Multiple Positive Stationary Solutions to Some Reaction-Diffusion Equations

反応拡散方程式に対する正値定常解の多重性

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Contents

Introduction

In this thesis, we study some reaction-diffusion equations. In general. reaction-diffusion equations are written in the form:

$$\frac{\partial}{\partial t}\boldsymbol{U} = \Delta \boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}),$$

which often arise in various models for some ecological systems (e.g., prey-predator, competition or symbiosis relationships), conduction of nerves, epidemics and oscillating chemical reactions. Here $\boldsymbol{U}(x,t) = (u_1(x,t), u_2(x,t), \ldots,\ u_k(x,t))$ is an unknown function with respect to space variable $x = (x_1, x_2, \ldots,\ x_N)$ and time variable $t > 0$; $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\boldsymbol{F}(\boldsymbol{U})$ is a nonlinear vector-valued function of $\boldsymbol{U}$ (the reaction terms). The reaction terms are derived from interactions between the components of $\boldsymbol{U}$; for example, when u_1 and u_2 denote the population densities of prey species and predator species, respectively, $\boldsymbol{F}(\boldsymbol{U})$ represents the effect of prey-predator relationships. The diffusion term means some random movement of individuals. Mathematically, the main problem is to study space and time depending behaviors of $\boldsymbol{U}$ under the effects of diffusion and reaction. There are lots of studies for various combinations between diffusion and reaction.

Concerning to reaction-diffusion equations, it is also important to discuss the corresponding stationary problem ;

$$\Delta \boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}) = 0.$$

This is because the structure of the sets of stationary solution possibly yields useful information on behaviors of time-depending solutions. Among other things, understanding the multiplicity and the stability of stationary solutions is a very important problem.

Actually, for lots of nonlinear reactions, stationary problems have been extensively discussed by many mathematicians from various view points, e.g., existence of solutions, uniqueness or multiplicity of solutions and stability analysis. In particular for single reaction-diffusion equations ($k = 1$), such studies on stationary problems have made remarkable progress for this decade, see e.g., [56], [57] and references therein.

In this two decades, reaction-diffusion equations with $\Delta \boldsymbol{U}$ replaced by some nonlinear diffusion terms have become a prosperous theme in the mathematical field. Some nonlinear diffusion terms make mathematical models more realistic than the linear diffusions. For example, in ecological models, it is natural to observe that diffusion of an individual depends on population densities of another species in addition to population densities of its own species.

This thesis consists of two parts (Part I and Part II). In Part I, we will study a single diffusion equation with a *concave-convex reaction*. Our main aim in this part is to reveal the relationship between the exact number of stationary solutions and the concave-convex reaction. In Part II, we will discuss a prey-predator system with a nonlinear *cross-diffusion*. Our main result will show that multiple positive steady-states appear by large cross-diffusion effects.

Part I.

In **Part I**, we will mainly discuss the following reaction-diffusion equation with *concave-convex* nonlinearities;

$$\text{(P1)} \begin{cases} u_t = \Delta u + \lambda |u|^{q-1}u + |u|^{p-1}u, & (x,t) \in B_1 \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_1 \times (0,T), \\ u(x,0) = u_0(x), & x \in B_1, \end{cases}$$

where B_1 is a unit ball in $\boldsymbol{R}^N$; i.e., $B_1 = \{x \in \boldsymbol{R}^N : |x| < 1\}$, p and q satisfy $0 < q < 1 < p$ and λ is a positive parameter. For the case $\lambda = 0$, (P1) has been studied by many authors from various viewpoints. On the other hand, it seems that there are few studies on (P1) with $\lambda \neq 0$ (except for [12]). Here we observe that if $\lambda > 0$ (resp. $\lambda < 0$), then the sublinear reaction term $\lambda |u|^{q-1}u$ gives a strong source (resp. absorption) effect for small $u > 0$. The main purpose of this part is to investigate the structure of positive steady-states of (P1) under the presence of source or absorption effects.

This type of differential equation was firstly discussed by Ambrosetti-Brézis-Cerami [6]: To be precise, they have studied the following semilinear elliptic equation with a homogeneous Dirichlet boundary condition;

$$\begin{cases} -\Delta\phi = \lambda|\phi|^{q-1}\phi + |\phi|^{p-1}\phi, & x \in \Omega, \\ \quad \phi = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain of $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$. In [6], they have obtained a $\supset$-shaped branch of positive solutions of (0.1); for a certain positive λ_1, (0.1) admits at least two positive solutions if $0 < \lambda < \lambda_1$, at least one positive solution if $\lambda = \lambda_1$ and no positive solution if $\lambda > \lambda_1$.

Here it is natural to ask whether the number of positive solutions is *exactly* two for $0 < \lambda < \lambda_1$ (in the special ball domain case). Ambrosetti et.al. have proposed this problem for the one-dimensional case in [6, Problem (d) in p. 542].

In **Chapter 1**, we treat (P1) for $N = 1$. We study the stationary problem associated with (P1):

$$\text{(SP1)} \begin{cases} -\phi_{xx} = \lambda|\phi|^{q-1}\phi + |\phi|^{p-1}\phi, & x \in (0,1), \\ \phi(0) = \phi(1) = 0. \end{cases}$$

We will give a complete answer to this problem in Chapter 1. More precisely, we will obtain the complete structure of the solution set $S(\lambda)$ to (SP1) to the parameter range $\lambda \in \boldsymbol{R}$. Furthermore, we will make the stability analysis for the obtained stationary solutions.

Our result is summarized as follows: For each $n \in \boldsymbol{N}$, we define subsets $S_n^+(\lambda)$ and $S_n^-(\lambda)$ of $S(\lambda)$ by

$$S_n^+(\lambda) = \{\phi \in S(\lambda) : \phi \text{ has exactly } (n-1)\text{-zero points in } (0,1) \text{ and } \phi'(0) > 0\},$$
$$S_n^-(\lambda) = \{\phi \in S(\lambda) : \phi \text{ has exactly } (n-1)\text{-zero points in } (0,1) \text{ and } \phi'(0) < 0\}.$$

Then, there exist two sequences $\{\lambda_n\}$ and $\{\lambda_{-n}\}$ with

$$\cdots < \lambda_{-n} < \cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

such that

$$S_n^+(\lambda) = \begin{cases} \{\overline{\phi}_n(\,\cdot\,;\lambda),\, \underline{\phi}_n(\,\cdot\,;\lambda)\} & \text{if } \lambda \in (0, \lambda_n\,], \\ \{\overline{\phi}_n(\,\cdot\,;\lambda)\} & \text{if } \lambda \in (\lambda_{-n}, 0\,], \\ \emptyset & \text{if } \lambda \in (-\infty, \lambda_{-n}] \cup (\lambda_n, \infty), \end{cases} \tag{0.2}$$

where $\overline{\phi}_n(\,\cdot\,;\lambda)$ and $\underline{\phi}_n(\,\cdot\,;\lambda)$ satisfy $\|\overline{\phi}_n(\,\cdot\,;\lambda)\|_\infty > \|\underline{\phi}_n(\,\cdot\,;\lambda)\|_\infty$ if $\lambda \in (0, \lambda_n)$, $\overline{\phi}_n(\,\cdot\,;\lambda_n) \equiv \underline{\phi}_n(\,\cdot\,;\lambda_n)$ and $\lim_{\lambda \downarrow 0} \underline{\phi}_n(\,\cdot\,;\lambda) = 0$ in $C^2[\,0,1\,]$. Moreover, $S_n^-(\lambda) = \{-\phi : \phi \in S_n^+(\lambda)\}$. From a viewpoint of global bifurcation structure, (0.2) is understood as follows: For each $n \in \boldsymbol{N}$, set $\Gamma_n^+ = \{(\lambda, \phi) \in \boldsymbol{R} \times C^2[\,0,1\,] \,:\, \phi \in S_n^+(\lambda)\}$ then Γ_n^+ is a branch of solutions of (SP1) bifurcating from $\phi \equiv 0$ at $\lambda = 0$ and has a turning point at $(\lambda_n, \overline{\phi}_n(\,\cdot\,;\lambda_n))$. This result is identical to the solution-set diagram expected in [6, Fig. 4]. We now note that if $\lambda \in (\lambda_{-1}, \infty)$, then $S(\lambda) = \{0\} \cup \cup_{n=1}^\infty S_n^\pm(\lambda)$, while if $\lambda \in (-\infty, \lambda_{-n})$, then $S(\lambda)$ includes a continuum of n-peak solutions which vanish in a set of positive measures. Such a set is called a *vanishing zone*. We will show that Γ_n^+ connects to a suitable component of continua of n-peak solutions with vanishing zones at $\lambda = \lambda_{-n}$. We will also show the λ-dependence of support of n-peak solutions. It should be noted that solutions with vanishing zone have been found in some nonlinear elliptic problems. We refer to Aronson-Crandall-Peletier [7], Cortázer-Elgueta-Felmer [16], Fukagai [30] and Yoshida [72]. Moreover, Díaz-Hernández [23] have recently determined a set of nonnegative solutions of (SP1) with the case $\lambda < 0$ and $0 < q < p < 1$.

Next, we analyze time-depending behaviors of solutions of (P1) in cases $\lambda > 0$ and $\lambda < 0$, respectively. In particular for $\lambda > 0$, we should note that the non-uniqueness result for solutions of (P1) is well-known: Fujita-Watanabe [29] showed that, in case $\lambda > 0$ for $u_0 \equiv 0$, (P1) has a local positive solution in addition to a zero solution. In this sense, we can say that the zero solution is very unstable. On the other hand, Cazenave-Dickstein-Escobedo [12] have recently obtained the comparison theorem for nonnegative solutions of (P1) with $u_0 \not\equiv 0$. By applying their theorem to our result for the stationary problem, we obtain the following result for behaviors of the nonnegative solution $u(\,\cdot\,, t; u_0)$ of (P1): If $\lambda \in (0, \lambda_1]$, then the maximal stationary solution $\overline{\phi}_1(\,\cdot\,;\lambda)$ plays a role as a separatrix in the following sense that, if $u_0 < \overline{\phi}_1(\lambda)$, then $\lim_{t\to\infty} u(t; u_0) = \underline{\phi}_1$ while, if $u_0 > \overline{\phi}_1(\lambda)$, then $u(t; u_0)$ blows up in a finite time. More precisely, the minimal stationary solution $\underline{\phi}_1(\,\cdot\,;\lambda)$ is attractive:

(i) If $\lambda \in (0, \lambda_1]$ and $u_0 \le k\overline{\phi}_1(\lambda)$ in $(0, 1)$ with some $0 < k < 1$, then $u(x, t; u_0) \le \overline{\phi}_1(x;\lambda)$ for $(x, t) \in (0, 1) \times (0, \infty)$ and $\lim_{t\to\infty} \|u(t; u_0) - \underline{\phi}_1(\lambda)\|_{C^1} = 0$.

(ii) If $\lambda \in (0, \lambda_1]$ and $u_0 \ge k\overline{\phi}_1(\lambda)$ in $(0, 1)$ with some $k > 1$, then $u(x, t; u_0)$ blows up in a finite time with respect to L^∞-norm; so that there exists a positive T_m such that $\lim_{t\uparrow T_m} \|u(t, u_0)\|_\infty = \infty$, and satisfies $u(x, t; u_0) \ge \overline{\phi}_1(x;\lambda)$ for $(x, t) \in (0, 1) \times (0, T_m)$.

(iii) If $\lambda \in (\lambda_1, \infty)$, then $u(x, t; u_0)$ blows up in a finite time.

For the case $\lambda < 0$, we can show the uniqueness and existence for solutions of (P1) in a standard way. We should observe that, in this case, the strong absorption term $\lambda|u|^{q-1}u$ causes extinction phenomena for solutions of (P1) with small initial data. (We refer to [28].) We will show that positive stationary solution $\overline{\phi}_1(\,\cdot\,;\lambda)$ and nonnegative stationary solutions with vanishing zone give separatrixes between blowing up and extinction.

Chapter 2 is concerned with (P1) in a higher dimensional case ($N \ge 3$). We will show an analogous result to the one-dimensional case and give information on the exact number of positive stationary solutions. Since B_1 is a unit ball, the well-known Gidas-Ni-Nirenberg's result ([31]) implies that all stationary solutions are necessarily radially symmetric. So any positive stationary solution $\psi = \psi(r), r = |x|$ must satisfy

$$\begin{cases} (r^{N-1}\psi_r)_r + r^{N-1}(\lambda|\psi|^{q-1}\psi + |\psi|^{p-1}\psi), \quad 0 < r < 1, \\ \psi_r(0) = \psi(1) = 0. \end{cases}$$

For a solution ψ of the above equation, let

$$\phi(r) := \lambda^{1/(q-p)}\psi(\lambda^{(1-p)/2(p-q)}r) \quad \text{and} \quad R := \lambda^{(p-1)/2(p-q)}. \tag{0.3}$$

Then ϕ satisfies

$$\text{(E)} \quad \begin{cases} (r^{N-1}\phi_r)_r + r^{N-1}(|\phi|^{q-1}\phi + |\phi|^{p-1}\phi) = 0, \quad 0 < r < R, \\ \phi_r(0) = \phi(R) = 0. \end{cases}$$

In Chapter 2, we will focus on (E). (To be precise, we will treat more general *concave-convex* nonlinearities.) In a series of works [56]-[58], Ouyang-Shi have established the method of counting the exact number of positive radial solutions of some semilinear elliptic equations. From the view-point of bifurcation, they have proved that (E) has exactly two positive solutions if $R < R_1$ with some positive R_1 depending on p and q; a unique positive solution if $R = R_1$; no positive solution if $R > R_1$ under a restriction $p \leq N/(N-2)$ ([57], [58]). Furthermore, Adimurthi-Pacella-Yadava [1] have shown that (E) has exactly two positive solutions if R is sufficiently small. As the boundary value problem in a general bounded domain, (0.1) has been discussed by many authors (e.g., [5]-[11]). Owing to their results, we know that (E) has at least two positive solutions if $R < R_1$ with some positive R_1, and no positive solution if $R > R_1$. So it would be natural to ask whether (E) has *exactly* two positive solutions if $R < R_1$ without the restriction $p \leq N/(N-2)$.

The main purpose of Chapter 2 is to determine the complete structure for the set of positive solutions of (E). Among other things, we extend the result of Ouyang-Shi to case $N/(N-2) < p < (N+2)/(N-2)$. Actually, as an important application of our main result (Theorem 2.2), we will prove the following exact multiplicity of positive solutions for all $0 < q < 1 < p < (N+2)/(N-2)$ and $R > 0$; there exists a positive number R_1 such that

(i) if $R < R_1$, then (E) has exactly two positive solutions and, moreover, these solutions are strictly ordered in $[0, R)$;

(ii) if $R = R_1$, then (E) has a unique positive solution ;

(iii) if $R > R_1$, then (E) has no positive solution.

By (0.3), the above result immediately implies a similar structure to the one-dimensional case about the set of positive stationary solutions of (P1).

Our approach is based on the shooting argument for the related initial value problem;

$$\begin{cases} (r^{N-1}\phi_r)_r + r^{N-1}(|\phi|^{q-1}\phi + |\phi|^{p-1}\phi) = 0, \quad r > 0, \\ \phi(0) = \alpha, \quad \phi_r(0) = 0. \end{cases} \tag{0.4}$$

Let $u(r,\alpha)$ be the solution of (0.4) and let $z_1(\alpha)$ be a first zero of $u(r,\alpha)$:

$$z_1(\alpha) = \inf\{r > 0 \,:\, u(r,\alpha) = 0\}.$$

(0.4). We will prove that $\lim_{\alpha\downarrow 0} z_1(\alpha) = \lim_{\alpha\uparrow\infty} z_1(\alpha) = 0$ and $z_1(\alpha)$ has a unique critical point in $(0,\infty)$. The above exact multiplicity result can be shown from this fact by letting $R_1 =$ the critical (maximum) value of $z_1(\alpha)$. It is the most crucial part of our proof to get the uniqueness of critical points of $z_1(\alpha)$. In this part, the Sturm oscillation theory and the Pohožaev type identity are basic tools in addition to the bifurcation method developed by Ouyang-Shi ([56]-[58]). Although their method needs the restriction $p \le N/(N-2)$, our combination approach is applicable even in the case $N/(N-2) < p < (N+2)/(N-2)$.

Furthermore, we will discuss the stability for the above two positive stationary solutions. In this stage, the comparison theorem due to Cavenave et.al ([12]) is a very useful tool. We will get the following result:

(i) If $R \le R_1$ and $u_0 \le k\overline{\phi}(R)$ in B_R with some $k \in (0,1)$, then $u(x,t;u_0) \le \overline{\phi}(x;R)$ for $(x,t) \in B_R \times (0,\infty)$ and $\lim_{t\to\infty} \|u(t;u_0) - \underline{\phi}(R)\|_{C^1} = 0$.

(ii) If $R \le R_1$ and $u_0 \ge k\overline{\phi}(R)$ in B_R with some $k \in (1,\infty)$, then $u(x,t;u_0)$ blows up in a finite time; so that there exists a positive T_m such that

$$\lim_{t\uparrow T_m} \|u(t;u_0)\|_\infty = \infty.$$

Moreover, $u(x,t;u_0) \ge \overline{\phi}(x;R)$ for $(x,t) \in B_R \times (0,T_m)$.

(iii) If $R > R_1$, then $u(x,t;u_0)$ blows up in a finite time.

Part II.

In this part, we discuss the following reaction-diffusion system:

$$
\text{(P3)} \quad \begin{cases} d_1^{-1} u_t = \Delta[(1+\alpha v)u] + u(a-u-cv) & \text{in } \ \Omega \times (0,T), \\ d_2^{-1} v_t = \Delta[(1+\beta u)v] + v(b+du-v) & \text{in } \ \Omega \times (0,T), \\ u = v = 0 & \text{on } \ \partial\Omega \times (0,T) \\ u(\,\cdot\,,0) = u_0 \geq 0, \quad v(\,\cdot\,,0) = v_0 \geq 0 & \text{in } \ \Omega. \end{cases}
$$

Here Ω is a bounded domain in $\boldsymbol{R}^N$; with smooth boundary $\partial\Omega$; d_i, a, c, d $(i = 1,2)$ are all positive constants, b is a real constant and α, β are nonnegative constants. (P3) arises form a prey-predator population model. From the view-point of ecological model, unknown functions u and v represent, respectively, population densities of prey and predator species which are interacting and migrating in the same habitat Ω. In diffusion terms, d_i represents natural dispersive force of movement of each individual, while α and β describe mutual interferences between individuals; α and β are usually referred as cross-diffusion pressures. Such a density-dependent population model was first proposed by Shigesada, Kawasaki and Teramoto [67] to investigate the habitat segregation phenomena. Since their pioneer work, many mathematicians have discussed various population models with cross-diffusion effects (e.g., [26], [36], [52], [53], [54], [55]). However, many important problems are still remain open in such cross-diffusion systems.

In **Chapter 3**, we will investigate nonnegative steady-state solutions of (P3). Thus we will concentrate on the following strongly-coupled elliptic system

$$
\text{(SP3)} \quad \begin{cases} \Delta[(1+\alpha v)u] + u(a-u-cv) = 0 & \text{in } \ \Omega, \\ \Delta[(1+\beta u)v] + v(b+du-v) = 0 & \text{in } \ \Omega, \\ u = v = 0 & \text{on } \ \partial\Omega. \end{cases}
$$

We are mainly interested in positive solutions of (SP3). It is said that (u,v) is a positive solution of (SP3) if $u > 0$ and $v > 0$ in Ω. We look for conditions such that this system admits multiple positive steady-state solutions.

We will prove that when (α, β, b, c, d) belongs to a certain range, the positive solution set $\{(u, v, a)\}$ of (SP3) contains an unbounded S-shaped curve with respect to a, while when (α, β, b, c, d) falls into another range, the positive solution set $\{(u, v, a)\}$ contains a bounded S or $\supset$-shaped curve. Our proof is based on the bifurcation theory and the Lyapunov-Schmidt reduction.

In order to introduce our results, we need some notation. Let λ_1 be the least eigenvalue for the following eigenvalue problem

$$-\Delta u = \lambda u \;\text{ in }\; \Omega, \qquad u = 0 \;\text{ on }\; \partial\Omega. \tag{0.5}$$

It is well known that the problem

$$\Delta u + u(a-u) = 0 \;\text{ in }\; \Omega, \qquad u = 0 \;\text{ on }\; \partial\Omega \tag{0.6}$$

has a unique positive solution θ_a if $a > \lambda_1$ (see, e.g. [10]); moreover, $a \to \theta_a : [\lambda_1, \infty) \to C(\overline{\Omega})$ is continuous and strictly increasing function. Here $C(\overline{\Omega})$ is equipped with the uniform convergence topology in $\overline{\Omega}$. According to these facts, it is possible to show that (SP3) has two semitrivial solutions

$$(\theta_a, 0) \;\text{ for }\; a > \lambda_1 \quad \text{ and } \quad (0, \theta_b) \;\text{ for }\; b > \lambda_1$$

in addition to the trivial solution $(0, 0)$.

Regarding a as a bifurcation parameter, we set

$$\mathcal{S} := \{(u, v, a) \,:\, (u, v) \text{ is a positive solution of (SP3)}, \; a > \lambda_1\}.$$

Our main results are concerned with the global structure of $\mathcal{S}$. The first result discussed the case $\min\{\beta b, d\} > \beta\lambda_1$, and asserts that for some (α, β, b, c, d) with $\min\{\beta b, d\} > \beta\lambda_1$, $\mathcal{S}$ contains an unbounded S-shaped curve (with respect to a) which bifurcates from the semitrivial solution curve $\{(0, \theta_b, a) \,:\, a > 0\}$:

Theorem 0.1. *Assume* $\min\{\beta b, d\} > \beta\lambda_1$. *For any* $c > 0$, *there exist a large number* M *and an open set*

$$O_1 = O_1(c) \subset \{(\alpha, \beta, b, d) \,:\, \beta \geq M, \, 0 < \alpha, d/\beta - \lambda_1, b - \lambda_1 \leq M^{-1}\}$$

such that $\partial O_1 \cap \{(\alpha, \beta, b, d) \,:\, d/\beta = \lambda_1\}$ *is not empty and, if* $(\alpha, \beta, b, d) \in O_1$, *then* $\mathcal{S}$ *contains an unbounded smooth curve*

$$\Gamma^1 = \{(u(s), v(s), a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty) \,:\, s \in (0, \infty)\},$$

which possesses the following properties:

(i) $(u(0), v(0)) = (0, \theta_b), \quad a(0) > \lambda_1, \quad a'(0) > 0.$

(ii) $a(s) > a(0)$ *for all* $s \in (0, \infty)$ *and* $\lim_{s \to \infty} a(s) = \infty$.

(iii) $a(s)$ *attains a strict local maximum and a strict local minimum at some* $s = \overline{s}$ *and* $s = \underline{s}$ $(0 < \overline{s} < \underline{s})$, *respectively.*

For some $(\alpha, \beta, a, b, c, d)$ with $\beta b > \beta\lambda_1 > d$, $\mathcal{S}$ contains a bounded S or $\supset$-shaped curve, which bifurcates from the semitrivial solution curve $\{(0, \theta_b, a) \,:\, a > 0\}$ and connects the other semitrivial solution curve $\{(\theta_a, 0, a)\,;\, a > \lambda_1\}$:

Theorem 0.2. *Assume* $\beta b > \beta\lambda_1 > d$. *For any* $c > 0$, *there exist a large number* M *and an open set*

$$O_2 = O_2(c) \subset \{(\alpha, \beta, b, d) \,:\, \beta \geq M,\, 0 < \alpha, \lambda_1 - d/\beta, b - \lambda_1 \leq M^{-1}\}$$

such that, if $(\alpha, \beta, b, d) \in O_2$, *then* $\mathcal{S}$ *contains a bounded smooth curve*

$$\Gamma^2 = \{(u(s), v(s), a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty) \,:\, s \in (0, C)\},$$

which possesses the following properties:

(i) $(u(0), v(0)) = (0, \theta_b), \quad a(0) > \lambda_1, \quad a'(0) > 0.$

(ii) $(u(C), v(C)) = (\theta_{a(C)}, 0), \quad a(C) > \lambda_1.$

(iii) Γ^2 *has at least one turning point with respect to* a. *Furthermore, there exists an open subset* $O_2' \subset O_2$ *such that* $\partial O_2' \cap \{(\alpha, \beta, b, d) \,:\, d/\beta = \lambda_1\}$ *is not empty and, if* $(\alpha, \beta, b, d) \in O_2'$, *then* Γ^2 *has at least two turning points with respect to* a.

In the linear diffusion case ($\alpha = \beta = 0$), there is a conjecture that the uniqueness for positive steady-states of (SP3) holds. Actually, if the spatial dimension is one ($N = 1, \alpha = \beta = 0$), uniqueness of positive steady-states was proved by López-Gómez & Pardo [51]. On the other hand, the above our results assert that the multiple existence of positive steady-states occurs under the presence of cross-diffusion effects.

A crucial part of proofs for Theorems 0.1 and 0.2 is to construct a positive solution curve of (SP3) in the extreme case $\alpha = 0$. The analysis is based on the bifurcation theory and the Lyapunov-Schmidt reduction procedure. If β is large and both $b - \lambda_1 > 0$ and $|d/\beta - \lambda_1|$ are small, then this reduction enables us to find a close relationship to a suitable limit problem. Further, by making use of the perturbation theory developed by Du and Lou [25], we will be able to depict precise solution curves Γ_i of (SP3) near the set of limiting solutions.

The main purpose of **Chapter 4** is to discuss the stability of the above multiple positive steady-states in the sense of the evolution problem (P3) with $\alpha = 0$. We will give some criteria on the stability of these positive steady-states. Furthermore, we will observe the Hopf bifurcation on the steady-state solution branch in some cases.

We will prove that the stability of steady-states depends strongly upon the coefficient ratio d_1/d_2. In particular, when d_1/d_2 is large, the Hopf bifurcation phenomenon can occur on the steady-state solution branch.

Before stating main results, we divide Γ^2 at every turning point with respect to a. In case $\alpha = 0$, put $\tilde{O}_2 := \{(\beta, b, d) \,:\, (0, \beta, b, d) \in O_2\}$. Further, for each $(\beta, b, d) \in \tilde{O}_2$, let

$$0 < s_1 < s_2 < \cdots < s_{k-1} < C$$

be all strict local maximum or minimum points of $a(s)$. For each $1 \le i \le k$, we set

$$\Gamma_i^2 := \{(u(s), v(s), a(s)) \in \Gamma^2 : s \in (s_{i-1}, s_i)\},$$

where $s_0 := 0$ and $s_k := C$. Further, we define $X := \left[\, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \,\right]^2$ for $p > N$.

We are ready to state our results. Our first result in this chapter can be stated as follows:

Theorem 0.3. *For almost every* $(\beta, b, d) \in \tilde{O}_2$*, there exists a small positive constant* δ *such that if* $d_1/d_2 \leq \delta$*, then all steady-state solutions on* $\Gamma^2_{2j-1}\,(j = 1, 2, \ldots, [(k+1)/2])$ *are asymptotically stable in the topology of* X*, while all steady-state solutions on* $\Gamma^2_{2j}\,(j = 1, 2, \ldots, [k/2])$ *are unstable.*

In the above case, we remark that $(u(0), v(0)) = (0, \theta_b)$ and $(u(C), v(C)) = (\theta_{a(C)}, 0)$ by Theorem 0.2. So Theorem 0.3 implies that stable positive steady-states bifurcate from the semitrivial solution $(0, \theta_b)$ and that, the stability on Γ^2 changes at every turning point with respect to a. On the other hand, we obtain the following result in case d_1/d_2 is large:

Theorem 0.4. *For any* $(\beta, b, d) \in \tilde{O}_2$*, there exists a large positive* D *such that, if* $d_1/d_2 \geq D$*, then the Hopf bifurcation occurs at some point* $(u(s^*), v(s^*), a(s^*)) \in \Gamma^2_1$*. In this case, there exists a periodic solution of* (P3) *if* a *lies in a right-side neighborhood of* $a(s^*)$*.*

For the unbounded steady-states branch Γ^1, similar results to Theorems 0.3 and 0.4 also hold true if $s \in (0, C)$ with some $C > \underline{s}$:

Theorem 0.5. *Assume* $(0, \beta, b, d) \in O_1$*. Let* $0 < s_1 < s_2 < \cdots < s_{k-1} < C$ *be all strict local maximum or minimum points of* $a(s)$ *in* $(0, C)$ *and define* $\Gamma^1_i := \{(u(s), v(s), a(s)) \in \Gamma^1 : s \in (s_{i-1}, s_i)\}$ $(1 \leq i \leq k)$*, where* $s_0 := 0$ *and* $s_k := C$*. Then the following properties hold true:*

(i) *For almost every* $(\beta, b, d) \in O_1$*, there exists positive constants* δ *and* $C > \underline{s}$ *such that, if* $d_1/d_2 \leq \delta$ *and* $s \in (0, C)$*, then steady-state solutions on* $\Gamma^1_{2j-1}\,(j = 1, 2, \ldots, [(k+1)/2])$ *are asymptotically stable in the topology of* X*, while steady-state solutions on* $\Gamma^1_{2j}\,(j = 1, 2, \ldots, [k/2])$ *are unstable.*

(ii) *There exists a large positive* D *such that, if* $d_1/d_2 \geq D$*, then the Hopf bifurcation occurs at some point* $(u(s^*), v(s^*), a(s^*)) \in \Gamma^1_1$*.*

Our strategy for the proofs of Theorems 0.3-0.5 is to study the distribution of the eigenvalues for the linearized problems associated with the steady-states constructed in Theorems 0.1 and 0.2. A crucial step in the proofs is to determine the sign of real parts of the principal two eigenvalues as perturbations of eigenvalues of a 2×2 regular matrix characterized by the limit problem.

Part I

Diffusion Equations with Concave-Convex Reaction

Chapter 1

One-Dimensional Case

1.1 Problem

In this chapter we consider the following initial and boundary value problem for the reaction-diffusion equation;

$$
\text{(P1)}\begin{cases} u_t = u_{xx} + \lambda |u|^{q-1}u + |u|^{p-1}u, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases}
$$

where p, q satisfy $0 < q < 1 < p$ and λ is a real parameter.

The present chapter consists of five sections. In sections 2 and 3, we treat the stationary problem associated with (P1);

$$
\text{(SP1)}\begin{cases} \phi_{xx} + \lambda |\phi|^{q-1}\phi + |\phi|^{p-1}\phi = 0, & x \in (0,1), \\ \phi(0) = \phi(1) = 0. \end{cases}
$$

In sections 4 and 5, we study the non-stationary problem (P1). We denote $L^p(0,1)$-norm by $\| \cdot \|_p$. In particular, $\| \cdot \|_2$ is simply denoted by $\| \cdot \|$.

1.2 Stationary Solution Set

The stationary problem (SP1) gives much useful information in order to analyze behaviors for solutions of (P1), and moreover has an interesting structure itself. In this section we state our results on the structure of the set of solutions of (SP1). Let $S(\lambda)$ denote the set of solutions of (SP) and define

$$
\begin{aligned}
S_n^+(\lambda) &= \{\phi \in S(\lambda) \,:\, \phi \text{ has exactly } (n-1)\text{-zero points in } (0,1) \text{ and } \phi'(0) > 0\},\\
S_n^-(\lambda) &= \{\phi \in S(\lambda) \,:\, \phi \text{ has exactly } (n-1)\text{-zero points in } (0,1) \text{ and } \phi'(0) < 0\}
\end{aligned}
\tag{1.1}
$$

and

$$
L = L(p,q) = \frac{1}{\sqrt{2}(p-q)}\left\{(q+1)^{p-1}(p+1)^{1-q}\right\}^{1/2(p-q)} B\left(\frac{1-q}{2(p-q)}, \frac{1}{2}\right),
$$

where $B(s,t)$ denotes the beta function; $B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}$. The next theorem give a complete structure of $S_n^{\pm}(\lambda)$.

Theorem 1.1. *For each $n \in \boldsymbol{N}$, define $\lambda_{-n} = -(2nL)^{2(p-q)/(p-1)}$. There exists a monotone increasing sequence of positive numbers $\{\lambda_n\} \uparrow \infty$ as $n \uparrow \infty$ such that*

$$
S_n^+(\lambda) = \begin{cases}
\emptyset & \text{if } \lambda \in (-\infty, \lambda_{-n}],\\
\{\overline{\phi}_n(\,\cdot\,;\lambda)\} & \text{if } \lambda \in (\lambda_{-n}, 0\,],\\
\{\overline{\phi}_n(\,\cdot\,;\lambda),\, \underline{\phi}_n(\,\cdot\,;\lambda)\} & \text{if } \lambda \in (0, \lambda_n\,],\\
\{\phi_n^*(\,\cdot\,;\lambda)\} & \text{if } \lambda = \lambda_n,\\
\emptyset & \text{if } \lambda \in (\lambda_n, \infty),
\end{cases}
\tag{1.2}
$$

where $\overline{\phi}_n(\,\cdot\,;\lambda)$ (resp. $\underline{\phi}_n(\,\cdot\,;\lambda)$) is a continuous with respect to $\lambda \in (\lambda_{-n}, \lambda_n)$ (resp. $\lambda \in (0, \lambda_n)$) in the topology of $C^2[0,1]$. Here, $\overline{\phi}_n(x;\lambda)$ and $\underline{\phi}_n(x;\lambda)$ sat-

isfy

$$\overline{\phi}_n(i/n;\lambda) = \underline{\phi}_n(i/n;\lambda) = 0 \quad for \quad i = 1, 2, \dots, n-1,$$

$$|\overline{\phi}_n(x;\lambda)| > |\underline{\phi}_n(x;\lambda)| \quad for \quad x \in \cup_{i=1}^{n}((i-1)/n, i/n),$$

$$\lim_{\lambda \downarrow 0} \underline{\phi}_n(\,\cdot\,;\lambda) = 0 \quad in \quad C^2[\,0,1\,] \quad and$$

$$\lim_{\lambda \uparrow \lambda_n} \underline{\phi}_n(\,\cdot\,;\lambda) = \lim_{\lambda \uparrow \lambda_n} \overline{\phi}_n(\,\cdot\,;\lambda) = \phi_n^*(\,\cdot\,;\lambda_n) \quad in \quad C^2[0,1].$$

Moreover, $S_n^-(\lambda) = \{-\phi \,:\, \phi \in S_n^+(\lambda)\}$.

For $\lambda < \lambda_{-1}$, we can find nontrivial solutions of (SP1) with vanishing zone. Let $D_1^+(\lambda)$ be the set of nonnegative solutions with vanishing zone. Its structure is given by the following theorem.

Theorem 1.2. *For* $\lambda \in (-\infty, \lambda_{-1}]$ *set* $M(\lambda) = L|\lambda|^{-(p-1)/2(p-q)}$. *Then for any* $x_* \in [M(\lambda), 1-M(\lambda)]$, (SP) *has a nonnegative solution* $\phi(x;\lambda,x_*)$ *with vanishing zone such that*

$$\|\phi(\,\cdot\,;\lambda,x_*)\|_\infty = \phi(x_*;\lambda,x_*) = \left\{\frac{|\lambda|(p+1)}{q+1}\right\}^{1/(p-q)} \quad and$$

$$\operatorname{supp}\phi(\,\cdot\,;\lambda,x_*) = [\,x_* - M(\lambda), x_* + M(\lambda)\,] \subset [\,0,1\,],$$

where $\operatorname{supp}\phi(\,\cdot\,;\lambda,x_*)$ *means the support of* $\phi(x;\lambda,x_*)$. *Moreover,* $D_1^+(\lambda)$ *is generated by* $\phi(\,\cdot\,;\lambda,x_*)$; *that is*

$$D_1^+(\lambda) = \{\phi(\,\cdot\,;\lambda,x_*)\,|\,M(\lambda) \le x_* \le 1-M(\lambda)\}\,.$$

Observe that the length of support of $\phi(x;\lambda,x_*)$ is $2M(\lambda)$ and $M(\lambda)$ is a decreasing function of $|\lambda|$. Therefore we can construct multi-peak solutions with vanishing zone in the following result.

Corollary 1.3. *Let* n *and* j *be positive integers such that* $j \le n$. *For* $\lambda \in (-\infty, \lambda_{-n})$, *assume that* $x_1, x_2, \dots, x_j$ *satisfy*

$$\begin{cases} 0 < x_1 < x_2 < \cdots < x_j < 1 \text{ and} \\ x_i - x_{i-1} \ge 2M(\lambda) \text{ for } 1 \le i \le j+1, \end{cases}$$

with $x_0 = 0$, $x_{j+1} = 1$. *Then any function of the form*

$$\sum_{i=1}^{j} \alpha_i \phi(x; \lambda, x_i) \quad \textit{with } \alpha_i = 1 \textit{ or } -1,$$

satisfies (SP).

For $\lambda \le 0$ and $n \in \boldsymbol{N}$, we set

$$D_n(\lambda) = \begin{cases} \left\{ \sum_{i=1}^{n} \alpha_i \phi(\,\cdot\,; \lambda, x_i) \;\middle|\; \begin{array}{l} 0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1 \text{ and} \\ |x_i - x_j| > 2M(\lambda) \text{ for } i \ne j \end{array} \right\} \\ \text{if } \lambda \in (-\infty, \lambda_{-n}), \\ \emptyset \;\text{ if }\; \lambda \in [\,\lambda_{-n}, 0\,]. \end{cases}$$

We should observe that, for each fixed $\lambda \in (-\infty, \lambda_{-n}]$, $D_n(\lambda)$ is a continuum of solutions of (SP). This is a big contrast to $S_n^{\pm}(\lambda)$.

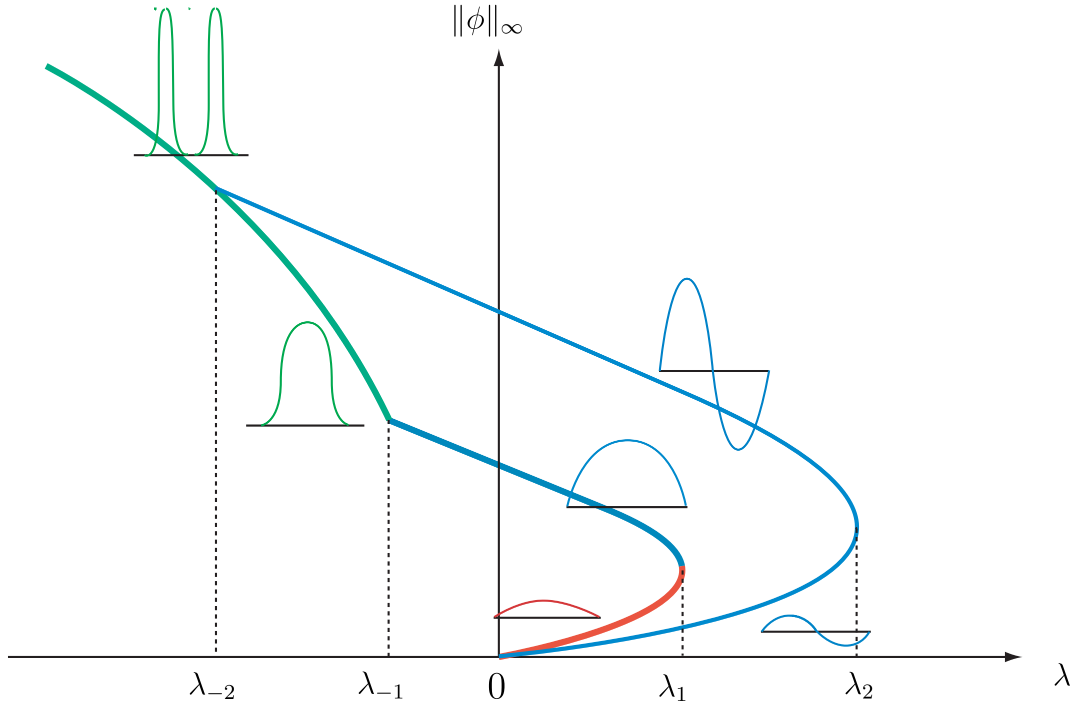


Fig. 1. Global bifurcation diagram

Summarizing Theorems 1.1 and 1.2 and Corollary 1.3, we can get complete information on the structure of (SP1);

Theorem 1.4. *For any* $\lambda \in \boldsymbol{R}$,

$$S(\lambda) = \{0\} \cup \bigcup_{n=1}^{\infty} S_n^{\pm}(\lambda) \cup \bigcup_{n=1}^{\infty} D_n(\lambda).$$

In particular for $\lambda \leq \lambda_n$, $S_n^{\pm}(\lambda) \neq \emptyset$ *(resp.* $D_n(\lambda) \neq \emptyset$*) if and only if* $D_n(\lambda) = \emptyset$ *(resp.* $S_n^{\pm}(\lambda) = \emptyset$*).*

1.3 Proofs of Theorems 1.1-1.4

In this section we give proofs of Theorems 1.1-1.4. Our strategy is to construct solutions of (SP1) by a standard shooting method (see e.g., Smoller-Wasserman [68]). In case $\lambda \neq 0$, we introduce a change of variable in (SP1);

$$\psi = |\lambda|^{-1/(p-q)}\phi, \tag{1.3}$$

then (SP1) is reduced to the problem;

$$\begin{cases} \psi'' + |\lambda|^{(p-1)/(p-q)}(|\psi|^{q-1}\psi \operatorname{sgn}\lambda + |\psi|^{p-1}\psi) = 0, \quad x \in (0,1), \\ \psi(0) = \psi(1) = 0, \end{cases} \tag{1.4}$$

where "$'$" means the differentiation with respect to x-variable and $\operatorname{sgn}\lambda = 1(\lambda > 0), = -1(\lambda < 0)$. We now introduce the related initial value problem;

$$\begin{cases} \psi'' + |\lambda|^{(p-1)/(p-q)} f(\psi) = 0, \quad x > 0, \\ \psi(0) = 0, \\ \psi_x(0) = \alpha, \end{cases} \tag{1.5}$$

where $f(\psi) = |\psi|^{q-1}\psi \operatorname{sgn}\lambda + |\psi|^{p-1}\psi$ and α is a positive parameter.

