyset \), \( \tilde{\Xi} \cap (\tilde{\sigma}' - \delta, \tilde{\sigma} + \delta) \neq \emptyset \) and \( \tilde{\Xi} \cap (\tilde{\sigma} + \delta, \tilde{\sigma}' - \delta) = \emptyset \). In this case, we set \( \tilde{\xi} := \sup\{y; y \in \tilde{\Xi} \cap (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta)\} \) and \( \tilde{\xi}' := \inf\{y; y \in \tilde{\Xi} \cap (\tilde{\sigma}' - \delta, \tilde{\sigma}' + \delta)\} \). We should note that Lemma 3.2.3 assures that there exists a unique critical point \( \tilde{\zeta} \in (\tilde{\xi}, \tilde{\xi}') \) of \( \tilde{u}_y^\varepsilon \). In the case that \( \tilde{u}_y^\varepsilon(y) > 1/2 \) in \( (\tilde{\xi}, \tilde{\xi}') \), Theorem 3.3.4 implies that there exist positive constants \( C \) and \( r \) such that

\[
1 - \tilde{u}_y^\varepsilon(\tilde{\zeta}) < C \exp\left(-r(\tilde{\zeta} - \tilde{\xi}) \varepsilon\right) \quad \text{and} \quad 1 - \tilde{u}_y^\varepsilon(\tilde{\zeta}) < C \exp\left(-r(\tilde{\xi}' - \tilde{\zeta}) \varepsilon\right).
\]

It should be noted that there exists a positive constant \( \delta^* \) satisfying

\[
\max\{\tilde{\zeta} - \tilde{\xi}, \tilde{\xi}' - \tilde{\zeta}\} > \frac{\tilde{\sigma}' - \tilde{\sigma}}{2} - \delta \geq \delta^*.
\]
Therefore, it holds that
\[ 1 - \tilde{u}^\varepsilon(\tilde{\zeta}) < C \exp \left( -\frac{r\delta^*}{\varepsilon} \right). \]

Similarly, in the case that \( \tilde{u}^\varepsilon(y) < 1/2 \) in \( (\tilde{\xi}, \tilde{\xi}') \), we obtain that
\[ \tilde{u}^\varepsilon(\tilde{\zeta}) < C \exp \left( -\frac{r\delta^*}{\varepsilon} \right). \]

Now we will study multi-layers.

**Lemma 3.5.3.** For \( \tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon} \), the following assertions hold true:

(i) If \( \tilde{u}^\varepsilon \) has a multi-layer, then it appears only in a neighborhood of a point in \( \tilde{\Sigma}^- \).

(ii) If \( \tilde{u}^\varepsilon \) has a transition layer in a neighborhood of a point in \( \tilde{\Sigma}^+ \), then it must be a single-layer.

**Proof of Lemma 3.5.3.** We first remark that (ii) is a contraposition of (i). Therefore, it suffices to show (i).

We will prove (i) by using a contradiction method. For this purpose, we suppose that there exists a multi-layer which includes \( m \) transition layers in a neighborhood of a point \( \tilde{\sigma} \in \tilde{\Sigma}^- \). Here, \( m \) is a natural number with \( 2 \leq m \leq n \).

In this case, taking a positive constant \( \delta \) independently of \( \varepsilon \) which satisfies \( \gamma_y(\tilde{\zeta}) > 0 \) in \( (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) \), we assume that
\[ \tilde{\Xi} \cap (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) = \{ \tilde{\xi}_k \}_{k=1}^m \]
(3.5.4)

with \( \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_m \). Then, Lemma 3.2.3 tells us that \( \tilde{u}^\varepsilon \) has a unique set of critical points \( \{ \tilde{\xi}_k \}_{k=0}^m \) satisfying \( \tilde{\zeta}_0 < \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_m < \tilde{\zeta}_m \) with \( \tilde{\zeta}_0 := \sup \{ y ; y < \tilde{\xi}_1 \text{ and } \tilde{u}^\varepsilon_y(y) = 0 \} \) and \( \tilde{\zeta}_m := \inf \{ y ; y > \tilde{\xi}_m \text{ and } \tilde{u}^\varepsilon_y(y) = 0 \} \).

