Open projections and suprema in the Cuntz semigroup

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Abstract

We provide a new and concise proof of the existence of suprema in the Cuntz semigroup using the open projection picture of the Cuntz semigroup initiated in [**12**]. Our argument is based on the observation that the supremum of a countable set of open projections in the bidual of a C*-algebra A is again open and corresponds to the generated hereditary C*-subalgebra of A.

Introduction

The Cuntz semigroup, first introduced in the late 70's by Cuntz (cf. [**9**], [**10**]), has over the years emerged as an important tool in the classification of simple C*-algebras. Motivated by the possible lack of projections Cuntz defined the semigroup $\mathrm{W}(A)$ as certain equivalence classes of positive elements in $M_\infty(A)$. More recently, the stabilized Cuntz semigroup $\mathsf{Cu}(A)$ given by classes of positive elements in $A \otimes \mathcal{K}$, has been considered. It has a more abstract category-theoretical description put forward in the remarkable work [**8**], where it is shown that $\mathsf{Cu}(A)$ can be described as equivalence classes of countably generated Hilbert modules. This description is used to establish the existence of suprema in $\mathsf{Cu}(A)$ and the continuity of the natural functor $\mathsf{Cu}(\,\cdot\,)$ from the category of C*-algebras to the category Cu. This fact has turned out to be very important and has been exploited in many other works, e.g. [**2**, **4**, **7**]. The proof in [**8**] however appears rather involved. An alternative, but still involved proof which is based on the positive element picture of $\mathsf{Cu}(A)$ can be found in [**16**]. But also this proof seems to us to take the reader away from the underlying algebraic structure that is leading to the construction of a suitable representative for the class of the supremum.

Recently a new approach to the Cuntz semigroup $\mathsf{Cu}(A)$ has been proposed in [**12**] based on the notion of open projections and a comparison theory for those projections introduced by Peligrad and Zsidó [**13**]. Note that in the stable and separable case there is a natural correspondence between open projections, hereditary subalgebras, countably generated Hilbert modules and positive elements of a given C*-algebra.

In this paper we give a proof of the existence of suprema in the Cuntz semigroup of a separable C*-algebra based on the open projection picture of $\mathsf{Cu}(A)$ which appears very natural and transparent. It stands in between the module picture and the positive element picture, and provides a constructive proof for one of the main properties of $\mathsf{Cu}(A)$, complementing the results in [**12**]. Along the way we observe that for stable algebras every class in the Cuntz semigroup can be represented by a projection in the multiplier algebra. Essentially, all we need is the very natural concept of compact subequivalence for open projections (cf. Definition 1·7) and the fact that increasing strong limits of open projections are open, a fact already known to Akemann (cf. [**1**, Proposition II.5]). However, we observe that if a family of hereditary subalgebras is directed by inclusion (equivalently, the family of open projections is increasing with respect to the usual order in the positive cone of a C*-algebra), then the hereditary subalgebra associated to the limit projection is simply the inductive limit of the system of hereditary subalgebras, where the connecting maps are the natural inclusions (Lemma 2·1).

The paper is organized as follows. In the first section we provide some background and well-known results for the Cuntz semigroup, its different definitions i.e. the positive element, open projection, Hilbert module and hereditary subalgebra picture. In Section 2 we state and prove our results aimed at the construction of suprema of arbitrary sequences of open projections and their relation with the associated hereditary subalgebras. We finish proving the existence of suprema in the Cuntz semigroup in Section 3.

1. *Open Projections and the Cuntz Semigroup*

In this section we briefly recall the definition of the Cuntz semigroup based on comparison of positive elements in a C*-algebra as well as alternative descriptions based on Hilbert modules ([**8**]), hereditary subalgebras and corresponding open projections ([**12**]). Throughout, A always denotes a separable C*-algebra. Our notation follows [**5**], to which we also refer for general background results.

DEFINITION 1·1 (Cuntz comparison of positive elements). *Let a, b be two positive elements from a C*-algebra A. We say that a is Cuntz-subequivalent to b, in symbols $a \precsim b$, if there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ such that*

$$\|x_n^* b x_n - a\| \to 0.$$

Cuntz equivalence arises as the antisymmetrization of the above pre-order relation, i.e. $a \sim b$ if and only if $a \precsim b$ and $b \precsim a$.

In the commutative setting, the Cuntz equivalence relation just defined reduces to comparison of the support of positive functions (cf. e.g. [**4**, Proposition 2.5]). Hence, equivalence classes are somehow *parametrized* by some open subset of the topological space X.

The (stabilized) Cuntz semigroup of a C*-algebra A is defined as the set of equivalence classes

$$\mathsf{Cu}(A) := (A \otimes \mathcal{K})_+ / \sim$$

equipped with the binary Abelian operation $+$ defined by

$$[a] + [b] := [a \oplus b],$$

whereas the classical Cuntz semigroup is obtained by replacing $A \otimes \mathcal{K}$ by $M_\infty(A)$. It was shown in [**8**] that $\mathsf{Cu}(A)$ belongs to a richer category than just that of Abelian monoids, also denoted by Cu, and that the natural functor $\mathsf{Cu}(\,\cdot\,)$ from C*-algebras to Cu is sequentially continuous. In fact, it is shown in [**3**] that this functor is continuous, i.e. the continuity of $\mathsf{Cu}(\,\cdot\,)$ holds for arbitrary inductive limits.