Let $\psi(x;\alpha)$ denote a unique solution of (1.5). Multiplying the both sides of (1.5) by ψ' and integrating with respect to x over $(0,1)$, we obtain

$$\frac{1}{2}\psi'(x;\alpha)^2 + |\lambda|^{(p-1)/(p-q)} F(\psi(x;\alpha)) = \frac{1}{2}\alpha^2, \tag{1.6}$$

where

$$F(\psi) = \int_0^{\psi} f(\xi)\,d\xi = \frac{\mathrm{sgn}\lambda}{q+1}|\psi|^{q+1} + \frac{1}{p+1}|\psi|^{p+1}.$$

We define $\psi_{\alpha,\lambda} > 0$ by $2|\lambda|^{(p-1)/(p-q)}F(\psi_{\alpha,\lambda}) = \alpha^2$ and set

$$X(\alpha,\lambda) := \inf\{x > 0 \,:\, \psi(x;\alpha) = \psi_{\alpha,\lambda}\},$$

which is sometimes called a time-map. Since $\psi' = \{2|\lambda|^{(p-1)/(p-q)}(F(\psi_{\alpha,\lambda}) - F(\psi))\}^{1/2}$ for $0 \le x \le X(\alpha,\lambda)$ from (1.6), we see

$$X(\alpha,\lambda) = \frac{1}{\sqrt{2}}|\lambda|^{-(p-1)/2(p-q)} \int_0^{\psi_{\alpha,\lambda}} \frac{d\psi}{\sqrt{F(\psi_{\alpha,\lambda}) - F(\psi)}}.$$

By a change of variable $\psi = \psi_{\alpha,\lambda}v$,

$$X(\alpha,\lambda) = \frac{1}{\sqrt{2}}|\lambda|^{-(p-1)/2(p-q)}\psi_{\alpha,\lambda}^{(1-q)/2} \int_0^1 \Phi(v,\psi_{\alpha,\lambda})^{-1/2}dv, \tag{1.7}$$

where

$$\Phi(v,a) = \frac{\mathrm{sgn}\,\lambda}{q+1}(1-v^{q+1}) + \frac{a^{p-q}}{p+1}(1-v^{p+1}). \tag{1.8}$$

We should observe that the existence of $\phi \in S_n^+(\lambda)$ with $\phi'(0) = |\lambda|^{1/(p-q)}\alpha$ is equivalent to the existence of $\alpha > 0$ such that $X(\alpha,\lambda) = 1/2n$. In what follows, we will study qualitative properties of of $X(\alpha,\lambda)$.

We begin with the case $\lambda > 0$. Since the mapping $\alpha \mapsto \psi_{\alpha,\lambda}$ is strictly monotone increasing for any fixed $\lambda > 0$, we will analyze

$$I(a) = a^{(1-q)/2}\int_0^1 \Phi(v,a)^{-1/2}dv, \quad a \ge 0 \tag{1.9}$$

in view of (1.7). We set

$$c_0 = \left(\frac{1-q}{p-1}\right)^{1/(p-q)} \quad \text{and} \quad c_1 = \left\{\frac{(p+1)(1-q)}{(p-1)(q+1)}\right\}^{1/(p-q)}.$$

Lemma 1.5. *For $\lambda > 0$, $I(\,\cdot\,)$ is continuous in $[\,0,\infty)$ and continuously differentiable in $(0,\infty)$ and satisfies $\lim_{a\downarrow 0} I(a) = \lim_{a\uparrow\infty} I(a) = 0$. Moreover, there exists $a^* \in [\,c_0, c_1\,]$ such that $I(\,\cdot\,)$ is strictly monotone increasing in $(0,a^*)$ and strictly monotone decreasing in (a^*,∞).*

Proof. Formally differentiating the both sides of (1.9), we have

$$I'(a) = \frac{1}{2} a^{-(q+1)/2} \int_0^1 \Phi^{-3/2} \Psi \, dv, \tag{1.10}$$

where

$$\Psi(v,a) = \frac{1-q}{q+1}(1 - v^{q+1}) \operatorname{sgn} \lambda - \frac{p-1}{p+1}(1 - v^{p+1}) a^{p-q}. \tag{1.11}$$

Let any $A > 0$ fixed. For $a \in [\,0, A\,]$ and $v \in [\,0, 1\,]$, $|\Phi^{-3/2}\Psi| < C(1-v)^{-1/2}$ with some positive C independent. Then one can justify (1.10) by Lebesgue's dominated convergence theorem. It is easy to see that $I(\,\cdot\,) \in C[\,0,\infty) \cap C^1(0,\infty)$ and $\lim_{a \downarrow 0} I(a) = \lim_{a \uparrow \infty} I(a) = 0$.

We observe that

$$\begin{aligned} \Psi_v(v,a) &= v^q\{(p-1)v^{p-q}a^{p-q} - (1-q)\} \\ &< v^q\{(p-1)c_0^{p-q} - (1-q)\} = 0 \end{aligned}$$

for $(v,a) \in (0,1) \times (0, c_0)$. Thus $\Psi(\,\cdot\,, a)$ is strictly monotone decreasing in $(0,1)$ for any fixed $a \in (0, c_0)$. Therefore, since $\Psi(1,a) = 0$,

$$\Psi(v,a) > 0 \quad \text{for } (v,a) \in (0,1) \times (0, c_0),$$

which, together with (1.10), implies

$$I'(a) > 0 \quad \text{for } a \in (0, c_0). \tag{1.12}$$

If $a \in (c_1, \infty)$, then

$$\begin{aligned} \Psi(v,a) &< (1 - v^{p+1}) \left(\frac{1-q}{q+1} - \frac{p-1}{p+1} a^{p-q} \right) \\ &< (1 - v^{p+1}) \left(\frac{1-q}{q+1} - \frac{p-1}{p+1} c_1^{p-q} \right) = 0 \end{aligned}$$

for $v \in (0,1)$. Therefore, we see from (1.10) that

$$I'(a) < 0 \quad \text{for } a \in (c_1, \infty). \tag{1.13}$$

It follows from (1.12) and (1.13) that $I(\,\cdot\,)$ has at least one critical point in $[\,c_0, c_1\,]$. To accomplish the proof, it is sufficient to show that $I''(a^*) < 0$ at any critical point $a* \in [\,c_0, c_1\,]$ of $I(a)$. From (3.8),

$$a^{(q+1)/2} I'(a) = \frac{1}{2}\int_0^1 \Phi^{-3/2}\Psi \, dv.$$

Differentiation of the above equality yields

$$\frac{q+1}{2}a^{-(1-q)/2}I'(a) + a^{(q+1)/2}I''(a) = \frac{1}{2}\int_0^1 \Theta(v,a)\Phi^{-5/2}\Phi_a \, dv, \tag{1.14}$$

where

$$\Theta(v,a) = \frac{p-1}{2(p+1)}(1-v^{p+1})a^{p-q} - \frac{1}{q+1}\left\{\frac{3}{2}(1-q) + (p-1)\right\}(1-v^{q+1}).$$

Clearly,

$$\Theta(v,a) < \Theta(v,c_1) = \frac{g(v)}{2(q+1)} \quad \text{for } (v,a) \in (0,1)\times(0,c_1), \tag{1.15}$$

where $g(v) = (2p-3q+1)v^{q+1} - (1-q)v^{p+1} - 2(p-q)$. Since one can easily see $g'(v) > 0$ for $v \in (0,1)$, it follows from (1.15) that

$$\Theta(v,a) < \Theta(v,c_1) = \frac{g(1)}{2(q+1)} = 0 \quad \text{for } (v,a) \in (0,1)\times(0,c_1).$$

Then, by putting $a = a^*$ in (1.14), we obtain $I''(a^*) < 0$. Thus the proof of Lemma 3.1 is accomplished.

□

Next we treat the case $\lambda < 0$. It is easily seen that the mapping $\alpha \mapsto \psi_{\alpha,\lambda}$ is also monotone increasing in $(0,\infty)$. However, it must be noted that

$$\lim_{\alpha\downarrow 0}\psi_{\alpha,\lambda} = \left(\frac{p+1}{q+1}\right)^{1/(p-q)} =: A \quad \text{and} \quad \lim_{\alpha\uparrow\infty}\psi_{\alpha,\lambda} = \infty \tag{1.16}$$

for any $\lambda < 0$. In what follows, we will analyze $I(a)$ for $a \in [\,A,\infty)$. Here we should observe from (1.8) that,

$$\Phi(v,a) \geq \left(\frac{a^{p-q}}{p+1} - \frac{1}{q+1}\right)(1-v^{q+1}) > 0 \tag{1.17}$$

for $(v,a) \in (0,1)\times(A,\infty)$.

Lemma 1.6. *For $\lambda < 0$, the following properties* (i) *and* (ii) *hold true:*

(i) $I(\,\cdot\,)$ *is continuously differentiable in* (A, ∞).

(ii) $I(\,\cdot\,)$ *is strictly monotone decreasing in* (A, ∞). *Moreover,* $I(\,\cdot\,)$ *satisfies*

$$\lim_{a \downarrow A} I(a) = \sqrt{2}L \quad \textit{and} \quad \lim_{a \uparrow \infty} I(a) = 0,$$

where L is a positive constant defined by (1.1).

Proof. In view of (1.17), the continuity of $I(\,\cdot\,)$ follows from Lebesgue's theorem. Moreover, as in the proof of Lemma 3.1 it is possible to show that (1.10) is valid. Therefore, $I(\,\cdot\,) \in C^1(A, \infty)$.

In order to show (ii), we use (1.10) again. Since $\Psi < 0$ for $\lambda < 0$ by (1.11), we see that $I(\,\cdot\,)$ is strictly monotone decreasing in (A, ∞). It is easy to see that $\lim_{a \uparrow \infty} I(a) = 0$. Moreover, letting $a \downarrow A$ in (1.9), we have

$$\begin{aligned}\lim_{a \downarrow A} I(a) &= \sqrt{q+1}\, A^{(1-q)/2} \int_0^1 v^{-(q+1)/2}(1 - v^{p-q})^{-1/2}\, dv \\ &= \frac{1}{p-q}\sqrt{q+1}\, A^{(1-q)/2} B\left(\frac{1-q}{2(p-q)}, \frac{1}{2}\right) = \sqrt{2}L.\end{aligned}$$

Thus the proof of (ii) is complete.

□

Proof of Theorem 1.1. We now recall that $\phi \in S_n^+(\lambda)$ with $\phi'(0) = |\lambda|^{1/(p-q)}\alpha$ if and only if $X(\alpha, \lambda) = 1/2n$.

(i) The case $\lambda > 0$: By Lemma 1.5 and the phase plane analysis, any nontrivial solution ϕ of (SP1) satisfies $\phi \in S_n^{\pm}(\lambda)$ for some n. It follows from (1.7), (1.9) and Lemma 1.5 that

$$\begin{aligned}&\lim_{\alpha \downarrow 0} X(\alpha, \lambda) = \lim_{\alpha \uparrow \infty} X(\alpha, \lambda) = 0 \quad \text{and} \\ &M(\lambda) := \sup_{\alpha > 0} X(\alpha, \lambda) = \frac{1}{\sqrt{2}} \lambda^{-(p-1)/2(p-q)} I(a^*)\end{aligned}$$

for any $\lambda > 0$, where a^* is the positive number defined in Lemma 1.5. Therefore, for each $n \in \boldsymbol{N}$, there exists a unique number $\lambda = \lambda_n$ such that $M(\lambda_n) = 1/2n$. Hence $\{\lambda_n\}$ satisfies $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \longrightarrow \infty$ as $n \to \infty$.

If $\lambda \in (0, \lambda_n)$, then $M(\lambda) > 1/2n$. Therefore, the following equation

$$X(\alpha, \lambda) = 1/2n \tag{1.18}$$

has exactly two solutions α_n and β_n such that

$$0 < \beta_n < \alpha_n. \tag{1.19}$$

In view of (1.3), if we set

$$\overline{\phi}_n(x;\lambda) = \lambda^{1/(p-q)}\psi(x;\alpha_n) \ \text{ and } \ \underline{\phi}_n(x;\lambda) = \lambda^{1/(p-q)}\psi(x;\beta_n), \tag{1.20}$$

then we can see $S_n^+(\lambda) = \{\overline{\phi}_n(\,\cdot\,;\lambda), \underline{\phi}_n(\,\cdot\,;\lambda)\}$ with $\overline{\phi}_n(i/n;\lambda) = \underline{\phi}_n(i/n;\lambda) = 0$, $\|\overline{\phi}_n(\,\cdot\,;\lambda)\|_\infty = \overline{\phi}_n(i/n - 1/2n;\lambda)$ and $\|\underline{\phi}_n(\,\cdot\,;\lambda)\|_\infty = \underline{\phi}_n(i/n - 1/2n;\lambda)$ for $i = 1, 2, \ldots, n$. Moreover, we will show that

$$|\overline{\phi}_n(x;\lambda)| > |\underline{\phi}_n(x;\lambda)| \ \text{ for } x \in \bigcup_{i=1}^{n}\left(\frac{i-1}{n}, \frac{i}{n}\right).$$

Clearly, it is sufficient to show

$$\overline{\phi}_n(x;\lambda) > \underline{\phi}_n(x;\lambda) \ \text{ for } x \in \left(0, \frac{1}{2n}\right). \tag{1.21}$$

Note that by (1.19)

$$\overline{\phi}_n(x;\lambda) > \underline{\phi}_n(x;\lambda) \ \text{ nearby } \ x = 0. \tag{1.22}$$

Now assume that (1.21) does not hold. Then, there exists

$$x_0 = \inf\left\{x \in \left(0, \frac{1}{2n}\right) \middle| \overline{\phi}_n(x;\lambda) \le \underline{\phi}_n(x;\lambda)\right\}.$$

It follows from (1.22) that $x_0 \neq 0$. Hence,

$$\overline{\phi}_n(x_0;\lambda) = \underline{\phi}_n(x_0;\lambda) \ \text{ and } \ \overline{\phi}_n'(x_0;\lambda) \le \underline{\phi}_n'(x_0;\lambda).$$

Thus it immediately follows from (1.20) that

$$\psi(x_0;\alpha_n) = \psi(x_0;\beta_n) \ \text{ and } \ \psi'(x_0;\alpha_n) \le \psi'(x_0;\beta_n). \tag{1.23}$$

On the other hand, it follows from (1.6) that

$$\begin{aligned}\frac{1}{2}\psi'(x;\alpha_n)^2 + \lambda^{(p-1)/(p-q)}F(\psi(x;\alpha_n)) &= \frac{1}{2}\alpha_n^2 \quad \text{and} \\ \frac{1}{2}\psi'(x;\beta_n)^2 + \lambda^{(p-1)/(p-q)}F(\psi(x;\beta_n)) &= \frac{1}{2}\beta_n^2\end{aligned} \tag{1.24}$$

for $x \in (0,1)$. Since $F(\psi(x_0;\alpha_n)) = F(\psi(x_0;\beta_n))$ by (1.23), it follows from (1.24) that

$$\frac{1}{2}\psi'(x_0;\alpha_n)^2 - \frac{1}{2}\psi'(x_0;\beta_n)^2 = \frac{1}{2}(\alpha_n^2 - \beta_n^2) > 0; \tag{1.25}$$

that is $\psi'(x_0;\alpha_n) > \psi'(x_0;\beta_n)$, which contradicts (1.23). Thus (1.21) holds true.

If $\lambda = \lambda_n$, then $M(\lambda) = 1/2n$. Therefore, (1.18) has a unique solution $\alpha = \alpha_n^*$; so that $S_n^+(\lambda_n) = \{\phi_n^*(\,\cdot\,;\lambda_n)\}$, where $\phi_n^*(x;\lambda_n) := \lambda_n^{1/(p-q)}\psi(x;\alpha_n^*)$.

Finally, if $\lambda \in (\lambda_n, \infty)$, then (1.18) has no solution. Therefore, $S_n^+(\lambda) = \emptyset$.

(ii) The case $\lambda < 0$: It follows from (1.7), (1.9) and Lemma 1.6 that for any fixed $\lambda < 0$, $X(\alpha,\lambda)$ is monotone decreasing with respect to $\alpha \in (0,\infty)$,

$$\begin{aligned}&M(\lambda) := \sup_{\alpha>0} X(\alpha,\lambda) = \lim_{\alpha\downarrow 0} X(\alpha,\lambda) = |\lambda|^{-(p-1)/2(p-q)}L \quad \text{and} \\ &\lim_{\alpha\uparrow\infty} X(\alpha,\lambda) = 0.\end{aligned} \tag{1.26}$$

Then, $M(\lambda) = 1/2n$ is equivalent to $\lambda = -(2nL)^{2(p-q)/(p-1)} (=: \lambda_{-n})$.

If $\lambda \in (\lambda_{-n}, 0)$, then $M(\lambda) > 1/2n$. Therefore, (1.18) has a unique solution $\alpha = \alpha_n$. Thus as in the case $\lambda > 0$, we can prove that $S_n^+(\lambda) = \{\overline{\phi}_n(\,\cdot\,;\lambda)\}$, where $\overline{\phi}_n(x;\lambda) := |\lambda|^{1/(p-q)}\psi(x;\alpha_n)$, and $\overline{\phi}_n$ satisfies $\overline{\phi}_n(i/n;\lambda) = 0$ and $\|\overline{\phi}_n(\,\cdot\,;\lambda)\|_\infty = \overline{\phi}_n(i/n - 1/2n;\lambda)$ for $i = 1, 2, \ldots, n$.

If $\lambda \in (-\infty, \lambda_{-n}]$, then $M(\lambda) \le 1/2n$; so that $X(\alpha;\lambda) > 1/2n$ for any $\alpha > 0$. Then, (1.18) has no positive solution. This implies $S_n^+(\lambda) = \emptyset$.

(iii): For the case $\lambda = 0$, it is shown by a standard argument that $S_n^+(0)$ has a unique element $\overline{\phi}_n(\,\cdot\,;0)$ for each $n \in \boldsymbol{N}$, that is $S_n^+(0) = \{\overline{\phi}_n(\,\cdot\,;0)\}$. Moreover, we can prove the continuity of two functions; $\lambda \mapsto \overline{\phi}_n(\,\cdot\,;\lambda)$ from $(\lambda_{-n}, \lambda_n)$ to $C^2[\,0,1\,]$ and $\lambda \mapsto \underline{\phi}_n(\,\cdot\,;\lambda)$ from $(0,\lambda_n)$ to $C^2[\,0,1\,]$, for each $n \in \boldsymbol{N}$ in a standard way. Finally, we note that f is an odd function, then clearly $S_n^-(\lambda) = \{-\phi \,:\, \phi \in S_n^+(\lambda)\}$. Thus the proof is complete. $\square$

Proof of Theorem 1.2. In case $\lambda < 0$, it is proved by (1.26) that for $\alpha = 0$, (1.5) has two nontrivial solutions $\psi(x;+0) := \lim_{\alpha\downarrow 0}\psi(x;\alpha)$ and $\psi(x;-0) := \lim_{\alpha\uparrow 0}\psi(x;\alpha)$. For $\lambda \le \lambda_{-1}$, we recall that $M(\lambda) = \lim_{\alpha\downarrow 0} X(\alpha,\lambda) \le 1/2$. We define $\phi(x;\lambda,M(\lambda))$ for $x \in [\,0,1\,]$ by

$$\phi(x;\lambda,M(\lambda)) := \begin{cases} |\lambda|^{1/(p-q)}\psi(x;+0) & \text{if } x \in [\,0, 2M(\lambda)\,], \\ 0 & \text{if } x \in [\,2M(\lambda), 1\,]. \end{cases}$$

Then, by virtue of (1.16), we see that

$$\|\phi(\,\cdot\,;\lambda,M(\lambda))\|_\infty = \phi(M(\lambda);\lambda,M(\lambda)) = A|\lambda|^{1/(p-q)}.$$

Hence $\phi'(0;\lambda,M(\lambda)) = \phi'(2M(\lambda);\lambda,M(\lambda)) = 0$; so that $\phi(\,\cdot\,;\lambda,M(\lambda)) \in C^1[\,0,1\,]$. Furthermore, in view of (SP1), $\phi(0;\lambda,M(\lambda)) = \phi(2M(\lambda);\lambda,M(\lambda)) = 0$ yields $\phi''(x;\lambda,M(\lambda)) = 0$ at $x = 0$ and $x = 2M(\lambda)$. Thus $\phi(x;\lambda,M(\lambda))$ is a classical solution of (SP). This consideration allows us to take a suitable translation of $\phi(x;\lambda,M(\lambda))$ as a solution of (SP). Indeed, let x_* be any point in $[\,M(\lambda), 1 - M(\lambda)\,]$. We define $\phi(x;\lambda,x_*)$ in $[\,0,1\,]$ by

$$\phi(x;\lambda,x_*) = \begin{cases} \phi(x - x_* + M(\lambda);\lambda,M(\lambda)) & \text{if } x \in [\,x_* - M(\lambda), x_* + M(\lambda)\,], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\phi(x;\lambda,x_*)$ is also a solution of (SP) with vanishing zone and satisfies

$$\|\phi(\,\cdot\,;\lambda,x_*)\|_\infty = \phi(x_*;\lambda,x_*) = A|\lambda|^{1/(p-q)},$$
$$\operatorname{supp}\phi(\,\cdot\,;\lambda,x_*) = [\,x_* - M(\lambda), x_* + M(\lambda)\,] \subset [\,0,1\,].$$

Thus the proof of Theorem 1.2 is complete. □

Observe that the length of the support of $\phi(\,\cdot\,;\lambda,x_*)$ is $2M(\lambda)$. Therefore, we can construct multi-peak solutions with vanishing zone as in Corollary 1.3.

Proof of Corollary 1.3. If $\lambda \le \lambda_{-n}$, then $M(\lambda) \le 1/2n$. Therefore for any $j \le n$, we can take j points; $x_1, x_2, \cdots, x_j \in [\,M(\lambda), 1 - M(\lambda)\,]$ such that

$$|x_i - x_k| \ge 2M(\lambda), \text{ for } i \ne k \text{ with } x_0 = 0 \text{ and } x_{n+1} = 1\ .$$

Hence Theorem 1.2 implies that any function of the form

$$\sum_{i=1}^{j} \alpha_i \phi(x; \lambda, x_i) \quad \text{with } \alpha_i = 1 \text{ or } -1,$$

satisfies (SP1). □

It follows from the above proofs that (SP1) has no solution except for solutions obtained in Theorems 1.1 and 1.2 and Corollary 1.3. Therefore, we get Theorem 1.4.

1.4 Non-Stationary Problem

In this section we introduce preliminary results for (P1) in order to study stability of solutions of (SP1).

As to the existence of solutions of (P1), we should note that the contraction mapping theorem does not work because the nonlinear reaction term $\lambda|u|^{q-1}u + |u|^{p-1}u$ breaks the Lipschitz continuity at $u = 0$. However, we can show the following existence theorem by using Schauder's fixed point theorem (see e.g., Pazy [60]).

Theorem 1.7. *For any* $u_0 \in L^\infty$, (P) *has at least one solution*

$$u \in C([0, T_m); L^2) \cap C^1((0, T_m); L^2) \cap C((0, T_m); H^2 \cap H_0^1) \cap L^\infty((0,1) \times (0,T)),$$

for any $T < T_m$, *where* T_m *is a maximal existence time of* u ;

$$T_m := \sup\{T > 0 \,:\, \|u(t)\|_\infty < +\infty\}.$$

Next we will give some results about the uniqueness of solutions of (P1). First of all, we should note that a non-uniqueness result has been established by Fujita-Watanabe [29] for $\lambda > 0$.

Theorem 1.8 ([29]). *Assume that* g *satisfies the following hypotheses:*

(A1) g is continuous, monotone-increasing and concave in $[\,0, M\,]$ with some $M > 0$;

(A2) $g(0) = 0$;

(A3) $\displaystyle\int_0^M \frac{d\xi}{g(\xi)} < +\infty.$

Then the initial-boundary value problem

$$\begin{cases} u_t = \Delta u + g(u), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = 0, & x \in \Omega \end{cases}$$

has a positive local solution. Here Ω *is a bounded domain in* $\boldsymbol{R}^N (N \geq 1)$ *with smooth boundary* $\partial\Omega$*.*

It is easily verified that if $\lambda > 0$, then $\lambda|u|^{q-1}u + |u|^{p-1}u$ satisfies (A1)-(A3). Therefore, in case $u_0 = 0$, (P1) has a positive local solution $u(x,t;0)$ in addition to the trivial solution. Hence $-u(x,t;0)$ also satisfies (P1). On the other hand, Cazenave-Dickstein-Escobedo [12] have recently obtained a very useful comparison theorem for nonnegative solutions of (P1). Before stating their theorem, we define a super- and sub-solution for (P1).

Definition 1.1. Let u satisfy

$$u \in C([\,0,T); L^2) \cap C^1((0,T); L^2) \cap C((0,T); H^2 \cap H_0^1) \cap L^\infty((0,1) \times (0,T)).$$

Then u is called a *super-solution* of (P1) if it satisfies

$$\begin{cases} u_t \geq u_{xx} + f_\lambda(u), & (x,t) \in (0,1) \times (0,T), \\ u(0,t) \geq 0, \quad u(1,t) \geq 0, & t \in (0,T). \end{cases} \tag{1.27}$$

If u satisfies (1.27) with "$\geq$" replaced by "$\leq$", then it is called a *sub-solution*.

Theorem 1.9 ([12]). *Let* $\lambda > 0$*. Assume that* $\lambda > 0$*,* u *is a nonnegative super-solution for* (P1) *in* $(0,1) \times (0,T)$ *and that* v *is a nonnegative sub-solution for* (P1) *in* $(0,1) \times (0,T)$*. If* $u(x,0) \not\equiv 0$ *and* $u(x,0) \geq v(x,0)$ *for all* $x \in (0,1)$*, then* $u(x,t) \geq v(x,t)$ *for all* $(x,t) \in (0,1) \times (0,T)$*.*

This theorem also implies uniqueness for nonnegative solutions of (P1) with $u_0 \not\equiv 0$. On the other hand, for the case $\lambda \leq 0$, the comparison theorem holds in a standard form:

Theorem 1.10. *For $\lambda \leq 0$ assume that $\lambda \leq 0$, u is a super-solution for* (P1) *in* $(0,1) \times (0,T)$ *and that v is a sub-solution for* (P1) *in* $(0,1) \times (0,T)$. *If $u(x,0) \geq v(x,0)$ for all $x \in (0,1)$, then $u(x,t) \geq v(x,t)$ for all $(x,t) \in (0,1) \times (0,T)$.*

As a summary of this section, one can see the following existence and uniqueness property of solutions of (P1) in case $\lambda > 0$ and $\lambda \leq 0$, respectively.

Corollary 1.11. (i) *Let $\lambda > 0$. Assume that $u_0 \in L^\infty$ satisfies $u_0 \geq 0$ in $(0,1)$ and $u_0 \not\equiv 0$. Then* (P) *has a unique nonnegative solution $u(x,t;u_0)$ in the sense of Theorem 1.7.*
(ii) *Let $\lambda \leq 0$ and $u_0 \in L^\infty$. Then* (P) *has a unique solution $u(x,t;u_0)$ in the sense of Theorem 1.7.*

1.5 Stability Analysis

In this section, we analyze time-depending behaviors of solutions for (P1) and discuss the stability of solutions for (SP1).

First, we consider the case $\lambda > 0$. In section 3, we have proved that, if $\lambda \in (0,\lambda_1)$, then (SP1) has exactly two positive solutions $\overline{\phi}_1(\lambda)$ and $\underline{\phi}_1(\lambda)$ such that $\overline{\phi}_1(\lambda) > \underline{\phi}_1(\lambda)$ in $(0,1)$ and if $\lambda \in (\lambda_1,\infty)$, then (SP1) has no positive solution. Correspondingly to this structure of positive stationary solutions, we can show the following theorem.

Theorem 1.12. *Assume that $u_0 \in L^\infty$ satisfies $u_0 \geq 0$ in $(0,1)$ and $u_0 \not\equiv 0$. Then the nonnegative solution $u(x,t;u_0)$ of* (P) *has the following properties*:
(i) *If $\lambda \in (0,\lambda_1]$ and $u_0 \leq k\overline{\phi}_1(\lambda)$ in $(0,1)$ with some $0 < k < 1$, then $u(x,t;u_0) \leq \overline{\phi}_1(x;\lambda)$ for $(x,t) \in (0,1) \times (0,\infty)$ and $\lim_{t\to\infty} \|u(t;u_0) - \underline{\phi}_1(\lambda)\|_{C^1} = 0$.*
(ii) *If $\lambda \in (0,\lambda_1]$ and $u_0 \geq k\overline{\phi}_1(\lambda)$ in $(0,1)$ with some $k > 1$, then $u(x,t;u_0)$ blows*

up in a finite time with respect to L^∞*-norm; so that there exists a positive* T_m *such that* $\lim_{t \uparrow T_m} \|u(t, u_0)\|_\infty = \infty$. *Furthermore,* $u(x, t; u_0) \geq \overline{\phi}_1(x; \lambda)$ *for* $(x, t) \in (0, 1) \times (0, T_m)$.

(iii) *If* $\lambda \in (\lambda_1, \infty)$*, then* $u(x, t; u_0)$ *blows up in a finite time.*

Proof. (i) Suppose that $\lambda \in (0, \lambda_1)$ and $k \in (0, 1)$. It follows from Theorem 1.1 that if $\overline{\lambda} \in (\lambda, \lambda_1)$ is sufficiently close to λ, then

$$u_0(x) \leq k\overline{\phi}_1(x; \lambda) \leq \overline{\phi}_1(x; \overline{\lambda}) \quad \text{for} \quad x \in (0, 1).$$

It is easily verified that $\overline{\phi}_1(\overline{\lambda})$ is a super-solution for (SP1). Then by Theorem 1.9, we see

$$u(x, t; u_0) \leq u(x, t; \overline{\phi}_1(\overline{\lambda})) \leq \overline{\phi}_1(x; \overline{\lambda}) \quad \text{for} \quad (x, t) \in (0, 1) \times (0, \infty). \tag{1.28}$$

Letting $\overline{\lambda} \downarrow \lambda$ in (1.28), we obtain $u(x, t; u_0) \leq \overline{\phi}_1(x; \lambda)$ for $(x, t) \in (0, 1) \times (0, \infty)$. For any fixed $\tau \geq 0$, we put $v(x, t) := u(x, t + \tau; \overline{\phi}_1(\overline{\lambda}))$; then v satisfies

$$\begin{cases} v_t = v_{xx} + \lambda |v|^{q-1} v + |v|^{p-1} v, & (x, t) \in (0, 1) \times (0, \infty), \\ v(0, t) = v(1, t) = 0, & t \in (0, \infty), \\ v(x, 0) = u(x, \tau; \overline{\phi}_1(\overline{\lambda})), & x \in (0, 1). \end{cases}$$

By (1.28), $v(x, 0) = u(x, \tau; \overline{\phi}_1(\overline{\lambda})) \leq \overline{\phi}_1(x; \overline{\lambda})$ for $(x, t) \in (0, 1) \times (0, \infty)$. Then it follows from Theorem 1.9 again that

$$u_t(x, t; \overline{\phi}_1(\overline{\lambda})) \leq 0 \quad \text{for} \quad (x, t) \in (0, 1) \times (0, \infty). \tag{1.29}$$

We observe that $\lim_{\lambda \downarrow 0} \underline{\phi}_1(\lambda) = 0$ in $C^2[0, 1]$ by Theorem 1.1. Thus it follows from the strong maximum principle for parabolic equations (see for instance [62, Chapter 3]) that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\lim_{\varepsilon \downarrow 0} \delta(\varepsilon) = 0$ and $u(x, \varepsilon; u_0) \geq \underline{\phi}_1(x; \delta)$ for $x \in (0, 1)$. Since $\underline{\phi}_1(\delta)$ is a sub-solution for (SP1) if *delta* $< \lambda$, then by Theorem 1.9 we have

$$u(x, t; u_0) \geq \underline{\phi}_1(x; \delta) \quad \text{for} \quad (x, t) \in (0, 1) \times (\varepsilon, \infty) \tag{1.30}$$

and

$$u_t(x,t;\underline{\phi}_1(\delta)) \geq 0 \quad \text{for} \quad (x,t) \in (0,1)\times(\varepsilon,\infty). \tag{1.31}$$

It follows from (1.28)-(1.31) there exist $u^*(x) := \lim_{t\to\infty} u(x,t;\overline{\phi}_1(\overline{\lambda}))$ and $u_*(x) := \lim_{t\to\infty} u(x,t;\underline{\phi}_1(\delta))$ for all $x \in (0,1)$. Clearly, $\underline{\phi}_1(x;\delta) \leq u_*(x) \leq u^*(x) \leq \overline{\phi}_1(x;\overline{\lambda})$ for $x \in (0,1)$. We now note that, by virtue of Theorem 1.1, if ϕ is a solution of (SP1) and satisfies $\underline{\phi}_1(\delta) \leq \phi \leq \overline{\phi}_1(\overline{\lambda})$ in $(0,1)$, then ϕ must be identical with $\underline{\phi}(\lambda)$. Because, $\|\underline{\phi}_1(\lambda)\|_\infty < \|\overline{\phi}_1(\overline{\lambda})\|_\infty < \|\overline{\phi}_1(\lambda)\|_\infty$. Therefore by a monotone method (see Sattinger [66]), we can deduce that $u^* \equiv u_* \equiv \underline{\phi}_1(\lambda)$ and $\lim_{t\to\infty} u(x,t;\underline{\phi}_1(\delta)) = \lim_{t\to\infty} u(x,t;\overline{\phi}_1(\overline{\lambda})) = \underline{\phi}_1(x;\lambda)$ for all $x \in (0,1)$. Moreover, by applying dynamical theory, it can be proved that $\lim_{t\to\infty} u(t;\underline{\phi}_1(\delta)) = \lim_{t\to\infty} u(t;\overline{\phi}_1(\overline{\lambda})) = \underline{\phi}_1(\lambda)$ in $C^1[\,0,1\,]$ (see e.g., [34], [45]). Then, together with (1.28)-(1.31), we obtain $\lim_{t\to\infty} \|u(t;u_0) - \underline{\phi}_1(\lambda)\|_{C^1} = 0$. It is easily verified that the statement of (i) holds for $\lambda = \lambda_1$.

(ii) Let $\lambda \in (0,\lambda_1]$ and $k \in (1,\infty)$. It follows from Theorem 1.1 that if $\underline{\lambda} \in (0,\lambda)$ is sufficiently close to λ, then $u_0(x) \geq k\overline{\phi}_1(x;\lambda) \geq \overline{\phi}_1(x;\underline{\lambda})$ for $x \in (0,1)$. Since $\overline{\phi}_1(\underline{\lambda})$ is a sub-solution for (SP1), then by Theorem 1.9, we have

$$u(x,t;u_0) \geq u(x,t;\overline{\phi}_1(\underline{\lambda})) \geq \overline{\phi}_1(\underline{\lambda}) \quad \text{for} \quad (x,t) \in (0,1)\times(0,\overline{T}) \tag{1.32}$$

and $u_t(x,t;\overline{\phi}_1(\underline{\lambda})) \geq 0$ for $(x,t) \in (0,1)\times(0,\overline{T})$, where $\overline{T}$ is a maximal existence time of $u(x,t;\overline{\phi}_1(\underline{\lambda}))$. Assume that $\|u(t;\overline{\phi}_1(\underline{\lambda}))\|_\infty$ is uniformly bounded for $t \in (0,\infty)$. Thus we see, by a similar way to the proof of (i), that there exists $u^* \in S_1^+(\lambda)$ such that $\lim_{t\to\infty} u(t;u_0) = u^*$ in $C^1[\,0,1\,]$. This clearly contradicts to Theorem 1.1. Therefore, it follows from (1.32) that $u(x,t;u_0)$ blows up or grows up; so that we put $T_m := \sup\{T > 0 : \|u(t;u_0)\|_\infty < \infty\}$, then (b) $T_m < \infty$ and $\lim_{t\uparrow T_m} \|u(t;u_0)\|_\infty = \infty$ or (g) $T_m = \infty$ and $\lim_{t\to\infty} \|u(t;u_0)\|_\infty = \infty$. Since $\|u(t;u_0)\|_\infty \leq \|u_x(t;u_0)\|$ for $t \in (0,T_m)$, then it follows that

$$T_m < \infty \quad \text{and} \quad \lim_{t\uparrow T_m} \|u_x(t;u_0)\| = \infty \quad \text{or} \tag{1.33}$$

$$T_m = \infty \quad \text{and} \quad \lim_{t\to\infty} \|u_x(t;u_0)\| = \infty. \tag{1.34}$$

We will exclude (1.34) by contradiction. Suppose that (1.34) is satisfied. Set for $v \in H_0^1$

$$\begin{aligned} &\varphi_1(v) = \frac{1}{2}\|v_x\|^2,\ \varphi_2(v) = \frac{1}{p+1}\|v\|_{p+1}^{p+1},\ \varphi_3(v) = \frac{1}{q+1}\|v\|_{q+1}^{q+1} \\ &\text{and}\ \ J(v) := \varphi_1(v) - \varphi_2(v) - \lambda\varphi_3(v). \end{aligned} \tag{1.35}$$

Multiplication of (P1) by $u_t(x,t;u_0)$ and integration with respect to x over $(0,1)$ give

$$\frac{d}{dt}J(u(t;u_0)) = -\|u_t(t;u_0)\|^2 \ \text{ for } \ t \in (0,\infty). \tag{1.36}$$

Since $J(u(t;u_0))$ is monotone decreasing in $t \in (0,\infty)$ by (1.36), then for any $\varepsilon > 0$

$$\begin{aligned} J(u(t;u_0)) &= \varphi_1(u(t;u_0)) - \varphi_2(u(t;u_0)) - \lambda\varphi_3(u(t;u_0)) \\ &\le J(u(\varepsilon;u_0)) =: C \ \text{ for } \ t \in [\,\varepsilon,\infty). \end{aligned} \tag{1.37}$$

We observe that

$$\varphi_3(v) \le d\varphi_2(v)^{(q+1)/(p+1)} \ \text{ for all } \ v \in L^{p+1} \tag{1.38}$$

with $d = (p+1)^{(q+1)/(p+1)}/(q+1)$ by Hölder's inequality. It follows from (1.37) and (1.38) that

$$\varphi_1(u(t;u_0)) - \varphi_2(u(t;u_0)) - \lambda d\varphi_2(u(t;u_0))^{(q+1)/(p+1)} \le C \ \text{ for } \ t \in [\,\varepsilon,\infty). \tag{1.39}$$

Multiplying (P1) by $u(x,t;u_0)$ and integrating with respect to x over $(0,1)$, we see

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t;u_0)\|^2 + 2\varphi_1(u(t;u_0)) &= \lambda(q+1)\varphi_3(u(t;u_0)) + (p+1)\varphi_2(u(t;u_0)) \\ &\ge (p+1)\varphi_2(u(t;u_0)) \ \text{ for } \ t \in [\,\varepsilon,\infty). \end{aligned} \tag{1.40}$$

By virtue of (1.34) and (1.39), we can show that for any $\delta \in (0,p-1)$ there exists $T_1 > 0$ such that

$$(p+1)\varphi_2(u(t;u_0)) - 2\varphi_1(u(t;u_0)) \ge \delta\varphi_2(u(t;u_0)) \ \text{ for } \ t \in [\,T_1,\infty). \tag{1.41}$$

Therefore, it follows from (1.40) and (1.41) that

$$\frac{1}{2}\frac{d}{dt}\|u(t;u_0)\|^2 \geq \delta\varphi_2(u(t;u_0)) = \frac{\delta}{p+1}\|u(t;u_0)\|_{p+1}^{p+1} \geq \frac{\delta}{p+1}\|u(t;u_0)\|^{p+1}. \tag{1.42}$$

for $t \in [T_1, \infty)$. In the last inequality of (1.42), we have used Hölder's inequality. It immediately follows that there exists $T_2 > T_1$ such that $\lim_{t\uparrow T_2}\|u(t;u_0))\| = \infty$, which contradicts to $T_m = \infty$. Then, (1.34) is excluded. It follows from (1.33) and (1.39) that $\lim_{t\uparrow T_m}\|u(t;u_0)\|_\infty = \infty$. In a similar way, we obtain (iii). Thus the proof is complete. □

Next we treat the stability analysis for (SP1) in case $\lambda < 0$. In this case, we note that $\lambda|u|^{q-1}u$ works as a strong absorption term. We will show that the zero solution is asymptotically stable and any other nonnegative stationary solutions give separatrixes between blowing up and extinction properties of solutions of (P1). The following lemma states an extinction property of solutions of (P1) with small initial data.

