Complete Noncompact CMC Surfaces in Hyperbolic 3-Space

by

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Contents

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Declarations

I declare that the work presented here is entirely my own except where acknowledged in the text. This thesis has not been submitted for a degree at another university.

Abstract

In this thesis we study the asymptotic Plateau problem for surfaces with constant mean curvature (CMC) in hyperbolic 3-space $\mathbb{H}^3$. We give a new, geometrically transparent proof of the existence of a CMC surface spanning any given Jordan curve on the sphere at infinity of $\mathbb{H}^3$, for mean curvature lying in the range (-1,1). Our proof does not require methods from geometric measure theory, and yields an immersed disk as solution. We then study the dependence of the solution surface on the boundary data. We view the set of H-surfaces (CMC surfaces with mean curvature equal to H) as consisting of the conformal H-harmonic maps. We therefore begin by showing smooth dependence on boundary data for H-harmonic maps (with $|H| < 1$) which solve a Dirichlet problem at infinity. This is achieved by showing that the linearised H-harmonic map operator is invertible as a map between appropriate function spaces. Finally we show smooth dependence on boundary data for H-surfaces which lie in a neighbourhood of the totally umbilic spherical caps $\{\Sigma_H\}$. This is achieved by studying the mapping properties of the so-called conformality operator. We use methods from complex geometry to show that the linearisation of this operator at a cap Σ_H is an isomorphism for all $H \in (-1, 1)$.

For Mum, Dad, Elise, Matthew, Lorraine and Peter.

Chapter Breakdowns and Preliminaries

Chapter Breakdowns

Chapter 1 is introductory in nature. In it we survey the relevant results on constant mean curvature (CMC) surfaces in the literature and place our new results in context. We identify the role played by Yau's isoperimetric inequality for negatively curved manifolds in the existence theory for CMC surfaces in such manifolds. Finally we describe what we believe to be some flaws in a paper which has some overlap with the present work.

In Chapter 2 we solve the (parametric) asymptotic Plateau problem for constant mean curvature surfaces, using purely PDE methods. Our approach yields an immersion of the unit disk as a solution.

In Chapter 3 we study the space of H-harmonic maps between hyperbolic 2-space $\mathbb{H}^2$ and hyperbolic 3-space $\mathbb{H}^3$ that solve a Dirichlet problem at infinity. Using an implicit function theorem-type argument we prove a perturbation result for such maps.

In Chapter 4 we prove a perturbation result for complete, noncompact CMC surfaces in $\mathbb{H}^3$ whose asymptotic boundaries lie in a neighborhood of the unit circle.

Preliminaries

We now set some sign conventions and give some basic definitions. Let (N, h) denote a Riemannian n-manifold, with associated Levi-Civita connection d^∇, and covariant derivative ∇. Let $\Sigma \subset N$ be an immersed hypersurface in N. At a point $p \in \Sigma$ we define the second fundamental form of Σ to be the symmetric bilinear mapping $A : T_p\Sigma \times T_p\Sigma \to (T_pN)^\perp$ defined by

$$A(X, Y) := (\nabla_X Y)^\perp,$$

for X, $Y \in T_p\Sigma$, and where $\perp$ denotes projection onto the orthogonal complement of $T_p\Sigma$ in T_pN. We define the vector valued mean curvature of Σ in M at p to be

$$\vec{H}(p) := -\frac{1}{n-1}\mathrm{tr}A,$$

where the trace is taken with respect to the restriction of the metric h to the subbundle $T\Sigma$. The (scalar valued) mean curvature $H(p)$ of Σ is defined by the relation $\vec{H}(p) = H(p)n(p)$, where $n(p)$ is a unit normal to Σ at p. Σ is said to have *constant mean curvature* (from here onwards abbreviated to *CMC*) equal to $H \in \mathbb{R}$ if $H(p) = H$ for all $p \in \Sigma$.

Remark 0.1. With this definition of mean curvature, the geodesic spheres in hyperbolic 3-space have constant mean curvature strictly *greater* than 1 with respect to the *outward* pointing unit normal.

We adopt the convention that the curvature tensor $\mathcal{R}$ of (N, h) at a point $q \in N$ is given by

$$\mathcal{R}(X, Y)Z := \nabla_Y\nabla_X Z - \nabla_X\nabla_Y - \nabla_{[Y,X]}Z,$$

for $X, Y, Z \in T_qN$. Then for two linearly independent tangent vectors X and Y in T_qN, the sectional curvature of the 2-plane spanned by X and Y is given by

$$K(X, Y) := \frac{g(\mathcal{R}(X, Y)X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

We set the rough Laplacian Δ associated to d^∇ to be

$$\Delta Z := \mathrm{tr}_h\nabla^2 Z,$$

where ∇^2 is the second covariant derivative, defined by $\nabla^2 Z(X, Y) := \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z$. Thus Δ has negative spectrum, and this will be true of all our Laplace operators (i.e. they will all be “geometer’s Laplacians”).

Now consider a smooth map $u : (M, g) \to (N, h)$ between Riemannian manifolds, where $\dim M = m$, $\dim N = n$. Let M have local coordinates $x = (x^1, \ldots, x^m)$. The energy density of u is defined to be the function on M given by

$$e(u)(x) := \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij}(x) h_{\alpha\beta}(u(x)) \frac{\partial u^\alpha}{\partial x^i}(x) \frac{\partial u^\beta}{\partial x^j}(x).$$

The tension field of u is the smooth section of the pullback bundle u^*TN defined by

$$\tau(u)(x) := \mathrm{tr}_g d^{\tilde{\nabla}} du,$$

where $d^{\tilde{\nabla}}$ is the induced connection on $T^*M \otimes u^*TN$.

Chapter 1

CMC Immersions and the Isoperimetric Inequality

We begin by motivating the work carried out in this thesis, and placing our new results in context. We revisit the solution for the Plateau problem for CMC surfaces on the interior of a negatively curved Riemannian manifold and show how Yau's [52] isoperimetric inequality for such manifolds lies at the heart of the existence result. Apart from describing the relevant known results in the literature we also provide a critique of a paper [10] which has some overlap with our present work, but appears to contain a number of serious flaws.

1.1 The Basic Models

The canonical examples of complete CMC surfaces in $\mathbb{H}^3$ can be grouped into four families which we now describe. As a unifying principle we shall view each family as consisting of the level sets of a specific function. We presently have the unit ball model of $\mathbb{H}^3$ in mind.

