On the probability of positive-definiteness in the gGUE via semi-classical Laguerre polynomials

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Abstract

In this paper, we compute the probability that an $N \times N$ matrix from the generalised Gaussian Unitary Ensemble (gGUE) is positive definite, extending a previous result of Dean and Majumdar [14]. For this purpose, we work out the large degree asymptotics of semi-classical Laguerre polynomials and their recurrence coefficients, using the steepest descent analysis of the corresponding Riemann–Hilbert problem.

1 Introduction and main results

The *Gaussian Unitary Ensemble* (GUE) is the most classical and studied example of a unitarily invariant Hermitian random matrix ensemble. Given the set of $N \times N$ Hermitian matrices $\mathcal{H}_N$, one defines a probability density

$$dP(M_N) = \frac{1}{\mathcal{Z}_N} e^{-N\,\mathrm{Tr} V(M_N)} dM_N, \tag{1.1}$$

where dM_N is the usual Lebesgue measure on $\mathcal{H}_N$, and $\mathcal{Z}_N$ is the partition function:

$$\mathcal{Z}_N = \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr} V(M_N)} dM_N. \tag{1.2}$$

The potential V is a smooth function with sufficient growth at infinity, so that (1.2) is well defined, and the GUE corresponds to the quadratic case $V(x) = x^2$, see references [3, 20, 23] for relevant background.

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In this paper we are interested in the probability that matrices drawn at random from (1.1) are positive definite, denoted here by $\mathbb{P}(M_N > 0)$. As well as being a natural question within random matrix theory, in several situations in the physics literature M_N is used to model the Hessian matrix of random high-dimensional *energy surfaces*, see *e.g.* [1, 10, 14, 21] and references therein. In such contexts $\mathbb{P}(M_N > 0)$ provides important information on the stability (maxima and minima) of such energy surfaces.

In the GUE case, the earliest investigations of this probability appeared in the string theory and cosmology literature [1], where it was argued that $\mathbb{P}(M_N > 0)$ decays exponentially in N^2 (at least implicitly, this already followed from the large deviations principle of Ben Arous and Guionnet [4]). However, the multiplicative constant in these asymptotics remained unknown until the work of Dean and Majumdar [14], who showed using Coulomb gas techniques that

$$\log \mathbb{P}(M_N > 0) = -c_1 N^2 + o(N^2) \tag{1.3}$$

where

$$c_1 = \frac{\log 3}{2}. \tag{1.4}$$

Then in subsequent work [8], further terms in the asymptotic expansion of (1.3) were computed using the technique of loop equations, where it was shown that

$$\log \mathbb{P}(M_N > 0) = -c_1 N^2 + c_2 \log(N) + c_3 + o(1) \tag{1.5}$$

where

$$c_2 = -\frac{1}{12}, \qquad c_3 = \frac{\log 3}{8} - \frac{\log 2}{6} + \zeta'(-1), \tag{1.6}$$

and $\zeta(s)$ is the Riemann-Zeta function. Our aim will be to give a proof of (1.5) using the firmly established methods of orthogonal polynomials. Our results also apply to the *generalised GUE*, leading to a 1-parameter extension of the asymptotics (1.5), which as far as we are aware have not appeared in either the mathematics or physics literature.

Like the ordinary GUE, the generalised GUE is defined on the set of $N \times N$ Hermitian matrices, but now the probability measure has the form

$$dP(M_N) = \frac{|\det(M_N)|^\lambda}{\mathcal{Z}_N^{\mathrm{gGUE}}} \exp\left(-N \operatorname{Tr}(M_N^2)\right) dM_N, \tag{1.7}$$

where we assume $\operatorname{Re}\lambda > -1$ to ensure finiteness of the normalizing constant $\mathcal{Z}_N^{\mathrm{gGUE}}$, which depends implicitly on λ although for simplicity of notation we do not emphasize it. Ensembles of the form (1.7), with extra algebraic terms in the density, were studied extensively in the literature on matrix models under the name *Gauss-Penner model*, see [16, 2] for details and applications.

The main purpose of this paper is to prove the following

Theorem 1.1. *Let $\mathbb{P}(M_N^{(\lambda)} > 0)$ denote the probability that a random matrix from ensemble* (1.7) *is positive definite. Then for any fixed λ with $\operatorname{Re}\lambda > -1$, we have the asymptotic expansion as $N \to \infty$,*

$$\begin{aligned}\log \mathbb{P}(M_N^{(\lambda)} > 0) = -c_1 N^2 - \frac{\lambda \log(3)}{2} N + \left(c_2 + \frac{\lambda^2}{4}\right)\log(N) + c_3 \\ + \frac{3\lambda^2}{4}\log(2) - \frac{\lambda^2}{2}\log(3) - \log \frac{G(\frac{3}{2})G(\frac{1}{2})G(\lambda+1)}{G(\frac{\lambda+3}{2})G(\frac{\lambda+1}{2})G(1)} + \mathcal{O}(N^{-1}),\end{aligned} \tag{1.8}$$

where c_1, c_2 and c_3 are the explicit constants defined above and $G(z)$ is the Barnes G function [17, §5.17].