Lemma 1.13. *For any $\delta \in (0, |\lambda|^{1/(p-q)})$, there exists $T = T(\delta) > 0$ such that, if $u_0 \in L^\infty$ satisfies $\|u_0\|_\infty < \delta$, then the solution of* (P1) *satisfies $u(t;u_0) \equiv 0$ for $t \geq T$.*

Proof. Define a function w in $\boldsymbol{R}^+$ by

$$w(t) = \begin{cases} \{(1-q)(\lambda + \delta^{p-q})t + \delta^{1-q}\}^{1/(1-q)}, & \text{if } 0 \leq t \leq T, \\ 0, & \text{if } t \geq T, \end{cases} \tag{1.43}$$

where $T = \delta^{1-q}/\{(1-q)(|\lambda| - \delta^{p-q})\}$. It is seen by a direct calculation that w satisfies

$$\begin{cases} w_t = (\lambda + \delta^{p-q})w^q, & \text{for } t \geq 0, \\ w(0) = \delta. \end{cases}$$

Since $w(t) \leq \delta$ for $t \geq 0$ by (1.43), it follows that $w_t \geq \lambda w^q + w^p$ for $t \geq 0$; so that w is a super-solution for (P1). Hence, $-w$ is a sub-solution for (P1). Therefore, if

$u_0 \in L^\infty$ satisfies $\|u_0\|_\infty < \delta$, then Theorem 1.10 yields $-w(t) \le u(x,t;u_0) \le w(t)$ for $(x,t) \in (0,1) \times (0,\infty)$. Hence, (1.43) implies that $u(t;u_0) = 0$ in $(0,1)$ for $t \ge T$. The proof is complete. □

Theorem 1.14. *For any $u_0 \in L^\infty$, the solution $u(x,t;u_0)$ of* (P1) *satisfies the following properties.*

(i) *If $\lambda \in (\lambda_{-1}, 0)$ and $|u_0| \le k\overline{\phi}_1(\lambda)$ in $(0,1)$ with some $0 < k < 1$, then $|u(x,t;u_0)| < \overline{\phi}_1(x;\lambda)$ for $(x,t) \in (0,1) \times (0,\infty)$ and $u(x,t;u_0) = 0$ for $(x,t) \in (0,1) \times (T,\infty)$ with some $T > 0$.*

(ii) *If $\lambda \in (\lambda_{-1}, 0)$ and $u_0 \ge k\overline{\phi}_1(\lambda)$ (resp. $u_0 \le -k\overline{\phi}_1(\lambda)$) in $(0,1)$ with some $k > 1$, then $u(x,t;u_0)$ blows up in a finite time with respect to L^∞-norm and satisfies $u(x,t;u_0) \ge \overline{\phi}_1(x;\lambda)$ (resp. $u(x,t;u_0) \le -\overline{\phi}_1(x;\lambda)$) for $(x,t) \in (0,1) \times (0,T_m)$, where T_m is a maximal existence time of $u(x,t;u_0)$.*

(iii) *If $\lambda \in (-\infty, \lambda_{-1}]$ and $-k\phi(x;\lambda,x^*) \le u_0(x) \le k\phi(x;\lambda,x^{**})$ for $x \in (0,1)$ with some $0 < k < 1$, then $-\phi(x;\lambda,x^*) < u(x,t;u_0) < \phi(x;\lambda,x^{**})$ for $(x,t) \in (0,1) \times (0,\infty)$ and $u(x,t;u_0) = 0$ for $(x,t) \in (0,1) \times (T',\infty)$ with some $T' > 0$. Here, x^*, $x^{**} \in [M(\lambda), 1 - M(\lambda)]$ and $\phi(\,\cdot\,;\lambda,x^*)$, $\phi(\,\cdot\,;\lambda,x^{**}) \in D_1^+(\lambda)$.*

(iv) *If $\lambda \in (-\infty, \lambda_{-1}]$ and $u_0(x) \ge k\phi(x;\lambda,x^*)$ (resp. $u_0(x) \le -k\phi(x;\lambda,x^*)$) for $x \in (0,1)$ with some $k > 1$, then $u(x,t;u_0)$ blows up in a finite time with respect to L^∞-norm and satisfies $u(x,t;u_0) \ge \phi(x;\lambda,x^*)$ (resp. $u(x,t;u_0) \le -\phi(x;\lambda,x^*)$) for $(x,t) \in (0,1) \times (0,T_m)$.*

Proof. (i) Suppose that $\lambda \in (\lambda_{-1}, 0)$ and $0 < k < 1$, then $-(k\overline{\phi}_1(\lambda))_{xx} = k(\lambda\overline{\phi}_1(\lambda)^q + \overline{\phi}_1(\lambda)^p) \ge \lambda(k\overline{\phi}_1(\lambda))^q + (k\overline{\phi}_1(\lambda))^p$ in $(0,1)$; so that $k\overline{\phi}_1(\lambda)$ is a super-solution of (SP1). Similarly, $-k\overline{\phi}_1(\lambda)$ is a sub-solution of (SP1). Therefore, as in the proof of Theorem 1.12, we can show by Theorem 1.10, that if $|u_0| \le k\overline{\phi}_1(\lambda)$ in $(0,1)$, then $|u(x,t;u_0)| < \overline{\phi}_1(x;\lambda)$ for $(x,t) \in (0,1) \times (0,\infty)$. We observe that, if ϕ is a solution of (SP1) such that $-k\overline{\phi}_1(\lambda) \le \phi \le k\overline{\phi}_1(\lambda)$ in $(0,1)$, then $\phi \equiv 0$. Then, by the monotone method, we deduce that $\lim_{t\to\infty} \|u(t;u_0)\|_{C^1} = 0$. Thus there exists $T_1 = T_1(u_0) > 0$ such that $\|u(T_1;u_0)\|_\infty < |\lambda|^{1/(p-q)}$. Moreover, it follows from Lemma 1.13 that there exists $T_2 > T_1$ such that $u(x,t;u_0) = 0$ for

$(x,t) \in (0,1) \times (T_2, \infty)$. Thus the proof of (i) is complete. We can also show (iii) in the same way.

(ii) Let $\lambda \in (\lambda_{-1}, 0)$ and $k > 1$, then $k\overline{\phi}_1(\lambda)$ is a sub-solution for (SP1). Thus Theorem 1.10 implies that $u_t(x,t;k\overline{\phi}_1(\lambda)) \geq 0$ for $(x,t) \in (0,1) \times (0, T_k)$, where T_k is a maximal existence time of $u(x,t;k\overline{\phi}_1(\lambda))$. Observe that $\overline{\phi}_1(\lambda)$ is a unique solution of (SP1). Therefore, as the proof of (ii) of Theorem 1.12, we can show that $u(x,t;u_0)$ satisfies (b) $T_m < \infty$ and $\lim_{t \uparrow T_m} \|u(t;u_0)\|_\infty = \infty$ or (g) $T_m = \infty$ and $\lim_{t\to\infty} \|u(t;u_0)\|_\infty = \infty$. Here, T_m is a maximal existence time of $u(x,t;u_0)$. Moreover, it follows that

$$T_m < \infty \quad \text{and} \quad \lim_{t \uparrow T_m} \|u_x(t;u_0)\| = \infty \quad \text{or} \tag{1.44}$$

$$T_m = \infty \quad \text{and} \quad \lim_{t \to \infty} \|u_x(t;u_0)\| = \infty. \tag{1.45}$$

We will exclude (1.45) by contradiction. Suppose that (1.45) is satisfied. Then, it follows from (1.37) that

$$\varphi_1(u(t;k\overline{\phi}_1(\lambda))) - \varphi_2(u(t;k\overline{\phi}_1(\lambda))) \leq J(u(\varepsilon, u_0)) =: C \quad \text{for} \quad t \in [\,\varepsilon, \infty). \tag{1.46}$$

Multiplying (P1) by $u(x,t;u_0)$ and integrating with respect to x over $(0,1)$, we see

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(t;u_0)\|^2 + 2\varphi_1(u(t;u_0))\\ &= -|\lambda|(q+1)\varphi_3(u(t;u_0)) + (p+1)\varphi_2(u(t;u_0)) \quad \text{for} \quad t \in (0,\infty).\end{aligned}$$

It follows from (1.38) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(t;u_0)\|^2 + 2\varphi_1(u(t;u_0))\\ &\geq -|\lambda|(q+1)d\varphi_2(u(t;u_0))^{(q+1)/(p+1)} + (p+1)\varphi_2(u(t;u_0))\end{aligned} \tag{1.47}$$

for $t \in (0,\infty)$. By observing $\lim_{t\to\infty} \varphi_1(u(t;u_0) = \infty$ and (1.46), we can show that for any $\delta \in (0, p-1)$ there exists $T_1 > 0$ such that

$$\begin{aligned}&2\varphi_1(u(t;u_0)) + |\lambda|(q+1)d\varphi_2(u(t;u_0))^{(q+1)/(p+1)}\\ &\leq (p+1-\delta)\varphi_2(u(t;u_0)) \quad \text{for} \quad t \in [\,T_1, \infty).\end{aligned} \tag{1.48}$$

Therefore, by virtue of (1.47) and (1.48), we obtain (1.42) for $t \in [T_1, \infty)$. It follows that there exists $T_2 > T_1$ such that $\lim_{t \uparrow T_2} \|u(t; u_0)\| = \infty$, which contradicts to $T_m = \infty$. Then, (1.45) is excluded. It follows from (1.44) and (1.46) that $\lim_{t \uparrow T_m} \|u(t; u_0)\|_{p+1} = \infty$. This implies $\lim_{t \uparrow T_m} \|u(t; u_0)\|_\infty = \infty$. Thus the proof of (ii) is complete. We can also show (iv) in the same way. $\square$

Chapter 2

Higher-Dimensional Case

2.1 Problem

In this paper we are concerned with the following reaction-diffusion equation

$$(\mathrm{P2})\quad \begin{cases} u_t = \Delta u + f(u), & (x,t) \in B_R \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,\infty), \\ u(x,0) = u_0(x), & x \in B_R, \end{cases}$$

where $B_R = \{x \in \mathbf{R}^N \mid |x| < R\}$ and $N \geq 3$. We assume that f satisfies the *concave-convex* condition. A typical example is

$$(\mathrm{f1})\quad f(u) = |u|^{q-1}u + |u|^{p-1}u \quad (0 < q < 1 < p < (N+2)/(N-2)).$$

The associate stationary problem to (P) is

$$\begin{cases} \Delta \phi + f(\phi) = 0, & x \in B_R, \\ \phi = 0, & x \in \partial B_R. \end{cases} \tag{2.1}$$

For a radially symmetric solution $\phi = \phi(r)$, $r = |x|$, (2.1) is reduced to

$$(\mathrm{E})\quad \begin{cases} (r^{N-1}\phi_r)_r + r^{N-1}f(\phi) = 0, & 0 < r < R, \\ \phi_r(0) = \phi(R) = 0. \end{cases}$$

It is shown that our assumption allows us to apply Gidas-Ni-Nirenberg's result [31] to (2.1); so that any positive solution of (2.1) is radially symmetric and satisfies (E).

For the case (f1), as a problem in a general bounded domain, (2.1) was studied by Ambrosetti-Brézis-Cerami [6], Ambrosetti-Azorero-Peral [5], Boccardo-Escobedo-Peral [9], Bartsh-Willem [8] and Cabré-Majer [11]. Further, Adimurthi-Pacella-Yadava [1] have studied (E) for case (f1). By their results, in the case when $p \in (1, (N+2)/(N-2)$, it is known that (E) with nonlinearity (f1) has at least two positive solutions which involve a minimal solution if $R < R_1$ with some $R_1 > 0$ and infinitely many sign changing solutions for any $R > 0$. Recently, Ouyang-Shi have obtained, in the series of studies [56]-[58], the *exact* multiplicity for positive solutions under the additional condition $p \in (1, N/(N-2))$; (E) possesses *exactly* two positive solutions if $R < R_1$; unique positive solution if $R = R_1$ and no positive solution if $R > R_1$. Our main purpose is to show the exact number of positive solutions for every $p \in (1, (N+2)/(N-2))$.

This chapter consists of six sections. In section 2, we describe some conditions on f and give our main result (Theorem 2.2). In Section 3, we collect preliminary results about some properties for solutions of the related initial value problem. Section 4 is devoted to the proof of Theorem 2.2. In Section 5, we discuss the structure of n-nodal solutions of (E). In Section 6, we state the relations between the positive stationary solution set and time-depending behaviors of nonnegative solutions (global existence or blowing up) to the non-stationary problem (P2).

2.2 Assumptions and Main Result

2.2.1 Assumptions on f

Throughout this chapter, the nonlinearity f is assumed to fulfill the following conditions:

Assumption (A)

(A1) $f \in C(\boldsymbol{R}) \cap C^2(\boldsymbol{R} \setminus \{0\})$;

(A2) $f(u) > 0$ and $f(u) = -f(-u)$ for $u > 0$;

(A3) there exists $a > 0$ such that $f''(u) < 0$ for $u \in (0, a)$ and $f''(u) > 0$ for $u \in (a, \infty)$;

(A4) there exists $p \in \left(1, \dfrac{N+2}{N-2}\right)$ such that $\lim_{u \to \infty} \dfrac{f(u)}{u^p} > 0$;

(A5) for some $\mu > \dfrac{N-2}{2}$,

$$k(u, \mu) := (N - \mu) f(u) - \mu f'(u) u$$

is a non-decreasing positive function with respect to $u > 0$ and satisfies

$$k(Cu, \mu) \le C k(u, \mu) \quad \text{for all} \quad u > 0 \text{ and } C \ge 1.$$

As typical examples of f, we can give

(f1) $f(u) = |u|^{q-1}u + |u|^{p-1}u \quad (0 < q < 1 < p < (N+2)/(N-2))$,

(f2) $f(u) = \sum_{i=1}^{j} a_i |u|^{p_i - 1} u$
$(a_i > 0,\ 0 < p_1 < p_2 < \cdots < p_{j-1} < 1 < p_j < (N+2)/(N-2))$,

(f3) $f(u) = \dfrac{u}{|u|+1} + |u|^{p-1}u \quad (2 < p < (N+2)/(N-2),\ N \le 5)$.

Indeed, (A1)-(A4) are clearly satisfied. In particular for (f1), since

$$k(u, \mu) = \{N - \mu(q+1)\} u^q + \{N - \mu(p+1)\} u^p \quad \text{for} \quad u > 0,$$

letting $\mu = N/(p+1)$ we see

$$k\left(u, \frac{N}{p+1}\right) = \frac{(p-q)N}{p+1} u^q \quad \text{for} \quad u > 0.$$

Thus (f1) satisfies (A5). Similarly, it is possible to verify that (f2) and (f3) also satisfy (A5).

2.2.2 Main Result

Let $\phi(r,\alpha)$ be the solution of the related initial value problem;

$$\begin{cases} (r^{N-1}\phi_r)_r + r^{N-1}f(\phi) = 0, \quad r > 0, \\ \phi(0) = \alpha, \quad \phi_r(0) = 0, \end{cases} \tag{2.2}$$

where α is a positive parameter since f is an odd function. Our strategy is to investigate the n-th zero $z_n(\alpha)$ of $\phi(r,\alpha)$. Before stating our results, consider the case $f(\phi) = |\phi|^{p-1}\phi$ with $p \in (0,(N+2)/(N-2))$. In this case, it is well known that $\phi(r,\alpha)$ has infinitely many zeros $\{z_n(\alpha)\}$ and easily verified that $\phi(r,\alpha) = \alpha\phi(\alpha^{(p-1)/2}r, 1)$ for any $\alpha > 0$. Therefore, for each $n \in \boldsymbol{N}$,

$$z_n(\alpha) = \alpha^{(1-p)/2} z_n(1) \quad \text{for} \quad \alpha > 0. \tag{2.3}$$

It follows from (2.3) that; for the sublinear case $p \in (0,1)$,
(i) $z_n(\,\cdot\,)$ is monotone increasing in $(0,\infty)$, $\lim_{\alpha\downarrow 0} z_n(\alpha) = 0$ and $\lim_{\alpha\uparrow\infty} z_n(\alpha) = +\infty$;
while, for the superlinear case $p \in (1,(N+2)/(N-2))$,
(ii) $z_n(\,\cdot\,)$ is monotone decreasing in $(0,\infty)$, $\lim_{\alpha\downarrow 0} z_n(\alpha) = +\infty$ and $\lim_{\alpha\uparrow\infty} z_n(\alpha) = 0$.
These facts (i) and (ii) imply that for any $R > 0$ and $n \geq 1$, (E) has a unique n-nodal solution with $\phi(0) > 0$ in case $p \in (0,1)$ and $p \in (1,(N+2)/(N-2))$, respectively. Recently, Kajikiya [35] has given a necessary and sufficient condition on f for the validity of (i). Moreover, Yanagida [71] has shown (ii) in case $f(r,\phi) = K(r)|\phi|^{p-1}\phi$ when $p > 1$ and $K(\,\cdot\,)$ satisfies a suitable condition.

So it would be natural to ask how $\{z_n(\alpha)\}$ behaves when f is a *concave-convex* function like (f1). Under Assumption (A), we will show the following behavior for the first zero $z_1(\alpha)$:

Proposition 2.1. *There exists a positive number α^* such that $z_1(\,\cdot\,)$ is strictly monotone increasing in $(0,\alpha^*)$ and strictly monotone decreasing in (α^*,∞). Moreover, $\lim_{\alpha\uparrow\infty} z_1(\alpha) = 0$ and $\lim_{\alpha\downarrow 0} z_1(\alpha) = C/\sqrt{f'(+0)}$, where C is a positive constant independent of f and $f'(+0) := \lim_{u\downarrow 0} f(u)/u$.*

It follows from Proposition 2.1 that if $R \in (C/\sqrt{f'(+0)}, z_1(\alpha^*))$, then $z_1(\alpha) = R$ has exactly two solutions $\alpha = \overline{\alpha}, \underline{\alpha}$ ($\overline{\alpha} > \underline{\alpha}$); so that (E) has exactly two positive solutions $\overline{\phi}(r; R) := \phi(r, \overline{\alpha})$ and $\underline{\phi}(r; R) := \phi(r, \underline{\alpha})$. To be precise, we obtain complete information on the structure of

$$S^+(R) := \{\phi \in C^2([\,0, R\,]) \,:\, \phi \text{ is a positive solution of (E)}\}.$$

Theorem 2.2. *There exists a positive number R_1 such that*

$$S^+(R) = \begin{cases} \{\overline{\phi}(\,\cdot\,; R)\} & \text{if } \; R \in (0, C/\sqrt{f'(+0)}\,], \\ \{\overline{\phi}(\,\cdot\,; R), \underline{\phi}(\,\cdot\,; R)\} & \text{if } \; R \in (C/\sqrt{f'(+0)}, R_1), \\ \{\phi^*(\,\cdot\,; R)\} & \text{if } \; R = R_1, \\ \emptyset & \text{if } \; R \in (R_1, \infty). \end{cases}$$

Here, $\overline{\phi}(r; R)$, $\underline{\phi}(r; R)$ and $\phi^*(r; R)$ are strictly monotone decreasing for $r \in (0, R)$, and satisfy

$$\lim_{R \downarrow 0} \overline{\phi}(0; R) = \infty, \qquad \lim_{R \downarrow C/\sqrt{f'(+0)}} \underline{\phi}(0; R) = 0,$$

$$\lim_{R \uparrow R_1} \overline{\phi}(0; R) = \lim_{R \uparrow R_1} \underline{\phi}(0; R) = \phi^*(0; R_1).$$

Furthermore, if $R \in (C/\sqrt{f'(+0)}, R_1)$, then

$$\overline{\phi}(r; R) > \underline{\phi}(r; R) \quad \text{for} \quad r \in [\,0, R). \tag{2.4}$$

In Theorem 2.2, $R_1 = z_1(\alpha^*)$. Though (2.4) does not follow from Proposition 2.1, we will give a proof anew.

Remark 2.1. For $f(\phi) = \phi^q + \phi^p \quad (1 \le q < p < (N+2)/(N-2))$, (E) has been discussed by many authors and the structure of positive solution set is quite different from the preceding case $q < 1$. In case $1 = q < p < (N+2)/(N-2)$, it is known that (E) has a unique positive solution if $R < \sqrt{\lambda_1}$, where λ_1 is the least eigenvalue of $-\Delta$ in B_1 with a homogeneous Dirichlet boundary condition (see Zhang [73], Kwong-Li [43], Srikanth [69] and Adimurthi-Yadava [2]). For $1 < q < p < (N+2)/(N-2)$, Erbe-Tang [27] have proved (E) has a unique positive solution for any R with an additional condition on p and q (see also Zhang [74]).

Remark 2.2. In view of (0.3), Theorem 2.2 enables us to get a similar $\supset$-shaped solution branch of (SP1) in a higher dimensional case.

2.3 Preliminary Results

First we collect fundamental properties of solutions of (2.2).

Lemma 2.3. *Suppose that* (A1)-(A4) *hold. Then the following properties are satisfied.*
(i) *For any* $\alpha \in \boldsymbol{R}$, (2.2) *has a unique solution* $\phi(r,\alpha)$. *Furthermore, if* $\alpha \neq 0$, *then*

$$|\phi(r,\alpha)| < \alpha \quad for \quad r \in (0,\infty). \tag{2.5}$$

(ii) *For any* $\alpha \neq 0$, $\phi(r,\alpha)$ *has infinitely many zeros* $\{z_n(\alpha)\}$ *such that*

$$0 < z_1(\alpha) < z_2(\alpha) < \cdots < z_n(\alpha) < \cdots \uparrow \infty \quad as \quad n \uparrow \infty.$$

(iii) *For any* $\alpha \neq 0$, $\phi_r(r,\alpha)$ *has infinitely many zeros* $\{t_n(\alpha)\}$ *such that* $t_0(\alpha) = 0$ *and* $t_n(\alpha) \in (z_n(\alpha), z_{n+1}(\alpha))$.

Proof. (i) If f is locally Lipschitz continuous in $\boldsymbol{R}$, then we can show the local existence and uniqueness of solutions of (2.2) by applying the contraction mapping theorem to the integral equation from (2.2)

$$\phi(r) = \alpha - \frac{1}{N-2}\int_0^r s\left\{1 - \left(\frac{s}{r}\right)^{N-2}\right\} f(\phi(s))ds.$$

When f does not satisfy the local Lipschitz continuity at the origin, (A1) and (A2) are enough to show the above result, see e.g., [35, Lemma 2.1]. Multiplying by ϕ_r the both sides of

$$\phi_{rr}(r,\alpha) + \frac{N-1}{r}\phi_r(r,\alpha) + f(\phi(r,\alpha)) = 0 \tag{2.6}$$

and integrating with respect to r, we have

$$\frac{1}{2}\phi_r(r,\alpha)^2 + F(\phi(r,\alpha)) = F(\alpha) - \int_0^r \frac{N-1}{s}\phi_r(s,\alpha)^2 ds \tag{2.7}$$

for $r > 0$, where $F(\phi) = \int_0^\phi f(s)ds$. Hence (2.5) follows from (2.7).

(ii) By (A1)-(A4), there exists $m > 0$ such that $f(\phi)/\phi \geq m$ for $\phi \in \boldsymbol{R} \setminus \{0\}$. Thus the property (ii) can be shown by applying Sturm's oscillation theorem (see e.g., [15, Chapter 8]) to

$$(r^{N-1}\phi_r)_r + r^{N-1}\frac{f(\phi)}{\phi}\phi = 0, \quad r > 0 \tag{2.8}$$

and

$$(r^{N-1}\phi_r)_r + r^{N-1}m\phi = 0, \quad r > 0.$$

(iii) It follows from (2.6) that $\phi_{rr}(t) = -f(\phi(t))$ at any positive critical point $r = t$ of $\phi(r,\alpha)$. Thus by (A2), $\operatorname{sign}\phi_{rr}(t) = \operatorname{sign}(-\phi(t))$, which implies (iii). □

For the case $f \in C^1(\boldsymbol{R})$, let $\varPhi(r,\alpha)$ be the solution of the linearized equation of (2.2) at $\phi(r,\alpha)$:

$$\begin{cases} (r^{N-1}\varPhi_r)_r + r^{N-1}f'(\phi(r,\alpha))\,\varPhi = 0, \quad r > 0, \\ \varPhi(0) = 1 \quad \varPhi_r(0) = 0. \end{cases} \tag{2.9}$$

Differentiating (2.2) with respect to α, we see that the unique solution of (2.9) is given by $\varPhi(r,\alpha) = \phi_\alpha(r,\alpha)$. Even if f is not differentiable at the origin, our assumptions enable us to obtain the uniqueness and existence of *weak* solutions of (2.9) in the following sense.

Lemma 2.4. *Assume* (A1)-(A4)*. Then* (2.9) *has a unique weak solution* $\varPhi(r,\alpha)$ *for any* $\alpha \in \boldsymbol{R}$ *in the sense of*

$$\varPhi(\,\cdot\,,\alpha) \in C^2([\,0,\infty) \setminus \cup_{n=1}^{\infty}\{z_n(\alpha)\}) \cap C^1([\,0,\infty)).$$

Proof. Suppose that f breaks the Lipschitz continuity at the origin. By (A1), $f'(\phi(r,\alpha))$ is continuous except for $z_n(\alpha)$ ($n = 1, 2, \dots$); so that (2.9) has a unique classical solution $\Phi(r)$ in $[0, z_1(\alpha))$.

As the first step of the proof, we show that $\Phi(r)$ is bounded in the topology of $C^1([0, z_1(\alpha)])$. By integrating (2.9) with respect to r, we have

$$\Phi_r(r) = -\int_0^r \left(\frac{s}{r}\right)^{N-1} f'(\phi(s,\alpha))\Phi(s)\,ds, \tag{2.10}$$

and moreover,

$$\Phi(r) = 1 - \frac{1}{N-2}\int_0^r s\left\{1 - \left(\frac{s}{r}\right)^{N-2}\right\} f'(\phi(s,\alpha))\Phi(s)\,ds. \tag{2.11}$$

Observe that $f'(\phi(r,\alpha))$ is integrable in a neighborhood of $r = z_1(\alpha)$ for any $\alpha \neq 0$. Indeed, let r_1 and r_2 be any positive numbers such that $r_1 < z_1(\alpha) < r_2 < t_1(\alpha)$. Since $|\phi_r(r,\alpha)| > 0$ for $r \in [r_1, r_2]$, then $\delta := \min_{r\in[r_1,r_2]} |\phi_r(r,\alpha)| > 0$. Thus it follows that

$$\begin{aligned}
\int_{r_1}^{r_2} f'(\phi(r,\alpha))\,dr &= \int_{\phi(r_1,\alpha)}^{\phi(r_2,\alpha)} f'(v)\frac{1}{\phi_r(r,\alpha)}\,dv \\
&\le \frac{1}{\delta}\left|\int_{\phi(r_1,\alpha)}^{\phi(r_2,\alpha)} f'(v)\,dv\right| \\
&= \frac{1}{\delta}\,|f(\phi(r_2,\alpha)) - f(\phi(r_1,\alpha))| < \infty.
\end{aligned}$$

Consequently, we can apply Gronwall's inequality to (2.11). So we get the boundness for $\Phi(r)$ in $C([0, z_1(\alpha)])$. Furthermore, by virtue of (2.10), $\Phi(r)$ is also bounded in $C^1([0, z_1(\alpha)])$.

Next we show that this $\Phi(r)$ can be uniquely extended as a weak solution of (2.9) in $C^2([0, z_1(\alpha)) \cup (z_1(\alpha), z_2(\alpha))) \cap C^1([0, z_2(\alpha)])$. Consider the following initial value problem

$$\begin{cases} (r^{N-1}\Phi_r)_r + r^{N-1} f'(\phi(r,\alpha))\,\Phi = 0, & r > z_1(\alpha), \\ \Phi(z_1(\alpha)) = \beta, \quad \Phi_r(z_1(\alpha)) = \gamma, \end{cases} \tag{2.12}$$

where $\beta := \lim_{r \uparrow z_1(\alpha)} \varPhi(r)$ and $\gamma := \lim_{r \uparrow z_1(\alpha)} \varPhi_r(r)$. The corresponding integral equation is

$$\begin{aligned} \varPhi(r) = \beta + \frac{1}{N-2}\Bigg[\gamma z_1(\alpha)\left\{1 - \left(\frac{z_1(\alpha)}{r}\right)^{N-2}\right\} \\ - \int_{z_1(\alpha)}^{r} s\left\{1 - \left(\frac{s}{r}\right)^{N-2}\right\} f'(\phi(s,\alpha))\varPhi(s)\, ds\Bigg]. \end{aligned} \tag{2.13}$$

For each continuous function $\varPhi(r)$, we define $(\Phi\varPhi)(r)$ by the right hand side of (2.13). So we can see, by standard arguments, that Φ becomes a strictly contraction map in the following bounded closed set

$$\{\varPhi \in C([\, z_1(\alpha), z_1(\alpha)+\varepsilon\,]) \; : \; \sup_{r\in[\, z_1(\alpha), z_1(\alpha)+\varepsilon\,]} |\varPhi(r) - \beta| \le M\}$$

for suitable positive numbers M and ε. Then, (2.13) has a unique solution $\varPhi(r)$ in $[\, z_1(\alpha), z_1(\alpha)+\varepsilon\,]$; so does (2.12). Obviously, this $\varPhi(r)$ can be uniquely extended as a solution of (2.12) in $[\, z_1(\alpha), z_2(\alpha)\,]$. Then, together with the result in the first step, (2.9) has the unique weak solution

$$\varPhi(r) \in C^2([\,0, z_1(\alpha)) \cup (z_1(\alpha), z_2(\alpha))) \cap C^1([\,0, z_2(\alpha)\,]).$$

Repeating the above arguments in each interval $[\, z_n(\alpha), z_{n+1}(\alpha)\,]$, we can obtain a unique solution of (2.9) in the class

$$\varPhi(r) \in C^2([\,0,\infty) \setminus \cup_{n=1}^{\infty}\{z_n(\alpha)\}) \cap C^1([\,0,\infty)).$$

Thus the proof of Lemma 2.4 is accomplished. □

Remark 2.3. When f breaks the Lipschitz continuity at the origin, $\varPhi_{rr}(r,\alpha)$ is singular at $r = z_n(\alpha)$ if $\varPhi(z_n(\alpha),\alpha) \neq 0$. However, the Sturm oscillation argument is still applicable to weak solutions in the sense of Lemma 2.4.

Lemma 2.5. *Suppose that* (A1)-(A4) *hold. Then there exists* $b > a$ *such that*

$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{f(s)}{s}\right) < 0 \quad \textit{for} \quad s \in (0,b), \\ \dfrac{d}{ds}\left(\dfrac{f(s)}{s}\right) > 0 \quad \textit{for} \quad s \in (b,\infty). \end{cases}$$

Proof. Recall that $f''(s) < 0$ for $s \in (0, a)$ by (A3), one can see $f'(s) < f(s)/s$ for $s \in (0, a)$, or equivalently

$$\left(\frac{f(s)}{s}\right)' = \frac{f'(s)s - f(s)}{s^2} < 0 \quad \text{for} \quad s \in (0, a).$$

Thus if we set

$$b := \sup\{c > 0 \ : \ f'(s)s < f(s) \quad \text{for all} \quad s \in (0, c)\}, \tag{2.14}$$

then $a < b < \infty$. Indeed, if $b = \infty$, then $f(s) \leq Ms$ for all $s \in (a, \infty)$ with some positive M, which contradicts (A4). Accordingly, we see $f'(b)b - f(b) = 0$. Together with

$$(f'(s)s - f(s))' = f''(s)s > 0 \quad \text{for} \quad s \in (b, \infty),$$

we obtain $f'(s)s - f(s) > 0$ for $s \in (b, \infty)$; so that

$$\left(\frac{f(s)}{s}\right)' > 0 \quad \text{for} \quad s \in (b, \infty).$$

Thus the proof of Lemma 2.5 is complete. □

2.4 Proof of Theorem 2.2

2.4.1 Proof of Proposition 2.1

In this subsection, we will give a proof of Proposition 2.1. To begin with, we introduce the following lemmas.

Lemma 2.6. *Suppose that* (A1)-(A4) *hold. Then the following properties hold true.*

(i) *For each $n \in \boldsymbol{N}$, $z_n'(\alpha) > 0$ if $\alpha \in (0, b)$, where b is the positive number obtained in Lemma 2.5.*

(ii) $\lim_{\alpha\downarrow 0} z_n(\alpha) = \sqrt{\lambda_n/f'(+0)}$, *where* $f'(+0) := \lim_{\phi\to 0} f(\phi)/\phi$ *and* λ_n *is the* n*-th eigenvalue of*

$$\begin{cases} (r^{N-1}w_r)_r + \lambda r^{N-1}w = 0, & r \in (0,1), \\ w_r(0) = w(1) = 0. \end{cases}$$

Lemma 2.7. *Suppose that* (A1)-(A4) *hold. Then* $\lim_{\alpha\to\infty} z_n(\alpha) = 0$ *for each* $n \in \boldsymbol{N}$.

Though (i) of Lemma 2.6 follows from [35, Theorem 1], we will give the proof for later arguments. The proof of Lemma 2.7 can be found in McLeod-Troy-Weissler [44]; so we omit it.

Proof of Lemma 2.6. (i) Differentiating the both sides of $\phi(z_n(\alpha),\alpha) = 0$, we have

$$\phi_r(z_n(\alpha),\alpha)z_n'(\alpha) + \Phi(z_n(\alpha),\alpha) = 0.$$

Since $\phi_r(z_n(\alpha),\alpha) \neq 0$,

$$z_n'(\alpha) = -\frac{\Phi(z_n(\alpha),\alpha)}{\phi_r(z_n(\alpha),\alpha)}. \tag{2.15}$$

In order to see $z_n'(\alpha) > 0$ for $\alpha \in (0,b)$, it is sufficient to show that

$$z_n(\alpha) < Z_n(\alpha) < t_n(\alpha) \quad \text{for} \quad \alpha \in (0,b),\ n \in \boldsymbol{N}, \tag{2.16}$$

where $Z_n(\alpha)$ is the n-th zero of $\Phi(r,\alpha)$. Indeed, in case $\alpha \in (0,b)$, if n is an odd number, then $\phi_r(z_n(\alpha),\alpha) < 0$ and $\Phi(z_n(\alpha),\alpha) > 0$ by (2.16). On the other hand, if n is an even number, then $\phi_r(z_n(\alpha),\alpha) > 0$ and $\Phi(z_n(\alpha),\alpha) < 0$ by (2.16). Thus for each $n \in \boldsymbol{N}$, $z_n'(\alpha) > 0$.

We will show (2.16). First we prove that

$$\Phi(r,\alpha) \text{ has at least one zero point in } (t_{n-1}(\alpha), t_n(\alpha)) \tag{2.17}$$

for any $\alpha > 0$ and $n \in \boldsymbol{N}$. We set $\psi(r,\alpha) := \phi_r(r,\alpha)$. Since $\psi(r,\alpha)$ satisfies

$$(r^{N-1}\psi_r)_r + r^{N-1}\left(f'(\phi(r,\alpha)) - \frac{N-1}{r^2}\right)\psi = 0 \quad \text{for} \quad r > 0, \tag{2.18}$$

(2.17) is verified for any $\alpha > 0$ and $n \geq 2$ by applying Sturm's oscillation theorem to (2.9) and (2.18). We will show (2.17) for $n = 1$. Letting $r \downarrow 0$ in (2.6), we have $\phi_{rr}(0, \alpha) = \psi_r(0, \alpha) = -f(\alpha)/N < 0$. Consequently,

$$\lim_{r\downarrow 0} \frac{r^{N-1}\psi_r(r)}{\psi(r)} = \lim_{r\downarrow 0} \frac{r^{N-2}\phi_{rr}(r)}{\phi_r(r)/r} = \lim_{r\downarrow 0} r^{N-2}\frac{\phi_{rr}(0)}{\phi_{rr}(0)} = 0.$$

Together with $\varPhi(0) = 1$ and $\varPhi_r(0) = 1$, we have

$$\lim_{r\downarrow 0} \frac{r^{N-1}\psi_r(r)}{\psi(r)} = \lim_{r\downarrow 0} \frac{r^{N-1}\varPhi_r(r)}{\varPhi(r)} = 0.$$

Thus, Sturm's theorem implies $\psi(r) > 0$ for all $r \in [\,0, Z_1(\alpha)\,]$; so that $Z_1(\alpha) < t_1(\alpha)$.

Next we show that

$$\phi(r,\alpha) \text{ has at least one zero in } (Z_{n-1}(\alpha), Z_n(\alpha)), \text{ where } Z_0(\alpha) = 0 \tag{2.19}$$

for any $\alpha \in (0, b)$ and $n \in \boldsymbol{N}$. It follows from (2.5) and (2.14) that

$$\frac{f(\phi(r,\alpha))}{\phi(r,\alpha)} > f'(\phi(r,\alpha)) \quad \text{for} \quad r > 0,\ \alpha \in (0, b).$$

Moreover,

$$\lim_{r\downarrow 0} \frac{r^{N-1}\phi_r(r,\alpha)}{\phi(r,\alpha)} = \lim_{r\downarrow 0} \frac{r^{N-1}\varPhi_r(r,\alpha)}{\varPhi(r,\alpha)} = 0.$$

By Sturm's theorem (2.19) holds true. Hence (2.16) follows from (2.17) and (2.19).

(ii) First we treat the case $f \in C^1(\boldsymbol{R})$. Set $y(r,\alpha) := \alpha^{-1}\phi(r,\alpha)$, then (2.7) implies

$$\sup_{r>0} y(r,\alpha) = y(0,\alpha) = 1. \tag{2.20}$$

Furthermore, $y(r,\alpha)$ satisfies

$$\begin{cases} (r^{N-1}y_r)_r + r^{N-1}\dfrac{f(\phi(r,\alpha))}{\phi(r,\alpha)}y = 0, \quad r > 0, \\ y(0,\alpha) = 1, \quad y_r(0,\alpha) = 0\,; \end{cases} \tag{2.21}$$

so that

$$y(r,\alpha) = 1 - \frac{1}{N-2}\int_0^r s\left\{1 - \left(\frac{s}{r}\right)^{N-2}\right\}\frac{f(\phi(s,\alpha))}{\phi(s,\alpha)}y(s,\alpha)\,ds \tag{2.22}$$

for $r > 0$. Since

$$\lim_{\alpha \to 0} \frac{f(\phi(r,\alpha))}{\phi(r,\alpha)} = f'(0) \quad \text{uniformly for } r \in \boldsymbol{R},$$

then by (2.20) and (2.21), for any fixed $M > 0$, $y(r,\alpha)$ is uniformly bounded in $C^1([\,0, M\,])$ with respect to $\alpha > 0$. Thus letting $\alpha \to 0$ in (2.22), we see that there exists $y^\infty \in C^2([\,0,\infty))$ such that $\lim_{\alpha \to 0} y(\,\cdot\,,\alpha) = y^\infty$ in $C([\,0, M\,])$ and y^∞ satisfies

$$\begin{cases} (r^{N-1} y_r^\infty)_r + r^{N-1} f'(0)\, y^\infty = 0, \quad r > 0, \\ y^\infty(0) = 1, \quad y_r^\infty(0) = 0. \end{cases}$$

Therefore, if we denote by l_n the n-th zero point of y^∞, then $\lim_{\alpha \to 0} z_n(\alpha) = l_n$. Moreover, it is easily verified that $l_n = \sqrt{\lambda_n / f'(0)}$.