Every positive element $a \in A$ defines the hereditary subalgebra $A_a = \overline{aAa}$ and the Hilbert module $\overline{aA}$. For the class of algebras we consider, every closed hereditary subalgebra and every closed (right)-ideal in $A \otimes \mathcal{K}$ are of this form. The Cuntz semigroup can therefore also be based on equivalence classes of these objects.

1·1. *Open projections*

As it is well-known, open subsets of a compact Hausdorff space X can be characterized by (a restatement of) Urysohn's Lemma. In [**1**], Akemann used this property to generalize the notion of open subsets to non-commutative C*-algebras by replacing sets with projections, and therefore the non-commutative analogue of the above lemma leads naturally to the following.

Definition 1·2 (Open projection). *Let A be any C^*-algebra. A projection $p \in A^{**}$ is open if it is the strong limit of an increasing net of positive elements $\{a_\alpha\}_{\alpha \in I} \subseteq A_+$.*

Equivalently, a projection $p \in A^{**}$ is open if it belongs to the strong closure of the hereditary subalgebra $A_p \subseteq A$ (cf. [**1**]), where

$$A_p := pA^{**}p \cap A = pAp \cap A. \tag{1·1}$$

In accordance with [**12**], the set of all open projections in A^{**} will be denoted by $P_o(A^{**})$. These projections are in one-to-one correspondence with hereditary subalgebras.

Recall that the bidual A^{**} of a C^*-algebra can be identified with the closure in the strong operator topology of A in its universal representation. The von Neumann algebra generated by A in a specific representation is given by projecting onto the representation space. The multiplier algebra $\mathcal{M}(A)$ of A obtained as the strict closure of A acting on itself is smaller, since strict convergence implies strong convergence in every representation. In any faithful representation of A the multiplier algebra also acts faithfully which is not the case for A^{**} in general. Moreover, the projections in $\mathcal{M}(A)$ are the strict analogues of open projections in the following sense.

Proposition 1·3. *The projections in $\mathcal{M}(A)$ are those projections in A^{**} which are strict limits of increasing nets of positive elements. Thus every projection in $\mathcal{M}(A)$ is open, in particular* $\mathrm{Proj}\,(\mathcal{M}(A \otimes \mathcal{K})) \subseteq P_o((A \otimes \mathcal{K})^{**})$.

Proof. Any projection which is a strict limit of an increasing net in A is in $\mathcal{M}(A)$. On the other hand, if $P \in \mathcal{M}(A)$ is a projection, then $PAP \subseteq A$ is a hereditary subalgebra and any increasing approximate unit of PAP converges strictly to P. □

We will see below (Proposition 1·9) that if A is stable then every open projection in A^{**} is Cuntz equivalent to a projection in $\mathcal{M}(A)$.

Continuing with the topological analogy, a projection $p \in A^{**}$ is said to be *closed* if its complement $1 - p \in A^{**}$ is an open projection, and so the closure of an open projection can also be defined. To this end, observe that the supremum of an arbitrary family of open projections in A^{**} is still an open projection and, likewise, the infimum of an arbitrary family of closed projections is still a closed projection, by results in [**1**]. Therefore, the closure of an open projection $p \in A^{**}$ can be defined as

$$\overline{p} := \inf\{q^*q = q \in A^{**} \mid 1 - q \in P_o(A^{**}) \ \wedge \ p \leq q\}.$$

Let B be a C*-subalgebra of A. A closed projection $p \in A^{**}$ is said to be *compact* in B if there exists a positive contraction $b \in B^1_+$ such that $pb = p$. An important relation between open projections is the following.

Definition 1·4 (Compact containment). *Let p and q be two open projections in A^{**}. Then p is said to be compactly contained in q ($p \Subset q$ in symbols) if $\overline{p}$ is compact in A_q.*

A sequence of open projections $\{p_n\}_{n\in\mathbb{N}} \subseteq P_o(A^{**})$ is said to be *rapidly increasing* if $p_k \Subset p_{k+1}$ for any $k \in \mathbb{N}$. Compact containment is a quite restrictive relation for projections, since it requires one projection to contain the other. To make it slightly more flexible one can use of the following definition of an equivalence relation due to Peligrad and Zsidó in [**13**].

Definition 1·5 (PZ-equivalence). *Two open projections $p, q \in P_o(A^{**})$ are said to be PZ-equivalent, $p \sim_{\mathrm{PZ}} q$ in symbols, if there exists a partial isometry $v \in A^{**}$ such that*

$$p = v^*v, \qquad q = vv^*,$$

and

$$vA_p \subseteq A, \qquad v^*A_q \subseteq A.$$

Remark 1·6. It is clear from this definition that PZ-equivalence is, in general, stronger than Murray-von Neumann equivalence, although there are some cases where the two relations are known to coincide (cf. [**12**]).

Combining compact containment and PZ-subequivalence leads to what we coin compact subequivalence.

Definition 1·7 (Compact subequivalence). *Two open projections $p, q \in P_o(A^{**})$ are said to be compactly subequivalent, $p \lll q$ in symbols, if there exists $q' \Subset q$ such that $p \sim_{\mathrm{PZ}} q'$.*

Observe that the usual compact containment relation $\Subset$ is a special instance of compact subequivalence $\lll$.