We note that in the case $\lambda = 0$ we immediately recover the result (1.5) of [8] as a special case. We also mention the work [9] where the dependence of the leading term c_1 on growing $\lambda \sim N$ is investigated.

Figure 1 illustrates the accuracy of the asymptotic expansion (1.8) for increasing N and several values of λ. The comparison has been made with respect to brute force calculation of the Hankel determinant expression for the partition functions, see Appendix A, which is quite time consuming and needs a large number of digits in Maple.

To prove Theorem 1.1 we will study the partition function:

$$Z_N(s) = \int_0^\infty \dots \int_0^\infty \prod_{j=1}^N w(x_j; \lambda, s) \prod_{1 \le p < q \le N} (x_q - x_p)^2 \, dx_1 \dots dx_N \tag{1.9}$$

where $w(x; \lambda, s) = x^\lambda e^{-x+sx(1-x)}$. Note that this deformed weight interpolates between the classical Laguerre weight if $s = 0$ and the generalized GUE if $s = 1$. Diagonalizing M_N in (1.7) and integrating out the eigenvectors (see *e.g.* [3, 20, 23]) we see that

$$\begin{aligned}\log \mathbb{P}(M_N^{(\lambda)} > 0) &= \log\left(\frac{Z_N(1)}{Z_N^{\mathrm{gGUE}}}\right) \\ &= \int_0^1 \frac{Z_N'(s)}{Z_N(s)}\, ds + \log Z_N(0) - \log Z_N^{\mathrm{gGUE}}.\end{aligned} \tag{1.10}$$

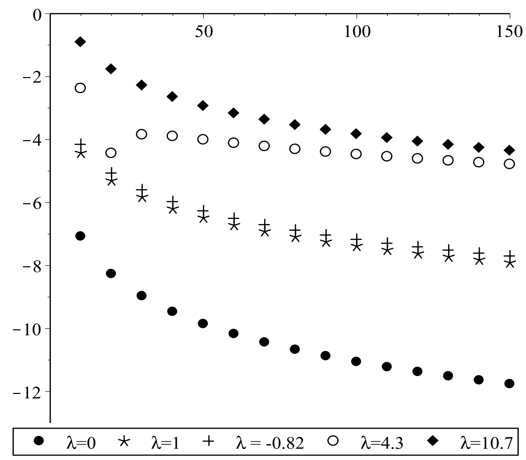

Figure 1: Absolute errors (in $\log_{10}$ scale) as a function of N and for different values of λ, taking all the terms in (1.8) up to order $\mathcal{O}(1)$ (included).

As the quantities $Z_N(0) = Z_N^{\mathrm{LUE}}$ and Z_N^{gGUE} turn out to have explicit evaluations in terms of Gamma functions (see Lemmas A.1 and A.2), our main task is to compute the integrand in (1.10).

1. We write $Z_N'(s)/Z_N(s)$ in terms of the recurrence coefficients $\alpha_N(s)$ and $\beta_N(s)$ of a suitable family of semiclassical Laguerre polynomials, orthogonal with respect to $w(x;\lambda,s)$ on $x \in [0,\infty)$.

2. We compute the first terms in the asymptotic expansion of $\alpha_N(s)$ and $\beta_N(s)$ as $N \to \infty$, using the corresponding Riemann–Hilbert problem and the Deift–Zhou method of steepest descent.

3. We show that such asymptotic expansions are uniform in $s \in [0,1]$ and we integrate term by term in (1.10).

2 Proof of Theorem 1.1

Semi-classical Laguerre orthogonal polynomials (OPs): $\pi_n(x) = \pi_n(x;\lambda,s)$ are defined by the orthogonality

$$\int_0^\infty \pi_n(x) x^k w(x;\lambda,s)dx = 0, \qquad k = 0,1,2,\dots,n-1, \tag{2.1}$$

where the weight function is

$$w(x;\lambda,s) = x^\lambda e^{-NV(x;s)}, \qquad \lambda > -1. \tag{2.2}$$

Here the potential is

$$V(x;s) = x + s(x^2 - x), \tag{2.3}$$

constructed in such a way that $V(x;0) = V(x) = x$ corresponds to the classical Laguerre OPs, while $V(x;1) = x^2$ is the potential that we are interested in. Such a deformation follows a similar idea as the construction by Bleher and Its in [6]. The quantities considered here were also recently investigated in the complementary regime of fixed N and large parameters by Clarkson and Jordaan [13]. Part of this interest stems from the fact that the recurrence coefficients for semi-classical Laguerre polynomials, with weight $w(x;\lambda,t) = x^\lambda \exp(-x^2+tx)$, satisfy deformation equations (in t) that are closely related to the Painlevé IV differential equation [7, 18, 13].