It should be noted that the definition of \( \delta \) implies that there exists a positive constant \( \delta^* \) satisfying
\[ \min \{ \tilde{\sigma} - \delta, L - (\tilde{\sigma} + \delta) \} > \delta^* \]
(3.5.5)
because both 0 and \( L \) belong to \( \tilde{\Sigma} \).

We first consider the case that \( \tilde{\zeta}_{m-1} > \tilde{\sigma} \). For definiteness, we assume \( \tilde{u}^\varepsilon_y(\tilde{\xi}_m) > 0 \). Then, by Key Lemma 3.4.3, there exists a positive constant \( K \) such that
\[ 1 - \tilde{u}^\varepsilon(\tilde{\zeta}_m) > K \varepsilon \quad \text{and} \quad \tilde{\zeta}_m - \tilde{\xi}_m = O(\varepsilon |\log \varepsilon|). \]
This together with (3.5.5) yields that \( \tilde{\zeta}_m < L \) when \( \varepsilon \) is sufficiently small. Hence, recalling the argument as in the proof of Lemma 3.5.1, we see that Lemma 3.2.3 and Theorem 3.3.4 assure the existence of a point \( \tilde{\xi}_{m+1} := \inf\{\tilde{\xi} \in \tilde{\Xi}; \tilde{\xi} > \tilde{\zeta}_m\} \) satisfying \( \tilde{\xi}_{m+1} \in (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) \). However, this fact contradicts (3.5.4) because \( \tilde{\xi}_{m+1} \neq \tilde{\xi}_k \) for \( k = 1, 2, \ldots, m \).

We next discuss the case of \( \tilde{\zeta}_{m-1} \leq \tilde{\sigma} \). In this case, it is clear that \( \tilde{\zeta}_1 < \tilde{\sigma} \) holds true. Using the same argument as above with replacing \( \tilde{\zeta}_m \) and \( \tilde{\zeta}_m \) by \( \tilde{\xi}_1 \) and \( \tilde{\xi}_0 \), respectively, one will find that \( \tilde{\xi}_0 := \sup\{\tilde{\xi} \in \tilde{\Xi}; \tilde{\xi} < \tilde{\xi}_1\} \) belongs to \( \tilde{\Xi} \cap (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) \). However this also contradicts (3.5.4).

Thus the proof is complete. \( \square \)

We will give a result concerning the precise profile of a multi-layer.

**Theorem 3.5.4.** Take \( \tilde{\sigma} \in \tilde{\Sigma}^- \) and a small positive constant \( \delta \). For \( \tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon} \) and \( m \in \mathbb{N} \) with \( 2 \leq m \leq n \), let \( \{\tilde{\xi}_k\}_{k=1}^m = \tilde{\Xi} \cap (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) \) and let \( \{\tilde{\zeta}_k\}_{k=0}^m \) be a set of critical points of \( \tilde{u}^\varepsilon \) satisfying \( \tilde{\zeta}_0 < \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_m < \tilde{\zeta}_m \), where \( \tilde{\zeta}_0 := \sup\{y; y < \tilde{\xi}_1 \} \) and \( \tilde{u}^\varepsilon(y) = 0 \} \) and \( \tilde{\zeta}_m := \inf\{y; y > \tilde{\xi}_m \} \) and \( \tilde{u}^\varepsilon(y) = 0 \}). If \( \tilde{u}^\varepsilon(\tilde{\xi}_1) < 0 \) (resp. \( \tilde{u}^\varepsilon(\tilde{\xi}_1) > 0 \)), then it holds that