Next we show $\lim_{\alpha \to 0} z_n(\alpha) = 0$ in the case where f is not differentiable at the origin. Let $\{\alpha_j\}$ be any positive sequence satisfying $\lim_{j \to \infty} \alpha_j = 0$. By (A1)-(A4), there exists a positive sequence $\{m_j\}$ such that $f(\phi)/\phi \geq m_j$ for any $\phi \in (0, \alpha_j)$ and $\lim_{j \to \infty} m_j = \infty$. Denote by $l_n(m_j)$ the n-th zero of the solution of

$$\begin{cases} (r^{N-1} w_r)_r + r^{N-1} m_j w = 0, \quad r > 0, \\ w(0) = 1 \quad w_r(0) = 0. \end{cases} \tag{2.23}$$

Thus by applying Sturm's oscillation theorem to (2.8) and (2.23), one can see

$$z_n(\alpha_j) < l_n(m_j) \quad \text{for} \quad j \in \boldsymbol{N}. \tag{2.24}$$

Since $\lim_{j \to \infty} l_n(m_j) = 0$ for each $n \in \boldsymbol{N}$, letting $j \to \infty$ in (2.24) yields $\lim_{j \to \infty} z_n(\alpha_j) = 0$. Thus the proof of Lemma 2.6 is complete. □

We set

$$K := \{\alpha \in (b,\infty) \;:\; z_1'(\alpha) = 0\}.$$

It follows from Lemmas 2.6 and 2.7 that K is not empty. To accomplish the proof of Proposition 2.1, it suffices to prove the nonexistence of local minimum elements of K. In order to get the nonexistence, we need the following lemma:

Lemma 2.8. *Suppose that* (A1)-(A5) *hold true. Then,*

$$z_1(\alpha) < Z_2(\alpha) \quad \textit{almost everywhere in} \quad (0,\infty).$$

The proof of Lemma 2.8 is the most technical part of the chapter. We will give it in the next subsection.

Proof of Proposition 2.1. We continue the proof of Proposition 2.1. Let α^* be any element in K. Then (2.15) implies $\Phi(z_1(\alpha^*),\alpha^*) = 0$. Hence it follows from Lemma 2.8 that $z_1(\alpha^*) = Z_1(\alpha^*)$ or $z_1(\alpha^*) = Z_2(\alpha^*)$. If $z_1(\alpha^*) = Z_2(\alpha^*)$, then by Lemma 2.8, $Z_1(\alpha) < z_1(\alpha) < Z_2(\alpha)$ for all $\alpha \in (\alpha^* - \delta, \alpha^*) \cup (\alpha^*, \alpha^* + \delta)$ with some positive δ. Therefore, $\Phi(z_1(\alpha),\alpha) < 0$ for $\alpha \in (\alpha^* - \delta, \alpha^*) \cup (\alpha^*, \alpha^* + \delta)$. By virtue of (2.15),

$$z_1'(\alpha) = -\frac{\Phi(z_1(\alpha),\alpha)}{\phi_r(z_1(\alpha),\alpha)} < 0 \quad \text{for all} \quad \alpha \in (\alpha^* - \delta, \alpha^*) \cup (\alpha^*, \alpha^* + \delta)\,;$$

so that $z_1(\alpha)$ is a strictly monotone decreasing function in $(\alpha^* - \delta, \alpha^* + \delta)$.

Suppose that $z_1(\alpha^*) = Z_1(\alpha^*)$. To accomplish the proof, it is sufficient to prove that $z_1(\alpha^*)$ is a strict local maximum. Twice differentiation with respect to α of $\phi(z_1(\alpha),\alpha) = 0$ yields

$$\begin{aligned}\phi_{rr}(z_1(\alpha),\alpha)z_1'(\alpha)^2 + \phi_r(z_1(\alpha),\alpha)z_1''(\alpha)&\\ + 2\,\Phi_r(z_1(\alpha),\alpha)z_1'(\alpha) + \Phi_\alpha(z_1(\alpha),\alpha) &= 0.\end{aligned}$$

Setting $\alpha = \alpha^*$ in the above identity leads to

$$z_1''(\alpha^*) = -\frac{\Phi_\alpha(z_1(\alpha^*),\alpha^*)}{\phi_r(z_1(\alpha^*),\alpha^*)}.$$

Then it suffices to verify that

$$\Phi_\alpha(z_1(\alpha^*),\alpha^*) = \phi_{\alpha\alpha}(z_1(\alpha^*),\alpha^*) < 0, \tag{2.25}$$

because $\phi_r(z_1(\alpha^*),\alpha^*) < 0$. We put $V(r,\alpha) := \Phi_\alpha(r,\alpha)$. Differentiating (2.9) with respect to α, we see that $V(r,\alpha)$ satisfies

$$\begin{cases} (r^{N-1}V_r)_r + r^{N-1}f'(\phi(r,\alpha))\,V = -r^{N-1}f''(\phi(r,\alpha))\,\Phi^2, \quad r > 0, \\ V(0,\alpha) = V_r(0,\alpha) = 0. \end{cases} \tag{2.26}$$

Then it follows from (2.9) and (2.26) that

$$(r^{N-1}\varPhi_r)_r V - (r^{N-1}V_r)_r \varPhi = r^{N-1} f''(\phi(r,\alpha))\, \varPhi^3. \tag{2.27}$$

Integration of (2.27) with $\alpha = \alpha^*$ over $(0, z_1(\alpha^*))$ yields

$$\begin{aligned}&\left[r^{N-1}\{\varPhi_r(r,\alpha^*)V(r,\alpha^*) - V_r(r,\alpha^*)\varPhi(r,\alpha^*)\}\right]_{r=0}^{r=z_1(\alpha^*)}\\ &= \int_0^{z_1(\alpha^*)} r^{N-1} f''(\phi(r,\alpha^*))\, \varPhi(r,\alpha^*)^3 dr.\end{aligned}$$

Noting that $z_1(\alpha^*) = Z_1(\alpha^*)$, we obtain

$$\begin{aligned}&V(z_1(\alpha^*),\alpha^*)\\ &= \frac{1}{z_1(\alpha^*)^{N-1}\varPhi_r(z_1(\alpha^*),\alpha^*)} \int_0^{z_1(\alpha^*)} r^{N-1} f''(\phi(r,\alpha^*))\, \varPhi(r,\alpha^*)^3 dr.\end{aligned}$$

Since $\varPhi_r(z_1(\alpha^*),\alpha^*) = \varPhi_r(Z_1(\alpha^*),\alpha^*) < 0$, we will show

$$\int_0^{z_1(\alpha^*)} r^{N-1} f''(\phi(r,\alpha^*))\, \varPhi(r,\alpha^*)^3 dr > 0 \tag{2.28}$$

to prove (2.25). In the following argument, we employ the technique developed by Ouyang-Shi [56]. We note that $\phi(r,\alpha^*)$ is monotone decreasing for $r \in (0, z_1(\alpha^*))$ by (iii) of Lemma 2.3. Since $\alpha^* > b > a$ by Lemmas 2.5, 2.6 and 2.7, it follows from (A3) that $r_0 \in (0, z_1(\alpha^*))$ such that $f''(\phi(r,\alpha^*)) > 0$ for $r \in (0, r_0)$ and $f''(\phi(r,\alpha^*)) < 0$ for $r \in (0, z_1(\alpha^*))$. We will show that there exists $k > 0$ such that

$$\begin{cases} k\varPhi(r,\alpha^*) > -\phi_r(r,\alpha^*) & \text{for} \ \ r \in (0, r_0) \ \ \text{and} \\ k\varPhi(r,\alpha^*) < -\phi_r(r,\alpha^*) & \text{for} \ \ r \in (r_0, z_1(\alpha^*)). \end{cases} \tag{2.29}$$

We put $w(r) := \varPhi(r,\alpha^*) + \phi_r(r,\alpha^*)$. Since $w(0) = 1$ and $w(z_1(\alpha^*)) < 0$, then $w(\,\cdot\,)$ has at least one zero point in $(0, z_1(\alpha^*))$. We will show the uniqueness of zero points of w. By (2.9) and (2.18), w satisfies

$$(r^{N-1}w_r)_r + r^{N-1} f'(\phi(r,\alpha^*))w = (N-1)r^{N-3}\phi_r(r,\alpha^*) < 0 \ \ \text{for} \ \ r \in (0, z_1(\alpha^*)). \tag{2.30}$$

It follows from (2.9) and (2.30) that

$$\{r^{N-1}\varPhi_r(r,\alpha^*)\}_r w - (r^{N-1}w_r)_r\varPhi(r,\alpha^*) = -(N-1)r^{N-3}\varPhi(r,\alpha^*)\phi_r(r,\alpha^*) > 0. \tag{2.31}$$

for $r \in (\,0, z_1(\alpha^*))$. Suppose that w has at least two zero points in $(0, z_1(\alpha^*))$. We denote by τ_1 and τ_2 ($\tau_1 < \tau_2$) the first two zero points of w. Integrating (2.31) over (τ_1, τ_2) leads to

$$\begin{aligned}\tau_1^{N-1}w_r(\tau_1)\,\varPhi(\tau_1,\alpha^*) - \tau_2^{N-1}w_r(\tau_2)\,\varPhi(\tau_2,\alpha^*) =\\ -(N-1)\int_{\tau_1}^{\tau_2} r^{N-3}\varPhi(r,\alpha^*)\phi_r(r,\alpha^*)dr > 0.\end{aligned}$$

On the other hand, by applying the strong maximum principle to (2.30), one can see $w_r(\tau_1) < 0 \le w_r(\tau_2)$; so that

$$\tau_1^{N-1}w_r(\tau_1)\varPhi(\tau_1,\alpha^*) - \tau_2^{N-1}w_r(\tau_2)\varPhi(\tau_2,\alpha^*) < 0.$$

Thus we meet a contradiction. Consequently, there exists a unique $\tau_0 \in (0, z_1(\alpha^*))$ such that $\varPhi(r,\alpha^*) > -\phi_r(r,\alpha^*)$ for $r \in (0,\tau_0)$ and $\varPhi(r,\alpha^*) < -\phi_r(r,\alpha^*)$ for $r \in (\tau_0, z_1(\alpha^*))$. Thus by choosing a suitable $k > 0$, we can show (2.29). To complete the proof, we use the following result due to Ouyang-Shi [56, Lemma 2.4] ;

$$\int_0^{z_1(\alpha^*)} r^{N-1}f''(\phi(r,\alpha^*))\phi_r(r,\alpha^*)^2\varPhi(r,\alpha^*)dr = 0. \tag{2.32}$$

It follows from (2.29) and (2.32) that

$$\begin{aligned}0 &= \int_0^{r_0} r^{N-1}f''(\phi(r,\alpha^*))\phi_r(r,\alpha^*)^2\varPhi(r,\alpha^*)dr\\ &\quad+\int_{r_0}^{z_1(\alpha^*)} r^{N-1}f''(\phi(r,\alpha^*))\phi_r(r,\alpha^*)^2\varPhi(r,\alpha^*)dr\\ &< k^2\int_0^{z_1(\alpha^*)} r^{N-1}f''(\phi(r,\alpha^*))\varPhi(r,\alpha^*)^3dr.\end{aligned}$$

Then, (2.28) holds true, which means $z_1''(\alpha^*) < 0$; so that $z_1(\alpha^*)$ is a strict local maximum. This finishes the proof of Proposition 2.1.

□

2.4.2 Proof of Lemma 2.8

In this subsection, we will give the proof of Lemma 2.8. We begin with the following lemma needed later.

Lemma 2.9. *Suppose that* (A1)-(A5) *hold. Suppose further that* $0 < \alpha_1 < \alpha_2$ *and* $z_1(\alpha_1) \le z_1(\alpha_2)$. *Then* $\phi(r, \alpha_1) < \phi(r, \alpha_2)$ *for all* $r \in [0, z_1(\alpha_1))$.

Proof. For simplicity, we put $\phi(r) := \phi(r, \alpha_1)$ and $w(r) := \phi(r, \alpha_2)$. We will prove Lemma 2.9 by a contradiction argument. Suppose that

$$\phi(r_1) = w(r_1) \quad \text{for some} \quad r_1 \in (0, z_1(\alpha_1)).$$

We may assume that

$$\phi(r) < w(r) \quad \text{for} \quad r \in [0, r_1)$$

without loss of generality. By the uniqueness of solutions for the initial value problem, $\phi_r(r_1) > w_r(r_1)$. It follows from $z_1(\alpha_1) \le z_1(\alpha_2)$ that there exists $r_2 \in (r_1, z_1(\alpha_1)]$ such that

$$\begin{cases} \phi(r) > w(r) \quad \text{for} \quad r \in (r_1, r_2), \\ \phi(r_2) = w(r_2) \quad \text{and} \quad \phi_r(r_2) < w_r(r_2). \end{cases}$$

According to (2.2),

$$(r^{N-1} w_r)_r + r^{N-1} f(w) = 0,$$
$$(r^{N-1} \phi_r)_r + r^{N-1} f(\phi) = 0.$$

Multiplying these equalities by ϕ and w, respectively, and integrating the resulting expressions over (τ, r), we have

$$\begin{aligned} 0 &= \left[s^{N-1} \{ w_r(s)\phi(s) - \phi_r(s) w(s) \} \right]_{s=\tau}^{s=r} \\ &\qquad\qquad + \int_\tau^r s^{N-1} \{ f(w(s))\phi(s) - f(\phi(s)) w(s) \} \, ds \\ &= r^{N-1} \{ w_r(r)\phi(r) - \phi_r(r) w(r) \} - \tau^{N-1} \{ w_r(\tau)\phi(\tau) - \phi_r(\tau) w(\tau) \} \\ &\quad + \int_\tau^r s^{N-1} \left(\frac{f(w(s))}{w(s)} - \frac{f(\phi(s))}{\phi(s)} \right) \phi(s) w(s) \, ds. \end{aligned}$$

So we obtain an identity

$$
\begin{aligned}
&r^{N-1}\{\phi_r(r)w(r)-w_r(r)\phi(r)\}-\tau^{N-1}\{\phi_r(\tau)w(\tau)-w_r(\tau)\phi(\tau)\}\\
&=\int_\tau^r s^{N-1}\left(\frac{f(w(s))}{w(s)}-\frac{f(\phi(s))}{\phi(s)}\right)\phi(s)w(s)\,ds.
\end{aligned}
\tag{2.33}
$$

Putting $\tau=r_1$ and $r=r_2$ in (2.33) leads to

$$
\begin{aligned}
&r_2^{N-1}\phi(r_2)(\phi_r(r_2)-w_r(r_2))-r_1^{N-1}\phi(r_1)(\phi_r(r_1)-w_r(r_1))\\
&=\int_{r_1}^{r_2} r^{N-1}\left(\frac{f(w(r))}{w(r)}-\frac{f(\phi(r))}{\phi(r)}\right)\phi(r)w(r)\,dr.
\end{aligned}
\tag{2.34}
$$

We will show

$$\phi(r_1)(=w(r_1))>b \tag{2.35}$$

from (2.34). Suppose that $\phi(r_1)(=w(r_1))\le b$ to the contrary. Since $0<w(r)<\phi(r)<b$ for $r\in(r_1,r_2)$, it follows from Lemma 2.5 that

$$\frac{f(w(r))}{w(r)}-\frac{f(\phi(r))}{\phi(r)}>0 \quad\text{for}\quad r\in(r_1,r_2).$$

Thus the right hand side of (2.34) is positive. On the other hand, because $\phi_r(r_2)<w_r(r_2)$ and $\phi_r(r_1)>w_r(r_1)$, the left hand side of (2.34) is negative, which is a contradiction.

Next we set $\tau=0$ in (2.33) to obtain

$$r^{N-1}\{\phi_r(r)w(r)-w_r(r)\phi(r)\}=\int_0^r s^{N-1}\left(\frac{f(w(s))}{w(s)}-\frac{f(\phi(s))}{\phi(s)}\right)\phi(s)w(s)\,ds. \tag{2.36}$$

Since $w(r)>\phi(r)>b$ for $r\in[\,0,r_1)$ by (2.35), Lemma 2.5 implies

$$\frac{f(w(r))}{w(r)}-\frac{f(\phi(r))}{\phi(r)}>0 \quad\text{for}\quad r\in[\,0,r_1).$$

Thus it follows from (2.36) that

$$\phi_r(r)w(r)-w_r(r)\phi(r)>0 \quad\text{for}\quad r\in(0,r_1\,]. \tag{2.37}$$

On the other hand, by virtue of $\phi(r_2)=w(r_2)$ and $\phi_r(r_2)<w_r(r_2)$,

$$\phi_r(r_2)w(r_2)-w_r(r_2)\phi(r_2)\le 0.$$

Here, the equality holds true if and only if $\phi(r_2) = 0$, or equivalently, $r_2 = z_1(\alpha_1) = z_1(\alpha_2)$. Hence, by the intermediate theorem, there exists $t \in (r_1, r_2\,]$ such that

$$\phi_r(r)w(r) - w_r(r)\phi(r) > 0 \quad \text{for} \quad r \in (0, t) \tag{2.38}$$

and

$$\phi_r(t)w(t) - w_r(t)\phi(t) = 0. \tag{2.39}$$

It follows from (2.38) that

$$\frac{d}{dr}\left(\frac{\phi(r)}{w(r)}\right) > 0 \quad \text{for} \quad r \in (0, t).$$

Put $C := \phi(t)/w(t) > 1$. (If $w(t) = 0$ or, equivalently, $t = r_2 = z_1(\alpha_1) = z_1(\alpha_2)$, then we take $C = \phi_r(r_2)/w_r(r_2) > 1$ by virtue of l'Hôpital's rule.) Since $\phi(r)/w(r)$ is strictly monotone increasing for $r \in (0, t)$, we obtain

$$\begin{cases} \phi(r) < Cw(r) \quad \text{for} \quad r \in (0, t), \\ \phi(t) = Cw(t). \end{cases} \tag{2.40}$$

Furthermore, in view of (2.39) and (2.40), we have $\phi_r(t) = Cw_r(t)$. Then it follows from (2.38) and (2.40) that

$$\frac{\phi_r(r)}{w_r(r)} < \frac{\phi(r)}{w(r)} < C \quad \text{for} \quad r \in (0, t). \tag{2.41}$$

We now observe the generalized Pohožaev identity (see e.g., Pucci-Serrin [63]);

$$\begin{aligned} P(\phi(r), \mu) :=& \int_0^r s^{N-1}K(\phi(s), \mu)\, ds + \left(\mu - \frac{N-2}{2}\right)\int_0^r s^{N-1}\phi_r(s)^2\, ds \\ =& \frac{1}{2}r^N\phi_r(r)^2 + \mu r^{N-1}\phi(r)\phi_r(r) + r^N F(\phi(r)), \end{aligned}$$

where μ is any real number, $F(\phi) := \int_0^\phi f(s)ds$ and $K(\phi, \mu) := NF(\phi) - \mu f(\phi)\phi$. Similarly,

$$P(w(r), \mu) = \frac{1}{2}r^N w_r(r)^2 + \mu r^{N-1}w(r)w_r(r) + r^N F(w(r)).$$

Recall $\phi(t) = Cw(t)$ and $\phi_r(t) = Cw_r(t)$; then one can deduce that

$$\begin{aligned}
&P(\phi(t), \mu) - C^2 P(w(t), \mu) \\
&= \left(\mu - \frac{N-2}{2}\right) \int_0^t r^{N-1}\{\phi_r(r)^2 - C^2 w_r(r)^2\}\, dr \\
&\qquad + \int_0^t r^{N-1}\{K(\phi(r), \mu) - C^2 K(w(r), \mu)\}\, dr \\
&= t^N \{F(\phi(t)) - C^2 F(w(t))\}.
\end{aligned} \tag{2.42}$$

Note that

$$\frac{d}{dr}\{K(\phi(r), \mu) - C^2 K(w(r), \mu)\} = \phi_r(r) k(\phi(r), \mu) - C^2 w_r(r) k(w(r), \mu), \tag{2.43}$$

where $k(\phi, \mu) = (N - \mu) f(\phi) - \mu f'(\phi)\phi$. It follows from (A5) and (2.40) that there exists $\mu > (N-2)/2$ such that

$$0 < k(\phi(r), \mu) \le k(Cw(r), \mu) \le Ck(w(r), \mu) \quad \text{for} \quad r \in (0, t).$$

Therefore, it follows from (2.43) that, for such μ,

$$\begin{aligned}
\frac{d}{dr}\{K(\phi(r), \mu) - C^2 K(w(r), \mu)\} &\ge C\phi_r(r) k(w(r), \mu) - C^2 w_r(r) k(w(r), \mu) \\
&= Ck(w(r), \mu)(\phi_r(r) - Cw_r(r))
\end{aligned}$$

for $r \in (0, t)$. Because $\phi_r(r) > Cw_r(r)$ from (2.41) and $k(w(r), \mu) > 0$, we get

$$\frac{d}{dr}\{K(\phi(r), \mu) - C^2 K(w(r), \mu)\} > 0 \quad \text{for} \quad r \in (0, t).$$

Then it follows from (2.42) that

$$\begin{aligned}
&\left(\mu - \frac{N-2}{2}\right) \int_0^t r^{N-1}(\phi_r(r)^2 - C^2 w_r(r)^2)\, dr \\
&\qquad + \frac{t^N}{N}\{K(\phi(t), \mu) - C^2 K(w(t), \mu)\} > t^N\{F(\phi(t)) - C^2 F(w(t))\};
\end{aligned}$$

so that

$$\begin{aligned}
&\left(\mu - \frac{N-2}{2}\right) \int_0^t r^{N-1}(\phi_r(r)^2 - C^2 w_r(r)^2)\, dr \\
&> t^N \left[\left(F(\phi(t)) - \frac{1}{N} K(\phi(t), \mu)\right) - C^2 \left(F(w(t)) - \frac{1}{N} K(w(t), \mu)\right)\right] \\
&= \frac{\mu}{N}\{\phi(t) f(\phi(t)) - C^2 w(t) f(w(t))\}.
\end{aligned}$$

Therefore, since $\mu > (N-2)/2$ and $\phi_r(r)^2 - C^2 w_r(r)^2 < 0$ for $r \in (0,t)$, we see

$$0 > \phi(t) f(\phi(t)) - C^2 w(t) f(w(t)) = C w(t) \{ f(C w(t)) - C f(w(t)) \}.$$

Then it follows that

$$w(t) > 0, \tag{2.44}$$

$$f(Cw(t)) - Cf(w(t)) < 0. \tag{2.45}$$

Consequently, by (2.39) and (2.44),

$$\begin{cases} \left(\dfrac{\phi}{w}\right)_r (r) > 0 \quad \text{for} \quad r \in (0,t), \\ \left(\dfrac{\phi}{w}\right)_r (t) = 0. \end{cases}$$

Because $\phi(r_1)/w(r_1) = \phi(r_2)/w(r_2) = 1$ if $r_2 < z_1(\alpha_1)$, (2.44) also implies $t < r_2$. Therefore, we obtain

$$\left(\frac{\phi}{w}\right)_{rr} (t) = \frac{\phi_{rr}(t) w(t) - w_{rr}(t)\phi(t)}{w(t)^2} \leq 0,$$

which, together with $\phi(t) = Cw(t)$, shows $\phi_{rr}(t) - C w_{rr}(t) \leq 0$. Recall $\phi_r(t) = C w_r(t)$ and

$$\begin{aligned} \phi_{rr}(t) + \frac{N-1}{t}\phi_r(t) + f(\phi(t)) = 0, \\ w_{rr}(t) + \frac{N-1}{t} w_r(t) + f(w(t)) = 0. \end{aligned}$$

Then

$$0 \geq \phi_{rr}(t) - C w_{rr}(t) = C f(w(t)) - f(C w(t)).$$

This evidently contradicts (2.45). Thus the proof of Lemma 2.9 is accomplished.

□

Proof of Lemma 2.8. Suppose for contradiction that the measure of the set

$$\{\alpha \in (0,\infty) \, : \, z_1(\alpha) \geq Z_2(\alpha)\}$$

is positive. We note that by (2.16), $z_1(\alpha) < Z_1(\alpha) < Z_2(\alpha)$ for all $\alpha \in (0, b)$. By taking account of the continuity of z_1 and Z_2 with respect to α, we can find positive numbers $\hat{\alpha} \geq b$ and δ such that

$$z_1(\hat{\alpha}) = Z_2(\hat{\alpha})$$

and

$$z_1(\alpha) \geq Z_2(\alpha) \quad \text{for all} \quad \alpha \in (\hat{\alpha}, \hat{\alpha} + \delta). \tag{2.46}$$

Observing that

$$\begin{cases} \phi(r, \alpha) - \phi(r, \hat{\alpha}) = \varPhi(r, \hat{\alpha})(\alpha - \hat{\alpha}) + o(|\alpha - \hat{\alpha}|) \quad \text{for} \quad r \in [0, z_1(\hat{\alpha})), \\ \varPhi(r, \hat{\alpha}) > 0 \quad \text{for} \quad r \in [0, Z_1(\hat{\alpha})), \\ \varPhi(r, \hat{\alpha}) < 0 \quad \text{for} \quad r \in (Z_1(\hat{\alpha}), z_1(\hat{\alpha})), \end{cases}$$

we see that $\phi(r, \alpha)$ intersects $\phi(r, \hat{\alpha})$ in a neighborhood of $r = Z_1(\hat{\alpha})$ provided $|\alpha - \hat{\alpha}|$ is sufficiently small. More precisely, for any sequence $\{\alpha_j\}$ satisfying $\alpha_j \downarrow \hat{\alpha}$ $(j \uparrow \infty)$, there exists a sequence $\{r_j\}$ such that

$$\begin{cases} \phi(r_j, \alpha_j) = \phi(r_j, \hat{\alpha}), \\ \phi_r(r_j, \alpha_j) < \phi_r(r_j, \hat{\alpha}) \end{cases} \tag{2.47}$$

for sufficiently large j. Moreover, one can show $\lim_{j \to \infty} r_j = Z_1(\hat{\alpha})$. Noting that $\varPhi(z_1(\alpha), \alpha) \geq 0$ for $\alpha \in (\hat{\alpha}, \hat{\alpha} + \delta)$ by (2.46), we see

$$z_1'(\alpha) = -\frac{\varPhi(z_1(\alpha), \alpha)}{\phi_r(z_1(\alpha), \alpha)} \geq 0 \quad \text{for} \quad \alpha \in (\hat{\alpha}, \hat{\alpha} + \delta).$$

This fact implies

$$z_1(\alpha_j) \geq z_1(\hat{\alpha}) \quad \text{for sufficiently large } j. \tag{2.48}$$

Hence (2.47) and (2.48) clearly contradict Lemma 2.9. Thus the proof of Lemma 2.8 is complete.

□

2.4.3 Proof of Theorem 2.2

In this subsection, we will finish the proof of Theorem 2.2 by making use of Proposition 2.1.

Proof of Theorem 2.2. By Proposition 2.1, if $R \in (0, \sqrt{\lambda_1/f'(+0)}\,]$, then $z_1(\alpha) = R$ has a unique solution $\alpha = \overline{\alpha}(R)$. We set $\overline{u}(r;R) := u(r,\overline{\alpha})$ for $0 \le r \le R$, then $\overline{u}(\,\cdot\,;R)$ is a unique positive solution of (E) and satisfies $\|\overline{u}(\,\cdot\,;R)\|_\infty := \max\limits_{0\le r\le R} \overline{u}(r;R) = \overline{\alpha}(R)$. It follows from $\lim\limits_{\alpha\to\infty} z_1(\alpha) = 0$ and the monotonicity of $z_1(\alpha)$ for $\alpha > \alpha^*$ that $\lim\limits_{R\downarrow 0} \|\overline{u}(\,\cdot\,;R)\|_\infty = \lim\limits_{R\downarrow 0} \overline{\alpha}(R) = \infty$, because $\overline{u}(r;R)$ is strictly monotone decreasing for $r \in (0,R)$ by Lemma 2.3.

We put $R_1 := \max\limits_{\alpha>0} z_1(\alpha)$. If $R \in (\sqrt{\lambda_1/f'(+0)}, R_1)$, then $z_1(\alpha) = R$ has exactly two solutions $\alpha = \overline{\alpha}(R), \underline{\alpha}(R)$ $(\overline{\alpha}(R) > \underline{\alpha}(R))$. If we set $\overline{u}(r;R) := u(r,\overline{\alpha})$ and $\underline{u}(r;R) := u(r,\underline{\alpha})$ for $0 \le r \le R$, then (E) has exactly two positive solutions $\overline{u}(\,\cdot\,;R)$ and $\underline{u}(\,\cdot\,;R)$ with $\|\overline{u}(\,\cdot\,;R)\|_\infty = \overline{\alpha}(R)$ and $\|\underline{u}(\,\cdot\,;R)\|_\infty = \underline{\alpha}(R)$. It follows from $\lim\limits_{\alpha\downarrow 0} z_1(\alpha) = \sqrt{\lambda_1/f'(+0)}$ that $\lim_{R\downarrow\sqrt{\lambda_1/f'(+0)}} \|\underline{u}(\,\cdot\,;R)\|_\infty = \lim_{R\downarrow\sqrt{\lambda_1/f'(+0)}} \underline{\alpha}(R) = 0$.

If $R = R_1$, then $z_1(\alpha) = R_1$ has a unique solution $\alpha = \alpha^*$. If we set $u^*(r;R_1) := u(r,\alpha^*)$ for $0 \le r \le R_1$, then $u^*(\,\cdot\,;R_1)$ is a unique positive solution of (E) and satisfies $\|u^*(\,\cdot\,;R_1)\|_\infty = \alpha^*$. Additionally, it obviously follows that $\|u^*(\,\cdot\,;R_1)\|_\infty = \lim\limits_{R\uparrow R_1} \|\overline{u}(\,\cdot\,;R)\|_\infty = \lim\limits_{R\uparrow R_1} \|\underline{u}(\,\cdot\,;R)\|_\infty$.

If $R \in (R_1,\infty)$, then $z_1(\alpha) < R$ for all $\alpha > 0$; so that (E) has no positive solutions.

Furthermore, the strict comparison (2.4) evidently follows from Lemma 2.9. Thus the proof of Theorem 2.2 is accomplished. □

2.5 Multiplicity of n-Nodal Solutions

In this section we discuss the structure of n-nodal solutions of (E). Here n-nodal solution means a solution which has exactly n zeros in $[0,R]$. For each $n \in \boldsymbol{N}$, we set

$$S_n^+(R) := \{u \in C^2([0,R]) \,:\, u \text{ is an } n\text{-nodal solution of (E) and } u(0) > 0\}.$$

It follows from Lemmas 2.6 and 2.7 that for each $n \geq 2$, $z_n(\,\cdot\,)$ is bounded and has at least one critical point. Thus in a similar way to the proof of Theorem 2.2, we obtain the following result for $S_n^+(R)$:

Theorem 2.10. *There exists a sequence of positive numbers $\{R_n\}_{n\geq 2} \uparrow \infty$ such that*

$$S_n^+(R) \supseteq \begin{cases} \{\overline{u}_n(\,\cdot\,;R)\} & \text{if } \ R \in (0, \sqrt{\lambda_n/f'(+0)}\,], \\ \{\overline{u}_n(\,\cdot\,;R), \underline{u}_n(\,\cdot\,;R)\} & \text{if } \ R \in (\sqrt{\lambda_n/f'(+0)}, R_n), \\ \{u_n^*(\,\cdot\,;R_n)\} & \text{if } \ R = R_n \end{cases}$$

and $S_n^+(R)$ is empty if $R \in (R_n, \infty)$. Here, $\overline{u}_n(r;R)$ and $\underline{u}_n(r;R)$ satisfy

$$\lim_{R\downarrow 0} \|\overline{u}_n(\,\cdot\,;R)\|_\infty = \infty \quad \text{and} \quad \lim_{R\downarrow\sqrt{\lambda_n/f'(+0)}} \|\underline{u}_n(\,\cdot\,;R)\|_\infty = 0.$$

Furthermore, if $R \in (0, R_n)$, then

$$\|\overline{u}_n(\,\cdot\,;R)\|_\infty < \|\overline{u}_{n+1}(\,\cdot\,;R)\|_\infty < \cdots < \|\overline{u}_{n+k}(\,\cdot\,;R)\|_\infty < \cdots \uparrow \infty \quad \text{as} \quad k \uparrow \infty. \tag{2.49}$$

Proof. It follows from Lemmas 2.6 and 2.7 that if $R \in (0, \sqrt{\lambda_n/f'(+0)}\,]$, then $z_n(\alpha) = R$ has at least one solution. For $\overline{\alpha}_n(R) := \sup\{\alpha > 0 \,:\, z_n(\alpha) = R\}$, we set $\overline{u}_n(r;R) := u(r;\overline{\alpha}_n)$ in $[0,R]$. Then, it follows that $\overline{u}_n(\,\cdot\,;R) \in S_n^+(R)$. By virtue of $\lim_{\alpha\to\infty} z_n(\alpha) = 0$ and (2.5), we see $\lim_{R\downarrow 0} \|\overline{u}_n(\,\cdot\,;R)\|_\infty = \lim_{R\downarrow 0} \overline{\alpha}_n(R) = \infty$.

Set $R_n := \max_{\alpha>0} z_n(\alpha)$. Then $R_n < R_{n+1}$ for each $n \in \boldsymbol{N}$. Moreover, it follows from (ii) of Lemma 2.3 that $\lim_{n\to\infty} R_n = \infty$.

If $R \in (\sqrt{\lambda_n/f'(+0)}, R_n)$, then $z_n(\alpha) = R$ has at least two solutions. Define

$$\overline{\alpha}_n(R) := \sup\{\alpha > 0 \,:\, z_n(\alpha) = R\}, \quad \underline{\alpha}_n(R) := \inf\{\alpha > 0 \,:\, z_n(\alpha) = R\},$$

and moreover set $\overline{u}_n(r;R) := u_n(r;\overline{\alpha}_n)$ and $\underline{u}_n(r;R) := u_n(r;\underline{\alpha}_n)$ in $[0,R]$. Thus $\overline{u}_n(\,\cdot\,;R)$, $\underline{u}_n(\,\cdot\,;R) \in S_n^+(R)$. It follows from $\lim_{\alpha\downarrow 0} z_n(\alpha) = \sqrt{\lambda_n/f'(+0)}$ that

$\lim_{R\downarrow\sqrt{\lambda_n/f'(+0)}} \|\underline{u}_n(\,\cdot\,;R)\|_\infty = 0$. Hence, together with similar arguments for $R = R_n$ and $R > R_n$, we obtain all assertions except for (2.49).

We will show (2.49). Recall that $z_n(\alpha) < z_{n+1}(\alpha)$ for any $\alpha > 0$, $\overline{\alpha}_{n+k}(R) < \overline{\alpha}_{n+k+1}(R)$ for any $k \geq 0$ and $R < R_n$; so that

$$\|\overline{u}_{n+k}(\,\cdot\,;R)\|_\infty < \|\overline{u}_{n+k+1}(\,\cdot\,;R)\|_\infty \text{ for any } k \geq 0 \text{ and } R < R_n.$$

Moreover, it can be shown that $\lim_{n\to\infty} \|\overline{u}_n(\,\cdot\,;R)\|_\infty = \infty$. Suppose that $\|\overline{u}_n(\,\cdot\,;R)\|_\infty$ is bounded with respect to n. Then $\lim_{n\to\infty} \|\overline{u}_n(\,\cdot\,;R)\|_\infty =: \beta$ exists. This implies that for arbitrary large $j \in \boldsymbol{N}$, there exists a positive $\delta \in (0,\beta)$ such that $u(r;\alpha)$ has j-zeros in $[\,0,R\,]$ if $\alpha \in (\beta-\delta,\beta)$. On the other hand, it follows from Lemma 2.3 that $u(r;\beta)$ has a finite numbers of zeros in $[\,0,R\,]$. This clearly contradicts the continuity of $u(\,\cdot\,,\alpha)$ with respect to α. Thus the proof is complete. □

2.6 Stability Analysis

In this section, we study time-depending behaviors of solutions of (P2) in particular for the case where f breaks the Lipschitz continuity at the origin. As to the existence of solutions of (P2), we remark similar theorems to Theorems 1.7 and 1.9 hold true even in the higher-dimensional case.

Theorem 2.11. *For any $u_0 \in L^\infty$,* (P) *has at least one solution*

$$u \in C([\,0,T_m);L^2) \cap C^1((0,T_m);L^2) \cap C((0,T_m);H^2 \cap H_0^1) \cap L^\infty(B_R \times (0,T))$$

for any $T < T_m$, where T_m is a maximal existence time of u; $T_m := \sup\{T > 0 : \|u(t)\|_\infty < +\infty\}$.

On the other hand, as to the uniqueness, Fujita-Watanabe [29] gave a nonuniqueness result for solutions of (P2). However, Cazenave-Dickstein-Escobedo [12] have recently obtained a comparison theorem for nonnegative solutions of (P2):

Theorem 2.12 ([12]). *Assume that u is a nonnegative super-solution for* (P2) *in $B_R \times (0,T)$ and that v is a nonnegative sub-solution for* (P2) *in $B_R \times (0,T)$. If $u(x,0) \not\equiv 0$ and $u(x,0) \geq v(x,0)$ for all $x \in B_R$, then $u(x,t) \geq v(x,t)$ for all $(x,t) \in B_R \times (0,T)$.*

It follows from Theorems 2.11 and 2.12 that if $u_0 \geq 0$ in B_R and $u_0 \not\equiv 0$, then (P2) has a unique nonnegative solution. By applying Theorem 2.12, we obtain the following theorem for time-depending behaviours of solutions of (P2):

Theorem 2.13. *Assume that $u_0 \in L^\infty$ satisfies $u_0 \geq 0$ in B_R and $u_0 \not\equiv 0$. Then, the nonnegative solution $u(x,t;u_0)$ of* (P2) *satisfies the following properties.*
(i) *If $R \in (0, R_1]$ and $u_0 \leq k\overline{\phi}(R)$ in B_R with some $k \in (0,1)$, then $u(x,t;u_0) \leq \overline{\phi}(x;R)$ for $(x,t) \in B_R \times (0,\infty)$ and $\lim_{t\to\infty} \|u(t;u_0) - \underline{\phi}(R)\|_{C^1} = 0$.*
(ii) *If $R \in (0, R_1]$ and $u_0 \geq k\overline{\phi}(R)$ in B_R with some $k \in (1,\infty)$, then $u(x,t;u_0)$ blows up in a finite time; so that there exists a positive T_m such that*

$$\lim_{t\uparrow T_m} \|u(t;u_0)\|_\infty = \infty. \ \ \textit{Furthermore, } u(x,t;u_0) \geq \overline{\phi}(x;R)$$

for $(x,t) \in B_R \times (0,T_m)$.
(iii) *If $R \in (R_1,\infty)$, then $u(x,t;u_0)$ blows up in a finite time.*

Proof. (i) For any $R \in (0,R_1)$ and $\varepsilon \in (0, R_1 - R)$, we set $\overline{\phi}_\varepsilon(x;R) := \overline{\phi}((R+\varepsilon)x/R; R+\varepsilon)$ for $x \in B_R$. It follows from Theorem 2.2 that if $\varepsilon > 0$ is sufficiently small, then $u_0(x) \leq k\overline{\phi}(x;R) \leq \overline{\phi}_\varepsilon(x;R)$ for $x \in B_R$. It is easily verified that $\overline{\phi}_\varepsilon(x;R)$ is a super-solution for (2.1). Then, by Theorem 2.12, we see

$$u(x,t;u_0) \leq u(x,t;\overline{\phi}_\varepsilon(R)) \leq \overline{\phi}_\varepsilon(x;R) \ \text{ for } \ (x,t) \in B_R \times (0,\infty). \tag{2.50}$$

Letting $\varepsilon \downarrow 0$ in (2.50), we obtain $u(x,t;u_0) \leq \overline{\phi}(x;R)$ for $(x,t) \in B_R \times (0,\infty)$. For any fixed $\tau \geq 0$, we put $v(x,t) := u(x,t+\tau;\overline{\phi}_\varepsilon(R))$. Then v satisfies

$$\begin{cases} v_t = \Delta v + f(v), & (x,t) \in B_R \times (0,\infty), \\ v(x,t) = 0, & (x,t) \in \partial B_R \times (0,\infty), \\ v(x,0) = u(x,\tau;\overline{\phi}_\varepsilon(R)), & x \in B_R. \end{cases}$$

By (2.50), $v(x,0) = u(x,\tau;\overline{\phi}_\varepsilon(R)) \leq \overline{\phi}_\varepsilon(x;R)$ for $x \in B_R$. It follows from Theorem 2.12 again that $v(x,t) = u(x,t+\tau;\overline{\phi}_\varepsilon(R)) \leq u(x,t;\overline{\phi}_\varepsilon(R))$; so that

$$u_t(x,t;\overline{\phi}_\varepsilon(R)) \leq 0 \quad \text{for} \quad (x,t) \in B_R \times (0,\infty). \tag{2.51}$$

On the other hand, we set $\underline{\phi}_\varepsilon(x;R) := \underline{\phi}(\varepsilon x/R;\varepsilon)$. It follows from the strong maximum principle for parabolic equations that for any $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $\lim_{\delta\downarrow 0} \varepsilon(\delta) = 0$ and $u(x,\delta;u_0) \geq \underline{\phi}_\varepsilon(x;R)$ for $x \in B_R$. Since $\underline{\phi}_\varepsilon(R)$ is a sub-solution for (2.1), we have by Theorem 2.12

$$\begin{aligned} &u(x,t;u_0) \geq u(x,t;\underline{\phi}_\varepsilon(R)) \geq \underline{\phi}_\varepsilon(x;R) \quad \text{for} \quad (x,t) \in B_R \times (\delta,\infty) \quad \text{and} \\ &u_t(x,t;\underline{\phi}_\varepsilon(R)) \geq 0 \quad \text{for} \quad (x,t) \in B_R \times (0,\infty). \end{aligned} \tag{2.52}$$

It follows from (2.50)-(2.52) that there exist $u^*(x) := \lim_{t\to\infty} u(x,t;\overline{\phi}_\varepsilon(R))$ and $u_*(x) := \lim_{t\to\infty} u(x,t;\underline{\phi}_\varepsilon(R))$ for all $x \in B_R$. Clearly, $\underline{\phi}_\varepsilon(x;R) \leq u_*(x) \leq u^*(x) \leq \overline{\phi}_\varepsilon(x;R)$ for $x \in B_R$. We now note that if ϕ is a solution of (2.1) and satisfies $\underline{\phi}_\varepsilon(x;R) \leq \phi(x) \leq \overline{\phi}_\varepsilon(x;R)$ in B_R, then $\phi \equiv \underline{\phi}(R)$. Therefore by a parabolic monotone method (see e.g. [66]), we can deduce that $u^* \equiv u_* \equiv \underline{\phi}(R)$ and moreover $\lim_{t\to\infty} u(t;\underline{\phi}_\varepsilon(R)) = \lim_{t\to\infty} u(t;\overline{\phi}_\varepsilon(R)) = \underline{\phi}(R)$ in $C^1(\overline{B_R})$. In view of (2.50)-(2.52), we obtain $\lim_{t\to\infty} \|u(t;u_0) - \underline{\phi}(R)\|_{C^1} = 0$. It is easily verified that the statement of (i) also holds true for $R = R_1$.