As presented in [**12**], $\mathsf{Cu}(A)$ can also be described using open projections. If p, q are open projections in A^{**}, then p is said to be Cuntz-subequivalent to q, in symbols $p \precsim q$, or sometimes also $p \precsim_{\mathrm{Cu}} q$, if for every open projection $p' \Subset p$ there exists an open projection $q' \Subset q$ such that $p' \sim_{\mathrm{PZ}} q'$.

As shown by [**12**, Theorem 6.1] it turns out that Cuntz comparison of positive elements coincides with Cuntz comparison of the corresponding open support projections, namely

$$\mathsf{Cu}(A) \cong P_o((A \otimes \mathcal{K})^{**})/\sim_{\mathrm{Cu}},$$

as ordered Abelian semigroups. Fixing notation, we will denote by $[p]$ the Cuntz class of the open projection $p \in P_o((A \otimes \mathcal{K})^{**})$ in $\mathsf{Cu}(A)$.

Remark 1·8. Given open projections $p, p', q \in P_o((A \otimes \mathcal{K})^{**})$ such that $p \lll q$, and $p' \sim_{\mathrm{PZ}} p$, then $p' \lll q$ by definition. However, as we will point out below when discussing Hilbert modules $p' \sim_{\mathrm{Cu}} p$ does not always imply $p' \sim_{\mathrm{PZ}} p$. Thus it is not clear whether the weaker assumption $p \lll q$, and $p' \sim_{\mathrm{Cu}} p$ always implies $p' \lll q$.

The following discussion, summarized in Proposition 1·9, shows that every class in $\mathsf{Cu}(A)$ can be represented by a projection in $\mathcal{M}(A \otimes \mathcal{K})$. For this we will need to use the

Hilbert module picture for $\mathsf{Cu}(A)$ (see [**4**] for further details) and Kasparov's stabilization theorem.

Following [**8**], a Hilbert module E is compactly contained in a Hilbert module F, written $E \subset\subset F$, if there exists a positive element x in the compact operators $\mathcal{K}(F)$ on F such that $x\xi = \xi$ for all $\xi \in E$. Moreover, we denote by $\mathcal{L}(E)$ the algebra of adjointable operators on E (which is a C*-algebra) and by $\mathcal{L}(E, F)$ the Banach space of bounded adjointable operators from E to F. In this context, E is Cuntz-subequivalent to F, written $E \precsim_{\mathrm{Cu}} F$, if for every compactly contained Hilbert submodule $E' \subset\subset E$ there exists $F' \subset\subset F$ with $E' \cong F'$. As mentioned before, $\mathsf{Cu}(A)$ can be defined as equivalence classes of countably generated Hilbert modules under the equivalence relation $E \sim_{\mathrm{Cu}} F$ if $E \precsim_{\mathrm{Cu}} F$ and $F \precsim_{\mathrm{Cu}} E$. Note that isomorphic Hilbert modules are in particular Cuntz equivalent (cf. [**12**, Proposition 4.3]).

Recall that Kasparov's theorem implies that every countably generated Hilbert module over a C*-algebra A is a submodule of

$$\ell^2(A) = \left\{ (a_n) \in A^{\mathbb{N}} \;\middle|\; \sum_{n=1}^{\infty} a_n^* a_n \text{ converges in norm} \right\},$$

and that there is a natural isomorphism between the set of compact operators $\mathcal{K}(\ell^2(A))$ (the span of the 'rank one operators' $\Theta_{\xi,\eta}$, where $\xi, \eta \in \ell^2(A)$ and $\Theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle$) and $A \otimes \mathcal{K}$. Moreover, one also has an isomorphism between $\mathcal{M}(A \otimes \mathcal{K})$ and the A-linear bounded and adjointable operators on $\ell^2(A)$, denoted, as usual, by $\mathcal{L}(\ell^2(A))$. In this setting, an open projection $p \in P_o(A^{**})$ determines a right Hilbert module over A by $E_p := \overline{aA}$, where a is any positive element of $\mathcal{M}(A)$ whose support projection is p. Clearly $\overline{E_p E_p^*} = \overline{aAa} = A_p$

In connection with Remark 1·8, this Hilbert module picture of $\mathsf{Cu}(A)$ is useful to provide an example of open projections which are Cuntz equivalent but not PZ-equivalent. Indeed, consider the two non-isomorphic Hilbert A-modules (denoted by $E_p, E_{p'}$) that determine the same Cuntz class in $\mathsf{Cu}(A)$ described in the counterexample of [**6**]. Then, by [**12**, Proposition 4.3], it follows that the open projections associated to each of these Hilbert A-modules are not PZ-equivalent ($p \not\sim_{\mathrm{PZ}} p'$). Hence, it could happen that there exists $q' \subset\subset q$ such that $q' \sim_{\mathrm{PZ}} p$, but there is no $q'' \subset\subset q$ such that $q'' \sim_{\mathrm{PZ}} p'$, though we have no specific example.