\[
\begin{align*}
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2}) > \tilde{u}^\varepsilon(\tilde{\zeta}_{2k}) & \quad \text{if } \tilde{\zeta}_0 \leq \tilde{\zeta}_{2k-2} < \tilde{\zeta}_{2k} < \tilde{\sigma}, \\
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-1}) < \tilde{u}^\varepsilon(\tilde{\zeta}_{2k+1}) & \quad \text{if } \tilde{\zeta}_1 \leq \tilde{\zeta}_{2k-1} < \tilde{\zeta}_{2k+1} < \tilde{\sigma},
\end{align*}
\]

and

\[
\begin{align*}
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2}) < \tilde{u}^\varepsilon(\tilde{\zeta}_{2k}) & \quad \text{if } \tilde{\sigma} \leq \tilde{\zeta}_{2k-2} < \tilde{\zeta}_{2k} \leq \tilde{\zeta}_m, \\
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-1}) > \tilde{u}^\varepsilon(\tilde{\zeta}_{2k+1}) & \quad \text{if } \tilde{\sigma} \leq \tilde{\zeta}_{2k-1} < \tilde{\zeta}_{2k+1} \leq \tilde{\zeta}_m.
\end{align*}
\]

(resp.

\[
\begin{align*}
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2}) < \tilde{u}^\varepsilon(\tilde{\zeta}_{2k}) & \quad \text{if } \tilde{\zeta}_0 \leq \tilde{\zeta}_{2k-2} < \tilde{\zeta}_{2k} < \tilde{\sigma}, \\
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-1}) > \tilde{u}^\varepsilon(\tilde{\zeta}_{2k+1}) & \quad \text{if } \tilde{\zeta}_1 \leq \tilde{\zeta}_{2k-1} < \tilde{\zeta}_{2k+1} < \tilde{\sigma},
\end{align*}
\]

and

\[
\begin{align*}
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2}) > \tilde{u}^\varepsilon(\tilde{\zeta}_{2k}) & \quad \text{if } \tilde{\sigma} \leq \tilde{\zeta}_{2k-2} < \tilde{\zeta}_{2k} \leq \tilde{\zeta}_m, \\
\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-1}) < \tilde{u}^\varepsilon(\tilde{\zeta}_{2k+1}) & \quad \text{if } \tilde{\sigma} \leq \tilde{\zeta}_{2k-1} < \tilde{\zeta}_{2k+1} \leq \tilde{\zeta}_m.
\end{align*}
\]

Moreover, for \( k = 1, 2, \ldots, m - 1 \), there exists a positive constant \( C \) such that

\[
W(\tilde{u}^\varepsilon(\tilde{\zeta}_k)) > C\varepsilon^2.
\] (3.5.6)

**Proof.** We only show that, if \( \tilde{u}^\varepsilon(\tilde{\xi}_1) < 0 \) and \( \tilde{\zeta}_0 \leq \tilde{\zeta}_{2k-2} < \tilde{\zeta}_{2k} < \tilde{\sigma} \), then \( \tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2}) > \tilde{u}^\varepsilon(\tilde{\zeta}_{2k}) \) holds true. Multiplying (3.4.4) by \( \tilde{u}^\varepsilon \) and integrating the resulting expression over \( \tilde{\zeta}_{2k-2}, \tilde{\zeta}_{2k} \) with respect to \( y \), we obtain

\[
W(\tilde{u}^\varepsilon(\tilde{\zeta}_{2k})) - W(\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2})) = J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_{2k-2}, \tilde{\zeta}_{2k}) > J_{\varepsilon}(\tilde{u}^\varepsilon; \tilde{\zeta}_{2k-1}, \tilde{\zeta}_{2k}).
\] (3.5.7)
For the right-hand side of (3.5.7), the same argument as we have shown (3.4.21) leads us to
\[ J_2(\tilde{u}^\varepsilon; \tilde{\zeta}_{2k-1}, \tilde{\zeta}_{2k}) > K_1 \varepsilon^2 \]
with some \( K_1 > 0 \). Therefore, it holds that \( W(\tilde{u}^\varepsilon(\tilde{\zeta}_{2k})) > W(\tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2})) \). Taking account of the profile of \( W \), one can easily see that \( \tilde{u}^\varepsilon(\tilde{\zeta}_{2k-2}) > \tilde{u}^\varepsilon(\tilde{\zeta}_{2k}) \) holds true.