(ii) For any $R \in (0,R_1]$ and $\varepsilon \in (0,R)$, we set $\psi_\varepsilon(x;R) := \overline{\phi}((R-\varepsilon)x/R;R-\varepsilon)$. If ε is sufficiently small, then $u_0(x) \geq k\overline{\phi}(x;R) \geq \psi_\varepsilon(x;R)$ for $x \in B_R$ with $k > 1$. Since $\psi_\varepsilon(R)$ is a sub-solution for (2.1), we get from 2.12

$$u(x,t;u_0) \geq u(x,t;\psi_\varepsilon(R)) \geq \psi_\varepsilon(x;R) \quad \text{for} \quad (x,t) \in B_R \times (0,T_\varepsilon), \tag{2.53}$$

where T_ε is a maximal existence time of $u(x,t;\psi_\varepsilon(R))$. Assume that $\|u(t;u_0)\|_\infty$ is bounded for $t \in (0,\infty)$. Thus we see, by a similar way to the proof of (i), that there exists $u^* \in S(R)$ such that $\lim_{t\to\infty} u(t;\psi_\varepsilon(R)) = u^*$ in $C^1(B_R)$. This clearly contradicts to Theorem 2.2. Therefore, it follows from (2.53) that $u(x,t;u_0)$ blows up and grows up; If we put $T_m = \sup\{T > 0 : \|u(t;u_0)\|_\infty < \infty\}$, then (b)

$T_m < \infty$ and $\lim_{t \uparrow T_m} \|u(t; u_0)\|_\infty = \infty$ or (g) $T_m = \infty$ and $\lim_{t\to\infty} \|u(t; u_0)\|_\infty = \infty$. We will exclude (g) by contradiction. Suppose that (g) is satisfied. Then it is proved that there exists a positive function $\psi \in L^1(B_R)$ such that $f(\psi)d_{B_R} \in L^1(B_R)$ with $d_{B_R} := \text{dist}\,(x, \partial B_R)$ and

$$\int_{B_R} \psi(-\Delta\xi)dx = \int_{B_R} f(\psi)\xi dx \quad \text{for all } \xi \in C_0^2(\overline{B_R}) \text{ and}$$

$$\lim_{t\to\infty} u(t; u_0) = \psi \quad \text{in } L^1(B_R)$$

(See e.g. [12, Lemma 3.1]). In the above sense, ψ is a weak solution of (2.1). On the other hand, by a similar argument to [12, Corollary 2.5], it is shown that $\psi \in S^+(R)$, which is impossible since $\psi > \overline{\phi}(R)$ in B_R. Therefore, (g) is excluded. Thus the proof is complete. □

Part II

A Prey-Predator System with Cross-Diffusion

Chapter 3

Multiple Existence of Positive Steady-States

3.1 Introduction and Main Results

In this part we study nonnegative steady-state solutions of the following strongly-coupled parabolic system

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{12}v)u] + u(a_1 - b_1u - c_1v) & \text{in } \ \Omega \times (0, \infty), \\ v_t \ = \Delta[(d_2 + \rho_{21}u)v] + v(a_2 + b_2u - c_2v) & \text{in } \ \Omega \times (0, \infty), \\ u = v = 0 & \text{on } \ \partial\Omega \times (0, \infty), \\ u(\,\cdot\,, 0) = u_0 \geq 0, \quad v(\,\cdot\,, 0) = v_0 \geq 0 & \text{in } \ \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$; ρ_{12}, ρ_{21} are nonnegative constants; d_i, a_i, b_i, c_i $(i = 1, 2)$ are also constants, and they are all positive except for a_2 which may be nonpositive. The system (3.1) is known as the Lotka-Volterra prey-predator system with cross-diffusion effects. In (3.1), u and v, respectively, represent the population densities of prey and predator species which are interacting and migrating in the same habitat Ω. Such a density-dependent population model was first proposed by Shigesada, Kawasaki and Teramoto [67] to investigate the habitat segregation phenomena.

In diffusion terms, d_i represents natural dispersive force of movement of an individual, while ρ_{ij} describes mutual interferences between individuals; ρ_{12} and ρ_{21} are usually referred as cross-diffusion pressures. The above model means that, in addition to dispersive force, diffusion also depends on population pressure from other species. For details, see the monograph of Okubo and Levin [55]. First cross-diffusion pressure ρ_{12} means the tendency that the prey keeps away from the predator. In a certain kind of prey-predator relationships, a great number of prey species form a huge group to protect themselves from the attack of predator. So we assume that the population pressure due to the high density of prey induces the diffusion of the form $\Delta(\rho_{21}uv)$ in the second equation. This kind of prey-predator models has also been discussed in [26, 54, 65]. The boundary condition means that the habitat Ω is surrounded by a hostile environment. The system with the *aggregation* term $\nabla(d_2\nabla v - \rho_{21}v\nabla u)$ (instead of $\Delta[(d_2+\rho_{21}u)v]$ in (3.1)) is also an interesting model. We will discuss such a problem elsewhere. See also [55] for the biological background.

The purpose of the present chapter is to investigate nonnegative steady-state solutions of (3.1). Thus we will concentrate on the following strongly-coupled elliptic system

$$\text{(SP3)}\quad \begin{cases} \Delta[(1+\alpha v)u] + u(a-u-cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b+du-v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

which is obtained from (3.1) by the rescaling

$$\alpha = \frac{d_2\rho_{12}}{c_2 d_1},\ \beta = \frac{d_1\rho_{21}}{b_1 d_2},\ a = \frac{a_1}{d_1},\ b = \frac{a_2}{d_2},\ c = \frac{c_1 d_2}{c_2 d_1},\ d = \frac{b_2 d_1}{b_1 d_2},\ \tilde{u} = \frac{b_1 u}{d_1},\ \tilde{v} = \frac{c_2 v}{d_2}.$$

For simplicity, we have dropped the ‘ ˜ ’ sign in (SP3).

We are mainly interested in positive solutions of (SP3). It is said that (u, v) is a positive solution of (SP3) if $u > 0$ and $v > 0$ in Ω. Among other things, we will prove that when (α, β, b, c, d) belongs to a certain range, the positive solution set $\{(u, v, a)\}$ of (SP3) contains an unbounded S-shaped curve with respect to a, while when (α, β, b, c, d) falls into another range, the positive solution set $\{(u, v, a)\}$ contains a bounded S or $\supset$-shaped curve. These results not only confirm multiple existence of positive solutions for (SP3) but also suggest that the dynamical behavior of (3.1) is quite complicate. The stability analysis for the above coexistence steady-states will be treated in the next chapter.

When there are no cross-diffusion effect ($\alpha = \beta = 0$), (SP3) is reduced to the classical Lotka-Volterra prey-predator system. This system has been discussed extensively by many authors (e.g., [10], [20], [21], [22], [46], [47], [49], [50], [51], [59], [70]). In particular, we know the exact range of parameter (a, b, c, d) for the existence of a positive solution of (SP3) (see Li [46, Theorem 1.A] or López-Gómez and Pardo [50, Theorem 3.1]). So it is possible to determine completely the coexistence region in a parameter space (a, b) (see [50, Figure 1]). Furthermore, López-Gómez and Pardo [51] have proved the uniqueness of positive solutions for the special case when the spatial dimension is one ($N = 1$, $\alpha = \beta = 0$).

To discuss the case $(\alpha, \beta) \neq (0, 0)$, we need some notation. Let $\lambda_1(q)$ be the least eigenvalue for the following eigenvalue problem

$$-\Delta u + q(x)u = \lambda u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

where $q(x)$ is a continuous function in $\overline{\Omega}$. We simply denote λ_1 instead of $\lambda_1(0)$. It is well known that the problem

$$\Delta u + u(a - u) = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{3.3}$$

has a unique positive solution θ_a if $a > \lambda_1$ (see, e.g. [10]); moreover, $a \to \theta_a : [\,\lambda_1, \infty) \to C(\overline{\Omega})$ and $q \to \lambda_1(q) : C(\overline{\Omega}) \to \boldsymbol{R}$ are continuous and strictly increasing functions. Here $C(\overline{\Omega})$ is equipped with the uniform convergence topology in $\overline{\Omega}$. It is possible to show that (SP3) has two semitrivial solutions

$$(\theta_a, 0) \quad \text{for} \quad a > \lambda_1 \qquad \text{and} \qquad (0, \theta_b) \quad \text{for} \quad b > \lambda_1$$

in addition to the trivial solution $(0,0)$.

Concerning the problem (SP3), Nakashima and Yamada [54] have obtained a sufficient condition of parameter (α,β,a,b,c,d) for the existence of a positive solution with use of the index theory.

Theorem 3.1 (Nakashima and Yamada [54]). *For $a>\lambda_1$, there exists a positive solution for* (SP3) *if one of the following conditions is satisfied:*

$$\lambda_1\left(\frac{c\theta_b-a}{1+\alpha\theta_b}\right)<0 \quad \textit{and} \quad \lambda_1\left(\frac{-b-d\theta_a}{1+\beta\theta_a}\right)<0, \tag{3.4}$$

$$\lambda_1\left(\frac{c\theta_b-a}{1+\alpha\theta_b}\right)>0 \quad \textit{and} \quad \lambda_1\left(\frac{-b-d\theta_a}{1+\beta\theta_a}\right)>0, \tag{3.5}$$

where it is understood that $\theta_b\equiv 0$ for $b\le\lambda_1$.

For $a\le\lambda_1$, (SP3) *has no positive solution.*

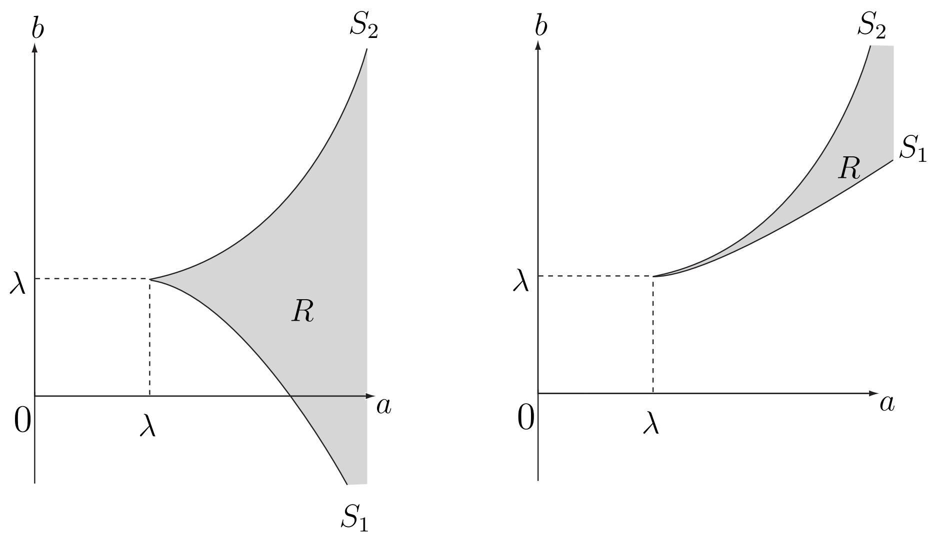
Before stating our results, we will explain the meaning of Theorem 3.1. Regarding a and b as parameters, we define

$$S_1:=\left\{(a,b)\in\boldsymbol{R}^2\ :\ \lambda_1\left(\frac{-b-d\theta_a}{1+\beta\theta_a}\right)=0\ \text{ for }\ a\ge\lambda_1\right\}, \tag{3.6}$$

$$S_2:=\left\{(a,b)\in\boldsymbol{R}^2\ :\ \lambda_1\left(\frac{c\theta_b-a}{1+\alpha\theta_b}\right)=0\ \text{ for }\ b\ge\lambda_1\right\}. \tag{3.7}$$

Lemma 3.20 in Section 3.7 implies that if $\beta\lambda_1<d$ (resp. $\beta\lambda_1>d$), then S_1 forms a monotone decreasing (resp. increasing) curve starting from (λ_1,λ_1). Lemma 3.22 asserts that S_2 is a monotone increasing curve which starts from (λ_1,λ_1). See Fig. 1. Combining these properties, one can deduce from Theorem 3.1 that if (a,b) lies in a region R surrounded by S_1 and S_2, then (SP3) has a positive solution. In case $\alpha=\beta=0$, this region R corresponds to the exact coexistence region shown by López-Gómez and Pardo [50]. Furthermore, Lemma 3.24 implies that S_1 curve is located below (resp. above) S_2 curve near (λ_1,λ_1) if $(\alpha\lambda_1+c)(\beta\lambda_1-d)<1$ (resp. $(\alpha\lambda_1+c)(\beta\lambda_1-d)>1$). From the view-point of the bifurcation theory, we can see that positive solutions bifurcate from $(\theta_a,0)$ when (a,b) crosses S_1 curve. Similarly positive solutions also bifurcate from $(0,\theta_b)$ when (a,b) moves across S_2.

We will give proofs of these bifurcation properties in Lemma 3.7. In this sense, Theorem 3.1 suggests that the structure to the positive solution set changes at $d/\beta = \lambda_1$.



A. Case $\beta\lambda_1 < d$ B. Case $\beta\lambda_1 > d$

Fig. 1. Coexistence region described in Theorem 3.1.

Once Theorem 3.1 is obtained, we are led to the next interesting problems; whether or not (SP3) has *multiple* positive solutions and whether (SP3) admits a positive solution even if (a, b) lies outside of R. Regarding a as a bifurcation parameter, we set

$$\mathcal{S} := \{(u, v, a) \,:\, (u, v) \text{ is a positive solution of (SP3)},\ a > \lambda_1\}.$$

Our main results are concerned with the global structure of $\mathcal{S}$. The first result asserts that for some (α, β, b, c, d) with $\min\{\beta b, d\} > \beta\lambda_1$, $\mathcal{S}$ contains an unbounded S-shaped curve (with respect to a) which bifurcates from the semitrivial solution curve $\{(0, \theta_b, a) \,:\, a > 0\}$:

Theorem 3.2. *Assume* $\min\{\beta b, d\} > \beta\lambda_1$. *For any* $c > 0$, *there exist a large number* M *and an open set*

$$O_1 = O_1(c) \subset \{(\alpha, \beta, b, d) \,:\, \beta \geq M,\, 0 < \alpha, d/\beta - \lambda_1, b - \lambda_1 \leq M^{-1}\}$$

such that $\partial O_1 \cap \{(\alpha, \beta, b, d) \,:\, d/\beta = \lambda_1\}$ *is not empty and, if* $(\alpha, \beta, b, d) \in O_1$, *then* $\mathcal{S}$ *contains an unbounded smooth curve*

$$\Gamma_1 = \{(u(s), v(s), a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty) \,:\, s \in (0, \infty)\},$$

which possesses the following properties:

(i) $(u(0), v(0)) = (0, \theta_b)$, $a(0) = a^* > \lambda_1$, $a'(0) > 0$, *where* a^* *is a unique number satisfying* $(a^*, b) \in S_2$.

(ii) $a(s) > a(0)$ *for all* $s \in (0, \infty)$ *and* $\lim_{s\to\infty} a(s) = \infty$;

(iii) $a(s)$ *attains a strict local maximum and a strict local minimum at some* $s = \overline{s}$ *and* $s = \underline{s}$ $(0 < \overline{s} < \underline{s})$, *respectively.*

Let $\overline{a} := a(\overline{s})$ and $\underline{a} := a(\underline{s})$. Theorem 3.2 implies that (SP3) has at least one positive solution if $a \in (a^*, \underline{a}) \cup (\overline{a}, \infty)$; at least two positive solutions if $a = \underline{a}$ or $a = \overline{a}$ and at least three positive solutions if $a \in (\underline{a}, \overline{a})$. Furthermore, we will show the nonexistence of positive solutions in $a \in (0, a^*]$. We remark that a^*, $\underline{a}$, $\overline{a}$ depend continuously on (α, β, b, c, d) and, moreover, (a^*, b) lies on S_2. Since Theorem 3.2 implies that (SP3) possesses multiple positive solutions for any (a, b) such that $(\alpha, \beta, b, d) \in O_1$ and $a \in [\underline{a}, \overline{a}]$, a multiple coexistence region can be formed in (a, b) space. Furthermore, this region is contained in R, because $a^* < \underline{a} < \overline{a}$ and $(a^*, b) \in S_2$, and S_2 is the left side boundary of R.

For some $(\alpha, \beta, a, b, c, d)$ with $\beta b > \beta\lambda_1 > d$, $\mathcal{S}$ contains a bounded S or $\supset$-shaped curve, which bifurcates from the semitrivial solution curve $\{(0, \theta_b, a) \,:\, a > 0\}$ and connects the other semitrivial solution curve $\{(\theta_a, 0, a)\,;\, a > \lambda_1\}$:

Theorem 3.3. *Assume* $\beta b > \beta\lambda_1 > d$. *For any* $c > 0$, *there exist a large number* M *and an open set*

$$O_2 = O_2(c) \subset \{(\alpha, \beta, b, d) \,:\, \beta \geq M,\, 0 < \alpha, \lambda_1 - d/\beta, b - \lambda_1 \leq M^{-1}\}$$

such that if $(\alpha, \beta, b, d) \in O_2$*, then* $\mathcal{S}$ *contains a bounded smooth curve*

$$\Gamma_2 = \{(u(s), v(s), a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty) \,:\, s \in (0, C)\},$$

which possesses the following properties:

(i) $(u(0), v(0)) = (0, \theta_b), \quad a(0) = a^* > \lambda_1, \quad a'(0) > 0$, *where* a^* *is a unique number satisfying* $(a^*, b) \in S_2$.

(ii) $(u(C), v(C)) = (\theta_{a(C)}, 0), \quad a(C) = a_* > \lambda_1$, *where* a_* *is a unique number satisfying* $(a_*, b) \in S_1$.

(iii) Γ_2 *has at least one turning point with respect to* a*. Furthermore, there exists an open set* $O_2' \subset O_2$ *such that* $\partial O_2' \cap \{(\alpha, \beta, b, d) \,:\, d/\beta = \lambda_1\}$ *is not empty and if* $(\alpha, \beta, b, d) \in O_2'$*, then* Γ_2 *has at least two turning points with respect to* a*.*

The above result asserts that if $(\alpha, \beta, b, d) \in O_2'$, then Γ_2 forms a bounded S-shaped positive solution branch. Further, it can be shown that $\hat{a} := \max_{s \in [0,C]} a(s) > \max\{a^*, a_*\}$ if $(\alpha, \beta, b, d) \in O_2 \setminus O_2'$. This fact means not only that (SP3) has multiple positive solutions for any $a \in (\max\{a_*, a^*\}, \hat{a})$ but also even in the right hand side of R, there exists a region where (SP3) admits multiple positive solutions, because $(\max\{a_*, a^*\}, b)$ lies on the right side boundary of R. In particular, for the one dimensional case ($N = 1$), the above multiple coexistence results in Theorems 3.2 and 3.3 are very different from the uniqueness result in the linear diffusion case $\alpha = \beta = 0$ (see [51]).

A crucial point of proofs for Theorems 3.2 and 3.3 is to construct a positive solution curve of (SP3) in the extreme case $\alpha = 0$. The analysis is based on the bifurcation theory and the Lyapunov-Schmidt reduction procedure. If β is large and $b - \lambda_1 > 0, |d/\beta - \lambda_1|$ are small, then this reduction enables us to find a relationship to a suitable limit problem. Further, by making use of the perturbation theory developed by Du and Lou [25], we will depict precise solution curves Γ_i of (SP3) near limit solution sets. In [25], they have obtained an S-shaped positive solution curve of a prey-predator system with the Holling-Tanner interaction terms.

The contents of the present paper are as follows: In Section 2, we first reduce (SP3) to the related semilinear problem (EP) by employing new unknown functions $U = (1+\alpha v)u$ and $V = (1+\beta u)v$. Next we give preliminary results about a priori estimates and bifurcation properties of positive solutions to (EP). In Section 3, we will introduce the perturbed problem (PP) for the Lyapunov-Schmidt reduction scheme. This problem (PP) can be reduced to (EP) with $\alpha = 0$ through some changing of variables and will play an important role in the proof. The solution set of (PP) will be investigated by way of a finite dimensional limit problem in Sections 4 and 5, where preliminary results obtained in Section 2 will give much important information. In Section 6, we will accomplish the proofs of our main results (by making use of results obtained in the preceding sections). Some basic properties of S_1 and S_2 defined by (3.6) and (3.7), which are needed in Sections 2-6, are given in Section 3.7.

Throughout the present part, the usual norms of the spaces $L^p(\Omega)$ for $p \in [1,\infty)$ and $C(\overline{\Omega})$ are defined by

$$\|u\|_p := \left(\int_\Omega |u(x)|^p\right)^{1/p}, \qquad \|u\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|.$$

In particular, we simply denote $\|u\|_2$ by $\|u\|$. Furthermore, we will denote by $\varPhi$ a unique positive solution of

$$-\Delta\varPhi = \lambda_1\varPhi \ \text{ in } \ \Omega, \quad \varPhi = 0 \ \text{ on } \ \partial\Omega, \quad \|\varPhi\| = 1.$$

3.2 Preliminaries

In this section, we first introduce a semilinear elliptic system equivalent to (SP3). Next we give some a priori estimates and local bifurcation properties of positive solutions to the semilinear system.

3.2.1 Reduction to the Semilinear Problem

Suppose $(\alpha, \beta) \neq (0, 0)$ in (SP3). Since we are interested in nonnegative solutions, it is convenient to introduce two unknown functions U and V by

$$U = (1 + \alpha v)u \quad \text{and} \quad V = (1 + \beta u)v. \tag{3.8}$$

There is a one-to-one correspondence between $(u, v) \geq 0$ and $(U, V) \geq 0$. It is possible to describe their relations by

$$u = u(U, V) = \frac{1}{2\beta}\left[\{(1 - \beta U + \alpha V)^2 + 4\beta U\}^{1/2} + \beta U - \alpha V - 1\right], \tag{3.9}$$

$$v = v(U, V) = \frac{1}{2\alpha}\left[\{(1 - \alpha V + \beta U)^2 + 4\alpha V\}^{1/2} + \alpha V - \beta U - 1\right]. \tag{3.10}$$

As far as we are concerned with nonnegative solutions, (SP3) is rewritten in the following equivalent form

$$\text{(EP)} \begin{cases} \Delta U + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta V + v(b + du - v) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega, \end{cases}$$

where $u = u(U, V)$ and $v = v(U, V)$ are understood to be functions of (U, V) defined by (3.9) and (3.10). It is easy to show that (EP) has two semitrivial solutions

$$(U, V) = (\theta_a, 0) \text{ for } a > \lambda_1 \quad \text{and} \quad (U, V) = (0, \theta_b) \text{ for } b > \lambda_1,$$

in addition to the trivial solution $(0, 0)$.

3.2.2 A Priori Estimates

We will derive some a priori estimates for positive solutions of (EP).

Lemma 3.4. *If* $a \leq \lambda_1$ *or* $\max\{\beta b, d\} \leq \beta\lambda_1$, *then* (EP) (*or equivalently,* (SP3)) *has no positive solution.*

Proof. Suppose for contradiction that (U, V) is a positive solution of (EP) for the case $\max\{\beta b, d\} \le \beta\lambda_1$. Observe that, if $d \le \beta b \le \beta\lambda_1$, then

$$\frac{V}{1+\beta u}(b+du-v) \le \frac{V}{1+\beta u}\{b(1+\beta u)-v\} < bV \quad \text{in} \quad \Omega,$$

while, if $\beta b < d \le \beta\lambda_1$, then

$$\frac{V}{1+\beta u}(b+du-v) \le \frac{d\,V}{\beta(1+\beta u)}\left(1+\beta u-\frac{\beta}{d}v\right) < \frac{d}{\beta}V \quad \text{in} \quad \Omega.$$

Then multiplying by V the second equation of (EP) and integrating the resulting expression in Ω, we obtain

$$\begin{cases} \|\nabla V\|^2 < b\,\|V\|^2 & \text{if} \quad d \le \beta b \le \beta\lambda_1, \\ \|\nabla V\|^2 < \dfrac{d}{\beta}\,\|V\|^2 & \text{if} \quad \beta b < d \le \beta\lambda_1. \end{cases} \tag{3.11}$$

Since $\|\nabla V\|^2 \ge \lambda_1\|V\|^2$ by Poincaré's inequality, (3.11) obviously yields a contradiction. By virtue of $U(a-u-cv)/(1+\alpha v) < aU$ in Ω, one can derive the assertion for the case $a \le \lambda_1$ in a similar manner. □

We will give a priori estimates for positive solutions in the case $a > \lambda_1$ and $\max\{\beta b, d\} > \beta\lambda_1$.

Lemma 3.5. *Let* (U, V) *be a positive solution of* (EP). *Then*

$$0 \le u(x) \le U(x) \le M(a) := \begin{cases} a & \textit{if} \quad \alpha a \le c, \\ \dfrac{(c+\alpha a)^2}{4\alpha c} & \textit{if} \quad \alpha a \ge c, \end{cases}$$

$$0 \le v(x) \le V(x) \le (1+\beta M(a))(b+dM(a))$$

for all $x \in \Omega$.

The following lemma gives other a priori estimates in the special cases.

Lemma 3.6. *Let* (U, V) *be a positive solution of* (EP). *If* $\alpha a \le c$, *then*

$$\theta_a \ge U \ge u \quad \textit{in} \quad \Omega. \tag{3.12}$$

If $\beta b \le d$, *then*

$$V \ge \theta_b \quad \textit{in} \quad \Omega.$$

For the proofs of Lemmas 3.5 and 3.6, see Lemmas 2 and 3 in [54].

3.2.3 Bifurcations from Semitrivial Solutions

In this subsection, we regard a as a bifurcation parameter with b fixed. We will derive local bifurcation properties for positive solutions of (EP) from the semitrivial solution curves

$$\{(U,V,a) : (U,V) = (\theta_a, 0),\ a > \lambda_1\} \ \text{ and } \ \{(U,V,a) : (U,V) = (0, \theta_b),\ a > \lambda_1\}.$$

Corollary 3.21 in Section 3.7 implies that, if $\beta b > \beta\lambda_1 > d$ or $d > \beta\lambda_1 > \beta b$, then there exists a unique constant $a_* \in (\lambda_1, \infty)$ such that

$$\lambda_1 \left(\frac{-b - d\theta_{a_*}}{1 + \beta\theta_{a_*}} \right) = 0. \tag{3.13}$$

Corollary 3.23 asserts that, if $b > \lambda_1$, then there exists a unique $a^* \in (\lambda_1, \infty)$ such that

$$\lambda_1 \left(\frac{c\theta_b - a^*}{1 + \alpha\theta_b} \right) = 0. \tag{3.14}$$

Let ϕ_* and ϕ^* denote the positive functions such that

$$-\Delta\phi_* - \frac{b + d\theta_{a_*}}{1 + \beta\theta_{a_*}}\phi_* = 0 \ \text{ in } \ \Omega, \quad \phi_* = 0 \ \text{ on } \ \partial\Omega, \quad \|\phi_*\| = 1$$

and

$$-\Delta\phi^* + \frac{c\theta_b - a^*}{1 + \alpha\theta_b}\phi^* = 0 \ \text{ in } \ \Omega, \quad \phi^* = 0 \ \text{ on } \ \partial\Omega, \quad \|\phi^*\| = 1.$$

By the definition of a_* and a^*, such ϕ_* and ϕ^* are uniquely determined from the above eigenvalue problems, respectively. Furthermore, for $p > N$, we define

$$\begin{cases} X := \left[\, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \,\right] \times \left[\, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \,\right], \\ Y := L^p(\Omega) \times L^p(\Omega). \end{cases} \tag{3.15}$$

Lemma 3.7. *Suppose $a > \lambda_1$. Then the following local bifurcation properties* (i) *and* (ii) *hold true:*

(i) *Let* $\beta b > \beta\lambda_1 > d$ *or* $d > \beta\lambda_1 > \beta b$. *Then positive solutions of* (EP) *bifurcate from the semitrivial solution curve* $\{(\theta_a, 0, a) \,:\, a > \lambda_1\}$ *if and only if* $a = a_*$. *To be precise, all positive solutions of* (EP) *near* $(\theta_{a_*}, 0, a_*) \in X \times \boldsymbol{R}$ *can be expressed as*

$$\{(\theta_{a_*} + s\psi + s\hat{U}(s),\, s\phi_* + s\hat{V}(s),\, a(s)) \,:\, 0 < s \le \delta\}$$

for some $\psi \in X$ *and* $\delta > 0$. *Here* $(\hat{U}(s), \hat{V}(s), a(s))$ *is a smooth function with respect to* s *and satisfies* $(\hat{U}(0), \hat{V}(0), a(0)) = (0, 0, a_*)$ *and* $\int_\Omega \hat{V}(s)\phi_* = 0$.

(ii) *Let* $b > \lambda_1$. *Then positive solutions of* (EP) *bifurcate from the semitrivial solution curve* $\{(0, \theta_b, a) \,:\, a > \lambda_1\}$ *if and only if* $a = a^*$. *More precisely, all positive solutions of* (EP) *near* $(0, \theta_b, a^*) \in X \times \boldsymbol{R}$ *are given by*

$$\{(s\phi^* + s\tilde{U}(s),\, \theta_b + s\chi + s\tilde{V}(s),\, a(s)) \,:\, 0 < s \le \delta\} \tag{3.16}$$

for some $\chi \in X$ *and* $\delta > 0$. *Here* $(\tilde{U}(s), \tilde{V}(s), a(s))$ *is a smooth function with respect to* s *and satisfies* $(\tilde{U}(0), \tilde{V}(0), a(0)) = (0, 0, a^*)$ *and* $\int_\Omega \tilde{U}(s)\phi^* = 0$.

Proof. For $a > \lambda_1$, set

$$f(u, v) = u(a - u - cv), \qquad g(u, v) = v(b + du - v),$$

where u, v are functions of U, V (see (3.9) and (3.10)). By Taylor's expansion at (U^*, V^*), we reduce differential equations of (EP) to the form

$$\begin{pmatrix} \Delta U \\ \Delta V \end{pmatrix} + \begin{pmatrix} f(u(U^*, V^*), v(U^*, V^*)) \\ g(u(U^*, V^*), v(U^*, V^*)) \end{pmatrix}$$
$$+ \begin{pmatrix} f_u^* & f_v^* \\ g_u^* & g_v^* \end{pmatrix} \begin{pmatrix} u_U^* & u_V^* \\ v_U^* & v_V^* \end{pmatrix} \begin{pmatrix} U - U^* \\ V - V^* \end{pmatrix} + \begin{pmatrix} \rho^1(U - U^*, V - V^*) \\ \rho^2(U - U^*, V - V^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.17}$$

where $f_u^* := f_u(u(U^*, V^*), v(U^*, V^*))$, $u_U^* := u_U(U^*, V^*)$ and other notations are defined by similar rules. Here $\rho^i(U - U^*, V - V^*)$ $(i = 1, 2)$ are smooth functions such that $\rho^i(0,0) = \rho^i_{(U,V)}(0,0) = 0$. Differentiation of (3.8) yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha v & \alpha u \\ \beta v & 1+\beta u \end{pmatrix} \begin{pmatrix} u_U & u_V \\ v_U & v_V \end{pmatrix}.$$

Since u and v are both nonnegative, we have

$$\begin{pmatrix} u_U & u_V \\ v_U & v_V \end{pmatrix} = \frac{1}{1+\alpha v+\beta u} \begin{pmatrix} 1+\beta u & -\alpha u \\ -\beta v & 1+\alpha v \end{pmatrix}. \tag{3.18}$$

Let $\beta b > \beta\lambda_1 > d$ or $d > \beta\lambda_1 > \beta b$. We note that

$$f(\theta_a, 0) = \theta_a(a-\theta_a) = -\Delta\theta_a, \quad g(\theta_a, 0) = 0.$$

So by virtue of (3.18), setting $(U^*, V^*) = (\theta_a, 0)$ and $\overline{U} := U - \theta_a$ in (3.17) yields

$$\begin{aligned} \begin{pmatrix} \Delta\overline{U} \\ \Delta V \end{pmatrix} + \frac{1}{1+\beta\theta_a} \begin{pmatrix} a-2\theta_a & -c\theta_a \\ 0 & b+d\theta_a \end{pmatrix} \begin{pmatrix} 1+\beta\theta_a & -\alpha\theta_a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{U} \\ V \end{pmatrix} \\ + \begin{pmatrix} \rho^1(\overline{U}, V; a) \\ \rho^2(\overline{U}, V; a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{3.19}$$

where $\rho^i(\overline{U}, V; a)$ $(i = 1, 2)$ are smooth functions satisfying

$$\rho^1_{(\overline{U},V)}(0,0;a) = \rho^2_{(\overline{U},V)}(0,0;a) = 0 \quad \text{for all} \quad a > \lambda_1. \tag{3.20}$$

Define a mapping $F : X \times \boldsymbol{R} \to Y$ by the left hand side of (3.19):

$$F(\overline{U}, V, a) = \begin{pmatrix} \Delta\overline{U} + (a-2\theta_a)\overline{U} - \dfrac{(\alpha a + c - 2\alpha\theta_a)\theta_a}{1+\beta\theta_a} V + \rho^1(\overline{U}, V, a) \\ \Delta V + \dfrac{b+d\theta_a}{1+\beta\theta_a} V + \rho^2(\overline{U}, V; a) \end{pmatrix}. \tag{3.21}$$

Since $(U, V) = (\theta_a, 0)$ is a semitrivial solution of (EP), it turns out $F(0, 0, a) = 0$ for $a > \lambda_1$. It follows from (3.20) and (3.21) that the Fréchet derivative of F at $(\overline{U}, V, a) = (0, 0, a)$ is given by

$$F_{(\overline{U},V)}(0,0,a) \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h + (a-2\theta_a)h - \dfrac{(\alpha a + c - 2\alpha\theta_a)\theta_a}{1+\beta\theta_a} k \\ \Delta k + \dfrac{b+d\theta_a}{1+\beta\theta_a} k \end{pmatrix}. \tag{3.22}$$

By virtue of (3.13), we see that $\mathrm{Ker}\, F_{(\overline{U},V)}(0,0,a)$ is nontrivial for $a = a_*$ and that

$$\mathrm{Ker}\, F_{(\overline{U},V)}(0,0,a_*) = \mathrm{span}\{\psi, \phi_*\},$$

with

$$\psi = -\left(-\Delta - a_* + 2\theta_{a_*}\right)^{-1} \left[\frac{(\alpha a_* + c - 2\alpha\theta_{a_*})\theta_{a_*}}{1+\beta\theta_{a_*}}\phi_*\right],$$

where $(-\Delta - a_* + 2\theta_{a_*})^{-1}$ is the inverse operator of $-\Delta - a_* + 2\theta_{a_*}$ with zero Dirichlet boundary condition on $\partial\Omega$. (Recall that $-\Delta - a_* + 2\theta_{a_*}$ is invertible; see, e.g., [20].) If ${}^t(\tilde{h}, \tilde{k}) \in \mathrm{Range}\, F_{(\overline{U},V)}(0,0,a^*)$, there must exist $(h,k) \in X$ such that

$$\begin{cases} \Delta h + (a_* - 2\theta_{a_*})h - \dfrac{(\alpha a_* + c - 2\alpha\theta_{a_*})\theta_{a_*}}{1+\beta\theta_{a_*}}k = \tilde{h} & \text{in } \Omega, \\ \Delta k + \dfrac{b + d\theta_{a_*}}{1+\beta\theta_{a_*}}k = \tilde{k} & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the second equation has a solution k if and only if $\int_\Omega \tilde{k}\phi_* = 0$. For such a solution k, the first equation has a unique solution h because of the invertibility of $-\Delta - a_* + 2\theta_{a_*}$. Then, it holds that $\mathrm{codimRange}\, F_{(\overline{U},V)}(0,0,a^*) = 1$. In order to use the local bifurcation theory by Crandall and Rabinowitz [17] at $(\overline{U}, V, a) = (0,0,a_*)$, we need to verify

$$F_{(\overline{U},V),a}(0,0,a_*)\begin{pmatrix} \psi \\ \phi_* \end{pmatrix} \notin \mathrm{Range}\, F_{(\overline{U},V)}(0,0,a_*).$$

Since $\rho^i_{(\overline{U},V),a}(0,0,a_*) = 0$ by (3.20), it follows from (3.21) that

$$\begin{aligned} &F_{(\overline{U},V),a}(0,0,a_*)\begin{pmatrix} \psi \\ \phi_* \end{pmatrix} \\ &= \begin{pmatrix} \left(1 - 2\left.\dfrac{d\theta_a}{da}\right|_{a=a_*}\right)\psi - \dfrac{\partial}{\partial a}\left(\dfrac{(\alpha a + c - 2\alpha\theta_a)\theta_a}{1+\beta\theta_a}\right)\Bigg|_{a=a_*}\phi_* \\ \dfrac{d-\beta b}{(1+\beta\theta_{a_*})^2}\left.\dfrac{d\theta_a}{da}\right|_{a=a_*}\phi_* \end{pmatrix}. \end{aligned}$$

Suppose for contradiction that there exists $k \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ such that

$$\Delta k + \frac{b + d\theta_{a_*}}{1+\beta\theta_{a_*}}k = \frac{d - \beta b}{(1+\beta\theta_{a_*})^2}\left.\frac{d\theta_a}{da}\right|_{a=a_*}\phi_*.$$

Multiplying the above equality by ϕ_* and integrating, we have

$$(d-\beta b)\int_\Omega \frac{1}{(1+\beta\theta_{a_*})^2}\frac{d\theta_a}{da}\bigg|_{a=a_*}\phi_*^2 = 0.$$

Thus it follows from the strict increasing property of θ_a that $d=\beta b$, which is impossible. Recall that $\overline{U}=U-\theta_a$, one can immediately obtain the assertion of (i) by applying the local bifurcation theorem ([17]). We note that the possibility of other bifurcation points except $a=a_*$ is excluded by virtue of the Krein-Rutman theorem.