We begin with the compact surfaces: these are the geodesic spheres, the level sets of the distance function to a point. Spheres of (hyperbolic) radius R have mean curvature everywhere equal to $\coth R > 1$. If we start with a geodesic sphere and allow the radius R to tend to infinity, whilst simultaneously shifting the centre of the sphere off to the ideal boundary of $\mathbb{H}^3$, we obtain a horosphere. Horospheres are level sets for the Busemann function associated to a unit-speed geodesic and have mean curvature everywhere equal to $\lim_{R\to\infty} \coth R = 1$. We note that horospheres have a single point as ideal boundary. If we consider the level sets for the distance function to a geodesic we obtain the hyperbolic cylinders. These have two points as ideal boundary and mean curvature equal everywhere to $\frac{1}{2}(\tanh R + \coth R) > 1$, where R is the distance to the geodesic. Once again, if we let $R \to \infty$, whilst simultaneously bringing the two points at infinity together we end up, in the limit, with a horosphere. Finally we consider the class of complete, noncompact CMC surfaces arising as the level sets for the distance function to a totally geodesic plane. These surfaces have mean curvature everywhere equal to $\tanh R \in [0, 1)$, and ideal boundary given by a circle. We shall refer to these surfaces as *spherical caps*, and denote by Σ_H the spherical cap with constant mean curvature H. We remark that this class of CMC surfaces has no Euclidean analogue: in $\mathbb{R}^3$ the surfaces equidistant to a plane are again planes with zero mean curvature. We will see other examples of how H-surfaces with $|H| < 1$ in $\mathbb{H}^3$ exhibit features which have no counterpart in Euclidean space (see, for example, the remark immediately following Theorem 1.4).

With these basic models in mind it is perhaps natural to investigate whether we have any "rigidity" results regarding CMC surfaces in $\mathbb{H}^3$. We now list some of the known results in this direction. For the compact and embedded case, Alexandrov's proof in the Euclidean setting [3] goes through unchanged and we have that

> *The only compact, embedded CMC surfaces in $\mathbb{H}^3$ are the geodesic spheres.*

More recently, Meeks & Tinaglia [37] showed that, furthermore

> *If Σ is a simply-connected and embedded H-surface, with $H > 1$, then Σ is properly embedded.*

By results of Korevaar, Kusner, Meeks & Solomon [26] such a Σ must necessarily be compact, and therefore a sphere by Alexandrov's theorem. If we extend the class to allow immersions, we again have a result akin to the situation in Euclidean space. We mention the work of Umehara & Yamada [48], where they obtain CMC tori in $\mathbb{H}^3$ by deforming Wente's construction in $\mathbb{R}^3$. Walter [49] also obtains related results; thus we have that

There exist immersed CMC tori in $\mathbb{H}^3$.

Note that by the maximum principle these tori must have mean curvature > 1. We now turn to the family of horospheres. In their 1983 paper do Carmo & Lawson [15] prove a number of Alexandrov-Bernstein type results, amongst them the following:

The only properly embedded CMC surfaces in $\mathbb{H}^3$ *with a single point as ideal boundary are the horospheres.*

Once again, the theorem is false if we allow immersions: counterexamples were constructed by Gomes in his PhD. thesis [19]. We remark that 1-surfaces have been the subject of much attention, especially since Bryant's discovery of a Weierstrass representation for such surfaces [8]. In particular, one can construct (embedded) 1-surfaces with two points at infinity. Focusing now on the family of cylindrical surfaces, we have the following relevant result of Korevaar, Kusner, Meeks & Solomon [26]:

Let $H > 1$*. The only complete, properly embedded* H*-surfaces with two ends in* $\mathbb{H}^3$ *are the surfaces of revolution.*

Finally we turn our attention to the class of hyperspheres, with mean curvature satisfying $|H| < 1$. As we shall see, we have a much richer existence theory for this class of H-surfaces. Let us first recall Bernstein's result for minimal surfaces, and a generalisation by Fischer-Colbrie & Schoen [17] and also due to do Carmo & Peng [14]:

Theorem 1.1 (Bernstein)**.** *If* $\Sigma \in \mathbb{R}^3$ *is a minimal surface given by the graph of a* C^2 *function defined on the whole of* $\mathbb{R}^2$*, then* Σ *is a plane.*

Theorem 1.2 (Fischer-Colbrie & Schoen, do Carmo & Peng)**.** *If* $\Sigma \in \mathbb{R}^3$ *is a stable minimal immersion then* Σ *is a plane.*

In the 1983 paper cited above, do Carmo & Lawson also prove a form of analogue of Bernstein's original result, by showing that

If $\Sigma \in \mathbb{H}^3$ *is a CMC surface which can be written as a (hyperbolic) graph over a totally geodesic plane, then* Σ *is a spherical cap,* here by "hyperbolic graph" we mean with respect to the orthogonal projection given by the exponential map. But what about analogues of the stronger Theorem 1.2? This is no longer true in $\mathbb{H}^3$: Uhlenbeck [47] and independently Wang & Wei [51] give constructions that show that

> *There exist complete, noncompact stable minimal surfaces in* $\mathbb{H}^3$ *that are not totally geodesic.*

We do, however, have the following result of da Silveira [13], which, for reasons explained below, we feel is the natural analogue of do Carmo & Peng's Bernstein-type result

> *Let* $H \geqslant 1$*. If* Σ *is a stable, complete and noncompact immersed* H*-surface, then* Σ *is a horosphere.*

The constructions of Uhlenbeck and Wang & Wei are carried out in a hyperbolic 3-manifold, and the minimal surfaces in $\mathbb{H}^3$ arise when one passes to the universal cover. The ideal boundary of these surfaces is given by a Jordan curve on the sphere at infinity. Thus we are led very naturally to an alternative method of construction, namely, by solving an asymptotic boundary version of the classical Plateau problem:

> **(I)** *Given a Jordan curve* Γ *on the sphere at infinity of* $\mathbb{H}^3$*, does there exist a CMC surface* Σ *whose ideal boundary* $\partial_\infty(\Sigma)$ *is given by* Γ*?*

The above problem shall form the focus of much of the work in the thesis. In the next section we describe the approaches to tackling **(I)**; we then go on to deal with the issue of continuous dependence of the solution surface on the given curve at infinity.

1.2 The Plateau Problem for H-Surfaces and the Isoperimetric Inequality

The systematic study of **(I)** was initiated by Anderson [5] in 1982, where he dealt with the minimal ($H = 0$) case. In 1996 Tonegawa [45] and independently Alencar & Rosenberg [1] extended the work of Anderson to cover the cases $|H| < 1$. This is sharp, in terms of the values of H for which we can hope to solve, as can be readily seen by using the maximum principle with horospheres as comparison surfaces. All of the works just cited employ the powerful machinery provided by geometric measure theory. As is the norm with GMT methods one is forced to forfeit control over the topology of the solution surface, and one needs to work harder to assert the existence of a minimal or CMC *disk*. Anderson successfully constructs complete embedded minimal disks asymptotic to a given Jordan curve at infinity in [6], by using the approach developed by Almgren & Simon in [4], in which they constrained the Jordan curve to lie on a convex set.