Similarly to the above framework, the Cuntz semigroup can also be described by classes of hereditary subalgebras of $A \otimes \mathcal{K}$. To this end, given a countably generated Hilbert module E, we associate to it any of the hereditary subalgebras of $A \otimes \mathcal{K}$ isomorphic to $\mathcal{K}(E)$. Conversely, given $p \in P_o((A \otimes \mathcal{K})^{**})$ one checks that $(A \otimes \mathcal{K})_p$ is isomorphic to $\mathcal{K}(E_p)$, by sending $aa_1a_2^*a \in (A \otimes \mathcal{K})_p$ to $\Theta_{aa_1,aa_2} \in \mathcal{K}(E_p)$, where a is any element in $(A \otimes \mathcal{K})_+$ whose support projection is p and $a_1, a_2 \in A \otimes \mathcal{K}$.

Combining both pictures described with Kasparov's stabilization theorem, we have

$$E_p \oplus \ell^2(A \otimes \mathcal{K}) \cong \ell^2(A \otimes \mathcal{K}),$$

for any $p \in P_o((A \otimes \mathcal{K})^{**})$. Fixing $\phi \in \mathcal{L}(E_p \oplus \ell^2(A \otimes \mathcal{K}), \ell^2(A \otimes \mathcal{K}))$ to be one of such isomorphisms, let $\tilde{P}$ be the projection onto E_p in the above orthogonal decomposition. Then, $P := \phi\tilde{P}\phi^{-1}$ is a projection in $\mathcal{L}(\ell^2(A \otimes \mathcal{K})) \cong \mathcal{M}(A \otimes \mathcal{K} \otimes \mathcal{K}) \cong \mathcal{M}(A \otimes \mathcal{K})$ with $E_P \cong E_p$, i.e. p and P determine isomorphic Hilbert modules. Thus, by [**12**, Proposition 4.3], the original open projection $p \in P_o((A \otimes \mathcal{K})^{**})$ is PZ-equivalent to the projection $P \in \mathcal{M}(A \otimes \mathcal{K})$. The choice of P depends on the isomorphism ϕ described above. However, all isomorphisms induce Cuntz equivalent projections in $\mathcal{M}(A\otimes\mathcal{K})$. Furthermore, $I-P$ is also open, with $(A\otimes\mathcal{K})_{I-P}\cong A\otimes\mathcal{K}$ and $I-P$ is Murray-von Neumann equivalent in $\mathcal{M}(A\otimes\mathcal{K})$ (in particular PZ-equivalent) to I. We say that P has large complement in this case.

Proposition 1·9. *Let A be any C^*-algebra. Every class $[p]\in\mathsf{Cu}(A)$, for $p\in P_o((A\otimes\mathcal{K})^{**})$, has a representative, denoted by P, in the set of projections in $\mathcal{M}(A\otimes\mathcal{K})$. Hence, $\mathsf{Cu}(A)$ can be thought of as the set of Cuntz equivalence classes of projections from $\mathcal{M}(A\otimes\mathcal{K})$. Moreover, p and P are PZ-equivalent, i.e. $p\sim_{\mathrm{PZ}}P$.*

Thus

$$\mathsf{Cu}(A)\cong\mathrm{Proj}\,(\mathcal{M}(A\otimes\mathcal{K}))/\sim_{\mathsf{Cu}}$$

with the Cuntz subequivalence relation we had in $P_o((A\otimes\mathcal{K})^{**})$, which a priori involves open projections not necessarily in $\mathcal{M}(A\otimes\mathcal{K})$.

By virtue of this last proposition, given an open projection $p\in P_o((A\otimes\mathcal{K})^{**})$, we will often use a capital P to denote one of their Cuntz equivalent projections in $\mathcal{M}(A\otimes\mathcal{K})$. Namely, given p, we fix an isomorphism given by Kasparov's theorem, and P is the orthogonal projection onto E_p in $E_p\oplus\ell^2(A\otimes\mathcal{K})\cong\ell^2(A\otimes\mathcal{K})$.

We finish this section by providing the main result that stems from our new equivalent characterization of $\mathsf{Cu}(A)$, Corollary 1·12. It shows the existence of a unitary element in $\mathcal{M}(A\otimes\mathcal{K})$ that implements the Cuntz subequivalence between two projections in $\mathcal{M}(A\otimes\mathcal{K})$. This is the analogue of the crucial result [**15**, Proposition 2.4] for stable algebras, where the existence of such unitary is shown under the stable rank one assumption. Before that, we need the following slight refinement of Kasparov's stabilization theorem. For a set $S\subseteq\mathbb{N}$ let

$$\ell^2(S,A)=\left\{(a_n)\in A^S\ \Big|\ \sum_{n\in S}a_n^*a_n \text{ converges in norm}\right\}.$$

Lemma 1·10. *Let $E\subseteq F$ be an inclusion of countably generated Hilbert modules over A and $S\subseteq\mathbb{N}$ infinite with infinite complement. Then there exists a unitary $U:\ell^2(\mathbb{N},A)\to\ell^2(\mathbb{N},A)\oplus F$ such that $U_{|\ell^2(S,A)}$ is a unitary mapping onto $\ell^2(S,A)\oplus E$.*

Proof. This is an easy modification of the standard proof of Kasparov's stabilization Theorem (e.g. [**5**, Theorem 13.6.2]) which we briefly indicate for convenience. By adjoining a unit we may assume that A is unital, so that $\ell^2(\mathbb{N},A)$ has the canonical basis (e_n). Let (η_n) be a bounded sequence of generators of F (e.g. a dense sequence in the unit ball) such that every sequence member appears infinitely often, and let $(\eta_j)_{j\in S}$ generating E such that each generator appears infinitely often too. Then the polar decomposition $U|T|$ of $T:\ell^2(\mathbb{N},A)\to\ell^2(\mathbb{N},A)\oplus F$ given by $T=\sum_n 2^{-n}\Theta_{2^{-n}e_n\oplus\eta_n,e_n}$ provides the required unitary U. □

Note that we cannot show with this construction that E is complemented in F (which is false in general), since the projections onto F and $\ell^2(S,A)$ do not commute in general.