Next assume $b>\lambda_1$. Setting $(U^*,V^*)=(0,\theta_b)$ and $\overline{V}=V-\theta_b$ in (3.17) implies

$$\begin{pmatrix} \Delta U - \dfrac{c\theta_b - a}{1+\alpha\theta_b}U + \rho^1(U,\overline{V};a) \\ \Delta\overline{V} - \dfrac{(\beta b - d - 2\beta\theta_b)\theta_b}{1+\alpha\theta_b}U + (b-2\theta_b)\overline{V} + \rho^2(u,\overline{V};a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.23}$$

where $\rho^i(U,\overline{V};a)\,(i=1,2)$ are smooth functions with $\rho^i_{(U,\overline{V})}(0,0;a)=0$ for all $a>\lambda_1$. Define a mapping $G(U,\overline{V},a)$ from $X\times\boldsymbol{R}$ to Y by the left hand side of (3.23). So it turns out that $G(0,0,a)=0$ for $a>\lambda_1$ and

$$G_{(U,\overline{V})}(0,0,a)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h - \dfrac{c\theta_b - a}{1+\alpha\theta_b}h \\ \Delta k - \dfrac{(\beta b - d - 2\beta\theta_b)\theta_b}{1+\alpha\theta_b}h + (b-2\theta_b)k \end{pmatrix}.$$

Thus it follows from (3.14) that $\operatorname{Ker} G_{(U,\overline{V})}(0,0,a)$ is nontrivial if $a=a^*$ and

$$\operatorname{Ker} G_{(U,\overline{V})}(0,0,a^*) = \operatorname{span}\{\phi^*,\chi\},$$

where

$$\chi = -(-\Delta - b + 2\theta_b)^{-1}\left[\frac{(\beta b - d - 2\beta\theta_b)\theta_b}{1+\alpha\theta_b}\phi^*\right]. \tag{3.24}$$

Furthermore, a similar procedure to the proof of (i) yields

$$G_{(U,\overline{V}),a}(0,0,a^*)\begin{pmatrix} \phi^* \\ \chi \end{pmatrix} \notin \operatorname{Range} G_{(U,\overline{V})}(0,0,a^*).$$

Hence the local bifurcation theory ensures the assertion of (ii). □

Remark 3.1. Corollary 3.21 in Section 3.7 also asserts that if $b > \lambda_1$ and $d \geq \beta\lambda_1$, then

$$\lambda_1\left(\frac{-b + d\theta_a}{1+\beta\theta_a}\right) < 0 \quad \text{for all} \quad a \in (\lambda_1, \infty).$$

So it follows from (3.22) that $F_{(\overline{U},V)}(0,0,a)$ is invertible for any $a \in (\lambda_1, \infty)$. By the implicit function theorem, we see that, if $b > \lambda_1$ and $d \geq \beta\lambda_1$, then no positive solution bifurcates from the semitrivial solution curve $\{(\theta_a, 0, a) \,:\, a > \lambda_1\}$.

3.3 The Lyapunov-Schmidt Reduction Scheme

We will carry out the Lyapunov-Schmidt reduction procedure in case $\alpha = 0$: Observe that, in $\alpha = 0$, (EP) is reduced to the problem

$$(\mathrm{EP})_0 \begin{cases} \Delta U + U\left(a - U - \dfrac{cV}{1+\beta U}\right) = 0 & \text{in } \Omega, \\ \Delta V + \dfrac{V}{1+\beta U}\left(b + dU - \dfrac{V}{1+\beta U}\right) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega. \end{cases}$$

We introduce the following change of variables in $(\mathrm{EP})_0$;

$$a = \lambda_1 + \varepsilon a_1, \;\; b = \lambda_1 + \varepsilon b_1, \;\; d/\beta = \lambda_1 + \varepsilon\tau, \;\; \beta = \gamma/\varepsilon, \;\; U = \varepsilon w, \;\; V = \varepsilon z, \quad (3.25)$$

where ε is a small positive parameter and τ is a constant which may be nonpositive. In what follows, we will mainly discuss the case that d/β and $b\,(> \lambda_1)$ are close to λ_1 and β is large. Through (3.25), $(\mathrm{EP})_0$ is rewritten in the form

$$(\mathrm{PP}) \begin{cases} \Delta w + \lambda_1 w + \varepsilon w\left(a_1 - w - \dfrac{cz}{1+\gamma w}\right) = 0 & \text{in } \Omega, \\ \Delta z + \lambda_1 z + \dfrac{\varepsilon z}{1+\gamma w}\left(b_1 + \tau\gamma w - \dfrac{z}{1+\gamma w}\right) = 0 & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that (3.25) maps semitrivial solutions

$$(U,V) = (\theta_a, 0)\,(a > \lambda_1) \quad \text{and} \quad (U,V) = (0, \theta_b)\,(b > \lambda_1)$$

of $(\mathrm{EP})_0$ to semitrivial ones

$$(w,z)=\left(\frac{1}{\varepsilon}\theta_{\lambda_1+\varepsilon a_1},0\right)\quad\text{and}\quad(w,z)=\left(0,\frac{1}{\varepsilon}\theta_{\lambda_1+\varepsilon b_1}\right)$$

of (PP), respectively. Furthermore, it follows from Lemma 3.7 and (3.25) that in case $\tau<0$, positive solutions of (PP) bifurcate from semitrivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1},0,a_1)\,:\,a_1>0\}$ if and only if

$$a_1=a_{1*}(\varepsilon):=\frac{1}{\varepsilon}(a_*-\lambda_1). \tag{3.26}$$

Similarly, positive solutions of (PP) bifurcate from other semitrivial solution curve $\{(0,\varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1},a_1)\,:\,a_1>0\}$ if and only if

$$a_1=a_1^*(\varepsilon):=\frac{1}{\varepsilon}(\lambda_1(c\theta_{\lambda_1+\varepsilon b_1})-\lambda_1). \tag{3.27}$$

In order to apply the Lyapunov-Schmidt reduction method, we will give a similar framework to that of Du and Lou [25]. For X and Y defined by (3.15), we introduce mappings $H:X\to Y$ and $B:X\times\boldsymbol{R}\to Y$ by

$$\begin{aligned}&H(w,z)=(\Delta w+\lambda_1 w,\Delta z+\lambda_1 z),\\&B(w,z,a_1)=\left(w\left(a_1-w-\frac{cz}{1+\gamma w}\right),\frac{z}{1+\gamma w}\left(b_1+\tau\gamma w-\frac{z}{1+\gamma w}\right)\right).\end{aligned} \tag{3.28}$$

Then (PP) is equivalent to

$$H(w,z)+\varepsilon B(w,z,a_1)=0. \tag{3.29}$$

Denote by X_1 and Y_1 the L^2-orthogonal complements of span$\{(\varPhi,0),(0,\varPhi)\}$ in X and Y respectively. Furthermore let $P:X\to X_1$ and $Q:Y\to Y_1$ be the L^2-orthogonal projections. Hence for each $(w,z)\in X$, there exists a unique $(s,t)\in\boldsymbol{R}^2$ such that

$$(w,z)=(s,t)\varPhi+\boldsymbol{u},\quad\text{where}\quad\boldsymbol{u}=P(w,z). \tag{3.30}$$

Additionally, (3.29) is decomposed as

$$\begin{cases} QH((s,t)\varPhi+\boldsymbol{u})+\varepsilon QB((s,t)\varPhi+\boldsymbol{u},a_1)=0, \\ (I-Q)H((s,t)\varPhi+\boldsymbol{u})+\varepsilon(I-Q)B((s,t)\varPhi+\boldsymbol{u},a_1)=0. \end{cases}$$

By virtue of $H((s,t)\varPhi)=0$ and $(I-Q)H(X_1)=0$, (3.29) (that is (PP)) is equivalent to

$$QH(\boldsymbol{u})+\varepsilon QB((s,t)\varPhi+\boldsymbol{u},a_1)=0 \tag{3.31}$$

and

$$(I-Q)B((s,t)\varPhi+\boldsymbol{u},a_1)=0.$$

In view of (3.31), we define a mapping $G:\boldsymbol{R}^4\times X_1\to Y_1$ by

$$G(s,t,a_1,\varepsilon,\boldsymbol{u})=QH(\boldsymbol{u})+\varepsilon QB((s,t)\varPhi+\boldsymbol{u},a_1).$$

Then it follows that

$$G(s,t,a_1,0,0)=0\ \text{ for any }\ (s,t,a_1)\in\boldsymbol{R}^3.$$

Furthermore, it is possible to verify that

$$G_{\boldsymbol{u}}(s,t,a_1,0,0)=QH\ \text{ for any }\ (s,t,a_1)\in\boldsymbol{R}^3;$$

so that $G_{\boldsymbol{u}}(s,t,a_1,0,0)$ is an isomorphism from X_1 onto Y_1. Therefore, the implicit function theorem implies that for any $(s',t',a_1')\in\boldsymbol{R}^3$ there exist a positive constant $\varepsilon'=\varepsilon'(s',t',a_1')$ and a neighborhood N' of $(w,z,a_1,\varepsilon)=(s'\varPhi,t'\varPhi,a_1',0)$ in $X\times\boldsymbol{R}^2$ such that all solutions of (3.31) in N' are expressed as

$$\{((s,t)\varPhi+\boldsymbol{u}(s,t,a_1,\varepsilon),a_1,\varepsilon)\ :\ |s-s'|,\,|t-t'|,\,|a_1-a_1'|,\,|\varepsilon|\le\varepsilon'\}.$$

Taking account for the compactness of $\{(s,t,a_1)\ :\ |s|,\,|t|,\,|a_1|\le C\}$, one can find a positive $\varepsilon_0=\varepsilon_0(C)$ and a neighborhood N_0 of $\{(s\varPhi,t\varPhi,a_1,0)\ :\ |s|,\,|t|,\,|a_1|\le C\}$ such that all solutions of (3.31) in N_0 are given by

$$\{((s,t)\varPhi+\boldsymbol{u}(s,t,a_1,\varepsilon),a_1,\varepsilon)\ :\ |s|,\,|t|,\,|a_1|\le C+\varepsilon_0,\,|\varepsilon|\le\varepsilon_0\}. \tag{3.32}$$

Here we note $\boldsymbol{u}(s,t,a_1,\varepsilon)$ is an X_1-valued smooth function with $\boldsymbol{u}(s,t,a_1,0)=0$. Hence if we put

$$\varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon)=\boldsymbol{u}(s,t,a_1,\varepsilon), \tag{3.33}$$

then $\boldsymbol{U}(s,t,a_1,\varepsilon)$ is also smooth for $|s|$, $|t|$, $|a_1|\le C+\varepsilon_0$ and $|\varepsilon|\le\varepsilon_0$. The above consideration gives the following lemma:

Lemma 3.8. *Suppose that* $|s|$, $|t|$, $|a_1|\le C+\varepsilon_0$ *and* $|\varepsilon|\le\varepsilon_0$. *Then any element of the set defined by* (3.32) ;

$$(w,z,a_1,\varepsilon)=((s,t)\Phi+\varepsilon\boldsymbol{U}(s,t,a_1,\varepsilon),a_1,\varepsilon)$$

becomes a solution of (3.29) (*or equivalently* (PP)) *in* N_0 *if and only if*

$$(I-Q)B((s,t)\Phi+\varepsilon\boldsymbol{U}(s,t,a_1,\varepsilon),a_1)=0.$$

Let $M=\{(s,t,a_1)\,:\,|s|,\,|t|,\,|a_1|\le C+\varepsilon_0\}$. For each $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$, define a mapping $F^\varepsilon : M\to\boldsymbol{R}^2$ by

$$F^\varepsilon(s,t,a_1)\Phi=(I-Q)B((s,t)\Phi+\varepsilon\boldsymbol{U}(s,t,a_1,\varepsilon),a_1).$$

It follows from (3.28) that, if we put $\boldsymbol{U}(s,t,a_1,\varepsilon)=(W(s,t,a_1,\varepsilon),Z(s,t,a_1,\varepsilon))$, then

$$F^\varepsilon(s,t,a_1) = \begin{pmatrix} \displaystyle\int_\Omega (s\Phi+\varepsilon W)\left[a_1-(s\Phi+\varepsilon W)-\frac{c(t\Phi+\varepsilon Z)}{1+\gamma(s\Phi+\varepsilon W)}\right]\Phi \\ \displaystyle\int_\Omega \frac{t\Phi+\varepsilon Z}{1+\gamma(s\Phi+\varepsilon W)}\left[b_1+\tau\gamma(s\Phi+\varepsilon W)-\frac{t\Phi+\varepsilon Z}{1+\gamma(s\Phi+\varepsilon W)}\right]\Phi \end{pmatrix}. \tag{3.34}$$

Lemma 3.8 asserts that for each $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$, the solution set of (3.29) in N_0 is identical to $\operatorname{Ker} F^\varepsilon$.

3.4 The Limit Solution Set of (PP)

In this section we investigate the structure of $\operatorname{Ker} F^0(s,t,a_1)$. It will give much important information on a set of positive solutions of (PP) when $\varepsilon>0$ is very small. It follows from (3.34) that

$$F^0(s,t,a_1) = \begin{pmatrix} s\left(a_1 - s\|\phi\|_3^3 - ct\displaystyle\int_\Omega \frac{\Phi^3}{1+\gamma s\Phi}\right) \\ t\left[b_1 - (b_1-\tau)\gamma s\displaystyle\int_\Omega \frac{\Phi^3}{1+\gamma s\Phi} - t\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2}\right] \end{pmatrix}. \tag{3.35}$$

Therefore, $\mathrm{Ker}\, F^0(s,t,a_1)$ is a union of the following four sets:

$$\begin{aligned} \mathcal{L}_0 &= \{(0,0,a_1) \,:\, a_1 \in \boldsymbol{R}\}, \\ \mathcal{L}_1 &= \{(a_1/\|\Phi\|_3^3, 0, a_1) \,:\, a_1 \in \boldsymbol{R}\}, \\ \mathcal{L}_2 &= \{(0, b_1/\|\Phi\|_3^3, a_1) \,:\, a_1 \in \boldsymbol{R}\}, \\ \mathcal{L}_p &= \{(s, \varphi(\gamma s), \psi(s)) \,:\, s \in \boldsymbol{R}\}, \end{aligned}$$

where

$$\begin{cases} \varphi(s) = \left[b_1 - (b_1-\tau)s\displaystyle\int_\Omega \frac{\Phi^3}{1+s\Phi}\right]\left(\int_\Omega \frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1}, \\ \psi(s) = s\|\Phi\|_3^3 + c\varphi(\gamma s)\displaystyle\int_\Omega \frac{\Phi^3}{1+\gamma s\Phi}. \end{cases} \tag{3.36}$$

By the identification $(s,t)\Phi$ with (s,t), $\mathcal{L}_1 \cap \overline{\boldsymbol{R}_+}^3$, $\mathcal{L}_2 \cap \overline{\boldsymbol{R}_+}^3$ and $\mathcal{L}_p \cap \overline{\boldsymbol{R}_+}^3$ can be regarded as the limit sets of semitrivial solution curves $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1) \,:\, a_1 > 0\}$, $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1) \,:\, a_1 > 0\}$ and the positive solution set of (PP) as $\varepsilon \to 0$, respectively. By virtue of (3.36),

$$(0, \varphi(0), \psi(0)) = (0, b_1/\|\phi\|_3^3, cb_1) \in \mathcal{L}_2. \tag{3.37}$$

It is easily verified that in case $\tau \geq 0$,

$$\varphi(s) > 0 \quad \text{for all} \quad s \in [\,0,\infty).$$

On the other hand, if $\tau < 0$, we can find a positive constant $s_0 = s_0(\tau/b_1)$ such that

$$\begin{cases} \varphi(s) > 0 \quad \text{for} \quad s \in [\,0, s_0), \\ \varphi(s) < 0 \quad \text{for} \quad s \in (s_0, \infty). \end{cases} \tag{3.38}$$

Thus it follows that

$$(s_0/\gamma, \varphi(s_0), \psi(s_0/\gamma)) = (s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma) \in \mathcal{L}_1 \tag{3.39}$$

provided $\tau < 0$.

We will study profiles of ψ:

Lemma 3.9. *The following properties of $\psi(s)$ hold true:*
(a) *If $\tau \geq 0$, then $\psi(s) > \psi(0) = cb_1$ for all $s \in (0, \infty)$ and $\lim_{s\to\infty} \psi(s) = \infty$,*
(b) *there exist positive constants $\tilde{\tau} = \tilde{\tau}(c, b_1)$ and $\tilde{\gamma} = \tilde{\gamma}(c, b_1)$ such that*

(i) *if $(\tau, \gamma) \in [\,0, \tilde{\tau}\,] \times [\,\tilde{\gamma}, \infty)$, then $\psi(s)$ attains a strict local maximum and a strict local minimum at some $s = \overline{s}$ and $s = \underline{s}$ $(0 < \overline{s} < \underline{s})$ respectively, which satisfy $\psi(\overline{s}) > \psi(\underline{s})$;*

(ii) *if $(\tau, \gamma) \in [\,-\tilde{\tau}, 0) \times [\,\tilde{\gamma}, \infty)$, then $\psi(s)$ achieves a strict local maximum in $(0, s_0/\gamma)$. Furthermore, there exists a continuous function $\hat{\gamma}(\tau)$ in $[\,-\tilde{\tau}, 0)$ with*

$$\tilde{\gamma} < \hat{\gamma}(\tau) \quad \text{for all} \quad \tau \in [\,-\tilde{\tau}, 0) \quad \text{and} \quad \lim_{\tau \uparrow 0} \hat{\gamma}(\tau) = \infty \tag{3.40}$$

such that, if $(\tau, \gamma) \in [\,-\tilde{\tau}, 0) \times [\,\tilde{\gamma}, \hat{\gamma}(\tau))$, then $\psi(s)$ attains a strict local minimum in $(0, s_0/\gamma)$ and, moreover, if $(\tau, \gamma) \in [\,-\tilde{\tau}, 0) \times [\,\hat{\gamma}(\tau), \infty)$, then $\max_{s \in [\,0, s_0/\gamma\,]} \psi(s) = \psi(\hat{s})$ for some $\hat{s} \in (0, s_0/\gamma)$.

Proof. In view of (3.36), if we define

$$\begin{aligned} h(s;\tau) &:= \varphi(s) \int_\Omega \frac{\Phi^3}{1+s\Phi} \\ &= \left[b_1 - (b_1 - \tau)s \int_\Omega \frac{\Phi^3}{1+s\Phi} \right] \int_\Omega \frac{\Phi^3}{1+s\Phi} \left(\int_\Omega \frac{\Phi^3}{(1+s\Phi)^2} \right)^{-1}, \end{aligned} \tag{3.41}$$

then

$$\psi(s) = s\|\Phi\|_3^3 + ch(\gamma s; \tau). \tag{3.42}$$

Recalling $\|\Phi\| = 1$, one can see

$$\lim_{s\to\infty} \frac{h(s;\tau)}{s} = \frac{\tau}{\|\Phi\|_1},$$

which immediately yields

$$\lim_{s\to\infty}\frac{\psi(s)}{s}=\|\Phi\|_3^3+\frac{c\gamma\tau}{\|\Phi\|_1}\quad\text{for any}\quad \tau\in\boldsymbol{R}.$$

In particular, observing that

$$\left(\int_\Omega\frac{\Phi^3}{1+s\Phi}\right)^2\le\int_\Omega\frac{\Phi^4}{(1+s\Phi)^2}\int_\Omega\Phi^2=\int_\Omega\frac{\Phi^4}{(1+s\Phi)^2}$$

from Schwarz' inequality, we see that

$$\begin{aligned}h(s;0)&=b_1\left(1-s\int_\Omega\frac{\Phi^3}{1+s\Phi}\right)\int_\Omega\frac{\Phi^3}{1+s\Phi}\left(\int_\Omega\frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1}\\&\ge b_1\left(\int_\Omega\frac{\Phi^3}{1+s\Phi}-s\int_\Omega\frac{\Phi^4}{(1+s\Phi)^2}\right)\left(\int_\Omega\frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1}\\&=b_1=h(0;0)\quad\text{for all}\quad s\in[0,\infty).\end{aligned}\tag{3.43}$$

It obviously follows from (3.41)-(3.43) that $\psi(s)>\psi(0)=cb_1$ for all $s\in(0,\infty)$ provided $\tau\ge0$. Furthermore, note

$$1-s\int_\Omega\frac{\Phi^3}{1+s\Phi}=\int_\Omega\frac{(1+s\Phi)\Phi^2}{1+s\Phi}-s\int_\Omega\frac{\Phi^3}{1+s\Phi}=\int_\Omega\frac{\Phi^2}{1+s\Phi}.$$

Hence

$$\begin{aligned}\lim_{s\to\infty}h(s;0)=&b_1\lim_{s\to\infty}\int_\Omega\frac{\Phi^2}{1+s\Phi}\int_\Omega\frac{\Phi^3}{1+s\Phi}\left(\int_\Omega\frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1}\\=&b_1=h(0;0).\end{aligned}\tag{3.44}$$

Thus (3.43) and (3.44) imply that $h(s;0)$ attains a global maximum at some point in $(0,\infty)$. Therefore, in view of (3.42), one can see that, if $\tau=0$ and γ is large enough, then $\psi(s)$ forms a '$\sim$'-shaped curve in the sense of the assertion (i). Hence this property of $\psi(s)$ is invariant for small $\tau>0$ and the proof of (i) is complete.

In case $\tau<0$, (3.38) and (3.41) yield

$$\begin{cases}h(s;\tau)>0 & \text{for}\quad s\in[0,s_0),\\h(s;\tau)<0 & \text{for}\quad s\in(s_0,\infty).\end{cases}$$

Hence, if $|\tau|$ is sufficiently small, then $h(s;\tau)$ achieves a global maximum at some point contained in $(0, s_0)$ because of (3.43). Thus by (3.42), we may assume that if $(\tau, \gamma) \in [\,-\tilde{\tau}, 0) \times [\,\tilde{\gamma}, \infty)$, then $\psi(s)$ possesses at least one strict local maximum in $(0, s_0/\gamma)$. Observe that g depends continuously on (τ, γ); so we get a continuous function $\hat{\gamma}(\tau)$ in $[\,-\tilde{\tau}, 0)$ with (3.40) such that, if $(\tau, \gamma) \in [\,-\tilde{\tau}, 0) \times [\,\tilde{\gamma}, \hat{\gamma}(\tau))$, then $\psi(s)$ forms a '$\sim$'-shaped curve in $(0, s_0/\gamma)$ and, if $(\tau, \gamma) \in [\,-\tilde{\tau}, 0) \times [\,\hat{\gamma}(\tau), \infty)$, then $\max\limits_{s \in [\,0, s_0/\gamma\,]} \psi(s) = \psi(\hat{s})$ for some $\hat{s} \in (0, s_0/\gamma)$. Thus the proof of Lemma 3.9 is accomplished. □

3.5 The Perturbed Solution Set of (PP)

3.5.1 Case $\tau \geq 0$

Let $\tau \geq 0$. By Lemma 3.9, there exist sufficient large numbers A_1 and C such that

$$A_1 = \psi(C) = \max_{s \in [\,0, C\,]} \psi(s). \tag{3.45}$$

In this subsection, we will prove that if $\varepsilon > 0$ is small enough, then all positive solutions of (PP) in the range of $a_1 \in [\,0, A_1\,]$ form a one-dimensional submanifold near

$$\{(w, z, a_1) = (s\Phi, \varphi(\gamma s)\Phi, \psi(s)) \,:\, 0 < s \leq C\}.$$

More precisely, we will prove the following proposition:

Proposition 3.10. *Let $\tau \geq 0$. Then there exist a positive constant $\varepsilon_0 = \varepsilon_0(A_1)$ and a family of bounded smooth curves*

$$\{S(\xi, \varepsilon) = (s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \boldsymbol{R}^3 \,:\, (\xi, \varepsilon) \in [\,0, C(\varepsilon)\,] \times [\,0, \varepsilon_0\,]\}$$

such that for each fixed $\varepsilon \in (0, \varepsilon_0\,]$, all positive solutions of (PP) *with $a_1 \in (0, A_1\,]$ can be parameterized as*

$$\begin{aligned} \Gamma^\varepsilon = \{(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) &= ((s, t)\Phi + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon), a_1) \,: \\ (s, t, a_1) &= (s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon)) \;\; \textit{for} \;\; \xi \in (0, C(\varepsilon)]\}, \end{aligned} \tag{3.46}$$

and

$$S(\xi,0)=(\xi,\varphi(\gamma\xi),\psi(\xi)),\quad S(0,\varepsilon)=(0,t(\varepsilon),a_1^*(\varepsilon)).$$

Here $C(\varepsilon)$ is a smooth positive function in $[0,\varepsilon_0]$ with $C(0)=C$ and $a_1(C(\varepsilon),\varepsilon)=A_1$, $t(\varepsilon):=\varepsilon^{-1}\int_\Omega \theta_{\lambda_1+\varepsilon b_1}\Phi$, $a_1^*(\varepsilon)$ is the positive number defined by (3.27) *and* $\boldsymbol{U}$ *is the X_1-valued function defined by* (3.33)*. Furthermore, Γ^ε can be extended to the range $a_1\in[A_1,\infty)$ as a positive solution curve of* (PP)*.*

As the first step to the proof of Proposition 3.10, we will express the nonnegative solution set of (3.29) (or equivalently (PP)) near the intersection point of $\mathcal{L}_p$ and $\mathcal{L}_2$; $(s,t,a_1)=(0,b_1/\|\Phi\|_3^3,cb_1)$.

Lemma 3.11. *Let F^ε be the mapping defined by* (3.34)*. Then there exist a neighborhood U_0 of $(0,b_1/\|\Phi\|_3^3,cb_1)$ and a positive constant δ_0 such that for any $\varepsilon\in[0,\delta_0]$,*

$$\begin{aligned}&\operatorname{Ker} F^\varepsilon\cap U_0\cap\overline{\boldsymbol{R}_+}^3\\ =&\{(s(\xi,\varepsilon),t(\xi,\varepsilon),a_1(\xi,\varepsilon))\,:\,\xi\in[0,\delta_0)\}\cup\{(0,t(\varepsilon),a_1)\in U_0\}\end{aligned}\tag{3.47}$$

with some smooth function $(s(\xi,\varepsilon),t(\xi,\varepsilon),a_1(\xi,\varepsilon))$ in $[0,\delta_0]\times[0,\delta_0]$ satisfying

$$(s(\xi,0),t(\xi,0),a_1(\xi,0))=(\xi,\varphi(\gamma\xi),\psi(\xi)),$$
$$(s(0,\varepsilon),t(0,\varepsilon),a_1(0,\varepsilon))=(0,t(\varepsilon),a_1^*(\varepsilon)).$$

Proof. By Lemma 3.7 and (3.27), we recall that for any $\varepsilon>0$, there exist a positive number $\delta=\delta(\varepsilon)$ and a neighborhood V_ε of the bifurcation point $(w,z,a_1)=(0,\varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1},a_1^*(\varepsilon))$ such that all positive solutions of (PP) in V_ε are given by

$$\begin{aligned}&(w,z,a_1)=(w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\\ =&(\xi\phi^*+\xi W(\xi,\varepsilon),\,\varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}+\xi\chi+\xi Z(\xi,\varepsilon),\,a_1(\xi,\varepsilon))\end{aligned}$$

for $\xi\in(0,\delta]$. Here χ is the function defined by (3.24), $(W(\xi,\varepsilon),Z(\xi,\varepsilon),a_1(\xi,\varepsilon))$ is a certain smooth function such that $a_1(0,\varepsilon)=a_1^*(\varepsilon)$ and $\int_\Omega W(\xi,\varepsilon)\phi^*=0$. We define an open set U_ε of $\boldsymbol{R}^3$ by

$$U_\varepsilon:=\left\{(s,t,a_1)\,:\,s=\int_\Omega w\Phi,\,t=\int_\Omega z\Phi,\,(w,z,a_1)\in V_\varepsilon\right\}.$$

and put

$$s(\xi,\varepsilon) := \int_\Omega w(\xi,\varepsilon)\Phi, \qquad t(\xi,\varepsilon) := \int_\Omega z(\xi,\varepsilon)\Phi.$$

By virtue of the equivalence of (PP) and (3.29), we can verify that, if $\varepsilon \in [0, \varepsilon_0]$, then

$$\begin{aligned}&\operatorname{Ker} F^\varepsilon \cap U_\varepsilon \cap \overline{\boldsymbol{R}_+}^3 \\ =&\{(s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)) \,:\, \xi \in [0,\delta]\} \cup \{(0, t(\varepsilon), a_1) \in U_\varepsilon\}.\end{aligned}$$

Since $(0, t(\varepsilon), a_1^*(\varepsilon))$ is a bifurcation point for any $\varepsilon \in [0, \varepsilon_0]$, it is possible to show that U_ε contains a neighborhood U_0 of $(0, b_1/\|\Phi\|_3^3, cb_1)$ if $\varepsilon > 0$ is sufficiently small. Thus the proof of Lemma 3.11 is complete. □

Lemma 3.12. *Assume $\tau \geq 0$ and let A_1, C be positive constants obtained in* (3.45)*. There exist $\varepsilon_0 = \varepsilon(A_1) > 0$ and a neighborhood U of $\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : 0 \leq s \leq C\}$ such that for each fixed $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of* (PP) *contained in $U \cap (X \times (0, A_1])$ can be expressed as* (3.46)*.*

Proof. We will prove this lemma along the perturbation theory by Du and Lou [25, Appendix]. Define $\mathcal{L}_p([\delta_0/2, C]) = \{(s, \varphi(\gamma s), \psi(s)) \,:\, s \in [\delta_0/2, C]\}$ for the positive constant δ_0 obtained in Lemma 3.11. By (3.35) and (3.36), direct calculations lead to

$$\det F^0_{(s,t)}(s, \varphi(\gamma s), \psi(s)) = s\varphi(\gamma s)\psi'(s) \int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2}. \tag{3.48}$$

Let $(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) \in \mathcal{L}_p([\delta_0/2, C])$ be any fixed point. Note that $\varphi(\gamma\overline{s}) > 0$ for $\tau \geq 0$. Thus (3.48) implies that, if $\psi'(\overline{s}) \neq 0$, then $F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s}))$ is invertible. In this case, the implicit function theorem gives a positive number $\delta = \delta(\overline{s})$ and neighborhood $W_{\overline{s}}$ of $(\overline{s}, \varphi(\gamma\overline{s}))$ such that for all $\varepsilon \in [0, \delta]$,

$$\operatorname{Ker} F^\varepsilon \cap U_{\overline{s}} = \{(s(a_1,\varepsilon), t(a_1,\varepsilon), a_1) \,:\, a_1 \in (\psi(\overline{s}) - \delta, \psi(\overline{s}) + \delta)\}, \tag{3.49}$$

where $U_{\overline{s}} = W_{\overline{s}} \times (\psi(\overline{s}) - \delta, \psi(\overline{s}) + \delta)$ and $(s(a_1,\varepsilon), t(a_1,\varepsilon))$ is a smooth function satisfying $(s(\psi(\overline{s}), 0), t(\psi(\overline{s}), 0)) = (\overline{s}, \varphi(\gamma\overline{s}))$.

On the other hand, if $\psi'(\overline{s}) = 0$, then (3.48) implies rank $F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = 1$; so that

$$\dim \operatorname{Ker} F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = \operatorname{codim} \operatorname{Range} F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = 1. \tag{3.50}$$

After some calculations, one can see

$$F^0_{a_1}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = \begin{pmatrix} \overline{s} \\ 0 \end{pmatrix} \notin \operatorname{Range} F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})). \tag{3.51}$$

According to the spontaneous bifurcation theory by Crandall and Rabinowitz [18, Theorem 3.2 and Remark 3.3], (3.50) and (3.51) enable us to get a positive number $\delta = \delta(\overline{s})$ and a neighborhood $U_{\overline{s}}$ of $(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s}))$ such that for each $\varepsilon \in [\,0, \delta\,]$,

$$\operatorname{Ker} F^\varepsilon \cap U_{\overline{s}} = \{(s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)) \,:\, \xi \in (-\delta, \delta)\} \tag{3.52}$$

with a suitable smooth function $(s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon))$ in $[\,-\delta, \delta\,] \times [\,0, \delta\,]$ with

$$(s(0,0), t(0,0), a_1(0,0)) = (\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})).$$

For each $U_{\overline{s}}$ satisfying (3.49) or (3.52), it clearly follows that

$$\mathcal{L}_p([\,\delta_0/2, C\,]) \subset \bigcup \{U_{\overline{s}} \,:\, \overline{s} \in [\,\delta_0/2, C\,]\}.$$

Since $\mathcal{L}_p([\,\delta_0/2, C\,])$ is compact, there are a finite number of points $\{s_j\}_{j=1}^k$ such that

$$\begin{cases} (s_j, \varphi(s_j), \psi(s_j)) \in \mathcal{L}_p([\,\delta_0/2, C\,]) \quad \text{for} \quad 1 \le j \le k, \\ \mathcal{L}_p([\,\delta_0/2, C\,]) \subset \displaystyle\bigcup_{j=1}^k U_j, \quad \text{where} \quad U_j := U_{s_j}. \end{cases}$$

We may assume $U_j \cap U_{j+1}$ are not empty for all $0 \le j \le k-1$. Here U_0 is the open set obtained in Lemma 3.11. Thus by (3.49) and (3.52), if we put $\delta_j = \delta(s_j)$, then for any $\varepsilon \in [\,0, \delta_j\,]$ $(1 \le j \le k)$,

$$\operatorname{Ker} F^\varepsilon \cap U_j = \{(s^j(\xi,\varepsilon), t^j(\xi,\varepsilon), a_1^j(\xi,\varepsilon)) \,:\, \xi \in (-\delta_j, \delta_j)\} =: J_j^\varepsilon$$

with some smooth functions $s^j(\xi,\varepsilon)$, $t^j(\xi,\varepsilon)$ and $a_1^j(\xi,\varepsilon)$ which satisfy

$$(s^j(0,0),t^j(0,0),a_1^j(0,0)) = (s_j,\varphi(\gamma s_j),\psi(s_j)).$$

Additionally in view of Lemma 3.11, if we set

$$J_0^\varepsilon = \{(s(\xi,\varepsilon),t(\xi,\varepsilon),a_1(\xi,\varepsilon)) \,:\, \xi\in(0,\delta_0\,]\}$$

and $U = \bigcup_{j=0}^{k} U_j$, then

$$\mathrm{Ker}\, F^\varepsilon \cap U \cap \boldsymbol{R}_+^3 = \bigcup_{j=0}^{k} J_j^\varepsilon \quad \text{for any} \quad \varepsilon\in[\,0,\min_{0\le j\le k}\delta_j\,]. \tag{3.53}$$

Clearly (3.53) implies that $\mathrm{Ker}\, F^\varepsilon\cap U\cap \boldsymbol{R}_+^3$ forms a one-dimensional submanifold. Indeed, with the aid of the procedure by Du and Lou [25, Proposition A3], it is possible to construct a smooth curve $S(\xi,\varepsilon) = (s(\xi,\varepsilon),t(\xi,\varepsilon),a_1(\xi,\varepsilon))$ such that

$$\begin{cases} \bigcup_{j=0}^{k} J_j^\varepsilon = S((0,C(\varepsilon)\,],\varepsilon), \\ (s(\xi,0),t(\xi,0),a_1(\xi,0)) = (\xi,\varphi(\gamma\xi),\psi(\xi)), \\ (s(0,\varepsilon),t(0,\varepsilon),a_1(0,\varepsilon)) = (0,t(\varepsilon),a_1^*(\varepsilon)) \end{cases} \tag{3.54}$$

for sufficiently small $\varepsilon>0$ and $\xi\in[\,0,C(\varepsilon)\,]$ with some smooth function $C(\varepsilon)$. In view of Lemma 3.8, one can get the conclusion from (3.54). □

The next lemma means that if $a_1\in(0,A_1\,]$ and $\varepsilon>0$ is small enough, then (PP) has no positive solution outside of U.

Lemma 3.13. *Assume $\tau\ge 0$ and V is any neighborhood of $\{(s\Phi,\varphi(\gamma s)\Phi,\psi(s)) : 0\le s\le C\}$. Then there exists a positive constant ε_1 such that for each $\varepsilon\in(0,\varepsilon_1\,]$, any solution of* (PP) *with $a_1\in(0,A_1\,]$ is given by*

$$(w,z) = (s,t)\Phi + \varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon) \quad \textit{for some} \quad (s\Phi,t\Phi,a_1)\in V.$$

Proof. We will prove this lemma by a contradiction argument. Suppose that for a certain sequence $\{(a_1^n, \varepsilon_n)\}$ satisfying $a_1^n \in (0, A_1]$ and $\lim_{n\to 0} \varepsilon_n = 0$, (PP) with $(a_1, \varepsilon) = (a_1^n, \varepsilon_n)$ have positive solutions (w_n, z_n) such that $(w_n, z_n, a_1^n) \notin V$ for all $n \in \boldsymbol{N}$. To derive a contradiction, it suffices to find a subsequence $\{(w_{n(k)}, z_{n(k)}, a_1^{n(k)}, \varepsilon_{n(k)})\}$ and a sequence $\{(s_k, t_k)\}$ such that

$$\begin{cases} (w_{n(k)}, z_{n(k)}) = (s_k, t_k)\Phi + \varepsilon_{n(k)} \boldsymbol{U}(s_k, t_k, a_1^{n(k)}, \varepsilon_{n(k)}) \ \text{ for all } \ k \in \boldsymbol{N}, \\ \lim_{k\to\infty} (s_k, t_k, a_1^{n(k)}) = (s, \varphi(\gamma s), \psi(s)) \ \text{ for some } \ s \in [0, C]. \end{cases} \tag{3.55}$$

We begin with a priori bounds for $\{w_n\}$ and $\{z_n\}$. It follows from (3.12) and (3.25) that

$$w_n \leq \frac{1}{\varepsilon_n} \theta_{\lambda_1 + \varepsilon_n A_1} \ \text{ in } \ \Omega$$

for all $n \in \boldsymbol{N}$. Recall that

$$\lim_{\lambda \downarrow \lambda_1} \frac{\theta_\lambda}{\lambda - \lambda_1} = \frac{\Phi}{\|\Phi\|_3^3} \ \text{ uniformly in } \ \Omega, \tag{3.56}$$

(see, e.g., Gui and Lou [33, Proposition 6.4]) one can see that

$$w_n \leq 1 + A_1 \frac{\|\Phi\|_\infty}{\|\Phi\|_3^3} =: M \ \text{ in } \ \Omega$$

for sufficiently large n. Therefore, $\{w_n\}$ is a bounded sequence in $C(\overline{\Omega})$. By virtue of the second equation of (PP), we see

$$\begin{aligned} -\Delta z_n &= \lambda_1 z_n + \frac{\varepsilon_n z_n}{1 + \gamma w_n} \left(b_1 + \tau\gamma w_n - \frac{z_n}{1 + \gamma w_n} \right) \\ &\leq \lambda_1 z_n + \varepsilon_n z_n \left[b_1 + \tau - \frac{z_n}{(1 + \gamma w_n)^2} \right] \\ &= z_n \left[\lambda_1 + \varepsilon_n (b_1 + \tau) - \frac{\varepsilon_n z_n}{(1 + \gamma M)^2} \right] \ \text{ in } \ \Omega \end{aligned}$$

for sufficiently large n. This fact implies that $\varepsilon_n z_n / (1 + \gamma M)^2$ is a subsolution of (3.3) with a replaced by $\lambda_1 + \varepsilon_n (b_1 + \tau)$. Thus by the well-known comparison result, one can obtain $\varepsilon_n z_n / (1 + \gamma M)^2 \leq \theta_{\lambda_1 + \varepsilon_n (b_1 + \tau)}$ in Ω; so that

$$z_n \leq (1 + \gamma M)^2 \, \frac{\theta_{\lambda_1 + \varepsilon_n (b_1 + \tau)}}{\varepsilon_n} \ \text{ in } \ \Omega.$$

Owing to (3.56), we get

$$z_n \le 1 + \frac{(1+\gamma M)^2(b_1+\tau)}{\|\Phi\|_3^3}\Phi \ \text{ in } \ \Omega \tag{3.57}$$

for sufficiently large n. Therefore $\{z_n\}$ is uniformly bounded in $C(\overline{\Omega})$.