A different approach to the problem was employed by Nelli & Spruck [40] and Guan & Spruck [21]. Here elliptic PDE methods were used to obtain graph-like solutions over the domain bounded by Γ in S^2_∞. Via this approach one can rather easily obtain solutions of the type of the disk, but some of the conditions imposed on Γ are somewhat unnatural, from the hyperbolic viewpoint. In particular, the requirement that Γ bound a starlike domain is a property which is not invariant under a Möbius transformation of S^2_∞: consider, for example, the fractional linear transformation given by

$$F : z \mapsto -\frac{1}{z}.$$

See Figure 1.1: F maps the starlike domain S to the non-starlike domain $F(S)$.

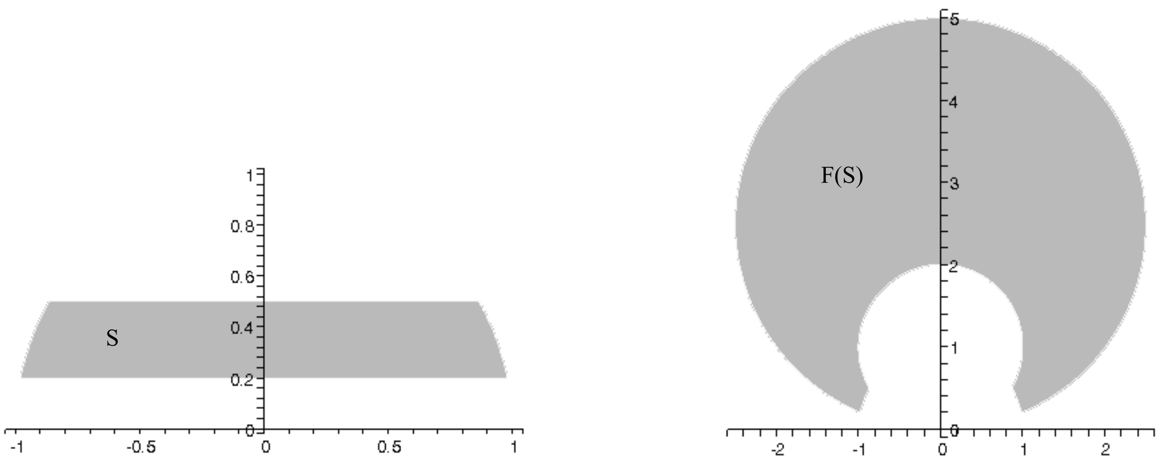

Figure 1.1: Action of $z \mapsto -\frac{1}{z}$ on a starlike domain

There is of course one very natural approach to the problem: consider a sequence of boundary curves $\Gamma_i \in \mathbb{H}^3$ converging to the given curve at infinity, solve the Plateau problem for H-surfaces for each Γ_i, and then extract a limit out of the solution surfaces. Indeed, in Chapter 2 we shall successfully employ this approach to prove the following:

Theorem A. *Let $H \in (-1,1)$ and suppose $\Gamma^i \subset \mathbb{H}^3$ is a sequence of Jordan curves converging to $\Gamma \subset S^2_\infty$. Suppose $u_i : \mathbb{B} \to \mathbb{H}^3$ is a sequence of conformal H-harmonic maps such that $u_i|_{\partial\mathbb{B}}$ is a parametrisation of Γ^i. Then, a subsequence converges uniformly on compact subsets of $\mathbb{B}$ to a conformal H-harmonic map $u : \mathbb{B} \to \mathbb{H}^3$ such that $\partial_\infty(u(\mathbb{B})) = \Gamma$.*

As a build up to this result, we now turn our attention to the Plateau problem for H-surfaces on the interior of a (strictly) negatively curved manifold, and identify the role played by Yau's isoperimetric inequality.

Our approach to constructing H-surfaces makes use of the fact that such objects have a variational characterisation: they arise as critical points for the functional $I_H(\cdot) := A(\cdot) - 2HV(\cdot)$, where A denotes the area of the surface and V the "enclosed" oriented volume. If the surfaces under consideration have boundary, we enclose a volume by adjoining a reference surface. We assume for the duration of this section that (N, h) is a complete, simply connected Riemannian 3-manifold of strict negative curvature, and denote by $\mathbb{B}$ the unit disk in $\mathbb{R}^2$ with Cartesian coordinates $z = (x, y)$. We identify an "origin" $o \in N$. In this setting it is convenient to use as reference surface the *geodesic cone* over the boundary curve, formed by joining every point on the curve to o via the connecting geodesic. The volume of the resulting domain can be readily calculated. Let $u : \mathbb{B} \to N$ be given, and define a map $F : \mathbb{B} \times [0,1] \to N$ by

$$F(z, \lambda) \mapsto \exp_o(\lambda \cdot \exp_o^{-1} \circ u(z)).$$

Then $F(\mathbb{B} \times [0,1])$ defines an oriented 3-chain, which call the geodesic cone over $u(\mathbb{B})$, and its volume is given by

$$V(u) = \int_{\mathbb{B}\times[0,1]} F^*(dV),$$

where dV denotes the volume form on N.

In the hope of applying the direct method we investigate whether I_H satisfies the crucial properties such as boundedness from below, coercivity and lower semi-continuity. If we restrict ourselves momentarily to bounded domains with smooth boundary, then in the setting of manifolds with negative curvature the basic result is Yau's isoperimetric inequality, which we now state. We also give the simple proof of this result, as it will serve to motivate some of the later discussion.

Theorem 1.3 (Yau, [52])**.** *Assume N is complete, simply connected and with all sectional curvatures bounded above by -1. Then for any domain Ω in N with compact closure and smooth boundary we have*

$$A(\partial\Omega) - 2V(\Omega) > 0.$$

Proof. The result comes from integrating the well known estimate for the Laplacian of the distance function $r(x) := d(o, x)$:

$$\Delta r \geqslant 2 \coth r > 2.$$

Let ν denote the unit outward normal to $\partial\Omega$. Using the divergence theorem we obtain

$$A(\partial\Omega) \geqslant \int_{\partial\Omega} (\nabla r, \nu) \, dA = \int_{\Omega} \Delta r \, dV > 2V(\Omega).$$

□

Thus for all such domains, and for $H \in [-1, 1]$,

$$A(\partial\Omega) - 2|H|V(\Omega) > 0.$$

We shall attempt to promote the notion that the above result not only motivates the study of H-surfaces in negatively curved manifolds, but also lies at the heart of any existence result. Furthermore, minimisers for the functional $Area - 2 \times Volume$ (i.e. $H = 1$) play a special role in $\mathbb{H}^3$, in that they are the borderline case, and we would expect some kind of rigidity result - such as the aforementioned result of da Silveira [13]. In particular, the minimal ($H = 0$) surfaces here do not distinguish themselves - the area functional by itself in hyperbolic space allows for an abundance of minimisers. Thus we see another, admittedly more heuristic, aspect of the correspondence between minimal surfaces in Euclidean space and 1-surfaces in hyperbolic space (the rigorous correspondence is of course given by Lawson's result [27] that every minimal surface in Euclidean 3-space is (canonically) locally isometric to a 1-surface in $\mathbb{H}^3$). Finally, we remark that the role of isoperimetric inequalities in existence proofs for H-surfaces has been highlighted before in the literature. We mention in particular the excellent survey article by Steffen [41].