Corollary 1·11. *Let E, F be countably generated Hilbert A-modules and let $v\in\mathcal{L}(E,F)$ be an isometry. Then there exists a unitary $u\in\mathcal{L}(E\oplus\ell^2(A),F\oplus\ell^2(A))\cong\mathcal{L}(\ell^2(A))\cong\mathcal{M}(A\otimes\mathcal{K})$ extending v.*

Proof. We may assume that v is an inclusion. By Lemma 1·10 we can extend an inclusion to the canonical inclusion $\ell^2(S,A) \hookrightarrow \ell^2(\mathbb{N},A)$, corresponding to the inclusion $S \hookrightarrow \mathbb{N}$. By adding another copy of $\ell^2(\mathbb{N},A)$ we can extend this inclusion to the required unitary of $\ell^2(A)$. □

Applying this to $A \otimes \mathcal{K}$ and using that $A \otimes \mathcal{K} \otimes \mathcal{K} \cong A \otimes \mathcal{K}$ we obtain the following.

COROLLARY 1·12. *Let P, Q be projections in $\mathcal{M}(A \otimes \mathcal{K})$ with $P \ll Q$, i.e. there exist $Q' \Subset Q$ and a partial isometry $v \in (A \otimes \mathcal{K})^{**}$ such that $P = v^*v, Q' = vv^*$, and $v(A \otimes \mathcal{K})_P \subseteq A \otimes \mathcal{K}, v^*(A \otimes \mathcal{K})_{Q'} \subseteq A \otimes \mathcal{K}$. Suppose moreover that P and Q have large complements (i.e. $I - P$ and $I - Q$ are both Murray-von Neumann equivalent to I in $\mathcal{M}(A \otimes \mathcal{K})$). Then there exists a unitary $u \in \mathcal{M}(A \otimes \mathcal{K})$ such that $uPu^* = Q'$.*

Proof. By [**12**, Proposition 4.3], because $P \sim_{\mathrm{PZ}} Q'$, one has that the Hilbert modules associated to P and Q' are isomorphic. Hence, we may regard v as an isometry from the Hilbert module E_P into E_Q. Applying Lemma 1·11, one obtains the unitary $u : E_P \oplus \ell^2(A) \to E_Q \oplus \ell^2(A)$ such that $uPu^* = Q'$, as desired. □

Remark 1·13. With the same notation as in Corollary 1·12, if $P \precsim Q$ and both have large complements, then for all $P' \Subset P$ there exists a unitary $u \in \mathcal{M}(A \otimes \mathcal{K})$ such that $uP'u^* \Subset Q$.

Note that a similar statement applies to sequences of open projections satisfying either $P_1 \ll P_2 \ll P_3 \ll \dots$ or $P_1 \precsim P_2 \precsim P_3 \precsim \dots$.

2. *Hereditary Subalgebras and Open Projections*

In this section we establish the hereditary C*-subalgebra analogue of the operation of taking suprema of countably many open projections in $P_o(A^{**})$. We start by observing that, given two open projections $p, q \in A^{**}$ such that $p \leq q$ (as positive elements), then q obviously acts as a unit on p, and $A_p \subseteq A_q$ (cf. [**12**, §4.5]). This property will be extensively used throughout this paper. The following very natural Lemma might be well-known to experts. Since we have not been able to find a proof in the literature we provide one.

LEMMA 2·1. *Let $\{p_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open projections in A^{**}. Then*

$$A_p = \overline{\bigcup_{k\in\mathbb{N}} A_{p_k}},$$

where $p := \mathrm{SOT}\lim_{n\to\infty} p_n$.

Proof. Let B denote the inductive limit on the right side which coincides with the union. By construction B is a hereditary subalgebra of A, and therefore, there exists a generator $a \in B$ such that $B = \overline{aAa}$ (recall that we assume A to be separable). It is then enough to show that the support projection $q \in A^{**}$ of a coincides with p. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive elements converging to a in norm such that $a_n \in A_{p_n}$ for any $n \in \mathbb{N}$. Let q be the support projection of a and q_n be the support projection of a_n for any $n \in \mathbb{N}$. It is clear that $q_n \leq p_n \leq q$ for any $n \in \mathbb{N}$ from which it follows that

$$\sup\{q_n\}_{n\in\mathbb{N}} \leq \mathrm{SOT}\lim_{n\to\infty} p_n \leq q.$$

Now suppose that q' is an open projection such that $q_n \leq q'$ for any $n \in \mathbb{N}$. This implies that $a_n q' = q' a_n = a_n$ for any $n \in \mathbb{N}$ and so $aq' = q'a = a$. Therefore, $q \leq q'$, which leads to $q = \sup\{q_n\}_{n\in\mathbb{N}}$, whence $p = q$. $\square$

A similar result, using the positive element picture, can be found in [**7**, Lemma 4.2], where this result is used to prove that the Cuntz semigroup of any C*-algebra of stable rank 1 admits suprema.