We next prove (3.5.6). For \( k \) with \( \tilde{\zeta}_0 \leq \tilde{\zeta}_{k-1} < \tilde{\zeta}_k < \tilde{\sigma} \), it is easy to show
\[ W(\tilde{u}^\varepsilon(\tilde{\zeta}_k)) > W(\tilde{u}^\varepsilon(\tilde{\zeta}_{k-1})). \]
(3.5.8)

Similarly to the estimate of the right-hand side of (3.5.7), there exists a positive number \( K_2 \) such that \( J_2(\tilde{u}^\varepsilon; \tilde{\zeta}_{k-1}, \tilde{\zeta}_k) > K_2 \varepsilon^2 \). This fact together with (3.5.8) implies the existence of a positive constant \( C \) such that
\[ W(\tilde{u}^\varepsilon(\tilde{\zeta}_k)) > C \varepsilon^2 \quad \text{for} \quad k \in \{ k \in \mathbb{N}; \tilde{\zeta}_0 \leq \tilde{\zeta}_{k-1} < \tilde{\zeta}_k < \tilde{\sigma} \}. \]

We should note that the essentially same argument as above is valid for the case of \( \tilde{\zeta}_k \geq \tilde{\sigma} \), so that the proof is complete.

We next consider transition layers near the boundary points.

**Lemma 3.5.5.** If \( \gamma_y(0) > 0 \) (resp. \( \gamma_y(L) > 0 \)), then \( \tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon} \) has no layer in a neighborhood of 0 (resp. \( L \)).

**Proof.** We will only prove that there exists no layer in a neighborhood of 0 when \( \gamma_y(0) > 0 \). Suppose that there is a transition layer in a neighborhood of 0. In this case, \( \tilde{\Xi} \cap (0, \delta) \neq \emptyset \) where \( \delta \) is a small positive constant which satisfies \( \tilde{u}^\varepsilon(y) > 0 \) in \( (0, \delta) \).

We set \( \tilde{\xi} := \sup \{ y; y \in \tilde{\Xi} \cap (0, \delta) \} \). For definiteness, we may assume \( \tilde{u}^\varepsilon(\tilde{\xi}) > 0 \). It follows from Lemma 3.2.3 that there exist critical points \( \tilde{\zeta}_0 \) and \( \tilde{\zeta}_1 \) of \( \tilde{u}^\varepsilon \) satisfying \( \tilde{\zeta}_0 < \tilde{\xi} < \tilde{\zeta}_1 \) with \( \tilde{\zeta}_0 := \sup \{ y; y < \tilde{\xi} \text{ and } \tilde{u}^\varepsilon(y) = 0 \} \) and \( \tilde{\zeta}_1 := \inf \{ y; y > \tilde{\xi} \text{ and } \tilde{u}^\varepsilon(y) = 0 \} \). Moreover, Remark 3.5.2 implies that there exist positive constants \( K_1 \) and \( K_2 \) such that
\[ 1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1) < K_1 \exp \left( -\frac{K_2}{\varepsilon} \right). \]
(3.5.9)

On the other hand, Key Lemma 3.4.3 implies that
\[ 1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1) > K_3 \varepsilon \]
with some positive constant \( K_3 \). This fact contradicts (3.5.9).

Thus, we complete the proof. □
The following lemma describes $\varepsilon$-dependence of the distance from a transition layer to the corresponding point in $\tilde{\Sigma}$.

**Lemma 3.5.6.** Take $\tilde{\sigma} \in \tilde{\Sigma}$. For $\tilde{\sigma} \in \tilde{\sigma}_n$, assume that $\tilde{\Xi} \cap (\tilde{\sigma} - \delta, \tilde{\sigma} + \delta) = \{\tilde{\xi}_k\}_{k=1}^m$ with some $\delta > 0$ and $m \in \mathbb{N}$. Then it holds that $|\tilde{\xi}_k - \tilde{\sigma}| = O(\varepsilon|\log \varepsilon|)$ for any $k = 1, 2, \ldots, m$.