Analytically the Plateau problem for H-surfaces is formulated as follows: given a Jordan curve $\Gamma \subset N$ and a real number H, find a map $u : \mathbb{B} \to N$ such that

(i) $\tau(u) + 2Hu_x \wedge u_y = 0$

(ii) $(u_x, u_x) - (u_y, u_y) = (u_x, u_y) = 0$

(iii) u maps $\partial\mathbb{B}$ homeomorphically onto Γ.

Here $(\cdot,\cdot)$ and $\wedge$ denote respectively the inner product and cross product with respect to the ambient metric h, and $\tau(u)$ denotes the tension field of u. Conditions (i) and (ii) together ensure that $u(\mathbb{B})$ is an H-surface. The solution of Plateau's problem for H-surfaces in a Riemannian manifold with an upper bound (not necessarily negative) on the sectional curvatures was obtained independently by Gulliver [22] and Hildebrandt & Kaul [23] in 1972.

We now revisit the proof of the result of Gulliver and Hildebrandt & Kaul in the setting of a negatively curved manifold, and show how all the necessary estimates follow from a suitable integration of the Laplacian of the distance function. Let $e(u)$ denote the energy density of u, given by

$$e(u) := \frac{1}{2}\left(|u_x|^2 + |u_y|^2\right),$$

where, for a tangent vector X, $|X|^2 = (X, X)$, and denote by

$$D(u) := \int_{\mathbb{B}} e(u)\, dxdy$$

the Dirichlet energy of u. We use the well established method of working with the Dirichlet functional rather than the area functional,

$$A(u) := \int_{\mathbb{B}} \sqrt{|u_x|^2|u_y|^2 - (u_x, u_y)^2}\, dxdy,$$

since the latter is invariant under any reparametrisation whereas $D(\cdot)$ is only invariant under conformal reparametrisations. We recall that $A(u) \leqslant D(u)$ for all admissable u, with equality if, and only if, u is conformal. Finally define

$$E_H(u) := D(u) - 2HV(u).$$

Theorem 1.4 ([22],[23]). *Let N be a complete, simply connected Riemannian 3-manifold with sectional curvatures all bounded above by -1. Let Γ be a curve satisfying the property that the space of maps $\{v : \mathbb{B} \to N \,|\, v$ maps $\partial\mathbb{B}$ continuously and monotonically onto Γ and $D(v)$ is finite$\}$ is non-empty. Then for all $H \in [-1, 1]$ there exists a map $u \in C^2(\mathbb{B}) \cap C(\overline{\mathbb{B}})$ satisfying $(i), (ii), (iii)$ above.*

Sketch proof. We assume we have normal coordinates centred around o. We shall prove a pointwise estimate at a point $p_0 = u(z_0)$, where $z_0 \in \mathbb{B}$. Let $r_0 = |u(z_0)| = d(o, p_0)$. We begin by writing $V(u)$ as an integral over $\mathbb{B}$:

$$V(u) = \int_{\mathbb{B}} \omega(u)\, dxdy,$$

where

$$\omega(u)(z_0) := \int_0^1 \lambda^2 J(\lambda r_0)\, d\lambda\, u(z_0) \cdot u_x(z_0) \times u_y(z_0),$$

and $\cdot$ and $\times$ denote respectively the inner and cross product with respect to the Euclidean metric. $J(\lambda r_0)$ denotes the Riemannian density in normal coordinates, evaluated at $F(z_0, \lambda)$. By definition of $\wedge$ we can write

$$\omega(u)(z_0) = \int_0^1 \lambda^2 \frac{J(\lambda r_0)}{J(r_0)}\, d\lambda\, (u(z_0), u_x(z_0) \wedge u_y(z_0)).$$

Now set

$$\mathcal{A}(r_0) := \int_0^1 \lambda^2 \frac{J(\lambda r_0)}{J(r_0)}\, d\lambda > 0,$$

so that

$$|\omega(u)|(z_0) = \mathcal{A}(r_0)|(u(z_0), u_x(z_0) \wedge u_y(z_0))| \leqslant \mathcal{A}(r_0) \cdot r_0 \cdot e(u)(z_0),$$

and by the estimate on the Laplacian of the distance function,

$$2\mathcal{A}(r_0) < \int_0^1 \Delta r(F(z_0, \lambda))\lambda^2 \frac{J(\lambda r_0)}{J(r_0)}\, d\lambda.$$

Now let a denote the Riemannian density in geodesic polar coordinates. Then J and a are related by $t^2 J(t) = a(t)$, and we can write

$$\int_0^1 \Delta r(F(z_0,\lambda))\lambda^2 \frac{J(\lambda r_0)}{J(r_0)}\, d\lambda = \int_0^1 \Delta r(F(z_0,\lambda)) \frac{a(\lambda r_0)}{a(r_0)}\, d\lambda.$$

But the Laplacian of the distance function is given by

$$\Delta r(F(z_0,\lambda)) = \frac{a'(\lambda r_0)}{a(\lambda r_0)},$$

(see, for example, [9], Chapter 6), and therefore

$$\begin{aligned} 2\mathcal{A}(r_0) &< \frac{1}{a(r_0)} \int_0^1 a'(\lambda r_0)\, d\lambda \\ &= \frac{1}{a(r_0)} \int_0^{r_0} a'(\rho)\, \frac{d\rho}{r_0} \\ &= \frac{1}{r_0}. \end{aligned}$$

We thus obtain $|\omega(u)| < \frac{1}{2} e(u)(z_0)$, and for any H in the range $[-1,1]$, we have

$$e(u) - 2H\omega(u) > 0.$$

The lower semi-continuity of the integral follows via an application of a lemma of Morrey ([39], Theorem 1.8.2) which reduces the issue to one of convexity of the integrand. It is easy to check that in our case positivity of the integrand implies the convexity (see [23], Lemma 4). With these estimates in place the rest of the proof proceeds via standard methods: one first fixes a parametrisation of the boundary, and minimises E_H within the Sobolev space of $W^{1,2}$ maps which agree (in a trace sense) with the given parametrisation on the boundary. This yields a *weak solution* to the H-harmonic map equation, and one must then resort to regularity theory to deduce that the solution is in fact smooth. The regularity results needed are discussed in more detail at the start of Chapter 2. Finally one varies the parametrisation of the boundary to obtain a map which is also conformal, and which therefore defines an H-surface. □

It is worth comparing this with the analogous result in Euclidean space: we can of course attempt to integrate the expression for the Laplacian of the distance function, but this now reads as $1/r$. This in turn forces us to make the following restrictions on the values of H: if Γ lies in a geodesic ball of radius R, then Γ can be spanned by an H-surface as long as

$|H| \leqslant 1/R$. Thus manifolds with $sect_N \leqslant -1$ distinguish themselves in that we can continue to solve the Plateau problem for H-surfaces as the boundary curve Γ "tends to infinity", for any H in the range $(-1, 1)$. This fact provides much of the motivation for attempting to solve the asymptotic Plateau problem by limiting the Gulliver-Hildebrandt-Kaul solution.