COROLLARY 2·2. *Let $\{p_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open projections in A^{**}. Then*

$$\overline{A_p}^{\text{SOT}} = \overline{\bigcup_{k\in\mathbb{N}} \overline{A_{p_k}}^{\text{SOT}}}^{\text{SOT}},$$

where $p := \text{SOT}\lim_{n\to\infty} p_n$.

Proof. From Lemma 2·1 one has $\overline{A_p}^{\text{SOT}} = \overline{\bigcup_{n\in\mathbb{N}} A_{p_n}}^{\text{SOT}}$. Therefore, by using that $\bigcup_{n\in\mathbb{N}} \overline{A_{p_n}}^{\text{SOT}} \subseteq \overline{\bigcup_{n\in\mathbb{N}} A_{p_n}}^{\text{SOT}}$ and $A_p \subseteq \overline{\bigcup_{n\in\mathbb{N}} \overline{A_{p_n}}^{\text{SOT}}}^{\text{SOT}}$, it follows that

$$A_p \subseteq \overline{\bigcup_{n\in\mathbb{N}} \overline{A_{p_n}}^{\text{SOT}}}^{\text{SOT}} \subseteq \overline{\bigcup_{n\in\mathbb{N}} A_{p_n}}^{\text{SOT}} = \overline{A_p}^{\text{SOT}}.$$

$\square$

The result that now follows is an example of an application of Lemma 2·1. The construction of the supremum, i.e. the join of an arbitrary family of projections in the bidual A^{**} of a C*-algebra A can be carried out by relying on the lattice structure on the set of projections in A^{**}. In the case of an *increasing* sequence of projections, Lemma 2·1 shows that the hereditary C*-subalgebra associated to the supremum coincides with the inductive limit of the increasing sequence of hereditary C*-subalgebras associated to each projection in the considered subset of $P_o(A^{**})$. For the general case we then have the following

PROPOSITION 2·3. *Let $\{p_n\}_{n\in\mathbb{N}} \subseteq P_o(A^{**})$ be an arbitrary sequence of open projections in A^{**}, and let $p := \sup\{p_n\}_{n\in\mathbb{N}}$. Then*

$$A_p = \bigvee_{n\in\mathbb{N}} A_{p_n},$$

i.e. A_p coincides with the hereditary C-subalgebra of A generated by the family of hereditary C*-subalgebras $\{A_{p_n} \mid n \in \mathbb{N}\}$.*

Proof. Consider the new sequence of open projections $\{q_n\}_{n\in\mathbb{N}}$ defined by

$$q_1 := p_1, \qquad q_{n+1} := q_n \vee p_{n+1},$$

for any $n \in \mathbb{N}$. This clearly defines an increasing sequence of open projections, and moreover $p := \sup\{p_n\}_{n\in\mathbb{N}} = \text{SOT}\lim_{n\to\infty} q_n$. Therefore, using Lemma 2·1, one has the identification $A_p = \overline{\bigcup_{k\in\mathbb{N}} A_{q_k}}$.

By definition, A_{p_k} is clearly contained in A_{q_k} for any $k \in \mathbb{N}$, so $\bigvee_{k\in\mathbb{N}} A_{p_k} \subseteq \overline{\bigcup_{k\in\mathbb{N}} A_{q_k}}$. On the other hand, A_{q_k} is contained in $\bigvee_{n\in\{1,\dots,k\}} A_{p_n}$, so $\overline{\bigcup_{k\in\mathbb{N}} A_{q_k}} \subseteq \bigvee_{k\in\mathbb{N}} A_{p_k}$, which shows equality. $\square$

3. *Suprema in the Cuntz semigroup*

In this section we show that the existence of suprema in the stabilized Cuntz semigroup can be proven by just referring to the open projection picture, using the results discussed in the previous sections.

LEMMA 3·1. *Let p be the strong limit of an increasing sequence of open projections $p_1 \leq p_2 \leq \cdots$. Then, for every $q \Subset p$, there is an $n \in \mathbb{N}$ and an open projection $q' \Subset p_n$ such that $q \sim_{\mathrm{PZ}} q'$.*

Proof. By the definition of the relation $q \Subset p$ there exists a positive element a in the unit ball of A_p such that $\overline{q}a = \overline{q}$, and by the same argument as in [**8**] (cf. [**4**, Proposition 4.11]), one can find $a' \in C^*(a)$ and $\epsilon > 0$ such that $\overline{q}(a' - \epsilon)_+ = \overline{q}$.

Let $a_n \in A_{p_n}$ be such that $\|a_n - a'\| < \epsilon$, which exists by Lemma 2·1. By [**11**, Lemma 2.2] there is a contraction $d \in A_p$ such that $da_nd^* = (a' - \epsilon)_+$, and it follows by [**13**, Theorem 1.4] that

$$\overline{q} \leq p_{x^*x} \sim_{\mathrm{PZ}} p_{xx^*} \leq p_n,$$

where $x = a_n^{1/2}d^*$.