**Proof.** For $\tilde{\sigma} \in \tilde{\sigma}_n$ and $\{\tilde{\xi}_k\}_{k=1}^m$, we define a set of critical points $\{\tilde{\zeta}_k\}_{k=0}^m$ of $\tilde{u}^\varepsilon$ satisfying $\tilde{\zeta}_0 < \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_m$ with $\tilde{\zeta}_0 := \sup\{y; y < \tilde{\xi}_1 \text{ and } \tilde{u}^\varepsilon(y) = 0\}$ and $\tilde{\zeta}_m := \inf\{y; y > \tilde{\xi}_m \text{ and } \tilde{u}^\varepsilon(y) = 0\}$.

We first consider the case of $m = 1$. This corresponds to the case that $\tilde{u}^\varepsilon$ has a single-layer in a neighborhood of $\tilde{\sigma}$. We only give a proof for the case of $\tilde{\gamma}_y(\tilde{\sigma}) > 0$. We may assume $\tilde{u}^\varepsilon(\tilde{\xi}_1) > 0$ and $\tilde{\xi}_1 > \tilde{\sigma}$. Remark 3.5.2 implies that there exists a positive constant $K_1$ satisfying

$$u^\varepsilon(\tilde{\zeta}_0) = O \left( \exp \left( -\frac{K_1}{\varepsilon} \right) \right) \quad \text{and} \quad 1 - u^\varepsilon(\tilde{\xi}_1) = O \left( \exp \left( -\frac{K_1}{\varepsilon} \right) \right). \tag{3.5.10}$$

It is now standard to get

$$W(u^\varepsilon(\tilde{\xi}_1)) - W(u^\varepsilon(\tilde{\zeta}_0)) = J_\varepsilon(u^\varepsilon; \tilde{\zeta}_0, \tilde{\xi}_1).$$

Therefore, it holds that

$$-J_\varepsilon(u^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}) + W(u^\varepsilon(\tilde{\xi}_1)) - W(u^\varepsilon(\tilde{\zeta}_0)) = J_\varepsilon(u^\varepsilon; \tilde{\sigma}, \tilde{\xi}_1). \tag{3.5.11}$$

For the left-hand side of (3.5.11), it follows from (3.5.10) and Lemma 3.2.8, we see that there exists a positive constant $K_2$ satisfying

$$-J_\varepsilon(u^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma}) + W(u^\varepsilon(\tilde{\xi}_1)) - W(u^\varepsilon(\tilde{\zeta}_0)) \leq |J_\varepsilon(u^\varepsilon; \tilde{\zeta}_0, \tilde{\sigma})| + O \left( \exp \left( -\frac{K_1}{\varepsilon} \right) \right) \leq K_2 \varepsilon u^\varepsilon(\tilde{\sigma}) + O \left( \exp \left( -\frac{K_1}{\varepsilon} \right) \right).$$

On the other hand, with use of the same argument as in the proof of Key Lemma 3.4.3, one can also prove that the right-hand side of (3.5.11) satisfies $J_\varepsilon(u^\varepsilon; \tilde{\sigma}, \tilde{\xi}_1) > K_3 \varepsilon^2$ with some $K_3 > 0$. Hence there exists a positive constant $K_4$ satisfying

$$u^\varepsilon(\tilde{\sigma}) > K_4 \varepsilon. \tag{3.5.12}$$

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Moreover, it follows from Theorem 3.3.5 that there are positive constants $C$ and $r$ satisfying
\[ \tilde{u}^\varepsilon(\tilde{\sigma}) < C \exp\left(-\frac{r(\tilde{\xi}_1 - \tilde{\sigma})}{\varepsilon}\right). \]

This fact together with (3.5.12) enables us to see that $\tilde{\xi}_1 - \tilde{\sigma} = O(\varepsilon \log \varepsilon)$.