The proof of Theorem A involves two main elements: the first is the control of the conformal factor for a conformal H-harmonic map, which in turn leads to a uniform gradient estimate for such maps; this result holds in any manifold of strictly negative sectional curvature. The second is the use of *barrier surfaces* to essentially "trap" the sequence of H-surfaces within a subdomain of $\mathbb{H}^3$, and here we will be forced to restrict ourselves to the constant sectional curvature case. The full construction is carried out in Chapter 2.

1.3 Perturbation Results

In Chapters 3 and 4 we focus our attention on the issue of continuous dependence of the solution surface on the boundary curve. We continue to work in the parametric setting. Following the work of Tromba [46] and Tomi & Tromba [44] on the global analysis approach to the Plateau problem in $\mathbb{R}^3$, we view the set of H-surfaces with prescribed ideal boundary as lying inside the set of proper H-harmonic maps (defined below) which solve a Dirichlet problem at infinity. Thus we treat separately the issues of H-harmonicity and conformality, and begin by obtaining a perturbation result for H-harmonic maps into hyperbolic 3-space which solve a Dirichlet problem at infinity.

Perturbation of H-Harmonic Maps

Let $(\mathbb{U}^2, g)$ and $(\mathbb{U}^3, h)$ denote respectively the upper half space models for hyperbolic space in 2 and 3 dimensions respectively. A C^2 map $u : \mathbb{U}^2 \to \mathbb{U}^3$ is said to be *H-harmonic* if

$$\tau_H(u) := \tau(u) + 2H du(e_1) \wedge du(e_2) = 0,$$

where $\tau(u)$ denotes the tension field of u and (e_1, e_2) is an orthonormal frame field on $(\mathbb{U}^2, g)$. We study the linearisation of $\tau_H(u)$, given by the Jacobi operator $J_{H,u}$:

$$\begin{aligned} J_{H,u} : C^\infty(\mathbb{U}^2, u^*T\mathbb{U}^3) &\rightarrow C^\infty(\mathbb{U}^2, u^*T\mathbb{U}^2) \\ J_{H,u} : \phi &\mapsto \Delta\phi + \operatorname{tr}_g \mathcal{R}(du, \phi)du + 2H(d^\nabla \phi \wedge du)(e_1, e_2), \end{aligned}$$

where $C^\infty(\mathbb{U}^2, u^*T\mathbb{U}^3)$ denotes the space of smooth sections of the pullback bundle, Δ is the rough Laplacian on $u^*T\mathbb{U}^3$, d^∇ and $\mathcal{R}$ denote the connection and curvature tensor of $(\mathbb{U}^3, h)$ respectively, and we have

$$(d^\nabla \phi \wedge du)(e_1, e_2) = d^\nabla \phi(e_1) \wedge du(e_2) + du(e_1) \wedge d^\nabla \phi(e_2).$$

Let (s, t), $t > 0$ denote the usual rectangular coordinates on $\mathbb{U}^2$. A simple calculation shows us that in these coordinates $J_{H,u}$ takes the form of a *uniformly degenerate* operator, meaning that it can be written as a system of polynomials in $t\partial_s$ and $t\partial_t$ whose coefficients are at least continuous up to the boundary $\partial\mathbb{U}^2$. Such operators have been often studied in the past, and their mapping properties become apparent when one works in the setting of appropriately weighted function spaces, where the weight is given by t raised to a power that reflects the degeneracy. We define these function spaces rigorously in Chapter 3, but for the time being it shall suffice to think of the space $C^{k,\alpha}_\delta$ as consisting of sections of $u^*T\mathbb{U}$ of the form $t^\delta \phi$, where ϕ is a section with locally $C^{k,\alpha}$ coefficients, equipped with an appropriate norm. The main result of Chapter 3 is the following

Theorem B. *Suppose $u : (\mathbb{U}^2, g) \rightarrow (\mathbb{U}^3, h)$ is a proper H-harmonic map with nowhere vanishing energy density on $\partial\mathbb{U}^2$. Then the Jacobi operator associated to u, $J_{H,u}$, extends to be an isomorphism $J_{H,u} : C^{k,\alpha}_\delta \rightarrow C^{k-2,\alpha}_\delta$ for all $k \geq 2$, and for all δ satisfying $0 < \delta < 3$.*

This in turn leads to the desired perturbation result:

Theorem C. *Assume $|H| < 1$. Let $u : (\mathbb{U}^2, g) \rightarrow (\mathbb{U}^3, h)$ be a proper H-harmonic immersion that extends to be a $C^{k,\alpha}$ map from $\overline{\mathbb{U}^2}$ to $\overline{\mathbb{U}^3}$, for some $k \geq 2$ and $0 < \alpha < 1$. Assume that the boundary map $f_0 := u|_{\partial\mathbb{U}^2} : \mathbb{R} \rightarrow \mathbb{R}^2$ has nowhere vanishing energy density. Then there exists a neighborhood $\mathcal{N}$ of f_0 in $C^{k,\alpha}(\mathbb{R}, \mathbb{R}^2)$ such that for every $f \in \mathcal{N}$, there exists a proper*

H-harmonic extension of f, $u_f \in C^{k,\gamma}(\overline{\mathbb{U}^2}, \overline{\mathbb{U}^3})$, and furthermore the map $f \mapsto u_f$ is $C^{k,\gamma}$ smooth.

Remark 1.5. The case $H = 0$ here corresponds to u being a harmonic map from $\mathbb{H}^2$ to $\mathbb{H}^3$. The asymptotic Dirichlet problem for harmonic maps between hyperbolic spaces was studied by Li & Tam in [31] and [32], though the issue of continuous dependence on the boundary data was not addressed. In the current work we restrict ourselves to studying the $\mathbb{H}^2$ to $\mathbb{H}^3$ case since we are ultimately interested in H-surfaces, but our methods extend to deal with perturbations of proper harmonic maps between hyperbolic spaces of any dimension. The details will be available in a forthcoming publication [12].