Since $\leq$ and $\sim_{\mathrm{PZ}}$ are special instances of $\precsim_{\mathsf{Cu}}$ and $\sim_{\mathsf{Cu}}$ respectively, using [**12**, Proposition 4.10] one also has

$$q \Subset p_{x^*x} \sim_{\mathsf{Cu}} p_{xx^*} \precsim_{\mathsf{Cu}} p_n.$$

Therefore there must exist an open projection $q' \Subset p_n$ such that $q \sim_{\mathrm{PZ}} q'$. $\square$

PROPOSITION 3·2. *If $p_1 \Subset p_2 \Subset \cdots$ is a rapidly increasing sequence of open projections in $P_o(A^{**})$, then $p_1 \leq p_2 \leq \cdots$ and*

$$\sup[p_n] = [\mathrm{SOT}\lim_{n\to\infty} p_n].$$

Proof. Let p be the strong limit of the p_ns. This, as explained before is still an element in $P_o(A^{**})$, so let us show that it is the least upper (Cuntz) bound. Suppose that $[q]$ is such that $[p_n] \leq [q]$ for any $n \in \mathbb{N}$. By Lemma 3·1, for every $p' \Subset p$ there is an $n \in \mathbb{N}$ and an open projection q' such that $p' \sim_{\mathrm{PZ}} q' \Subset p_n$. But, since $[p_n] \leq [q]$, there exists a $q'' \Subset q$ such that $q'' \sim_{\mathrm{PZ}} q'$. Therefore, $[p] \leq [q]$. Since $[q]$ is arbitrary, it follows that $[p] = \sup[p_n]$. $\square$

To prove that every increasing sequence, in the Cuntz sense, has a supremum in the open projection picture one of course needs a more general result. If one naïvely tries to tackle this problem inside A directly, one runs into the following problem. Let $\{p_n\}_{n\in\mathbb{N}}$ be any sequence of open projections in $P_o(A^{**})$ with the property that $p_n \lll p_{n+1}$ for every $n \in \mathbb{N}$. By assumption there are open projections $\{q_n\}_{n\in\mathbb{N}}$ such that $q_k \Subset p_{k+1}$ and $p_k \sim_{\mathrm{PZ}} q_k$. These determine an inductive sequence $(A_{p_k}, \phi_k)_{k\in\mathbb{N}}$ of hereditary subalgebras of A, where the connecting maps are given by the adjoint action of partial isometries $\{v_n\}_{n\in\mathbb{N}}$ satisfying $p_k = v_k^*v_k$, $q_k = v_kv_k^*$ and $vA_{p_k} \subseteq A$, $v^*A_{q_k} \subseteq A$, i.e. $\phi_k(a) = v_k^*av_k$ for any $a \in A_{p_k}$. Denoting by $\tilde{A}$ the inductive limit of such a sequence, one gets maps $\{\rho_n\}_{n\in\mathbb{N}}$ that make the following diagram

$$\begin{array}{ccc} A_{p_k} & \xrightarrow{\phi_k} & A_{p_{k+1}} \\ & \searrow^{\rho_k} & \downarrow{\scriptstyle \rho_{k+1}} \\ & & \tilde{A} \end{array}$$

commutative. But unless $\tilde{A} \subseteq A$, nothing more can be said about this sequence to conclude the existence of the supremum of $\{p_n\}_{n\in\mathbb{N}}$. Indeed, using [**14**, Example 1], one may show that $\tilde{A}$ is not always a subalgebra of A. To see this, let p, q be the corresponding open projections associated to the two Cuntz equivalent Hilbert modules, which do not embed one into the other, described in [**14**, Example 1]. Without loss of generality assume that p is the unit of A^{**}. Now, choose two rapidly increasing sequences of open projections (p_i) and (q_i) that converge to p and q respectively, and isomorphisms $\phi_i : A_{p_i} \to A_{q_i}$. Composing ϕ_i with the embedding $\iota_i : A_{q_i} \to A_{q_{i+1}}$ and ϕ_{i+1}^{-1}, we get a $\phi_{i+1}^{-1} \circ \iota_i \circ \phi_i : A_{p_i} \to A_{p_{i+1}}$. Taking the inductive limit with these maps, we get an algebra isomorphic to A_q which however does not embed in $A_p = A$, though all A_{p_i} are hereditary subalgebras of A.

On the other hand, by working with $A \otimes \mathcal{K}$ instead of A, one can extend the above partial isometries into unitaries (Corollary 1·12), in order to construct a *tower* rather than a *tunnel*.

LEMMA 3·3. *Every sequence $\{p_n\}_{n\in\mathbb{N}}$ of open projections in $P_o(A \otimes \mathcal{K})^{**}$ with the property that $p_n \prec\!\!\prec p_{n+1}$ for every $n \in \mathbb{N}$ has a supremum in $\mathsf{Cu}(A)$.*

Proof. By Proposition 1·9, one can replace the given sequence by one made of PZ-equivalent projections in $\mathcal{M}(A\otimes\mathcal{K})$ satisfying the same compact subequivalence relation and having large complements. Let us denote, as usual, this new sequence by $\{P_n\}$. Then, by Corollary 1·12, there exist a collection of unitaries $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(A \otimes \mathcal{K})$ such that $u_{k-1}P_{k-1}u_{k-1}^* = Q_k$ for all $k \in \mathbb{N}$. Hence, $P_{k-1} = u_{k-1}^* Q_k u_{k-1} \Subset u_{k-1}^* P_k u_{k-1}$ and therefore one has that

$$P_1 \Subset u_1^* P_2 u_1 \Subset u_1^* u_2^* P_3 u_2 u_1 \Subset u_1^* u_2^* u_3^* P_4 u_3 u_2 u_1 \Subset \cdots .$$