Since the weight function (2.2) is positive and integrable on $[0,\infty)$ for $s \in [0,1]$, it follows from the general theory [11, 22], that the orthogonal polynomials $\pi_n(x)$ exist uniquely for all $n \geq 0$ and $s \in [0,1]$, and they satisfy $\deg \pi_n = n$. Furthermore, they are solutions of a three term recurrence relation (written in monic form):

$$x\pi_n(x) = \pi_{n+1}(x) + \alpha_n \pi_n(x) + \beta_n \pi_{n-1}(x), \tag{2.4}$$

with initial data $\pi_{-1}(x) = 0$, $\pi_0(x) = 1$, and recurrence coefficients $\alpha_n = \alpha_n(\lambda,s)$ and $\beta_n = \beta_n(\lambda,s)$. Next, we consider $n = N$ and we write $Z_N'(s)/Z_N(s)$ in terms of these recurrence coefficients.

Lemma 2.1. *We have the following deformation equation*

$$\frac{Z_N'(s)}{Z_N(s)} = \beta_N c_{N,\lambda}(s) - N^2\left[(1-3s)E_N + 2sF_N\right], \tag{2.5}$$

where

$$c_{N,\lambda}(s) := N^2(3-s) + \lambda N \tag{2.6}$$

$$E_N := \beta_N(\alpha_N + \alpha_{N-1}) \tag{2.7}$$

$$F_N := \beta_N(\beta_{N+1} + \beta_N + \beta_{N-1} + \alpha_N^2 + \alpha_N\alpha_{N-1} + \alpha_{N-1}^2) \tag{2.8}$$

Proof. A simple computation shows that

$$\frac{Z_N'(s)}{Z_N(s)} = -N\int_0^\infty (x^2 - x)\rho_N(x)\,dx \tag{2.9}$$

where $\rho_N(x)$ is the so-called 'one-point correlation function' or 'eigenvalue density' corresponding to the weight (2.2), see *e.g.* [3, 23] for definitions and basic properties of this quantity. In particular it is known that $\rho_N(x)$ can be computed explicitly by means of the Christoffel-Darboux formula:

$$\rho_N(x) = \frac{\pi'_N(x)\pi_{N-1}(x) - \pi_N(x)\pi'_{N-1}(x)}{h_{N-1}} \tag{2.10}$$

Inserting (2.10) into (2.9) yields four different contributions which can all be written in terms of the recurrence coefficients α_N and β_N. One term vanishes due to

$$\int_0^\infty x\pi'_{N-1}(x)\pi_N(x)w(x)\,dx = 0, \tag{2.11}$$

a consequence of orthogonality. So (2.9) can be decomposed as $I = I_1 + I_2 + I_3$, where

$$I_1 := \frac{N}{h_{N-1}} \int_0^\infty x^2\pi'_{N-1}(x)\pi_N(x)w(x)\,dx \tag{2.12}$$

$$I_2 := -\frac{N}{h_{N-1}} \int_0^\infty x^2\pi'_N(x)\pi_{N-1}(x)w(x)\,dx \tag{2.13}$$

$$I_3 := \frac{N}{h_{N-1}} \int_0^\infty x\pi'_N(x)\pi_{N-1}(x)w(x)\,dx \tag{2.14}$$

First observe that $I_1 = N(N-1)\beta_N$ (as a consequence of $h_N/h_{N-1} = \beta_N$). An exercise in integration by parts shows that

$$I_2 = N(N+1+\lambda)\beta_N - N^2(1-s)E_N - 2sN^2F_N \tag{2.15}$$

$$I_3 = (1-s)N^2\beta_N + N^2 2sE_N \tag{2.16}$$

where

$$E_N := \frac{1}{h_{N-1}} \int_0^\infty \pi_N(x)\pi_{N-1}(x)x^2w(x)\,dx \tag{2.17}$$

$$F_N := \frac{1}{h_{N-1}} \int_0^\infty \pi_N(x)\pi_{N-1}(x)x^3w(x)\,dx \tag{2.18}$$

Combining all these terms yields (2.5). Finally the identities (2.7) and (2.8) follow from the three term recurrence relation (2.4). $\square$

The recurrence coefficients in Lemma 2.1 can be computed by solving the following coupled system of recurrence relations in the limit $N \to \infty$.

Proposition 2.2 (String equations)**.** *The recurrence coefficients* $\alpha_N(s)$ *and* $\beta_N(s)$ *in* (2.4) *satisfy*

$$\begin{aligned} 2s(\beta_{N+1}+\beta_N+\alpha_N^2)+(1-s)\alpha_N &= 2+\frac{\lambda+1}{N}, \\ s^2\beta_N\left(2\alpha_N+\frac{1-s}{s}\right)\left(2\alpha_{N-1}+\frac{1-s}{s}\right) &= (2s\beta_N-1)\left(2s\beta_N-1-\frac{\lambda}{N}\right). \end{aligned} \tag{2.19}$$

Proof. The string equations are known from [13, Lemma 4.2], and also [7, Theorem 1.1], [18] adapting the potential $V(x;t)=x^2-tx$ to the present one. □

We remark in passing that Boelen and Van Assche [7] have shown that (2.19) can be obtained from an asymmetric discrete Painlevé IV equation by a limiting process.