We next consider the case of $m \geq 2$. Note that Lemma 3.5.3 implies that $\tilde{\sigma} \in \tilde{\Sigma}^-$. It follows from (3.5.6) that there exists a positive constant $K_5$ satisfying
\[ W(\tilde{u}^\varepsilon(\tilde{\zeta}_k)) > K_5 \varepsilon^2 \quad \text{for} \quad k = 1, 2, \ldots, m-1. \quad (3.5.13) \]

Therefore, by using the same argument as in the proof of (3.5.3), we see that
\[ \tilde{\xi}_{k+1} - \tilde{\xi}_k = O(\varepsilon |\log \varepsilon|) \quad \text{for} \quad k = 1, 2, \ldots, m-1. \]

Hence, if $\tilde{\sigma} \in (\tilde{\xi}_l, \tilde{\xi}_{l+1})$ for some $l \in \{1, 2, \ldots, m-1\}$, then it is easy to see that $|\tilde{\xi}_k - \tilde{\sigma}| = O(\varepsilon |\log \varepsilon|)$ for any $k = 1, 2, \ldots, m$. Therefore, it suffices to consider the case that $\tilde{\xi}_1 > \tilde{\sigma}$ or $\tilde{\xi}_m < \tilde{\sigma}$. We only give a proof of the latter case, so that our goal is to show
\[ |\tilde{\xi}_m - \tilde{\sigma}| = O(\varepsilon |\log \varepsilon|). \quad (3.5.14) \]

For definiteness, we assume $\tilde{u}_y^\varepsilon(\tilde{\xi}_m) > 0$. We first show that $\tilde{\xi}_m > \tilde{\sigma}$. We will take a contradiction method. For this purpose, we assume $\tilde{\xi}_m \leq \tilde{\sigma}$. Since $\gamma(y) > 0$ in $(\tilde{\xi}_{m-1}, \tilde{\xi}_m)$, the same technique of getting (3.4.15) leads us to
\[ K_6 (1 - \tilde{u}^\varepsilon(\tilde{\xi}_m))^2 > W(\tilde{u}^\varepsilon(\tilde{\xi}_m)) > W(\tilde{u}^\varepsilon(\tilde{\xi}_m)) - W(\tilde{u}^\varepsilon(\tilde{\xi}_{m-1})) \]
\[ = J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\zeta}_{m-1}, \tilde{\xi}_m) > J_\varepsilon\left(\tilde{u}^\varepsilon; \tilde{\zeta}_m - \varepsilon, \tilde{\xi}_m - \frac{\varepsilon}{2}\right) > K_7 \varepsilon^2 \]
with some positive constants $K_6$ and $K_7$. On the other hand, Remark 3.5.2 implies that
\[ 1 - \tilde{u}^\varepsilon(\tilde{\xi}_m) = O\left(\exp\left(-\frac{K_8}{\varepsilon}\right)\right) \quad (3.5.15) \]
with some $K_8 > 0$. This is a contradiction. Hence, we can see that $\tilde{\xi}_m > \tilde{\sigma}$.

We will prove (3.5.14). By (3.5.13) and (3.5.15), using a similar argument to get (3.5.11), we obtain that
\[ -J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\sigma}, \tilde{\xi}_m) = W(\tilde{u}^\varepsilon(\tilde{\xi}_{m-1})) - W(\tilde{u}^\varepsilon(\tilde{\xi}_m)) + J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\xi}_{m-1}, \tilde{\sigma}) \]
\[ > K_5 \varepsilon^2 + O\left(\exp\left(-\frac{K_8}{\varepsilon}\right)\right) + J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\xi}_{m-1}, \tilde{\sigma}). \quad (3.5.16) \]
Moreover, by the same way of getting the estimate of the right-hand side of (3.5.11), we see that

\[ J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\zeta}_{m-1}, \tilde{\sigma}) > K_9 \varepsilon^2 \]

with some positive constant \( K_9 \). This fact together with (3.5.16) enables us to see that

\[ -J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\sigma}, \tilde{\zeta}_m) > K_{10} \varepsilon^2 \quad (3.5.17) \]

with some positive constant \( K_{10} \).