Let $\overline{w}_n = w_n/\|w_n\|_\infty$ and $\overline{z}_n = z_n/\|z_n\|_\infty$. Thus it follows from (PP) that $\overline{w}_n$ and $\overline{z}_n$ satisfy

$$\begin{cases} -\Delta \overline{w}_n = \lambda_1 \overline{w}_n + \varepsilon_n \overline{w}_n \left(a_1^n - w_n - \dfrac{c z_n}{1+\gamma w_n} \right) & \text{in } \ \Omega, \\ -\Delta \overline{z}_n = \lambda_1 \overline{z}_n + \dfrac{\varepsilon_n \overline{z}_n}{1+\gamma w_n} \left(b_1 + \tau\gamma w_n - \dfrac{z_n}{1+\gamma w_n} \right) & \text{in } \ \Omega, \\ \overline{w}_n = \overline{z}_n = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{3.58}$$

Since $\{(w_n, z_n, a_1^n)\}$ is uniformly bounded in $C(\overline{\Omega})^2 \times \boldsymbol{R}$, $\{a_1^n - w_n - c z_n/(1+\gamma w_n)\}$ and $\{b_1 + \tau\gamma w_n - z_n/(1+\gamma w_n)\}$ are also bounded in $C(\overline{\Omega})$. With the aid of the standard elliptic regularity theory, $\{w_n\}$, $\{z_n\}$ are uniformly bounded in $C^2(\overline{\Omega})$. So one can choose a subsequence $\{(w_{n(k)}, z_{n(k)}, a_1^{n(k)})\}$ such that

$$\lim_{k\to\infty} (\overline{w}_{n(k)}, \overline{z}_{n(k)}, a_1^{n(k)}) = (\overline{w}, \overline{z}, a_1^\infty) \ \text{ in } \ C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times \boldsymbol{R}$$

with some $(\overline{w}, \overline{z}, a_1^\infty)$. For simplicity, we rewrite this subsequence by $\{(w_n, z_n, a_1^n)\}$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, letting $n \to \infty$ in (3.58) implies that $\overline{w}$ and $\overline{z}$ satisfy

$$-\Delta \overline{w} = \lambda_1 \overline{w}, \quad -\Delta \overline{z} = \lambda_1 \overline{z} \ \text{ in } \ \Omega, \qquad \overline{w} = \overline{z} = 0 \ \text{ on } \ \partial\Omega.$$

Together with $\|\overline{w}\|_\infty = \|\overline{z}\|_\infty = 1$, we can deduce $\overline{w} = \overline{z} = \Phi/\|\Phi\|_\infty$. So the boundness of $\{(w_n, z_n)\}$ in $C^2(\overline{\Omega})^2$ yields

$$\lim_{n\to\infty} (w_n, z_n) = (s\Phi, t\Phi) \ \text{ in } \ C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \tag{3.59}$$

for some $s \ge 0$ and $t \ge 0$. By virtue of (3.32) and (3.59), for sufficiently large n, (w_n, z_n) must be given by

$$(w_n, z_n) = (s_n, t_n)\Phi + \varepsilon_n \boldsymbol{U}(s_n, t_n, a_1^n, \varepsilon_n)$$

with some sequence $\{(s_n, t_n)\}$ such that $\lim_{n\to\infty}(s_n, t_n) = (s, t)$.

To prove $t = \varphi(\gamma s)$, we multiply by Φ the second equation of (3.58) and integrate the resulting expression;

$$\int_\Omega \frac{\overline{z}_n \Phi}{1+\gamma w_n}\left(b_1 + \tau\gamma w_n - \frac{z_n}{1+\gamma w_n}\right) = 0.$$

By (3.59), letting $n \to \infty$ in the above equality yields

$$b_1 - (b_1 - \tau)\gamma s \int_\Omega \frac{\Phi^3}{1+\gamma s\Phi} = t\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2},$$

which, together with (3.36), implies $t = \varphi(\gamma s)$.

Finally we will prove $a_1^\infty = \psi(s)$. Multiply the first equation of (3.58) by Φ and integrate it;

$$\int_\Omega \overline{w}_n \Phi\left(a_1^n - w_n - \frac{cz_n}{1+\gamma w_n}\right) = 0.$$

Letting $n \to \infty$ in the above equality, we have

$$a_1^\infty - s\|\Phi\|_3^3 - ct\int_\Omega \frac{\Phi^3}{1+\gamma s\Phi} = 0,$$

which immediately leads to $a_1^\infty = \psi(s)$ by (3.36). Then we obtain (3.55), which completes the proof of Lemma 3.13. □

Proof of Proposition 3.10. We have already shown (3.46) by Lemmas 3.12 and 3.13. To accomplish the proof of Proposition 3.10, it remains to show that Γ^ε can be extended to the range $a_1 \in [A_1, \infty)$ as a positive solution curve of (PP). Let $\hat{\Gamma}^\varepsilon$ be a maximum extension of Γ^ε in the direction $a_1 \geq A_1$ as a solution curve of (PP). According to the global bifurcation theorem by Rabinowitz [64], the following (i) or (ii) must hold true;

(i) $\hat{\Gamma}^\varepsilon$ is unbounded in $X \times \boldsymbol{R}$;

(ii) $\hat{\Gamma}^\varepsilon$ meets the trivial or a semitrivial solution curve at some point except for $(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon_{b_1}}, a_1^*)$.

We introduce the following positive cone

$$P = \left\{(w,z)\,:\, w>0,\, z>0 \text{ in } \Omega \text{ and } \frac{\partial w}{\partial \nu} < 0, \frac{\partial z}{\partial \nu} < 0 \text{ on } \partial\Omega\right\}.$$

Suppose that $(\hat{w}, \hat{z}, \hat{a}_1) \in \hat{\Gamma}^\varepsilon$ satisfies $(\hat{w}, \hat{z}) \in \partial P$ at $\hat{a}_1 \in (A_1, \infty)$. Thus it follows that $\hat{w} \geq 0, \hat{z} \geq 0$ for all $x \in \Omega$ and

$$\hat{w}(x_0)\hat{z}(x_0) = 0 \quad \text{at some} \quad x_0 \in \Omega \tag{3.60}$$

or

$$\frac{\partial \hat{w}}{\partial \nu}(x_1)\frac{\partial \hat{z}}{\partial \nu}(x_1) = 0 \quad \text{at some} \quad x_1 \in \partial\Omega. \tag{3.61}$$

By applying the strong maximum principle to (PP), it is possible to prove that both (3.60) and (3.61) imply $\hat{w} \equiv 0$ or $\hat{z} \equiv 0$.

We now recall that positive solutions of (PP) bifurcate from the semitrivial solution curve $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1) \,:\, a_1 > 0\}$ if and only if $a_1 = a_1^*$ and no positive solution bifurcates from other semitrivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1) \,:\, a_1 > 0\}$ if $\tau \geq 0$ (see Remark 3.1). In addition, it is easily verified that the trivial solution is non-degenerate. Therefore, we can deduce that $(\hat{w}, \hat{z}, \hat{a}_1) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon_{b_1}}, a_1^*)$, which contradicts (ii). Thus (ii) is excluded and (i) must be satisfied. Lemma 3.5 and (3.25) imply the boundness of w and z

$$\begin{cases} w(x) \leq \dfrac{1}{\varepsilon}(\lambda_1 + \varepsilon a_1), \\ z(x) \leq \dfrac{1}{\varepsilon}\{1 + \beta(\lambda_1 + \varepsilon a_1)\}\{\lambda_1 + \varepsilon b_1 + \beta(\lambda_1 + \varepsilon\tau)(\lambda_1 + \varepsilon a_1)\} \end{cases}$$

for all $x \in \Omega$. Therefore, Γ^ε must be extended with respect to $a_1 \in [\,A_1, \infty)$ as a positive solution curve of (PP). Thus the proof of Proposition 3.10 is complete. □

Proposition 3.10 in combination with Lemma 3.9 implies that Γ^ε forms an unbounded S-shaped curve with respect to a_1 for the special case when $(\tau, \gamma) \in [\,0, \tilde{\tau}\,] \times [\,\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is small enough:

Corollary 3.14. *Suppose that $(\tau, \gamma) \in [\,0, \tilde{\tau}\,] \times [\,\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is sufficiently small. Then the positive solution set of* (PP) *contains an unbounded S-shaped curve Γ^ε which bifurcates from the semitrivial solution curve $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1) \,:\, a_1 > 0\}$ at $a_1 = a_1^*(\varepsilon)$. Furthermore, there exist two positive numbers $\overline{a}_1(\varepsilon) > \underline{a}_1(\varepsilon)\ (> a_1^*(\varepsilon))$ such that*

(i) *if* $a_1 \in (0, a_1^*(\varepsilon)\,]$, *then* (PP) *has no positive solution*;

(ii) *if* $a_1 \in (a_1^*(\varepsilon), \underline{a}_1(\varepsilon)) \cup (\overline{a}_1(\varepsilon), \infty)$, *then* (PP) *has at least one positive solution*;

(iii) *if* $a_1 = \underline{a}_1(\varepsilon)$ *or* $a_1 = \overline{a}_1(\varepsilon)$, *then* (PP) *has at least two positive solutions*;

(iv) *if* $a_1 \in (\underline{a}_1(\varepsilon), \overline{a}_1(\varepsilon))$, *then* (PP) *has at least three positive solutions.*

Proof. Let $S(\xi, \varepsilon) = (s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon))$ be the smooth curve obtained in Proposition 3.10. We recall that $S(\xi, 0) = (\xi, \varphi(\xi), \psi(\xi))$. Additionally it is possible to verify that $\psi'(0) > 0$ if $\tau \geq 0$ and

$$\lim_{\varepsilon \to 0}(t(\xi, \varepsilon), a_1(\xi, \varepsilon)) = (\varphi(\xi), \psi(\xi)) \ \text{ in } \ C^1([\,0, C\,]) \times C^1([\,0, C\,]),$$

where C is the positive constant defined in (3.45). Thus it follows from Lemma 3.9 that if $(\tau, \gamma) \in [\,0, \tilde{\tau}\,] \times [\,\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is small enough, then $\psi_\varepsilon(\xi) := a_1(\xi, \varepsilon)$ $(0 \leq \xi \leq C(\varepsilon))$ satisfies $\psi_\varepsilon'(0) > 0$, $\psi_\varepsilon(\xi) > \psi_\varepsilon(0) = a_1^*(\varepsilon)$ for all $\xi \in (0, C(\varepsilon)\,]$ and achieves a local minimum and a local maximum at some $\overline{\xi}(\varepsilon)$ and $\underline{\xi}(\varepsilon)$, respectively, which satisfy $\lim_{\varepsilon \to 0} \overline{\xi}(\varepsilon) = \overline{s}$ and $\lim_{\varepsilon \to 0} \underline{\xi}(\varepsilon) = \underline{s}$. Here, $\overline{s}$ and $\underline{s}$ are critical points of g obtained in Lemma 3.9. Define $\overline{a}_1(\varepsilon) := \psi_\varepsilon(\overline{\xi}(\varepsilon))$, $\underline{a}_1(\varepsilon) := \psi_\varepsilon(\underline{\xi}(\varepsilon))$ and

$$K_\varepsilon(a_1) := \{\xi \in (0, \infty) \ : \ \psi_\varepsilon(\xi) = a_1\}.$$

Obviously if $\varepsilon > 0$ is small enough, then $K_\varepsilon(a_1)$ has no element for $a_1 \in (0, a_1^*(\varepsilon)\,]$; at least one element for $a_1 \in (a_1^*(\varepsilon), \underline{a}_1(\varepsilon)) \cup (\overline{a}_1(\varepsilon), A_1\,]$; at least two elements for $a_1 = \underline{a}_1(\varepsilon)$ or $\overline{a}_1(\varepsilon)$; at least three elements for $a_1 \in (\underline{a}_1(\varepsilon), \overline{a}_1(\varepsilon))$. We observe that (3.46) implies that the number of elements of $K_\varepsilon(a_1)$ is equal to the number of positive solutions of (PP) provided $\varepsilon \in (0, \varepsilon_0\,]$ and $a_1 \in (0, A_1\,]$. Since the extension of Γ^ε implies that (PP) has at least one positive solution for $a_1 \in [\,A_1, \infty)$, we obtain the assertion. □

3.5.2 Case $\tau < 0$

For the case $\tau < 0$, let A_1 be a sufficiently large number. In this subsection, we will prove that all positive solutions (PP) with $a_1 \in [\,0, A_1\,]$ lie on a bounded curve near

$$\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) \,:\, 0 < s < s_0/\gamma\}$$

if $\varepsilon >$ is sufficiently small:

Proposition 3.15. *Let $\tau < 0$. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(A_1)$ such that for each $\varepsilon \in (0, \varepsilon_0\,]$, all positive solutions of* (PP) *with $a_1 \in (0, A_1\,]$ are given by*

$$\begin{aligned}\Gamma^\varepsilon = \{(w, z, a_1) = ((s,t)\Phi + \varepsilon\, \boldsymbol{U}(s, t, a_1, \varepsilon), a_1) \,:& \\ (s, t, a_1) \in \{S(\xi, \varepsilon) : 0 < \xi < C(\varepsilon)\}\},&\end{aligned} \tag{3.62}$$

where $S(\xi, \varepsilon) \in \boldsymbol{R}^3$ is a suitable smooth curve for $(\xi, \varepsilon) \in [\,0, C(\varepsilon)\,] \times [\,0, \varepsilon_0\,]$ satisfying

$$S(\xi, 0) = (\xi, \varphi(\xi), \psi(\xi)), S(0, \varepsilon) = (0, t(\varepsilon), a_1^*(\varepsilon)), S(C(\varepsilon), \varepsilon) = (s(\varepsilon), 0, a_{1*}(\varepsilon)).$$

Here, $t(\varepsilon) := \varepsilon^{-1} \int_\Omega \theta_{\lambda_1 + \varepsilon b_1} \Phi$, $s(\varepsilon) := \varepsilon^{-1} \int_\Omega \theta_{\lambda_1 + \varepsilon a_{1*}(\varepsilon)} \Phi$ and $C(\varepsilon)$ is a certain smooth function in $[\,0, \varepsilon_0]$ such that $C(0) = s_0/\gamma$.

It follows from (3.37) and (3.39) that if $\tau < 0$, then $\mathcal{L}_p$ intersects $\mathcal{L}_1$ and $\mathcal{L}_2$ at $(s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma)$ and $(0, b_1/\|\Phi\|_3^3, cb_1)$, respectively. Even if $\tau < 0$, Lemma 3.11 remains valid; so that one can obtain the expression (3.47) for all nonnegative solutions of (3.29) near $(0, b_1/\|\Phi\|_3^3, cb_1)$. We can also express all nonnegative solutions of (3.29) near $(s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma)$:

Lemma 3.16. *Let $\tau < 0$. Then there exist a positive number δ_e and a neighborhood U_e of $(s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma)$ such that for each $\varepsilon \in [\,0, \delta_e\,]$*

$$\mathrm{Ker}\, F^\varepsilon \cap U_e \cap \overline{\boldsymbol{R}_+}^3 = \{\tilde{S}(\xi, \varepsilon) : \xi \in [\,0, \delta_e)\} \cup \left\{ \left(\frac{1}{\varepsilon} \int_\Omega \theta_{\lambda_1 + \varepsilon a_1} \Phi, 0, a_1 \right) \in U_e \right\}$$

with a smooth curve $\tilde{S}(\xi, \varepsilon) \in \boldsymbol{R}^3$ $(0 \le \xi \le \delta_e)$ which satisfies

$$\tilde{S}(\xi, 0) = (s_0 - \xi/\gamma, \varphi(s_0 - \xi/\gamma), \psi(s_0/\gamma - \xi)) \quad \text{and} \quad \tilde{S}(0, \varepsilon) = (s(\varepsilon), 0, a_{1*}(\varepsilon)).$$

Recall that $(w, z, a_1) = (\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_{1*}(\varepsilon)}, 0, a_{1*}(\varepsilon))$ is a bifurcation point of positive solutions of (PP) on the semitrivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1) : a_1 > 0\}$. So the proof of Lemma 3.16 can be carried out by the same argument as in the proof of Lemma 3.11.

Lemma 3.17. *Let $\tau < 0$. Then there exists a neighborhood U of $\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : s \in [0, s_0/\gamma]\}$ such that, if $\varepsilon > 0$ is sufficiently small, then all positive solutions of* (PP) *contained in U are given by* (3.62).

Proof. Let δ_0 and δ_e be positive numbers in Lemmas 3.11 and 3.16, respectively. Put

$$\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2]) := \{(s, \varphi(\gamma s), \psi(s)) : s \in [\delta_0/2, s_0/\gamma - \delta_e/2]\}.$$

Hence $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2])$ is a compact set and both $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2]) \cap U_0$ and $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2]) \cap U_e$ are not empty. Here U_0 and U_e are open sets obtained in Lemmas 3.11 and 3.16. Therefore, when $\varepsilon > 0$ is small enough, a similar procedure to the proof of Lemma 3.12 enables us to construct the solution curve of (3.29) in a neighborhood U' of $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2])$. We note that both $U' \cap U_0$ and $U' \cap U_e$ are not empty. Therefore, together with Lemmas 3.11 and 3.16, we obtain the assertion. $\square$

Lemma 3.18. *Let $\tau < 0$ and assume that $A_1 > 0$ is sufficient large. Let V be any neighborhood of $\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : s \in [0, s_0/\gamma]\}$. If $\varepsilon > 0$ is sufficiently small, then any positive solution of* (PP) *with $a_1 \in (0, A_1]$ can be expressed by*

$$(w, z, a_1) = (s, t)\Phi + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon) \quad \textit{for some} \quad (s\Phi, t\Phi, a_1) \in V.$$

Proof. Let $\{(w_n, z_n)\}$ be any sequence of positive solutions of (PP) with $\varepsilon = \varepsilon_n \downarrow 0$ and $a_1 = a_1^n \in (0, A_1]$. It suffices to get a subsequence $\{(w_{n(k)}, z_{n(k)}, a_1^{n(k)}, \varepsilon_{n(k)})\}$ and a sequence $\{(s_k, t_k)\}$ satisfying (3.55) with C replaced by s_0/γ. The proof of this assertion is almost the same as that of Lemma 3.13. We have only to note that in case $\tau < 0$, (3.57) is replaced by $z_n \leq 1 + (1 + \gamma M)^2 b_1 \Phi / \|\Phi\|_3^3$ in Ω. $\square$

Proposition 3.15 follows from Lemmas 3.17 and 3.18. Furthermore, we can employ Lemma 3.9 to obtain the following corollary about the positive solution set of (PP) for the case when $(\tau,\gamma) \in [-\tilde{\tau},0) \times [\tilde{\gamma},\infty)$ and $\varepsilon > 0$ is sufficiently small.

Corollary 3.19. *Suppose that* $(\tau,\gamma) \in [-\tilde{\tau},0) \times [\tilde{\gamma},\infty)$ *and that* $\varepsilon > 0$ *is sufficiently small. Then the positive solution set of* (PP) *contains a bounded smooth curve*

$$\Gamma^{\varepsilon} = \{(w(\xi), z(\xi), a_1(\xi)) \,:\, \xi \in (0, C(\varepsilon))\},$$

which possesses the following properties;

(i) $(w(0), z(0), a_1(0)) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*(\varepsilon)),\ \ a_1'(0) > 0$;

(ii) $(w(C(\varepsilon)), z(C(\varepsilon)), a_1(C(\varepsilon))) = (\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_{1*}(\varepsilon)}, 0, a_{1*}(\varepsilon))$;

(iii) $a_1(\xi)$ *attains a strict local maximum in* $(0, C(\varepsilon))$. *In particular, if* $(\tau,\gamma) \in [-\tilde{\tau},0) \times [\tilde{\gamma},\hat{\gamma}(\tau))$, *then* $a_1(\xi)$ *attains a strict local minimum in* $(0, C(\varepsilon))$.

The proof of Corollary 3.19 is essentially the same as that of Corollary 3.14.

3.6 Proofs of Main Results

Proof of Theorem 3.2. We begin with the case $\alpha = 0$. The bifurcation point of Γ^{ε} obtained in Proposition 3.10; $(w,z,a_1) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*(\varepsilon))$ is mapped by (3.25) to the bifurcation point $(U,V,a) = (0, \theta_b, \lambda_1(c\theta_b))$ on the semitrivial solution curve $\{(0,\theta_b,a) \,:\, a > 0\}$ of $(\mathrm{EP})_0$. In view of (3.25), we define

$$O_1^0 := \{(\beta,b,d) = (\gamma/\varepsilon, \lambda_1+\varepsilon b_1, (\lambda_1+\varepsilon\tau)\gamma/\varepsilon) \,:\, (\tau,\gamma) \in [0,\tilde{\tau}\,] \times [\tilde{\gamma},\infty)\}$$

for sufficiently small $\varepsilon > 0$. It follows from Corollary 3.14 that if $(\beta,b,d) \in O_1^0$ and $\varepsilon > 0$ is small enough, then the positive solution set of $(\mathrm{EP})_0$ contains an unbounded S-shaped curve $\Gamma_{(\mathrm{EP})_0}$ which bifurcates from the semitrivial solution curve $\{(0,\theta_b,a) \,:\, a > 0\}$ at $a = \lambda_1(c\theta_b)$. To be precise, if we let

$$\overline{a} = \lambda_1 + \varepsilon\overline{a}_1(\varepsilon) \ \text{ and } \ \underline{a} = \lambda_1 + \varepsilon\underline{a}_1(\varepsilon),$$

then $(\mathrm{EP})_0$ has no positive solution for $a \in (0, \lambda_1(c\theta_b)\,]$; at least one positive solution for $a \in (\lambda_1(c\theta_b), \underline{a}) \cup (\overline{a}, \infty)$; at least two positive solutions for $a = \underline{a}$ or $\overline{a}$; at least three positive solutions for $a \in (\underline{a}, \overline{a})$. By virtue of the one-to-one correspondence between $(u, v) \geq 0$ and $(U, V) \geq 0$ through (3.8), define $\Gamma_1 = \{(u, v, a) : (U, V, a) \in \Gamma_{(\mathrm{EP})_0}\}$ for $\alpha = 0$ and $(\beta, b, d) \in O_1^0$. Hence Γ_1 is contained in the positive solution set of (SP3) and possesses at least two turning points with respect to a. Therefore, Theorem 3.2 is proved for the special case $\alpha = 0$.

Next we will justify that the above S-shaped property of Γ_1 allows a small perturbation with respect to α. Define $F : X \times \boldsymbol{R}^3 \to Y$ by

$$F(U, V, a, \alpha, \beta) = (\Delta U + u(a - u - cv), \Delta V + v(b + dv - v))$$

to study (EP). Here $u = u(U, V, \alpha, \beta)$ and $v = v(U, V, \alpha, \beta)$ are given by (3.9) and (3.10), respectively. For any $(\beta, b, d) \in O_1^0$, let (U_0, V_0) be any positive solution of $(\mathrm{EP})_0$. We note that all positive solutions of $(\mathrm{EP})_0$ near $(0, \theta_b, a^*)$ are given by (3.16). By following the procedure by Du and Lou [25, Lemma 3.14], we can prove that, if $\|U_0\|_{W^{2,p}} \geq \delta/2$ for the positive number δ in (3.16), then $F_{(U,V)}(U_0, V_0, a, 0, \beta)$ is a Fredholm operator with index 0; so that the following (i) or (ii) holds true alternatively.

(i) $F_{(U,V)}(U_0, V_0, a, 0, \beta) : X \to Y$ is an isomorphism;

(ii) $$\begin{cases} \dim \mathrm{Ker}\, F_{(U,V)}(U_0, V_0, a, 0, \beta) = \mathrm{codim}\, \mathrm{Range}\, F_{(U,V)}(U_0, V_0, a, 0, \beta) = 1, \\ F_a(U_0, V_0, a, 0, \beta) \notin \mathrm{Range}\, F_{(U,V)}(U_0, V_0, a, 0, \beta). \end{cases}$$

For sufficiently large A, denote by $\Gamma_{(\mathrm{EP})_0}|_{0<a\leq A}$ the restriction of $\Gamma_{(\mathrm{EP})_0}$ in the range $0 < a \leq A$. Thus, in the same way as the proof of Lemma 3.12, we can construct a positive solution curve $\Gamma_{(\mathrm{EP})}|_{0<a\leq A}$ of (EP) in a neighborhood W of $\Gamma_{(\mathrm{EP})_0}|_{0<a\leq A}$ if $\alpha > 0$ is sufficiently small. By taking account for the continuity of positive solutions of (EP) with respect to α, it can be verified that $\Gamma_{(\mathrm{EP})}|_{0<a\leq A}$ converges to $\Gamma_{(\mathrm{EP})_0}|_{0<a\leq A}$ in $C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times [\,0, A\,]$ as $\alpha \downarrow 0$. Furthermore, it is

also possible to prove that if $0 < a \le A$ and $\alpha > 0$ is small enough, then there is no positive solution of (EP) outside of W. With the aid of Lemma 3.5, we can extend $\Gamma_{(\mathrm{EP})}|_{0<a\le A}$ to the range $a \in [\,A, \infty)$ as a positive solution curve of (EP) by applying the global bifurcation theorem ([64]). By virtue of (3.8), we can get the S-shaped positive solution curve of (SP3) when $(\beta, b, d) \in O_1^0$ and $\alpha > 0$ is small. Thus the proof of Theorem 3.2 is complete. ☐

Proof of Theorem 3.3. In view of Corollary 3.19, the proof of Theorem 3.3 can be carried out in a similar way. ☐

3.7 Basic Properties of S_1 and S_2

In this section, we will give some properties of S_1 and S_2 defined by (3.6) and (3.7).

Lemma 3.20. *If $\beta\lambda_1 < d$ (resp. $\beta\lambda_1 > d$), then S_1 can be expressed as*

$$S_1 = \{(a, b) \,:\, a = f(b) \ \ \textit{for} \ \ b \le \lambda_1 \ (\textit{resp.}\, b \ge \lambda_1)\},$$

where $f(\,\cdot\,)$ is a continuous function with values in $(-\infty, \lambda_1\,]$ (resp. $[\,\lambda_1, \infty)$) and possesses the following properties:

(i) $f(\,\cdot\,)$ *is* $\begin{cases} \textit{strictly monotone decreasing if } \beta\lambda_1 < d\,, \\ \textit{strictly monotone increasing if } \beta\lambda_1 > d\,; \end{cases}$

(ii) $f(\lambda_1) = \lambda_1\,;$

(iii) $\lim_{b\to-\infty} f(b) = \infty$ *if* $\beta\lambda_1 < d$ *and* $\lim_{b\to\infty} f(b) = \infty$ *if* $\beta\lambda_1 > d$.

Proof. If we put

$$S(a, b) := \lambda_1 \left(\frac{-b - d\theta_a}{1 + \beta\theta_a} \right),$$

then

$$S(\lambda_1, b) = \lambda_1(-b) = \lambda_1 - b. \tag{3.63}$$

We note that for each compact set K in Ω

$$\lim_{a\to\infty} \frac{-b-d\theta_a}{1+\beta\theta_a} = -\frac{d}{\beta} \quad \text{uniformly in} \quad K$$

(see, e.g., Dancer [20, Lemma 1]). Therefore, it can be seen that for any $b \in \boldsymbol{R}$

$$\lim_{a\to\infty} S(a,b) = \lambda_1\left(-\frac{d}{\beta}\right) = \lambda_1 - \frac{d}{\beta}. \tag{3.64}$$

Recall that both of the mappings $q \to \lambda_1(q) : C(\overline{\Omega}) \to \boldsymbol{R}$ and $a \to \theta_a : [\,\lambda_1, \infty) \to C(\overline{\Omega})$ are strictly increasing. Therefore, by virtue of

$$\frac{\partial}{\partial a}\left(\frac{-b-d\theta_a}{1+\beta\theta_a}\right) = \frac{\beta b - d}{(1+\beta\theta_a)^2}\frac{d\theta_a}{da} \quad \text{in} \quad \Omega,$$

we know that

$$S(a,b) \text{ is } \begin{cases} \text{strictly decreasing with respect to } a \in (\lambda_1, \infty) \text{ if } \beta b < d, \\ \text{strictly increasing with respect to } a \in (\lambda_1, \infty) \text{ if } \beta b > d. \end{cases} \tag{3.65}$$

Suppose $\beta\lambda_1 < d$. By virtue of (3.63) and (3.64), $S(\lambda_1, b_0) \leq 0$ (resp. $S(\lambda_1, b_0) > 0$) if $b_0 \geq \lambda_1$ (resp. $b_0 < \lambda_1$) and $\lim_{a\to\infty} S(a, b_0) < 0$. Since $S(a, b_0)$ is a monotone function with respect to a by (3.65), we see that, if $b_0 \geq \lambda_1$, then $S(a, b_0) < 0$ for all $a \in (\lambda_1, \infty)$. On the other hand, if $b_0 < \lambda_1$, then the intermediate theorem gives a unique $a_0 \in (\lambda_1, \infty)$ such that $S(a_0, b_0) = 0$. Furthermore it follows from (3.65) that $S_a(a_0, b_0) < 0$. Therefore, by the implicit function theorem, there exists a smooth function $a = f(b)$ such that

$$\begin{cases} f(b_0) = a_0, \\ S(f(b), b) = 0 \quad \text{for all} \quad b \in [\,b_0 - \delta, b_0 + \delta\,] \end{cases}$$

with some $\delta > 0$. Since $b_0 \in (-\infty, \lambda_1)$ can be taken arbitrarily, we deduce that there exists a unique smooth function $a = f(b)$ such that

$$S(f(b), b) = 0 \quad \text{for} \quad b \in (-\infty, \lambda_1). \tag{3.66}$$

Differentiation of (3.66) with respect to b implies

$$S_a(f(b),b)f'(b) + S_b(f(b),b) = 0.$$

Note that $S_b(a,b) = -\lambda_1'((-b - d\theta_a)/(1+\beta\theta_a)) < 0$. Thus from (3.65) one can see $f'(b) < 0$ for $b \in (-\infty, \lambda_1)$. It is easy to see $f(\lambda_1) = \lambda_1$. Furthermore, we can show $\lim_{b\to-\infty} f(b) = \infty$. Indeed, if $\lim_{b\to-\infty} f(b) < \infty$,

$$\lim_{b\to-\infty} S(f(b),b) = \lim_{b\to-\infty} \lambda_1\left(\frac{-b - d\theta_{f(b)}}{1+\beta\theta_{f(b)}}\right) = +\infty,$$

which obviously contradicts (3.66). Thus the proof for $\beta b < \beta\lambda_1 < d$ is complete. For the case $\beta b > \beta\lambda_1 > d$, a similar argument is valid to get the conclusion. □

The following corollary immediately follows from Lemma 3.20

Corollary 3.21. *For each b satisfying $\beta b < \beta\lambda_1 < d$ or $\beta b > \beta\lambda_1 > d$, there exists a unique $a = a_* \in (\lambda_1, \infty)$ such that*

$$\lambda_1\left(\frac{-b - d\theta_{a_*}}{1+\beta\theta_{a_*}}\right) = 0.$$

If $b > \lambda_1$ and $d \geq \beta\lambda_1$, then

$$\lambda_1\left(\frac{-b + d\theta_a}{1+\beta\theta_a}\right) < 0 \quad \text{for all} \quad a \in (\lambda_1, \infty).$$

We can also give an analogous property for S_2:

Lemma 3.22. *If S_2 is defined by (3.7), it can be expressed as*

$$S_2 = \{(a,b) \,:\, b = g(a) \quad \text{for} \quad a \geq \lambda_1\},$$

where $g(\,\cdot\,)$ is a continuous function with values in $[\,\lambda_1, \infty)$ and possesses the following properties:

(i) $g(\,\cdot\,)$ *is strictly monotone increasing;*

(ii) $g(\lambda_1) = \lambda_1$;

(iii) $\lim_{a\to\infty} g(a) = \infty$.

The proof of Lemma 3.22 is essentially the same as Lemma 3.20; so we omit it. The following corollary immediately follows from Lemma 3.22.

Corollary 3.23. *For each $b > \lambda_1$, then there exists a unique $a = a^* \in (\lambda_1, \infty)$ such that*

$$\lambda_1 \left(\frac{c\theta_b - a^*}{1 + \alpha\theta_b} \right) = 0.$$

We can give more information about S_1 and S_2 near (λ_1, λ_1).

Lemma 3.24. (i) *The function $f(\cdot)$ defined in Lemma 3.20 satisfies*

$$f(b) = \lambda_1 + \frac{1}{\beta\lambda_1 - d}(b - \lambda_1) + o(b - \lambda_1) \quad \textit{near} \quad \lambda_1.$$

(ii) *The function $g(\cdot)$ defined in Lemma 3.22 satisfies*

$$g(a) = \lambda_1 + \frac{1}{\alpha\lambda_1 + c}(a - \lambda_1) + o(a - \lambda_1) \quad \textit{near} \quad \lambda_1.$$

The proof is based on the local bifurcation analysis and is accomplished in the same manner as [70, Lemmas 3.4 and 3.6].

Chapter 4

Stability Analysis for Positive Steady-States

4.1 Introduction

In this chapter, we discuss dynamical behaviors of the nonnegative solutions to

$$
(\mathrm{P})_0 \begin{cases} d_1^{-1}u_t = \Delta u + u(a - u - cv) & \text{in } \Omega \times (0,T), \\ d_2^{-1}v_t = \Delta[(1+\beta u)v] + v(b + du - v) & \text{in } \Omega \times (0,T), \\ u = v = 0 & \text{on } \partial\Omega \times (0,T) \\ u(\,\cdot\,,0) = u_0 \geq 0, \quad v(\,\cdot\,,0) = v_0 \geq 0 & \text{in } \Omega. \end{cases}
$$

Our main purpose is to treat the stability analysis for the steady-state solutions obtained in the previous chapter. Here we note that the local solvability of $(\mathrm{P})_0$ has been established by Amann [3], where a wide class of quasilinear parabolic systems is discussed. According to his result, (3.1) has a unique local solution provided $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ for $p > N$. However, many problems are left open about the global solvability for (3.1). We refer to [52], [13], [14] and references therein for the related problems.

In the previous chapter, we have found *multiple* positive stationary solutions of $(\mathrm{P})_0$ in some cases. Regarding a as a bifurcation parameter, we set

$$\mathcal{S}_0 := \{(u,v,a) \,:\, (u,v) \text{ is a positive stationary solution of } (\mathrm{P})_0,\ a > \lambda_1\}.$$

As the extreme case ($\alpha = 0$) of Theorems 3.2 and 3.3, it follows that $\mathcal{S}_0$ possibly forms an S or $\supset$-shaped curve with respect to a:

Corollary 4.1. *Assume $\lambda_1 < d/\beta$ and $b > \lambda_1$. For any $c > 0$, there exist a large number M and an open set*

$$O_1 = O_1(c) \subset \{(\beta,b,d) \,:\, \beta \ge M,\ 0 < d/\beta - \lambda_1, b - \lambda_1 \le M^{-1}\}$$

such that, if $(\beta,b,d) \in O_1$, then $\mathcal{S}_0$ contains an unbounded smooth curve

$$\Gamma^1 = \{(u(s),v(s),a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1,\infty) \,:\, s \in (0,\infty)\},$$

which possesses the following properties:

(i) $(u(0),v(0)) = (0,\theta_b), \quad a^* := a(0) > \lambda_1, \quad a'(0) > 0\,;$

(ii) $a(s) > a^*$ *for all* $s \in (0,\infty)$ *and* $\lim_{s\to\infty} a(s) = \infty$;

(iii) *$a(s)$ attains a strict local maximum and a strict local minimum at some $s = \overline{s}$ and $s = \underline{s}$ $(0 < \overline{s} < \underline{s})$ respectively, which satisfy $a(\overline{s}) > a(\underline{s})$.*

Corollary 4.2. *Assume $\beta b > \beta\lambda_1 > d$. For any $c > 0$, there exist a large number M and an open set*

$$O_2 = O_2(c) \subset \{(\beta,b,d) \,:\, \beta \ge M,\ 0 < \lambda_1 - d/\beta, b - \lambda_1 \le M^{-1}\}$$

such that if $(\beta,b,d) \in O_1$, then $\mathcal{S}_0$ contains a bounded smooth curve

$$\Gamma^2 = \{(u(s),v(s),a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1,\infty) \,:\, s \in (0,C)\},$$

which possesses the following properties:

(i) $(u(0),v(0)) = (0,\theta_b), \quad a(0) > \lambda_1, \quad a'(0) > 0\,;$

(ii) $(u(C),v(C)) = (\theta_{a(C)},0), \quad a(C) > \lambda_1\,;$

(iii) *$a(s)$ attains a strict local maximum in $(0,C)$. Furthermore, there exists an open set $O_2' \subset O_2$ such that, if $(\alpha,\beta,b,d) \in O_2'$, then $a(s)$ also attains a strict local minimum in $(0,C)$.*

Since multiple positive steady-states are obtained, it is natural to ask their stability or instability. In case d_1/d_2 is small, we will show that the stability of steady-states on Γ^2 changes at every turning point with respect to a. So in these cases, stabilities of all steady-states on Γ^2 can be determined. If d_1/d_2 becomes large enough, then the above stability property totally different. We will prove the Hopf bifurcation occurs at some point on Γ^2 when d_1/d_2 is sufficiently large. So there exist periodic solutions of (P) near the Hopf bifurcation point. Concerning the unbounded branch Γ^1, similar results still hold if s belongs to a certain compact range $(0, C)$.

Concerning the stability problem for cross-diffusion systems, to my knowledge, there is only one work by Kan-on [35], in which he gave some criteria on the stability of nonconstant steady-states to the competition model of singular perturbation type which is proposed by Mimura et al. [53]. It should be noted that our situation is quite different from their cases.

The content of the present chapter is as follows. In Section 2, we state our main results. For the sake of the stability analysis, a brief sketch of the proofs of Theorems 4.2 and 4.1 is given in Section 3. By way of the Lyapunov-Schmidt reduction, one can find a finite dimensional limit problem as $\beta \to \infty$ and $|d/\beta - \lambda_1|$, $b - \lambda_1 \to 0$. Section 4 is devoted to the proofs of main results. We will study the distribution of the eigenvalues for the linearized problems associated with the steady-states constructed in Theorems 4.2 and 4.1. A crucial point of the proof is to determine the sign of real parts of the principal two eigenvalues as perturbations of eigenvalues of a 2×2 regular matrix characterized by the limit problem.

4.2 Main Results

Before stating main results, we divide $\varGamma^2$ at every turning point with respect to a. In case $(\beta, b, d) \in O_2$, let

$$0 < s_1 < s_2 < \cdots < s_{k-1} < C$$

be all strict local maximum or minimum points of $a(s)$. For each $1 \le i \le k$, we set

$$\varGamma_i^2 := \{(u(s), v(s), a(s)) \in \varGamma^2 : s \in (s_{i-1}, s_i)\},$$

where $s_0 := 0$ and $s_k := C$. Furthermore for $p > N$, let X and Y be Banach spaces defined by (3.15). We note that $X \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ by the Sobolev embedding theorem.

We are ready to state main results. Our first result can be stated as follows:

Theorem 4.3. *For almost every $(\beta, b, d) \in O_2$, there exists a small positive constant δ such that if $d_1/d_2 \le \delta$, then all steady-state solutions on $\varGamma_{2j-1}^2$ $(j = 1, 2, \dots, [(k+1)/2])$ are asymptotically stable in the topology of X, while all steady-state solutions on $\varGamma_{2j}^2$ $(j = 1, 2, \dots, [k/2])$ are unstable.*

In the above case, we remark that $(u(0), v(0)) = (0, \theta_b)$ and $(u(C), v(C)) = (\theta_{a(C)}, 0)$ by Theorem 4.2. So Theorem 4.3 implies that stable positive steady-states bifurcate from the semitrivial solution $(0, \theta_b)$, the stability on $\varGamma^2$ changes at every turning point with respect to a. On the other hand, we obtain the following result in case d_1/d_2 is large:

Theorem 4.4. *For any $(\beta, b, d) \in O_2$, there exists a large positive D such that if $d_1/d_2 \ge D$, then the Hopf bifurcation occurs at some point $(u(s^*), v(s^*), a(s^*)) \in \varGamma_1^2$. In this case, there exists a periodic solution of* $(\mathrm{P})_0$ *if a lies in a right-side neighborhood of $a(s^*)$.*

For the unbounded steady-states branch $\varGamma^1$, similar results to Theorems 4.3 and 4.4 also hold true if $s \in (0, C)$ with some $C > \underline{s}$:

Theorem 4.5. *Assume* $(\beta, b, d) \in O_1$. *Let* $0 < s_1 < s_2 < \cdots < s_{k-1} < C$ *be all strict local maximum or minimum points of* $a(s)$ *in* $(0, C)$ *and define* $\Gamma_i^1 := \{(u(s), v(s), a(s)) \in \Gamma^1 : s \in (s_{i-1}, s_i)\}$ $(1 \leq i \leq k)$, *where* $s_0 := 0$ *and* $s_k := C$. *Then the following properties hold true:*

(i) *For almost every* $(\beta, b, d) \in O_1$, *there exists positive constants* δ *and* $C > \underline{s}$ *such that, if* $d_1/d_2 \leq \delta$ *and* $s \in (0, C)$, *then steady-state solutions on* Γ_{2j-1}^1 $(j = 1, 2, \ldots, [(k+1)/2])$ *are asymptotically stable in the topology of* X, *while steady-state solutions on* Γ_{2j}^1 $(j = 1, 2, \ldots, [k/2])$ *are unstable.*

(ii) *There exists a large positive* D *such that, if* $d_1/d_2 \geq D$, *then the Hopf bifurcation occurs at some point* $(u(s^*), v(s^*), a(s^*)) \in \Gamma_1^1$.