To solve this system of equations asymptotically as $N\to\infty$, we exploit the following fact, the proof of which is postponed to the next section.

Proposition 2.3. *Let* $q\in\mathbb{Z}$ *be fixed and set* $n=N+q$. *The recurrence coefficients* $\alpha_n(s)$ *and* $\beta_n(s)$ *in* (2.4) *have asymptotic expansions in inverse powers of* N*:*

$$\alpha_n(\lambda,s)\sim\sum_{k=0}^{\infty}f_k(\lambda,s)N^{-k},\qquad \beta_n(\lambda,s)\sim\sum_{k=0}^{\infty}g_k(\lambda,s)N^{-k}. \tag{2.20}$$

The coefficients $f_k(\lambda,s)$ *and* $g_k(\lambda,s)$ *depend implicitly on* q *and are analytic functions of* $s\in[0,1]$.

With these ingredients in hand, we can now prove Theorem 1.1. We insert the expansions (2.20) into the recurrence (2.19) and equate terms with equal powers of N. At leading order the solution that remains bounded as $s\to 0^+$ is

$$f_0=\frac{s-1+\Delta}{6s},\qquad g_0=\frac{s^2+10s+1+(s-1)\Delta}{72s^2}, \tag{2.21}$$

where $\Delta=\sqrt{s^2+22s+1}$. Next, we have

$$f_1=\frac{\lambda+1}{\Delta},\qquad g_1=\frac{(\Delta+s-1)\lambda}{12\Delta s}. \tag{2.22}$$

Higher order corrections can be computed systematically in MAPLE, but become quite cumbersome. If we substitute the terms up to order $\mathcal{O}(N^{-2})$ (included) into the right hand side of (2.5), we get

$$\frac{Z_N'(s)}{Z_N(s)} = A(s)N^2 + B(s)N + C(s) + \mathcal{O}(N^{-1}), \qquad N \to \infty, \tag{2.23}$$

where

$$A(s) = \frac{\Delta^3(s+1) + s^4 + 34s^3 - 216s^2 - 34s - 1}{432s^3}, \tag{2.24}$$

$$B(s) = \frac{\lambda(s^2 - 12s - 1 + (s+1)\Delta)}{24s^2}, \tag{2.25}$$

$$C(s) = \frac{\lambda^2(s+1)[s^2 + 6s + 1 + (s-1)\Delta]}{4s(s^2 - 10s + 1)\Delta} \tag{2.26}$$

$$- \frac{(s+1)^3\Delta + (s^2-1)(s^2+14s+1)}{12s(s^2-10s+1)\Delta^2}, \tag{2.27}$$

using again $\Delta = \sqrt{s^2 + 22s + 1}$. Now integrating from $s = 0$ to 1, we get

$$\begin{aligned}\int_0^1 \frac{Z_N'(s)}{Z_N(s)} ds &= N^2 \int_0^1 A(s)ds + N \int_0^1 B(s)ds + \int_0^1 C(s)ds + \mathcal{O}(N^{-1}) \\ &= N^2 \left(\frac{3}{4} - \frac{\log 6}{2}\right) + N \left(\frac{1}{2} - \frac{\log 6}{2}\right) \lambda \\ &+ \frac{\lambda^2 \log(2/3)}{2} + \frac{\log 3}{8} - \frac{\log 2}{6} + \mathcal{O}(N^{-1}).\end{aligned} \tag{2.28}$$

The integrals in (2.28) are easily calculated in any computer algebra package. Combining (2.28) with the known asymptotics for $\log Z_N(0)$ and $\log Z_N^{\mathrm{gGUE}}$ (see Lemmas A.1 and A.2 respectively) in (1.10) completes the proof of Theorem 1.1. In the next section we prove Proposition 2.3.

3 $1/N$ expansion for the recurrence coefficients

The main purpose of this section is to justify the Ansatz (2.20) which we inserted into the string equations. This is based on the fact that the recurrence coefficients can be computed in terms of the solution of an appropriate Riemann-Hilbert problem (RHP). Then their asymptotics can be analysed very precisely using the Deift–Zhou method of steepest descent.