The Conformality Operator and Perturbation of Spherical Caps

In Chapter 4 we prove the following result:

Theorem D. *Let $H \in (-1, 1)$. There exists a neighbourhood η in $H^5(S^1, \mathbb{R}^2)$ of the identity map $id : S^1 \to \mathbb{R}^2$ such that for every $f \in \mathcal{N}$ there exists a conformal, H-harmonic extension u_f satisfying $u_f|_{S^1} = f$.*

This is achieved by studying the linearisation of the conformality operator k, defined by the action

$$k : u \mapsto h(u_*\partial_z, u_*\partial_z),$$

where u is an H-harmonic map, and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ is a section of the complexified tangent bundle $T_{\mathbb{C}}\mathbb{B} = T\mathbb{B} \otimes \mathbb{C}$. $h(\cdot, \cdot)$ here is the complex bi-linear extension of the hyperbolic metric on $\mathbb{U}$. Our approach is very much in the spirit of the aforementioned work of Tomi and Tromba: we use methods from complex geometry to show directly that the linearisation of k at a spherical cap Σ_H is an isomorphism.

The work in Chapters 3 and 4 has some overlap with recent results of Alexakis & Mazzeo [2]. They prove the following structure theorem for embedded minimal surfaces in hyperbolic 3-manifolds:

Theorem 1.6. *Let M be a convex, co-compact hyperbolic 3-manifold, and let $\mathcal{M}_k(M)$ be the space of properly embedded minimal surfaces in M of genus k with asymptotic boundary curve a $C^{3,\alpha}$ embedded, closed (but possibly disconnected) curve in ∂M. Let $\mathcal{E}$ denote the space of all $C^{3,\alpha}$ closed embedded curves in ∂M. Then $\mathcal{M}_k(M)$ and $\mathcal{E}$ are both Banach manifolds and the projection map*

$$\Pi : \mathcal{M}_k(M) \to \mathcal{E}$$

is Fredholm of index zero.

1.4 Concluding Remarks

Related Results

We now list some results which are related to our present work, whilst being of a slightly different flavour. First we mention the work of Toda [42], who obtains an existence result for closed H-surfaces in the homotopy class of a given immersion, in a closed 3-manifold of strict negative curvature. His approach uses a combination of variational and heat-flow methods. In a more recent preprint [50], Wang uses the volume preserving mean curvature flow to show the existence of a CMC foliation of a quasi-Fuchsian 3-manifold that contains a minimal surface with principal curvatures < 1 in modulus. Finally, in other recent work Mazzeo & Pacard [35] construct CMC foliations (by compact hypersurfaces) in a neighbourhood of infinity in an asymptotically hyperbolic manifold (M, g). They use perturbative methods to deform the level sets of the boundary defining function, obtaining an interesting relation between the existence and uniqueness of such foliations, and the sign of the Yamabe invariant of the conformal infinity (conformal class of metrics on ∂M) associated to the metric g. We remark that the method of perturbing level sets of a suitable function to obtain CMC foliations has its origins in the pioneering work of Ye [53], [54], and has since been utilized to great success in a variety of settings; we mention, for example, other work of Mazzeo & Pacard [36], as well as work by Mahmoudi, Mazzeo & Pacard [33] and Fall & Mahmoudi [16].

Critique of a Paper of Coskunuzer [10]

Finally we wish to discuss briefly some work of Coskunuzer [10], who obtains a Banach manifold structure result for properly immersed minimal surfaces in $\mathbb{H}^3$ with asymptotic boundary curve on the sphere at infinity. We believe there to be some issues with the work, which we now highlight.

(1) *Differentiability of the identification map.* The starting point, and indeed the overall approach in [10] is similar to our own: Coskunuzer views the proper minimal immersions as the subset of the space of proper harmonic maps consisting of those maps which are also conformal. Through the results of Li & Tam, Coskunuzer identifies every C^1 map $f : S^1 \to S^2$ with its unique harmonic extension $\tilde{f} : \overline{\mathbb{H}}^2 \to \overline{\mathbb{H}}^3$, and considers the action of the conformality operator k:

$$k : f \mapsto \left(\frac{\partial \tilde{f}}{\partial r} \cdot \frac{\partial \tilde{f}}{\partial \theta} \right) \Bigg|_{S^1},$$

where r and θ are the usual radial and angular coordinates on $\mathbb{B}$ respectively, and $\cdot$ denotes the Euclidean inner product. Thus, the space of minimal immersions is precisely the zero set of k, and the structure of this space can be studied via the implicit function theorem. Our first objection is that the derivative of k involves differentiating the identification map $\Psi : f \mapsto \tilde{f}$, so that we are forced to deal with the issue of the regularity of Ψ.

(2) *The Hopf differential.* An essential element in work of this kind is the fact that the Hopf differential associated to a harmonic map is holomorphic. While it is true that a map into $\mathbb{H}^3$ is conformal with respect to the hyperbolic metric if, and only if, it is conformal with respect to the Euclidean metric, the Hopf differential constructed with respect to the Euclidean metric will *not*, in general, be holomorphic. In [10] the calculations are carried out with respect to this "Euclidean Hopf differential", and this point is overlooked. Furthermore, the compatibility conditions of Li & Tam clearly show that a proper harmonic map with non-vanishing energy density on the boundary is automatically "asymptotically conformal" with respect to the Euclidean inner product, so in particular the conformality operator k defined in [10] is identically 0.

(3) *The Jacobi operator.* Finally, in [10] there is also the erroneous assumption that a variation

vector field ϕ arising from a variation of harmonic maps u_t (i.e. $\phi = \frac{d}{dt}u_t|_{t=0}$) satisfies the equation

$$\Delta_{\mathbb{H}}\phi = 0,$$

where $\Delta_{\mathbb{H}}$ is the hyperbolic Laplacian on the unit ball. This is clearly incorrect, as the equation satisfied is of course

$$J_u\phi = 0,$$

where $u = u_0$, and J_u is the Jacobi operator associated to u, as described above (with $H = 0$). Thus the elliptic boundary value system studied in [10] in order to prove Fredholm properties is not the relevant one.

Chapter 2

The Asymptotic Plateau Problem for H-Surfaces

In this chapter we prove a compactness result for sequences of solutions to the Plateau problem for CMC surfaces on the interior of hyperbolic 3-space. As a consequence of this result we obtain an immersed disk-like solution to the asymptotic problem. We begin by discussing the regularity of weak solutions to the H-harmonic map equation. Then, in Section 2.2 we derive a uniform gradient estimate for conformal H-harmonic maps of the unit disk. In Section 2.3 we give the barrier surface construction, and conclude in Section 2.4 with the convergence result.

2.1 Regularity Theory

Let $u : \mathbb{B} \to N$ be a (smooth) immersion of the unit disk into a Riemannian 3-manifold (N, h). Assume we have coordinates (x^1, x^2) on $\mathbb{B}$, and let h_{ij} denote the coefficients of h in some choice of local coordinates on N. Then the H-harmonic map for u in local coordinates takes the form

$$\Delta u^l = 2H\sqrt{h}h^{lm}(D_1 u \wedge D_2 u)_m - \Gamma^l_{jk} D_\alpha u^j D_\alpha u^k,\ l = 1, 2, 3. \tag{2.1}$$

Here $D_\alpha X = \partial X/\partial x^\alpha$, the summation convention is in place, and Greek indices run from 1 to 2, Latin indices from 1 to 3. Also $\sqrt{h} = \det(h_{ij})$ and $h^{ij} = (h_{ij})^{-1}$. Assume now that (N, h) has everywhere strictly negative sectional curvature, so that in particular the cut-locus of N is empty, and we may identify any point in N with its normal coordinates. We thus have a simple way of defining the Sobolev spaces $W^{k,p}(\mathbb{B}, N)$ as subsets of the familiar $W^{k,p}(\mathbb{B}, \mathbb{R}^3)$.