In the left-hand side of (3.5.17), Lemma 3.2.8 implies that there exists a positive constant \( K_{11} \) satisfying

\[ -J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\sigma}, \tilde{\zeta}_m) \leq |J_\varepsilon(\tilde{u}^\varepsilon; \tilde{\sigma}, \tilde{\zeta}_m)| \leq K_{11} \varepsilon (1 - \tilde{u}^\varepsilon(\tilde{\sigma})). \quad (3.5.18) \]

Combining (3.5.17) and (3.5.18), we obtain that

\[ 1 - \tilde{u}^\varepsilon(\tilde{\sigma}) > K_{12} \varepsilon \]

with some \( K_{12} \). Applying Theorem 3.3.4 to the above inequality, we obtain (3.5.14). We complete the proof. □

Summarizing Lemmas 3.5.1, 3.5.3, 3.5.5 and 3.5.6, we have the following theorem:

**Theorem 3.5.7** (Location of transition layers for solutions of (SP2')). For \( \tilde{u}^\varepsilon \in \tilde{S}_{u,\varepsilon} \), if \( \tilde{u}^\varepsilon \) has a transition layer, then it appears only in an \( O(\varepsilon |\log \varepsilon|) \)-neighborhood of a point in \( \tilde{\Sigma} \). Moreover, the following assertions hold true:

(i) If \( \tilde{u}^\varepsilon \) has a multi-layer, then it appears only in a neighborhood of a point in \( \tilde{\Sigma}^{-} \).

(ii) If \( \tilde{u}^\varepsilon \) has a transition layer in a neighborhood of a point in \( \tilde{\Sigma}^{+} \), then it must be a single-layer.

(iii) If \( \gamma_y(0) > 0 \) (resp. \( \gamma_y(L) > 0 \)), then \( \tilde{u}^\varepsilon \) has no transition layer in a neighborhood of 0 (resp. \( L \)).

### 3.6 Summary

At the end of this chapter, we will state that Theorems 3.1.1 and 3.1.2 are corollaries of Theorems 3.5.7 and 3.5.4, respectively. For this purpose, we will clarify the relation of (SP2) and (SP2').
By the change of variables (3.1.2), we see that any $y \in [0, L]$ has one-to-one correspondence to an $x \in [0, 1]$ because $d(x) > 0$ and $h(x) > 0$ in $[0, 1]$. Recalling $\gamma(y) = \varphi_x(x)/h(x)^2$, we can obtain that each $y_0 \in \tilde{\Sigma}$ corresponds to an $x_0 \in \Sigma$; thus $\varphi_x(x_0) = 0$. In particular, if $y_0 \in \tilde{\Sigma}^-$, then it follows from

$$
\gamma_y(y) = \frac{d(x)\{\varphi_{xx}(x)h(x) - 2\varphi_x(x)h(x_x)\}}{h(x)^4}
$$

that $x_0$ also satisfies $\varphi_{xx}(x_0) < 0$. In other words, any $y_0 \in \tilde{\Sigma}^-$ corresponds to a local maximum point of $\varphi$. On the other hand, $y_0 \in \tilde{\Sigma}^+$ corresponds to a local minimum point of $\varphi$.

For $y_1, y_2 \in [0, L]$ with $y_1 \leq y_2$, let $x_1, x_2 \in [0, 1]$ be corresponding points with respect to the change of variables (3.1.2), respectively. Moreover set

$$
M_* := \min_{x \in [0, 1]} \frac{h(x)}{d(x)} \quad \text{and} \quad M^* := \max_{x \in [0, 1]} \frac{h(x)}{d(x)}.
$$

Then it holds that

$$
M_*(x_2 - x_1) \leq y_2 - y_1 = \int_{x_1}^{x_2} \frac{h(s)}{d(s)} ds \leq M^*(x_2 - x_1).
$$

Therefore, one can conclude that Theorems 3.1.1 and 3.1.2 hold true.
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