3.1 Equilibrium measure

In the steepest descent analysis, a key role is played by the equilibrium measure $d\mu_V$, which minimises the logarithmic energy

$$E(\nu) = \iint \log \frac{1}{x-y} d\nu(x) d\nu(y) + \int V(x;s) d\nu(x), \tag{3.1}$$

over all probability measures supported on $[0,\infty)$, where the external field $V(x;s)$ is given by (2.3). Such a problem has a unique solution, since $w(x;\lambda,s) = x^{\lambda} e^{-NV(x;s)}$ is an admissible weight function in the sense of Saff and Totik [25, Def. 1.1]. Moreover, in this case the support and density of this equilibrium measure can be worked out explicitly:

Lemma 3.1. *Let $s \in [0,1]$, the equilibrium measure corresponding to the weight function $w(x;\lambda,s) = x^{\lambda} e^{-NV(x;s)}$, with $V(x;s)$ given by (2.3) is supported on the interval $(0,c)$ where*

$$c = \frac{s-1+\sqrt{s^2+22s+1}}{3s}. \tag{3.2}$$

If we write $d\mu_V(x) = \psi_V(x)dx$, the density is given by

$$\psi_V(x) = -\frac{1}{\pi}(ax+b)\sqrt{\frac{c-x}{x}}, \tag{3.3}$$

with

$$a = -s, \qquad b = \frac{2s-2-\sqrt{s^2+22s+1}}{6}. \tag{3.4}$$

Proof. We apply the change of variables $x \mapsto x^2$, and then we take the potential

$$W(x;s) = \frac{1}{2} V(x^2) = \frac{1}{2}(x^2 + s(x^4 - x^2)).$$

Since $W''(x;s) = 6sx^2 + 1 - s \geq 0$ for all $x \in \mathbb{R}$ and $s \in [0,1]$, the potential is convex, and the equilibrium measure corresponding to this weight, say $d\mu_W(x) = \psi_W(x)dx$ is supported on a single interval $[-\sqrt{c}, \sqrt{c}]$. Consider the resolvent

$$\omega(z) = \int_{-\sqrt{c}}^{\sqrt{c}} \frac{d\mu_W(x)}{z-x},$$

which is analytic in $\mathbb{C} \setminus [-\sqrt{c}, \sqrt{c}]$ and satisfies

$$\begin{aligned} \omega(z) &= \frac{1}{z} + \mathcal{O}(z^{-2}), \qquad z \to \infty, \\ \omega_+(x) + \omega_-(x) &= W'(x), \end{aligned} \tag{3.5}$$

where $\omega_\pm(x)$ indicates the boundary values for $x \in (0, c)$, from above and below the real axis respectively. Consequently, we look for $\omega(z)$ of the form

$$\omega(z) = \frac{W'(z)}{2} + (az^2 + b)(z^2 - c)^{1/2},$$

with a branch cut on $[-\sqrt{c}, \sqrt{c}]$. The first equation in (3.5) gives the coefficients a, b and c in (3.2) and (3.4).

The density of the equilibrium measure is recovered as

$$\psi_W(x) = \frac{1}{2\pi i}(\omega_-(x) - \omega_+(x)) = -\frac{1}{\pi}(ax^2 + b)\sqrt{c - x^2}, \quad x \in [-\sqrt{c}, \sqrt{c}]. \tag{3.6}$$

See [5] for more details. Next, we apply a result of Claeys and Kuijlaars [12, Lemma 2.2] which gives the density of the equilibrium measure corresponding to $V(x)$ in terms of that corresponding to $W(x)$:

$$\psi_W(x) = |x|\psi_V(x^2),$$

so

$$\psi_V(x) = -\frac{1}{\pi}(ax + b)\sqrt{\frac{c - x}{x}}, \qquad x \in (0, c),$$

which is the form given in the lemma. □

We observe that the form of the equilibrium measure is uniform in $s \in [0, 1]$. This will be crucial in the asymptotic expansions obtained below.

3.2 RH problem

Following the original idea of Fokas, Its and Kitaev [19] in this context, the (monic) semiclassical Laguerre polynomials $\pi_n(x)$ are the $(1, 1)$ entry of a 2×2 matrix $Y(z) = Y_n(z; \lambda, s) : \mathbb{C} \mapsto \mathbb{C}^{2\times 2}$ that satisfies the following RH problem:

1. $Y(z)$ is analytic in $\mathbb{C} \setminus [0, \infty)$.

2. On $[0, \infty)$, oriented from left to right, the boundary values of Y satisfy

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & x^\lambda e^{-NV(x;s)} \\ 0 & 1 \end{pmatrix},$$

where $(\cdot)_\pm(x)$ indicates the boundary values from above and below the real axis respectively.

3. As $z \to \infty$, we have

$$Y(z) = \left(I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \tag{3.7}$$

4. As $z \to 0$, we have

$$Y(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(z^\lambda) \\ \mathcal{O}(1) & \mathcal{O}(z^\lambda) \end{pmatrix}, & \lambda < 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \lambda = 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, & \lambda > 0. \end{cases} \tag{3.8}$$

It is known [15] that the recurrence coefficients $\alpha_n(\lambda, s)$ and $\beta_n(\lambda, s)$ in (2.4) can be written as follows:

$$\alpha_n = \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22}, \qquad \beta_n = (Y_1)_{12}(Y_1)_{21} \tag{3.9}$$

where Y_1 and Y_2 are the matrices that appear in the asymptotic expansion (3.7), see [5, §3.2] or [15].