We say a map $u \in W^{1,2}(\mathbb{B}, N)$ is a *weak solution* of (2.1) if for any smooth test function with compact support ϕ, we have

$$\int_{\mathbb{B}} D_\alpha u^l D_\alpha \phi^l \, dx = \int_{\mathbb{B}} \left\{ \Gamma^l_{jk} D_\alpha u^j D_\alpha u^k - 2H\sqrt{h} h^{lm} (D_1 u \wedge D_2 u)_m \right\} \phi^l \, dx. \tag{2.2}$$

It is straightforward to check that the map obtained during the "fixed boundary parametrisation" minimisation stage described in the proof of Theorem 1.4 satisfies (2.2). The transition from weak $W^{1,2}$ solutions to smooth solutions is essentially a three step process: because the right hand side of (2.1) is quadratic in Dw, one must first show Hölder continuity for the first derivatives before one can apply standard elliptic regularity theory and the usual "bootstrapping" argument kicks in. We describe these steps below.

Remark 2.1. In the work of Gulliver and Hildebrandt & Kaul, the essential estimates on E_H (i.e. coercivity and lower semi-continuity) are obtained by working in a bounded region K of the ambient manifold N (in [23] K is referred to as a *gauge ball*). The minimisation process is thus carried out *only amongst those maps whose image lies within* K, and the minimiser w need not, a priori, be a weak solution of (2.1). To surmount this difficulty, one must first show continuity of w on $\overline{\mathbb{B}}$. This is achieved via Morrey's celebrated Dirichlet Growth Lemma ([39], Theorem 6.2). In the setting of strict negative curvature any sufficiently large ball will work as a gauge ball, but the point is that in such a setting one does not need the notion of a gauge ball at all (as we showed in our proof of Theorem 1.4). In particular we minimise amongst *all* $W^{1,2}$ maps into N, and our minimiser is automatically a weak solution to the H-harmonic map equation. Furthermore, working within a bounded subdomain K introduces the additional complication of requiring a inclusion principle that guarantees that the minimiser obtained also lies entirely within K. Once again, in the case of $sect_N < 0$, this additional complication does not arise.

Step 1 One first obtains Hölder continuity of the solution via an application of Morrey's celebrated *Dirichlet Growth Lemma* [39]. For $z \in \mathbb{B}$, let $B_R(z)$ denote the disk of radius R centred at z.

Theorem 2.2 (Morrey, [39]). *Let $w \in W1,p(B_R(z_0))$, $1 \leqslant p \leqslant n$. Suppose that for all $z \in B_R(z_0)$, and all $r \in (0, \rho(z)]$, where $\rho = R - |z - z_0|$,*

$$\int_r (z)|Dw|^p \, dx \leqslant C^p \left(\frac{r}{\rho}\right)^{n-p+p\delta} \tag{2.3}$$

holds with $\delta \in (0,1]$. Then $w \in C^{0,\delta}(B_s(z_0))$ for all $s < R$.

It can be readily shown that the weak solution obtained via the minimisation process described in Theorem 1.4 satisfies the estimate 2.3 above.

Step 2 One then deduces $C^{1,\alpha}$ regularity ($\alpha \in (0,1)$) by means of the following result of Tomi [43]:

Theorem 2.3 ([43]). *Suppose that w is a continuous, weak solution for the system*

$$\Delta w = f(z, w, Dw),$$

where f satisfies $|f(z,w,Dw)| \leqslant \mu(|w|)|Dw|^2$ for some monotonically nondecreasing function μ. Then $w \in C^{1,\delta}$ for all $\delta \in (0,1)$.

Step 3 Having ascertained that our weak solution u is in $C^{1,\delta}$, we define functions f^l by setting

$$f^l := 2H\sqrt{h}h^{lm}(D_1 u \wedge D_2 u)_m - \Gamma^l_{jk} D_\alpha u^j D_\alpha u^k, \; l = 1,2,3.$$

By Tomi's result, $f^l \in C^{0,\alpha}$. We now apply standard elliptic regularity theory to the equation

$$\Delta u^l = f^l,$$

to conclude that $u^l \in C^{2,\delta}$. The bootstrapping argument now kicks in, since we now know that $f^l \in C^{1,\delta}$, which implies that $u^l \in C^{3,\delta}$, and so on.

2.2 Uniform Gradient Bound

Assume now that $sect_N \leqslant -1$. We now obtain a uniform gradient bound for conformal H-harmonic maps on compact subsets of $\mathbb{B}$, by controlling the conformal factor.

Lemma 2.4. *Suppose $|H| < 1$, and let $u : \mathbb{B} \to N$ be a conformal, H-harmonic map. Then at any point $z \in \mathbb{B}$ we have*

$$|\nabla u|^2(z) < C_\rho < \infty,$$

where C_ρ is some constant depending only on $\rho = |z|$.

Proof. Since u is conformal and H-harmonic, $u(\mathbb{B})$ is an H-surface. By the Gauss equation we have that

$$\mathcal{R}_u = K_u + \mathcal{R}_h$$

where $\mathcal{R}_u$ and $\mathcal{R}_h$ are the sectional curvatures of $u(\mathbb{B})$ and (N, h) respectively, and K_u is the extrinsic curvature (i. e. product of principal curvatures) of $u(\mathbb{B})$. K_u and the mean curvature H are related by the inequality

$$K_u \leqslant H^2.$$

Therefore, since $|H| < 1$ and $\mathcal{R}_h \leqslant -1$ we have

$$\mathcal{R}_u \leqslant H^2 - 1 < 0. \tag{2.4}$$

We set $\epsilon := \sqrt{1 - H^2}$, so that $\mathcal{R}_u \leqslant -\epsilon^2$.