Denoting by $\overline{P_i} := (\prod_{i=1}^{n-1} u_i)^* P_n (\prod_{i=1}^{n-1} u_i)$, let $P := \textsc{sot}\lim_{n\to\infty} \overline{P_i}$. By Proposition 3·2 it follows that $[P] = \sup[\overline{P}_n]$ which implies that $[P] = \sup[p_i]$ since $[p_i] = [P_i] = [\overline{P_i}]$. □

Remark 3·4. The above could also be proven from the hereditary subalgebras point of view. In this case, using the same notation as in the above proof, one has that

$$A_{P_1} \subseteq u_1^* A_{P_2} u_1 \subseteq u_1^* u_2^* A_{P_3} u_2 u_1 \subseteq \ldots \subseteq (\prod_{i=1}^{n-1} u_i)^* A_{P_n} (\prod_{i=1}^{n-1} u_i) \subseteq \ldots \quad ,$$

where they belong to $A \otimes \mathcal{K}$ since $\{u_n\}_{n\in\mathbb{N}}$ are unitaries in $\mathcal{M}(A \otimes \mathcal{K})$ and A_{P_n} are hereditary subalgebras of $A \otimes \mathcal{K}$.

Denoting by P the open projection associated to the hereditary subalgebra

$$A_P = \overline{\bigcup_{n=1}^{\infty} (\prod_{i=1}^{n-1} u_i)^* A_{P_n} (\prod_{i=1}^{n-1} u_i)},$$

it follows that $[P] = \sup[P_n]$.

The above is an intermediate step towards the more general proof of the existence of suprema for arbitrary Cuntz-increasing sequences of open projections in $A \otimes \mathcal{K}$.

THEOREM 3·5. *Every Cuntz-increasing sequence $\{p_n\}_{n\in\mathbb{N}}$ of projections in $P_o(A \otimes \mathcal{K})^{**}$ admits a supremum in $\mathsf{Cu}(A)$.*

Proof. Without loss of generality we may assume that A is a stable C*-algebra.

By assumptions there are positive contractions $\{a_{n,k}\}_{n,k\in\mathbb{N}} \subseteq A_+^1$ such that $p_n = \textsc{sot}\lim_{k\to\infty} a_{n,k}$ and $a_{n,k} \leq a_{n,k+1}$ for any $k, n \in \mathbb{N}$.

These elements can be modified to yield rapidly increasing sequences of positive elements by setting

$$a'_{n,k} := \left(a_{n,k} - \tfrac{1}{k}\right)_+ .$$

Denoting by $q_{n,k}$ the support projections associated to these new elements $a'_{n,k}$, one has $q_{n,k} \Subset q_{n,k+1}$ for any $k, n \in \mathbb{N}$. Now, starting with e.g. $q_{1,1}$ and applying Lemma 3·1 to $q_{1,1} \Subset q_{1,2} \Subset p_1 \precsim p_2$, one gets $m_1 \in \mathbb{N}$ and $q_{1,1} \Subset p_{2,m_1}$ such that $q_{1,1} \sim_{\mathrm{PZ}} q'_{1,1}$. By iterating these steps one can construct a sequence of open projections $q_k := q_{k,m_{k-1}}$ that satisfies

$$q_1 \sim_{\mathrm{PZ}} q'_{1,1} \Subset q_2 \sim_{\mathrm{PZ}} q'_{2,m_1} \Subset q_3 \cdots ,$$

i.e.

$$q_1 \lll q_2 \lll q_3 \lll q_4 \lll \cdots .$$

Observe that $[q_k] \leq [p_k]$ for any $k \in \mathbb{N}$, and that for any $n, m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $[q_{n,m}] \leq [q_l]$. Therefore, $[p_n] \leq \sup_k [q_k] \leq \sup_k [p_n]$, which implies

$$\sup_n [p_n] \leq \sup_k [q_k] \leq \sup_n [p_n], \quad \text{i.e.} \quad \sup_n [p_n] = \sup_k [q_k].$$

The existence of the supremum follows from Lemma 3·3. □

En passant we observe that we have the following realization for suprema in the Cuntz semigroup.

Corollary 3·6. *Every element $[p] \in \mathsf{Cu}(A)$ can be written as the Cuntz class of the strong limit of an increasing sequence of projections in $\mathcal{M}(A \otimes \mathcal{K})$.*

Proof. Let p be an open projection in $P_o(A \otimes \mathcal{K})^{**}$. By assumptions there are positive contractions $\{a_n\}_{n\in\mathbb{N}} \subseteq A_+^1$ such that $p = \textsc{sot}\lim_{k\to\infty} a_n$ and $a_n \leq a_{n+1}$ for any $n \in \mathbb{N}$. Modify these positive elements to yield a rapidly increasing sequence, as done in proof of Theorem 3·5, and denote by p_n the support projections associated to the elements of this rapidly increasing sequence. In particular, one has that $p_n \lll p_{n+1}$.

Now, in the same fashion as in the proof of Lemma 3·3, one obtains the desired sequence of projections in $\mathcal{M}(A \otimes \mathcal{K})$ such that its strong limit is Cuntz equivalent to the strong limit of $\{p_n\}$, i.e. the initial open projection p. □

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