3.3 Steepest descent

The steepest descent of Deift and Zhou consists of a series of transformations that lead to a final RH problem that can be solved asymptotically as $N \to \infty$, uniformly in z in the complex plane. Since we are only using the steepest descent method in order to prove existence of an asymptotic expansion in powers of $1/N$ for the recurrence coefficients, and not to obtain the details of the coefficients therein, the presentation will be quite brief. We refer the reader to the work of Vanlessen [26], or Zhao et al. [27] for a more detailed explanation in a similar setting.

The basic steps in this case are the following:

$$Y \mapsto T \mapsto S \mapsto R. \tag{3.10}$$

The first step $Y \mapsto T$ is a normalization at infinity:

$$T = \begin{pmatrix} e^{-N\ell/2} & 0 \\ 0 & e^{N\ell/2} \end{pmatrix} Y \begin{pmatrix} e^{-N(g(z)-\ell/2)} & 0 \\ 0 & e^{N(g(z)-\ell/2)} \end{pmatrix}, \tag{3.11}$$

where ℓ is a constant (Lagrange multiplier of the equilibrium problem), and g is the logarithmic transform of the equilibrium measure:

$$g(z) = \int_0^c \log(z-x) d\mu_V(x), \tag{3.12}$$

which is analytic in $\mathbb{C}\setminus(-\infty, c]$, with c given by (3.2), and as $z \to \infty$ satisfies

$$g(z) = \log z - \frac{\mu_{1,s}}{z} - \frac{\mu_{2,s}}{2z^2} + \mathcal{O}(z^{-3}), \tag{3.13}$$

where $\mu_{k,s} = \int_0^c x^k d\mu_V(x)$, $k \geq 1$, are the moments of the measure $d\mu_V$, that can be computed explicitly. As a consequence, we have the expansion

$$e^{Ng(z)\sigma_3} = \begin{pmatrix} e^{Ng(z)} & 0 \\ 0 & e^{-Ng(z)} \end{pmatrix} = \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} \left(I + \frac{G_1}{z} + \frac{G_2}{z} + \mathcal{O}(z^{-3}) \right), \tag{3.14}$$

as $z \to \infty$, where G_1 and G_2 are diagonal matrices (and dependent of s and N).

The second step $T \mapsto S$ deforms the jump contours by opening a lens around the interval $[0, c]$. This step does not make any change away from a small neighbourhood of $[0, c]$, and since we will be using information as $z \to \infty$ for the recurrence coefficients, see (3.9), we can replace $T = S$.

The final step, $S \mapsto R$ uses both a global parametrix $P^{(\infty)}$, away from the endpoints $z = 0$ and $z = c$, and two local parametrices, P_{Airy} and P_{Bessel} built out of Airy functions in a neighbourhood $z = 0$ and Bessel functions in a neighbourhood of $z = c$. Then we construct

$$R = \begin{cases} S[P^{(\infty)}]^{-1}, & z \in \mathbb{C} \setminus \overline{D_\delta(0)} \cup \overline{D_\delta(c)}, \\ S[P_{\text{Airy}}]^{-1}, & z \in D_\delta(c), \\ S[P_{\text{Bessel}}]^{-1}, & z \in D_\delta(0), \end{cases}$$

where $D_\delta(0)$ and $D_\delta(c)$ are discs of fixed radius $\delta > 0$ around $z = 0$ and $z = c$ respectively. The RH problem for R can be solved iteratively, since R is normalized at infinity and all jumps are close to the identity, see [5, §11] or [15]. The consequence is an asymptotic expansion of the form:

$$R(z) \sim \sum_{m=0}^{\infty} \frac{R^{(m)}}{N^m}, \qquad N \to \infty, \tag{3.15}$$

uniformly in z away from a contour Σ_R around the interval $[0, c]$, see [15, Chapter 7] or [27]. It is at this stage that the uniform form of the equilibrium measure with respect to s is fundamentally important, since the parametrices depend on s but they have the same structure for $s \in [0,1]$, and then result (3.15) holds uniformly.

In addition, R has an asymptotic expansion as $z \to \infty$, that we write

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R_k}{z^m}, \tag{3.16}$$

and combining (3.16) with (3.15), each coefficient R_k can be expanded asymptotically in inverse powers of N.

Away from the interval $[0,c]$, we write $T = S = RP^{(\infty)}$ and replace this in (3.11):

$$Y = \begin{pmatrix} e^{N\ell/2} & 0 \\ 0 & e^{-N\ell/2} \end{pmatrix} RP^{(\infty)} \begin{pmatrix} e^{N(g(z)-\ell/2)} & 0 \\ 0 & e^{-N(g(z)-\ell/2)} \end{pmatrix}. \tag{3.17}$$