4.3 Proofs of Main Results

In this section, we will prove Theorems 4.3-4.5 by making use of the results in the last chapter.

4.3.1 Preliminaries

In $(\mathrm{P})_0$, we employ the change of variables (3.25). Then a pair of new unknown functions (w, z) satisfy

$$\begin{cases} d_1^{-1} w_t = \Delta w + \lambda_1 w + \varepsilon f(w, z, a_1) & \text{in } \Omega \times (0, T), \\ d_2^{-1} \left[-\dfrac{\gamma z}{(1+\gamma w)^2} w_t + \dfrac{z_t}{1+\gamma w} \right] = \Delta z + \lambda_1 z + \varepsilon g(w, z) & \text{in } \Omega \times (0, T), \\ w = z = 0 & \text{on } \partial\Omega \times (0, T), \\ w(\,\cdot\,, 0) = u_0/\varepsilon, \quad z(\,\cdot\,, 0) = (1 + \beta u_0) v_0/\varepsilon & \text{in } \Omega, \end{cases} \tag{4.1}$$

where f and g are nonlinear terms defined in (3.36). The steady-state problem associate with (4.1) is reduced to the semilinear elliptic equation (PP) introduced in Section 3.3.

By virtue of the regularity of (3.25), the stability of a steady-state (u^*, v^*) of $(\mathrm{P})_0$ coincides with that of the steady-state $(w^*, z^*) = (u^*/\varepsilon, (1+\beta u^*)z^*/\varepsilon)$ of (4.1). So we will concentrate on the stability analysis for the steady-states on Γ^ε given in Propositions 3.10 and 3.15.

4.3.2 Linearized Stability

By virtue of Propositions 3.10 and 3.15, all positive steady-states of (4.1) with $a_1 \in (0, A_1)$ are parameterized as

$$\Gamma^\varepsilon = \{(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) : \xi \in (0, C_\varepsilon)\}$$

when $\varepsilon > 0$ is sufficiently small. For each $(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \Gamma^\varepsilon$, we define a linear operator $L(\xi,\varepsilon) : X \to Y$ by

$$L(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix} := -H\begin{pmatrix} h \\ k \end{pmatrix} - \varepsilon B_{(w,z)}(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon))\begin{pmatrix} h \\ k \end{pmatrix},$$

where H, B are mappings defined by (3.28) and $B_{(w,z)}$ denotes the Fréchet derivative of B with respect to (w, z). Furthermore, in view of the left hand side of (4.1), we set

$$J(\xi,\varepsilon) := \begin{bmatrix} \dfrac{1}{d_1} & 0 \\ -\dfrac{\gamma z(\xi,\varepsilon)}{d_2(1+\gamma w(\xi,\varepsilon))^2} & \dfrac{1}{d_2(1+\gamma w(\xi,\varepsilon))} \end{bmatrix}.$$

Then the linearized eigenvalue problem associated with $(w(\xi,\varepsilon), z(\xi,\varepsilon))$ is

$$L(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix} = \mu J(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix}. \tag{4.2}$$

In this subsection, we study the linearized stability of steady-states on Γ^ε by the spectral analysis for (4.2). Put

$$\rho(\xi,\varepsilon) := \{\mu \in \boldsymbol{C} \,:\, (4.2) \text{ has no solution except for } h = k = 0\}.$$

We begin with the following lemma.

Lemma 4.6. *Suppose that* $\varepsilon > 0$ *is sufficiently small. Then there exist positive constants* κ_1, ω *independent of* (ξ,ε) *such that* $-\rho(\xi,\varepsilon) \supset \{z \in \boldsymbol{C} : |z| \geq \kappa_1 \text{ and } |\arg z| \leq \pi/2+\omega\}$. *On the other hand, all eigenvalues* $\{\mu_i(\xi,\varepsilon)\}_{i=1}^{\infty}$ (*counting multiplicity*) *of* (4.2) *satisfy*

$$\lim_{\varepsilon\downarrow 0}\mu_1(\xi,\varepsilon) = \lim_{\varepsilon\downarrow 0}\mu_2(\xi,\varepsilon) = 0 \tag{4.3}$$

and

$$\operatorname{Re}\mu_i(\xi,\varepsilon) > \kappa_2 \quad \textit{for all} \quad i \geq 3 \quad \textit{and} \quad \xi \in (0, C_\varepsilon)$$

for some positive constant κ_2 *independent of* (ξ,ε).

Proof. It follows from Proposition 3.10 that for any fixed $\xi \in (0, C_\varepsilon)$

$$\lim_{\varepsilon\downarrow 0}(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) = (\xi\Phi, \varphi(\gamma\xi)\Phi, \psi(\xi)) \;\text{ in }\; C^1(\overline{\Omega})\times C^1(\overline{\Omega})\times \boldsymbol{R}.$$

Thus letting $\varepsilon \downarrow 0$ in (4.2), we have

$$\begin{cases} -\Delta h - \lambda_1 h = \dfrac{\mu}{d_1} h & \text{in } \Omega, \\ -\Delta k - \lambda_1 k = \dfrac{\mu}{d_2}\left[-\dfrac{\gamma\varphi(\gamma\xi)\Phi}{(1+\gamma\xi\Phi)^2}h + \dfrac{k}{1+\gamma\xi\Phi}\right] & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Clearly, $\mu = 0$ is a double eigenvalue of (4.4). If $h \not\equiv 0$, then any eigenvalues of (4.4) is real and nonnegative by the first equation. If $h \equiv 0$, we are lead to the same result by the second equation. Consequently we see all eigenvalues of (4.4) are real and nonnegative. From this fact, we can obtain all assertions of Lemma 4.6 with the aid of the perturbation theory by T. Kato [37, Chapter 8]. □

We note that all eigenvalues $\{\mu_i(\xi,\varepsilon)\}$ form a symmetric set with respect to the real axis on the complex plane $\boldsymbol{C}$. Then $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ (with (4.3)) satisfy the following properties (i) or (ii);

(i) both of $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ are real numbers;

(ii) $\mu_1(\xi,\varepsilon)$ is a complex conjugate of $\mu_2(\xi,\varepsilon)$.

In what follows, we assume that $\mu_1(\xi,\varepsilon) \leq \mu_2(\xi,\varepsilon)$ in case (i), and $\operatorname{Im}\mu_1(\xi,\varepsilon) \geq \operatorname{Im}\mu_2(\xi,\varepsilon)$ in case (ii).

Definition 4.1 (Linearized stability). A steady-state $(w(\xi,\varepsilon), z(\xi,\varepsilon))$ of (4.1) is called *lineary stable* if $\mathrm{Re}\,\mu_1(\xi,\varepsilon) > 0$. If $\mathrm{Re}\,\mu_1(\xi,\varepsilon) < 0$, then it is called *lineary unstable*.

We define matrixes $J(s)$ and $M(s)$ by

$$\begin{aligned} J(s) &= \begin{bmatrix} \dfrac{1}{d_1} & 0 \\ -\dfrac{\gamma\varphi(\gamma s)}{d_2}\displaystyle\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2} & \dfrac{1}{d_2}\displaystyle\int_\Omega \frac{\Phi^2}{1+\gamma s\Phi} \end{bmatrix}, \\ M(s) &= -J(s)^{-1}F^0_{(s,t)}(s, \varphi(\gamma s), \psi(s)) \end{aligned} \tag{4.5}$$

for the mapping F^0 defined by (3.35). To determine the sign of $\mathrm{Re}\,\mu_1(\xi,\varepsilon)$, the following lemma plays an important role.

Lemma 4.7. *Let $\nu_1(s)$ and $\nu_2(s)$ be eigenvalues of $M(s)$ and satisfy $\mathrm{Re}\,\nu_1(s) \le \mathrm{Re}\,\nu_2(s)$ and $\mathrm{Im}\,\nu_1(s) \ge \mathrm{Im}\,\nu_2(s)$. Then for any $s \in (0, C_0)$, it holds true that*

$$\lim_{(\xi,\varepsilon)\to(s,0)} \frac{\mu_i(\xi,\varepsilon)}{\varepsilon} = \nu_i(s) \quad \text{for} \quad i = 1, 2. \tag{4.6}$$

Proof. For any sequence $\{(\xi_n, \varepsilon_n)\}_{n=1}^\infty$ such that $\lim_{n\to\infty}(\xi_n,\varepsilon_n) = (s, 0)$, we set

$$(w_n, z_n, a_1^n) := (w(\xi_n,\varepsilon_n), z(\xi_n,\varepsilon_n), a_1(\xi_n,\varepsilon_n)) \in \varGamma^{\varepsilon_n}, \quad \mu_i^n := \mu_i(\xi_n,\varepsilon_n) \quad (i = 1, 2).$$

Let (h_i^n, k_i^n) ($\|h_i^n\| + \|k_i^n\| = 1$) be eigenfunctions of (4.2) associated with eigenvalues μ_i^n. Then by virtue of (3.36) and (3.28), the eigenvalue problem (4.2) can be written as

$$\begin{cases} -\Delta h_i^n - \lambda_1 h_i^n - \varepsilon_n\,[\, f_w(w_n, z_n, a_1^n) h_i^n + f_z(w_n, z_n, a_1^n) k_i^n \,] = \dfrac{\mu_i^n}{d_1} h_i^n & \text{in } \Omega, \\ -\Delta k_i^n - \lambda_1 k_i^n - \varepsilon_n\,[\, g_w(w_n, z_n) h_i^n + g_z(w_n, z_n) k_i^n \,] & \\ \qquad = \dfrac{\mu_i^n}{d_2}\left[-\dfrac{\gamma z_n}{(1+\gamma w_n)^2} h_i^n + \dfrac{k_i^n}{1+\gamma w_n} \right] & \text{in } \Omega, \\ h_i^n = k_i^n = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking L^2-inner product of the above differential equations with $\varPhi$, we can deduce

$$\begin{cases} \displaystyle\int_\Omega f_w(w_n, z_n, a_1^n) h_i^n \varPhi + \int_\Omega f_z(w_n, z_n, a_1^n) k_i^n \varPhi = -\frac{\mu_i^n}{\varepsilon_n} \frac{1}{d_1} \int_\Omega h_i^n \varPhi, \\ \displaystyle\int_\Omega g_w(w_n, z_n) h_i^n \varPhi + \int_\Omega g_z(w_n, z_n) k_i^n \varPhi \\ \displaystyle\qquad\qquad = -\frac{\mu_i^n}{\varepsilon_n} \frac{1}{d_2} \left[-\gamma \int_\Omega \frac{z_n}{(1+\gamma w_n)^2} h_i^n \varPhi + \int_\Omega \frac{k_i^n}{1+\gamma w_n} \varPhi \right]. \end{cases} \tag{4.7}$$

From the proof of Lemma 4.6, we may assume that subject to a subsequence,

$$\lim_{n\to\infty} (h_i^n, k_i^n) = (p_i \varPhi, q_i \varPhi) \ \text{ in } \ C(\overline{\Omega}) \times C(\overline{\Omega})$$

for some (p_i, q_i) with $|p_i| + |q_i| = 1$ $(i = 1, 2)$. Thus together with

$$\lim_{n\to\infty} (w_n, z_n, a_1^n) = (s\varPhi, \varphi(\gamma s)\varPhi, \psi(s)) \ \text{ in } \ C(\overline{\Omega}) \times C(\overline{\Omega}) \times \boldsymbol{R}$$

we see

$$\lim_{n\to\infty} \begin{bmatrix} \displaystyle\int_\Omega f_w(w_n, z_n, a_1^n) h_i^n \varPhi & \displaystyle\int_\Omega f_z(w_n, z_n, a_1^n) k_i^n \varPhi \\ \displaystyle\int_\Omega g_w(w_n, z_n) h_i^n \varPhi & \displaystyle\int_\Omega g_z(w_n, z_n) k_i^n \varPhi \end{bmatrix}$$
$$= \begin{bmatrix} p_i \displaystyle\int_\Omega f_w(s\varPhi, \varphi(\gamma s)\varPhi, \psi(s)) \varPhi^2 & q_i \displaystyle\int_\Omega f_z(s\varPhi, \varphi(\gamma s)\varPhi, \psi(s)) \varPhi^2 \\ p_i \displaystyle\int_\Omega g_w(s\varPhi, \varphi(\gamma s)\varPhi) \varPhi^2 & q_i \displaystyle\int_\Omega g_z(s\varPhi, \varphi(\gamma s)\varPhi) \varPhi^2 \end{bmatrix}.$$

In view of (3.35), observe that

$$F^0_{(s,t)}(s, \varphi(\gamma s), \psi(s))$$
$$= \begin{bmatrix} \displaystyle\int_\Omega f_w(s\varPhi, \varphi(\gamma s)\varPhi, \psi(s)) \varPhi^2 & \displaystyle\int_\Omega f_z(s\varPhi, \varphi(\gamma s)\varPhi, \psi(s)) \varPhi^2 \\ \displaystyle\int_\Omega g_w(s\varPhi, \varphi(\gamma s)\varPhi) \varPhi^2 & \displaystyle\int_\Omega g_z(s\varPhi, \varphi(\gamma s)\varPhi) \varPhi^2 \end{bmatrix}.$$

Therefore, letting $n \to \infty$ in (4.7) leads us to

$$F^0_{(s,t)}(s, \varphi(\gamma s), \psi(s)) \begin{pmatrix} p_i \\ q_i \end{pmatrix} = -\left(\lim_{n\to\infty} \frac{\mu_i^n}{\varepsilon_n} \right) J(s) \begin{pmatrix} p_i \\ q_i \end{pmatrix}. \tag{4.8}$$

Since $|p_i| + |q_i| = 1$ $(i = 1, 2)$, then (4.8) implies (4.6). Since $\nu_i(s)$ is independent of subsequences, it is easily verified that μ_i^n/ε_n itself converges to $\nu_i(s)$ for each $i = 1, 2$. Thus the proof of Lemma 4.7 is accomplished. □

Lemma 4.8. *Suppose that $\varepsilon > 0$ is sufficiently small. Suppose further that $\xi \in (0, C_\varepsilon)$. Thus all zeros of $\mu_1(\xi, \varepsilon)$ coincide with all zeros of $\partial_\xi a_1(\xi, \varepsilon)$.*

The above lemma asserts that the degeneracy of steady-states on Γ^ε is equivalent to the criticality of $a_1(\xi, \varepsilon)$ with respect to ξ. We refer the proof of Lemma 4.8 to the perturbation theory for the Fredholm operator developed by Du and Lou [25, Theorem 3.13 and Appendix]

Since ψ is analytic, ψ' possesses at most a finite number of zeros in $(0, C_0)$. Furthermore, by virtue of (3.36), any zero of ψ' must be a strictly critical point of ψ for almost every $(\tau, \gamma) \in [-\tilde{\tau}, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$. For such $(\tau, \gamma) \in [-\tilde{\tau}, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and sufficiently small $\varepsilon > 0$, all zeros of $\partial_\xi a_1(\xi, \varepsilon)$ are denoted by

$$0 < \xi_1(\varepsilon) < \xi_2(\varepsilon) < \cdots < \xi_{k-1}(\varepsilon) < C_\varepsilon.$$

That is to say,

$$(w_i, z_i, a_1^i) := (w(\xi_i(\varepsilon), \varepsilon), z(\xi_i(\varepsilon), \varepsilon), a_1(\xi_i(\varepsilon), \varepsilon)) \in \Gamma^\varepsilon \quad (i = 1, 2, \ldots, k-1)$$

are all turning points on Γ^ε with respect to a_1. Here we remark that $\lim_{\varepsilon \downarrow 0} a_1(\,\cdot\,, \varepsilon) = \psi$ in $C^2([0, C_0])$ by Propositions 3.10 and 3.15 (see also the proof of Lemma 3.12). Additionally, for each $1 \leq i \leq k$ we set

$$\Gamma_i^\varepsilon := \{((w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) \,:\, \xi \in (\xi_{i-1}(\varepsilon), \xi_i(\varepsilon))\},$$

where $\xi_0(\varepsilon) := 0$ and $\xi_k(\varepsilon) = C_\varepsilon$. This implies $\bigcup_{i=1}^{k} \Gamma_i^\varepsilon = \Gamma^\varepsilon \setminus \bigcup_{i=1}^{k-1} \{(w_i, z_i, a_1^i)\}$.

Lemma 4.9. *For almost every $(\tau, \gamma) \in [-\tilde{\tau}, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$, there exist small positive constants δ, ε_0 such that if $d_1/d_2 \leq \delta$ and $\varepsilon \leq \varepsilon_0$, then all steady-state solutions on $\Gamma_{2j-1}^\varepsilon$ $(j = 1, 2, \ldots, [(k+1)/2])$ are linearized stable, while all steady-state solutions on Γ_{2j}^ε $(j = 1, 2, \ldots, [k/2])$ are linearized unstable.*

Proof. Taking the trace of $M(s)$, one can see

$$\begin{aligned}
&\nu_1(s)+\nu_2(s)\\
&= d_2\varphi(\gamma s)\left[\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2}\left(\int_\Omega \frac{\Phi^2}{1+\gamma s\Phi}\right)^{-1} - \frac{d_1 c\gamma s}{d_2}\int_\Omega \frac{\Phi^4}{(1+\gamma s\Phi)^2}\right] \qquad (4.9)\\
&+ d_1 s\|\Phi\|_3^3 + d_1 c\gamma s\varphi(\gamma s)\int_\Omega \frac{\Phi^3}{1+\gamma s\Phi}\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2}\left(\int_\Omega \frac{\Phi^2}{1+\gamma s\Phi}\right)^{-1}.
\end{aligned}$$

Here we set $h(s) := \int_\Omega s\Phi^4/(1+s\Phi)^2$. Since $h(0)=0$ and $h(s)=O(s^{-1})$ $(s\to\infty)$, $h(\hat{s}) = \sup_{s>0} h(s)$ for some $\hat{s}>0$. Then by (4.9), we obtain

$$\begin{aligned}
\nu_1(s)+\nu_2(s) &\geq d_2\varphi(\gamma s)\left[\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2}\left(\int_\Omega \frac{\Phi^2}{1+\gamma s\Phi}\right)^{-1} - \frac{d_1 c}{d_2}h(\gamma s)\right] + d_1 s\|\Phi\|_3^3\\
&> d_2\varphi(\gamma s)\left[\int_\Omega \frac{\Phi^3}{(1+\gamma C_0\Phi)^2} - \frac{d_1 c}{d_2}h(\hat{s})\right] + d_1 s\|\Phi\|_3^3
\end{aligned}$$

for all $s\in[\,0,C_0\,]$. Therefore it follows from $\varphi(\gamma s)>0$ $(s\in[\,0,C_0))$ that if

$$\frac{d_1}{d_2} < \frac{1}{2ch(\hat{s})}\int_\Omega \frac{\Phi^3}{(1+\gamma C_0\Phi)^2},$$

then $\nu_1(s)+\nu_2(s)>0$ for all $s\in[\,0,C_0\,]$. Thus we can see by Lemma 4.7 that for sufficiently small $\varepsilon>0$,

$$\mu_1(\xi,\varepsilon)+\mu_2(\xi,\varepsilon) > 0 \ \text{ for all } \ \xi\in[\,0,C_\varepsilon\,]. \qquad (4.10)$$

Hence (4.10) also implies $\operatorname{Re}\mu_2(\xi,\varepsilon)>0$ for all $\xi\in[\,0,C_\varepsilon\,]$. On the other hand, in view of (4.5), (3.35) and (3.36), direct calculations enable us to obtain

$$\nu_1(s)\nu_2(s) = \det M(s) = d_1 d_2 s\varphi(\gamma s)\psi'(s)\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2}\left(\int_\Omega \frac{\Phi^2}{1+\gamma s\Phi}\right)^{-1}. \qquad (4.11)$$

So it holds that $\operatorname{sign}\nu_1(s)\nu_2(s) = \operatorname{sign}\psi'(s)$ for all $s\in(0,C_0)$. Let $s_0\in(0,C_0)$ be any fixed point. If $\psi'(s_0)>0$, then Lemma 4.7 implies $\mu_1(\xi,\varepsilon)\mu_2(\xi,\varepsilon)>0$ if (ξ,ε) is sufficiently near $(s_0,0)$. Further, together with (4.10), we obtain $\operatorname{Re}\mu_1(\xi,\varepsilon)>0$. Similarly if $\psi'(s_0)<0$ and (ξ,ε) is close to $(s_0,0)$, then $\operatorname{Re}\mu_1(\xi,\varepsilon)<0$.

Additionally it follows from Lemma 4.8 that $\mu_1(\xi,\varepsilon)=0$ if and only if $\xi=\xi_i(\varepsilon)$ for some $1\le i\le k-1$ provided that $\varepsilon>0$ is sufficiently small. Since $\operatorname{Re}\mu_2(\xi,\varepsilon)>0$ for all $\xi\in[\,0,C_\varepsilon\,]$, consequently $\operatorname{Re}\mu_1(\xi,\varepsilon)=0$ holds if and only if $\xi=\xi_i(\varepsilon)$ for some $1\le i\le k-1$. We now remark $\psi'(0)>0$ if $(\tau,\gamma)\in[\,-\tilde\tau,\tilde\tau\,]\times[\,\tilde\gamma,\infty)$ (see Lemma 3.9). Therefore we obtain

$$\begin{cases}\operatorname{Re}\mu_1(\xi,\varepsilon)>0 \ \text{ if } \ (w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\in\mathit{\Gamma}^\varepsilon_{2j-1},\\ \operatorname{Re}\mu_1(\xi,\varepsilon)<0 \ \text{ if } \ (w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\in\mathit{\Gamma}^\varepsilon_{2j}.\end{cases}$$

Thus the proof of Lemma 4.9 is complete. □

4.3.3 Asymptotic Stability

In this subsection, we will prove that the linearized stability result (Lemma 4.9) is directly connected with the *asymptotic* stability in the sense of the problem (4.1). In order to accomplish it, we make use of the linerization principle for quasilinear parabolic equations developed by Potier-Ferry [61]. To introduce the principle, we need the interpolation spaces $[\,X,Y\,]_{\theta,p}$ $(0\le\theta\le1)$ in the sense of Lions-Peetre [48]. We note that $[\,X,Y\,]_{\theta,p}=[\,W^{2(1-\theta),p}(\Omega)\cap W^{1,p}_0(\Omega)\,]^2$ if $0\le\theta\le1/2$, while $[\,X,Y\,]_{\theta,p}=W^{2(1-\theta),p}(\Omega)^2$ if $1/2<\theta\le1$ (Grisvard [32]).

Lemma 4.10 (Potier-Ferry [61]). *Let $0<\theta\le1$ and $0\le\theta'<1$. For each u in a neighborhood of 0 in $[\,X,Y\,]_{\theta,p}$ let $T(u):X\to Y$ be a closed linear operator. Let f be a nonlinear map from a neighborhood of 0 in X into $[\,X,Y\,]_{\theta',p}$. Suppose that*

(i) *There exist positive numbers ω,κ and C_1 such that the resolvent set of $-T(0)$ contains $\Sigma(\omega,\kappa):=\{\operatorname{Re}z\ge-\kappa$ or $|\arg z|<\pi/2+\omega\}$ and*

$$\|(T(0)+\mu)^{-1}\|\le\frac{C_1}{1+|\mu|}\quad\textit{for all}\quad\mu\in\Sigma(\omega,\kappa).\tag{4.12}$$

(ii) *For any given $u\in X$, the map $u\mapsto T(u)x$ from a neighborhood of 0 in $[\,X,Y\,]_{\theta,p}$ into X is differentiable and there are positive constants η,C_2 such that*

$$\|[T'(u_1)v-T'(u_2)v]x\|_Y\le C_2(\|u_2-u_1\|_{\theta,p})^\eta\|v\|_{\theta,p}\|x\|_X.$$

(iii) *There is a positive constant* C_3 *such that* f *satisfies the Lipschitz condition*

$$\|f(u_1) - f(u_2)\|_{\theta',p} \le C_3 \|u_1 - u_2\|_X.$$

(iv) *There exists a positive constant* C_4 *such that* $\|f(u)\|_{\theta',p} \le C_4 \|u\|_X^2$.

Then for small $\|u_0\|_X$*, the initial value problem*

$$\frac{du}{dt} + T(u)u = f(u), \quad u(0) = u_0 \in X$$

has a unique global solution $u \in C([\,0,\infty), X) \cap C^1([\,0,\infty), Y)$ *and satisfies*

$$\|u(t)\|_X \le C_5 \|u_0\|_X \exp(-\kappa t)$$

for some positive constant C_5

Proposition 4.11. *For almost every* $(\tau,\gamma) \in [\,-\tilde{\tau}, \tilde{\tau}\,] \times [\,\tilde{\gamma}, \infty)$*, there exist small positive constants* δ, ε_0 *such that if* $d_1/d_2 \le \delta$ *and* $\varepsilon \le \varepsilon_0$*, then all steady-state solutions on* $\varGamma^{\varepsilon}_{2j-1}\,(j = 1, 2, \ldots, [(k+1)/2])$ *are asymptotically stable in the topology of* X*, while all steady-state solutions on* $\varGamma^{\varepsilon}_{2j}\,(j = 1, 2, \ldots, [k/2])$ *are unstable.*

Proof. For any $(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \varGamma^{\varepsilon}_{2j-1}$, we simply write

$$(w^{\xi,\varepsilon}, z^{\xi,\varepsilon}, a_1^{\xi,\varepsilon}) := (w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)), \quad \boldsymbol{w}^{\xi,\varepsilon} = (w^{\xi,\varepsilon}, z^{\xi,\varepsilon}).$$

Substituting $(w, z, a_1) = (W + w^{\xi,\varepsilon}, Z + z^{\xi,\varepsilon}, a_1^{\xi,\varepsilon})$ into (4.1), we see that $\boldsymbol{W} := (W, Z)$ satisfies the initial value problem

$$\frac{d\boldsymbol{W}}{dt} + T(\boldsymbol{W})\boldsymbol{W} = \boldsymbol{f}(\boldsymbol{W}), \quad \boldsymbol{W}(0) = (w_0 - w^{\xi,\varepsilon}, z_0 - z^{\xi,\varepsilon}),$$

where

$$T(\boldsymbol{W})\boldsymbol{W} = -J(\boldsymbol{W})^{-1}(H\boldsymbol{W} + \varepsilon B_{(w,z)}(\boldsymbol{w}^{\xi,\varepsilon}, a^{\xi,\varepsilon})\boldsymbol{W}),$$

$$J(\boldsymbol{W}) = \begin{bmatrix} \dfrac{1}{d_1} & 0 \\ -\dfrac{\gamma(z^{\xi,\varepsilon} + Z)}{d_2\{1 + \gamma(w^{\xi,\varepsilon} + W)\}^2} & \dfrac{1}{d_2\{1 + \gamma(w^{\xi,\varepsilon} + W)\}} \end{bmatrix},$$

$$\boldsymbol{f}(\boldsymbol{W}) =$$

$$\varepsilon J(\boldsymbol{W})^{-1} \left(\begin{pmatrix} f(\boldsymbol{w}^{\xi,\varepsilon} + \boldsymbol{W}, a_1^{\xi,\varepsilon}) - f(\boldsymbol{w}^{\xi,\varepsilon}, a_1^{\xi,\varepsilon}) \\ g(\boldsymbol{w}^{\xi,\varepsilon} + \boldsymbol{W}) - g(\boldsymbol{w}^{\xi,\varepsilon}) \end{pmatrix} - B_{(w,z)}(\boldsymbol{w}^{\xi,\varepsilon}, a^{\xi,\varepsilon})\boldsymbol{W} \right).$$

Thus for asymptotic stabilities in X, it suffices to verify all assumptions in Lemma 4.10. We first observe $W^{2(1-\theta),p}(\Omega) \subset C^{\nu}(\overline{\Omega})$ if $\nu = 2(1-\theta) - N/p$ and $\theta \in [0, 1/2)$ by the Sobolev embedding theorem. So letting $\theta = 1/4$ implies

$$X \subset [\,X, Y\,]_{1/4,p} = [\,W^{3/2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,]^2 \subset C^{1/2}(\overline{\Omega})^2.$$

Evidently (ii) of Lemma 4.10 is satisfied for $\theta = 1/4$. Furthermore, since

$$X \subset C^{1/2}(\overline{\Omega})^2 \subset W^{1/2,p}(\Omega)^2 = [\,X, Y\,]_{3/4,p},$$

then (iii) and (iv) of Lemma 4.10 hold true for $\theta' = 3/4$. Furthermore by virtue of

$$T(0) = -J(\xi, \varepsilon)^{-1}(H + \varepsilon B_{(w,z)}(\boldsymbol{w}^{\xi,\varepsilon}, a_1^{\xi,\varepsilon})),$$

it follows from (4.2) and Lemma 4.9 that if $(\boldsymbol{w}^{\xi,\varepsilon}, a_1^{\xi,\varepsilon}) \in \mathit{\Gamma}_{2j-1}^{\varepsilon}$, then the resolvent set of $-T(0)$ contains $\Sigma(\omega, \kappa)$ for some positives ω and κ. It is possible to verify (4.12) by the standard argument. Therefore, by Lemma 4.10 we see that if $\|\boldsymbol{W}(0)\|_X$ is small enough, then

$$\|\boldsymbol{W}(t)\|_X \le C_1 \|\boldsymbol{W}(0)\|_X \exp(-\kappa t) \quad \text{for all} \quad t > 0$$

with some positive C_1. This fact immediately implies the asymptotic stability of $\boldsymbol{w}^{\xi,\varepsilon}$. Furthermore, by Lemma 4.9, it is possible to prove the instability of $\boldsymbol{w}^{\xi,\varepsilon}$ with $(\boldsymbol{w}^{\xi,\varepsilon}, a_1^{\xi,\varepsilon}) \in \mathit{\Gamma}_{2j}^{\varepsilon}$. We refer to Drangeid [24, Theorem 4.1]. □

In view of (3.25), we immediately obtain Theorem 4.3 from Proposition 4.11.

4.3.4 The Hopf Bifurcation

In this subsection, we will give the proof of Theorem 4.4.

Proposition 4.12. *For any $(\tau, \beta) \in [\,-\tilde{\tau}, \tilde{\tau}\,] \times [\,\tilde{\gamma}, \infty)$, there exist a large $D > 0$ and a small $\varepsilon_0 > 0$ such that if $d_1/d_2 \ge D$ and $\varepsilon \le \varepsilon_0$, then the Hopf bifurcation occurs at a certain point on $\mathit{\Gamma}_1^{\varepsilon}$.*

Proof. To accomplish the proof, it suffices to find small positives ξ^*, ε such that $\mu_1(\xi^*, \varepsilon)$, $\mu_2(\xi^*, \varepsilon)$ are pure imaginary pair and satisfy $\partial_\xi \mathrm{Re}\mu_i(\xi^*, \varepsilon) < 0$ for $i = 1, 2$. We refer to Amann [4] for the abstract Hopf bifurcation theorem for strongly coupled parabolic equations (see also [19]).

Take $(\tau, \gamma) \in [-\tilde{\tau}, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$. Let $\nu_1(s)$ and $\nu_2(s)$ be eigenvalues of $M(s)$ defined by (4.5). We first remark that by (3.48) and $\psi'(0) > 0$, $\nu_1(s)\nu_2(s) > 0$ in $(0, s_1)$ with some $s_1 > 0$. If we set

$$k(s) := \int_\Omega \frac{\Phi^4}{(1+\gamma s\Phi)^2} - \int_\Omega \frac{\Phi^3}{1+\gamma s\Phi} \int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1+\gamma s\Phi} \right)^{-1} - \frac{\|\Phi\|_3^3}{c\gamma\varphi(\gamma s)}$$

then, (4.9) is rewritten as

$$\nu_1(s) + \nu_2(s) = d_2 \varphi(\gamma s) \left[\int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1+\gamma s\Phi} \right)^{-1} - \frac{d_1}{d_2} c\gamma s k(s) \right].$$

Thus direct calculations imply

$$\nu_1(0) + \nu_2(0) = d_2, \quad \nu_1'(0) + \nu_2'(0) = d_2 \left(\tilde{C} - \frac{d_1}{d_2} c\gamma k(0) \right) \tag{4.13}$$

for some constant $\tilde{C}$ independent of d_1 and d_2. By virtue of Schwarz' inequality and $\|\Phi\| = 1$, we see $\|\Phi\|_4^4 > \|\Phi\|_3^6$.

Thus it turns out that $k(0) = \|\Phi\|_4^4 - \|\Phi\|_3^6 - \|\Phi\|_3^3 (cb_1\gamma)^{-1} > 0$ if γ is large enough. It follows from (4.13) that if d_1/d_2 is sufficiently large, we can find a small positive $s_0 \in (0, s_1)$ such that

$$\begin{cases} \nu_1(s) + \nu_2(s) > 0 \ \text{ in } \ (0, s_0), \quad \nu_1(s_0) + \nu_2(s_0) = 0, \\ \nu_1'(s_0) + \nu_2'(s_0) < 0. \end{cases} \tag{4.14}$$

We will prove that for a certain (ξ^*, ε) near $(s_0, 0)$, eigenvalues $\mu_1(\xi^*, \varepsilon)$, $\mu_2(\xi^*, \varepsilon)$ are pure imaginary pair and satisfy $\partial_\xi \mathrm{Re}\mu_i(\xi^*, \varepsilon) < 0$ $(i = 1, 2)$. In order to show this fact, we construct eigenvalues μ and eigenfunctions (h, k) of (4.2) in the following form

$$\mu = \varepsilon\nu, \quad (h, k) = (1, \eta)\Phi + \varepsilon \boldsymbol{V} \quad (\boldsymbol{V} \in X_1)$$

by making use of the implicit function theorem. In view of (4.2), we define a mapping $G : \boldsymbol{R}^2 \times \boldsymbol{C}^2 \times X_1 \to Y$ by

$$
\begin{aligned}
&G(\xi, \varepsilon, \nu, \eta, \boldsymbol{V}) \\
:=&H((1, \eta)\Phi + \varepsilon \boldsymbol{V}) + \varepsilon \hat{B}(\xi, \varepsilon)((1, \eta)\Phi + \varepsilon \boldsymbol{V}) + \varepsilon \nu J(\xi, \varepsilon)((1, \eta)\Phi + \varepsilon \boldsymbol{V}),
\end{aligned}
$$

where $\hat{B}(\xi, \varepsilon) := B_{(w,z)}(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon))$. Thus (4.2) is equivalent to

$$G(\xi, \varepsilon, \nu, \eta, \boldsymbol{V}) = 0.$$

This equation can be decomposed as

$$
\begin{cases}
(I - Q)\hat{B}(\xi, \varepsilon)((1, \eta)\Phi + \varepsilon \boldsymbol{V}) + \nu(I - Q)J(\xi, \varepsilon)((1, \eta)\Phi + \varepsilon \boldsymbol{V}) = 0, \\
QH(\boldsymbol{V}) + Q\hat{B}(\xi, \varepsilon)((1, \eta)\Phi + \varepsilon \boldsymbol{V}) + \varepsilon \nu \, QJ(\xi, \varepsilon)((1, \eta)\Phi + \varepsilon \boldsymbol{V}) = 0.
\end{cases}
\tag{4.15}
$$

Define a mapping $G^1 : \boldsymbol{R}^2 \times \boldsymbol{C}^2 \times X_1 \to \operatorname{span}\{(\Phi, 0), (0, \Phi)\}$ by the left hand side of the above first equation. Denote by $G^2 : \boldsymbol{R}^2 \times \boldsymbol{C}^2 \times X_1 \to Y_1$ the left hand side of the second equation. Further, we denote by $\nu_i^0 (i = 1, 2)$ and $(1, \eta_i^0)$ the eigenvalues of $M(s_0)$ and the associate eigenvectors. By virtue of $(I - Q)\hat{B}(s_0, 0) = F_{(s,t)}(s_0, \varphi(\gamma s_0), \psi(s_0))$ and $(I - Q)J(s_0, 0) = J(s_0)$, we see that

$$G(s_0, 0, \nu_i^0, \eta_i^0, \boldsymbol{V}_i^0) = 0 \quad (i = 1, 2)$$

for $\boldsymbol{V}_i^0 := -(QH)^{-1}(Q\hat{B}(s_0, 0)[\,(1, \eta_i^0)\Phi\,] + \nu_i \, QJ(s_0, 0)[\,(1, \eta_i^0)\Phi\,])$. To use the implicit function theorem near $(s_0, 0, \nu_i^0, \eta_i^0, \boldsymbol{V}_i^0)$, we need to verify that

$$G_{(\nu,\eta,\boldsymbol{V})}(s_0, 0, \nu_i^0, \eta_i^0, \boldsymbol{V}_i^0) : \boldsymbol{C}^2 \times X_1 \to Y$$

is invertible. By direct calculations, we have

$$
\begin{aligned}
&G^1_{(\nu,\eta,\boldsymbol{V})}(s_0, 0, \nu_i^0, \eta_i^0, \boldsymbol{V}_i^0)\,[\,\overline{\eta}, \overline{\nu}, \overline{\boldsymbol{V}}\,] \\
=&F_{(s,t)}(s_0, \varphi(\gamma s_0), \psi(s_0))[\,(0, \overline{\eta})\Phi\,] + \overline{\nu} J(s_0)[\,(1, \eta_i)\Phi\,] + \nu_i^0 J(s_0)[\,(0, \overline{\eta})\Phi\,], \\
&G^2_{(\nu,\eta,\boldsymbol{V})}(s_0, 0, \nu_i^0, \eta_i^0, \boldsymbol{V}_i^0)\,[\,\overline{\nu}, \overline{\eta}, \overline{\boldsymbol{V}}\,] \\
=&\overline{\nu}\, QJ(s_0, 0)[\,(1, \eta_i)\Phi\,] + Q\hat{B}(s_0, 0)[\,(0, \overline{\eta})\Phi\,] + \nu_i^0 QJ(s_0, 0)[\,(0, 1)\Phi\,] + QH(\overline{\boldsymbol{V}}).
\end{aligned}
$$

By virtue of $\psi'(s_0) > 0$ and

$$\det F_{(s,t)}(s_0, \varphi(\gamma s_0)\psi(s_0)) = s_0 \varphi(\gamma s_0)\psi'(s_0) \int_\Omega \frac{\Phi^3}{(1+\gamma s_0 \Phi)^2},$$

we see $F_{(s,t)}(s_0, \varphi(\gamma s_0), \psi(s_0))$ is invertible. From this fact, it is easily verified that $G_{(\nu,\eta,\boldsymbol{V})}(s_0, 0, \nu_i^0, \eta_i^0, \boldsymbol{V}_i^0)$ is also invertible. Then the implicit function theorem enables us to get eigenvalues $\mu_i(\xi, \varepsilon) = \varepsilon \nu_i(\xi, \varepsilon)$ $(i = 1, 2)$ of (4.2) for certain smooth functions $\nu_i(\xi, \varepsilon)$ in a neighborhood of $(s_0, 0)$. By taking account for $\nu_i(s_0, 0) = \nu_i(s_0)$ and (4.14), it is possible to find a (ξ^*, ε) near $(s_0, 0)$ such that

$$\begin{cases} \mu_1(\xi,\varepsilon) + \mu_2(\xi,\varepsilon) > 0 \quad \text{for} \quad \xi \in (0, \xi^*), \qquad \mu_1(\xi^*,\varepsilon) + \mu_2(\xi^*,\varepsilon) = 0, \\ \partial_\xi \mu_1(\xi^*,\varepsilon) + \partial_\xi \mu_2(\xi^*,\varepsilon) < 0, \\ \mu_1(\xi,\varepsilon)\mu_2(\xi,\varepsilon) > 0 \quad \text{for} \quad \xi \in (0, \xi^*\,]. \end{cases}$$

Evidently this fact implies that $\mu_1(\xi^*, \varepsilon)$, $\mu_2(\xi^*, \varepsilon)$ are pure imaginary pair and satisfy $\partial_\xi \mathrm{Re}\, \mu_i(\xi^*, \varepsilon) < 0$ $(i = 1, 2)$. Therefore the Hopf bifurcation occurs at $(w(\xi^*, \varepsilon), z(\xi^*, \varepsilon), a_1(\xi^*, \varepsilon))$, which belongs to Γ_1^ε because ξ^* is sufficiently small. The proof of Proposition 4.12 is accomplished. □

By (3.25), Proposition 4.12 immediately implies Theorem 4.4 and (ii) of Theorem 4.5.

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