The global parametrix $P^{(\infty)}$ satisfies a RH problem analogous to the one presented in [26, Section 3.5], but on $[0,c]$ instead of $[0,1]$. Making the corresponding changes, we have

$$P^{(\infty)} = I + \frac{P_1^{(\infty)}}{z} + \frac{P_2^{(\infty)}}{z^2} + \mathcal{O}(z^{-3}), \tag{3.18}$$

as $z \to \infty$, with some matrices $P_1^{(\infty)}$ and $P_2^{(\infty)}$ that can be computed explicitly, but whose precise form is not relevant in the present discussion. Using (3.16) in (3.17) and identifying terms, we obtain

$$\begin{aligned} Y_1 &= e^{\frac{N\ell\sigma_3}{2}}(P_1^{(\infty)} + G_1 + R_1)e^{\frac{N\ell\sigma_3}{2}} \\ Y_2 &= e^{\frac{N\ell\sigma_3}{2}}(G_2 + P_1^{(\infty)}G_1 + P_2^{(\infty)} + R_1 + P_1^{(\infty)} + R_1G_1 + R_2)e^{-\frac{N\ell\sigma_3}{2}} \end{aligned} \tag{3.19}$$

From this, we can obtain an expression for the recurrence coefficients in terms of all the matrices involved. The terms in the expansion of $P^{(\infty)}$ are independent of N, and the ones for G contain only integer powers of N.

This result, together with (3.19), gives asymptotic expansions in powers of $1/N$ for the recurrence coefficients, as desired.

Finally, we note that this result applies to α_n and β_n with $n = N$, but similar expansions can be obtained for $n = N \pm q$, needed in the string equations (2.19). We can rewrite

$$NV(x;s) = n\frac{V(x;s)}{t}, \qquad t = \frac{n}{N},$$

and work with the potential $V(x;s)/t$. Since t will be close to 1 when N is large and n is in the regime $n = N \pm q$ with q fixed, and all quantities depend analytically on t, we get the same kind of asymptotic expansions in the steepest descent method.

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A Asymptotic expansions for LUE and gGUE partition functions

Lemma A.1. *The partition function of the Laguerre Unitary Ensemble:*

$$Z_N^{\mathrm{LUE}} = \int_0^\infty \cdots \int_0^\infty \prod_{1\le j<k\le N} (x_k - x_j)^2 \prod_{j=1}^N x_j^\lambda e^{-Nx_j} dx_j, \tag{A.1}$$

with $\lambda > -1$, can be written as

$$Z_N^{\mathrm{LUE}} = N^{-N(N+\lambda)} \prod_{j=1}^N \Gamma(j+1)\Gamma(j+\lambda), \tag{A.2}$$

and as $N \to \infty$ we have

$$\begin{aligned} \log Z_N^{\mathrm{LUE}} = &-\frac{3}{2}N^2 + N\log N + (\log(2\pi) - 1 - \lambda)\, N + \frac{3\lambda^2+2}{6}\log N \\ &+ \frac{1+3(\lambda+1)\log(2\pi)}{6} - 2\log A - \log G(\lambda+1) \\ &+ \frac{2\lambda^3 - \lambda + 1}{12N} + \mathcal{O}(N^{-2}), \end{aligned} \tag{A.3}$$

where G is the Barnes G-function, see [17, §5.17], and

$$A = \exp\left(\frac{1}{12} - \zeta'(-1)\right) \tag{A.4}$$

is the Glaisher–Kinkelin constant, $A = 1.2824271291\ldots$

Proof. The explicit formula (A.2) is a consequence of the fact that (A.1) can be written as a Selberg integral. See [3, Theorem 2.5.8, Corollary 2.5.9], and also [27] and the monograph by Mehta [23]. Next, we rewrite (A.2) as follows:

$$Z_N^{\mathrm{LUE}} = N^{-N(N+\lambda)} \frac{G(N+2)G(N+\lambda+1)}{G(2)G(\lambda+1)} \tag{A.5}$$

again in terms of the Barnes G-function. This function has a known asymptotic expansion:

$$\begin{aligned} \log G(z+1) \sim \tfrac{1}{4}z^2 + z\log\Gamma(z+1) - \left(\tfrac{1}{2}z(z+1) + \tfrac{1}{12}\right)\log z \\ - \log A + \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}}, \qquad z \to \infty, \end{aligned} \tag{A.6}$$

see for example [17, 5.17.5]. Here B_{2k+2} are Bernoulli numbers. Replacing this asymptotic expansion in (A.5) and using MAPLE, we obtain (A.3). □

Next, we consider the generalised GUE partition function:

$$Z_N^{\mathrm{gGUE}} := \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j\, |x_j|^{\lambda} e^{-Nx_j^2} \prod_{1\le k<j\le N} (x_k - x_j)^2 \tag{A.7}$$

Lemma A.2. *For fixed $\lambda > -1$, the partition function* (A.7) *can be written as*

$$\begin{aligned} Z_N^{\mathrm{gGUE}} &= (2N)^{-N^2/2}(2\pi)^{N/2}N^{-\lambda N/2}\prod_{j=1}^{N}\frac{\Gamma(\frac{\lambda+1}{2}+\lfloor\frac{j}{2}\rfloor)}{\Gamma(\frac{1}{2}+\lfloor\frac{j}{2}\rfloor)}j! \\ &= (2N)^{-N^2/2}(2\pi)^{N/2}N^{-\lambda N/2}\frac{G(\frac{3}{2})G(\frac{1}{2})}{G(\frac{\lambda+3}{2})G(\frac{\lambda+1}{2})} \\ &\quad\times \frac{G(N+2)G(\frac{\lambda+N+3}{2})G(\frac{\lambda+N+1}{2})}{G(\frac{N+3}{2})G(\frac{N+1}{2})} \end{aligned} \tag{A.8}$$

where G is the Barnes G-function. Here $\lfloor j/2 \rfloor$ denotes the largest integer less than or equal to $j/2$, and we assumed that N is even for simplicity. As $N \to \infty$, we have

$$\begin{aligned} \log Z_N^{\mathrm{gGUE}} &= \left(-\frac{3}{4} - \frac{\log 2}{2}\right)N^2 + N\log(N) \\ &\quad + \left(\log(2\pi) - \frac{\lambda(1+\log(2))+2}{2}\right)N \\ &\quad + \frac{3\lambda^2+5}{12}\log(N) + c_0 + \frac{c_1}{N} + \mathcal{O}(N^{-2}), \end{aligned} \tag{A.9}$$

where c_0 and c_1 are explicit constants

$$c_0 = \frac{1 - 3\lambda^2 \log(2) - 12 \log A + 6(\lambda+1)\log(2\pi)}{12} + \log \frac{G(\frac{3}{2})G(\frac{1}{2})}{G(\frac{\lambda+3}{2})G(\frac{\lambda+1}{2})}$$

$$c_1 = \frac{\lambda^3 + \lambda + 1}{12}. \tag{A.10}$$

Proof. The first equality in (A.8) was obtained by Mehta and Normand in [24]. For completeness we reproduce their derivation here. The Heine identity

$$Z_N(\lambda) = N!\, D_N(\lambda), \qquad D_N(\lambda) = \det\left[\mu_{j+k}\right]_{j,k=0}^{N-1}, \tag{A.11}$$

allows us to write the partition function in terms of the Hankel determinant, which is constructed with the moments of the weight function:

$$\mu_k = \mu_k(\lambda) = \int_0^\infty x^k x^\lambda e^{-x^2}\, dx, \qquad k \geq 0. \tag{A.12}$$

Thus, the partition function (A.7) can be written as

$$Z_N^{\mathrm{gGUE}} = c_N^{(\lambda)} \det\left\{ \int_{\mathbb{R}} x^{i+j} |x|^\lambda e^{-x^2}\, dx \right\}_{i,j=0}^{N-1} = c_N^{(\lambda)} \det\left\{ \Phi_{i,j} \right\}_{i,j=0}^{N-1} \tag{A.13}$$

where $c_N^{(\lambda)} = N^{-N(N+\lambda)/2} N!$ and $\Phi_{i,j} = \Gamma((\lambda+1+i+j)/2)$ if $i+j$ is even and $\Phi_{i,j} = 0$ if $i+j$ is odd. This determinant has a 'checkerboard structure' of zeros and by elementary row and column manipulations, it can be arranged so that all $\Phi_{i,j}$ with purely even indices appear in the top-left block and $\Phi_{i,j}$ with odd indices in the bottom-right. This allows us to write (A.13) as a product

$$Z_N^{\mathrm{gGUE}} = c_N^{(\lambda)} \det\{\Phi_{2i,2j}\}_{i,j=0}^{\lfloor (N-1)/2 \rfloor} \det\{\Phi_{2i+1,2j+1}\}_{i,j=0}^{\lfloor (N-2)/2 \rfloor}. \tag{A.14}$$

The latter determinants can be computed from the simple fact that for generic $z \in \mathbb{C}$ we have

$$\det\{\Gamma(z+i+j)\}_{i,j=0}^{M} = \prod_{j=0}^{M} j!\,\Gamma(z+j) \tag{A.15}$$

which is a simple exercise to prove from, say the classical Laplace expansion of the determinant. Applying (A.15) to (A.14) shows that

$$\frac{Z_N^{\mathrm{gGUE}}}{Z_N^{\mathrm{GUE}}} = N^{-\lambda N/2} \prod_{j=1}^{N} \frac{\Gamma(\frac{\lambda+1}{2} + \lfloor \frac{j}{2} \rfloor)}{\Gamma(\frac{1}{2} + \lfloor \frac{j}{2} \rfloor)} \tag{A.16}$$

where we used that the left-hand side must equal 1 when $\lambda = 0$. The first equality in (A.8) now follows from (A.16) and the well-known formula for $Z_N^{\mathrm{GUE}} := Z_N^{\mathrm{gGUE}}|_{\lambda=0}$ (see *e.g.* [23]). The second equality in (A.8) and the asymptotics follow from the general properties and corresponding asymptotic expansion (A.6) of the Barnes G-function.

□

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