Now, since u is conformal we can write

$$u^*(h) = \lambda^2 |dz|^2, \tag{2.5}$$

for some positive function λ on $\mathbb{B}$, where $|dz|$ denotes the flat metric on $\mathbb{B}$. In fact, $\lambda^2 = \frac{1}{2}|\nabla u|^2$. Set $f = \ln \lambda$. Note that f is bounded on $\overline{\mathbb{B}}$, by definition. By the formulae for a conformal change of metric we obtain the differential inequality

$$\Delta f \geq \epsilon^2 e^{2f} =: \eta(f).$$

We wish compare f with the solution to the differential equation

$$\Delta \phi = \eta(\phi),$$

given by

$$\phi(z) = \ln \frac{2}{\epsilon(1 - |z|^2)}.$$

Let $\psi := f - \phi$. We claim that $\psi(z) \leqslant 0$ for $|z| < 1$. Indeed, suppose $\psi > 0$ at some point. We note that $\psi \to -\infty$ as $|z| \to 1$, therefore ψ must take a positive maximum at some point $p \in \mathbb{B}$ with $|p| < 1$. By continuity, $\psi > 0$ on some neighbourhood $\mathcal{N}$ of p. Therefore, on $\mathcal{N}$, we have $\Delta(\psi) = \Delta(f) - \Delta(\phi) \geq \eta(f) - \eta(\phi) > 0$. Thus ψ is subharmonic on $\mathcal{N}$, contradicting the fact that it had a maximum at p. We conclude that

$$f(z) \leqslant \ln \frac{2}{\epsilon(1 - |z|^2)},$$

so that λ must satisfy

$$\lambda(z) \leqslant \frac{2}{\epsilon(1 - |z|^2)},$$

and

$$|\nabla u|^2(z) \leqslant \frac{8}{\epsilon^2(1 - |z|^2)^2} =: C_\rho.$$

□

2.3 Construction of Barriers

At this stage we are forced to restrict ourselves to the case $N = \mathbb{H}^3$. The reason is that our barriers are constructed as surfaces equidistant from a totally geodesic surface - a construction which simply cannot be carried out in a space of variable curvature. Thus from now on $S^2_\infty = \partial_\infty \mathbb{H}^3$, the sphere at infinity of hyperbolic 3-space, and we assume that we are working with the unit ball model. Given Γ, an (oriented) Jordan curved on S^2_∞ we consider a sequence Γ^i of (oriented) Jordan curves converging to Γ. For each i we solve the interior Plateau problem for H-surfaces to obtain a sequence $u_i(\mathbb{B})$ of surfaces of constant mean curvature H and with $\partial(u_i(\mathbb{B})) = \Gamma^i$. We now show that if the sequence of curves $\{\Gamma^i\}$ converges to Γ within some special, fixed region of $\mathbb{H}^3$, then the corresponding surfaces Σ^i also lie entirely within this region. Similar constructions have previously been described by Alencar & Rosenberg [1], Tonegawa [45] and most recently Coskunuzer [11].

We begin by noting that Γ separates S^2_∞ into two disjoint open disks. We fix and label these two regions as Ω^+ and Ω^-. The given orientation on Γ combined with some fixed orientation for $\mathbb{H}^3$ determines a normal for any surface spanning Γ, and we suppose that given some

$H \in (-1, 1)$, we wish to span Γ by a CMC surface whose mean curvature is H with respect to this normal. We now take a circle U contained entirely inside Ω^+, and we consider the totally geodesic surface spanning U. U also separates S^2_∞ into two regions, and we label the region contained entirely inside Ω^+ as Ω^+_U. Finally we construct the equidistant surface that has mean curvature $|H|$ with respect to the normal that points towards Ω^+_U. With our convention (mean curvature $= -\frac{1}{2}\text{tr } A$, $A =$ second fundamental form) this surface will be the one lying on the side towards Ω^+_U. We call these surfaces $|H|$-*caps*. In a similar manner we take a circle in Ω^-, set Ω^-_U to be that part of $S^2_\infty \setminus U$ entirely contained in Ω^- and take the cap of constant mean curvature $|H|$ with respect to the normal that points towards Ω^-_U. Each $|H|$-cap separates $\overline{\mathbb{H}^3}$ into two regions. Modifying the terminology of [11] we call these regions $|H|$-*shifted half-spaces*, and if Γ is contained in one of these half-spaces then we say the half-space is *supporting*.

Finally we perform this construction for all circles lying in $S^2_\infty \setminus \Gamma$, and make the following definition:

Definition 2.5. The $|H|$-*shifted convex hull* of Γ is defined to be the intersection of all supporting $|H|$-shifted half-spaces, and is denoted by $\mathcal{C}_H(\Gamma)$.

We claim:

Lemma 2.6. *If a sequence of curves $\{\Gamma^i\}$ converging to Γ lies entirely within $\mathcal{C}_H(\Gamma)$, then so too does the sequence of H-surfaces $\{u_i(\mathbb{B})\}$, where $\partial(u_i(\mathbb{B})) = \Gamma^i$.*

We shall use the following maximum principle:

Lemma 2.7. *Let Σ^1 and Σ^2 be two hypersurfaces in a Riemannian manifold that intersect tangentially at a common point p. Let H^i be the mean curvature of Σ^i at p. If Σ^1 lies on the positive side of Σ^2 then $H^1 > H^2$.*

By *positive side* we mean the side *opposite* to the direction indicated by the chosen reference normal at p. This is dictated by our convention for the mean curvature.

Remark 2.8. We note that since we are working with *immersed* surfaces, we cannot refine the region $\mathcal{C}_H(\Gamma)$ any further. The problem arises when trying to apply the maximum principle -

after choosing a reference normal ν, say, one cannot know beforehand whether the surface one is trying to control has mean curvature $+|H|$ or $-|H|$ with respect to ν.

Proof of Lemma 2.6: Let Γ^0 be a Jordan curve contained in $\mathcal{C}_H(\Gamma)$, and let Σ^0 be the H-surface with boundary Γ^0. We must show that Σ^0 does not intersect any non-supporting half-space. Let W be a non-supporting half-space, i. e. $W \subset \overline{\mathbb{H}^3}$ and $W \cap \Gamma = \emptyset$. Suppose Σ^0 enters W. We can foliate W by $|H|$-caps whose asymptotic boundaries lie in S^2_∞. Since $\Gamma^0 \cap W = \emptyset$, Σ^0 must intersect some $|H|$-cap K tangentially at some point p. We choose as reference normal at p the normal ν_p that places (according to our convention) Σ^0 on the positive side of K. The maximum principle then implies that the mean curvature of Σ^0 must be strictly greater than that of K. Our construction ensures that with respect to ν_p, K will have mean curvature $+|H|$, while Σ^0 will have mean curvature either $+|H|$ or $-|H|$. In either case we obtain a violation of the maximum principle. $\square$

2.4 Compactness

Theorem A. *Let $H \in (-1,1)$ and suppose $\Gamma^i \subset \mathbb{H}^3$ is a sequence of Jordan curves converging (in the Hausdorff distance) to $\Gamma \subset S^2_\infty$. Suppose $u_i : \mathbb{B} \to \mathbb{H}^3$ is a sequence of conformal H-harmonic maps such that $u_i|_{\partial\mathbb{B}}$ is a parametrisation of Γ^i. Then, a subsequence converges uniformly on compact subsets of $\mathbb{B}$ to a conformal H-harmonic map $u : \mathbb{B} \to \mathbb{H}^3$ such that $\partial_\infty(u(\mathbb{B})) = \Gamma$.*

Proof. Let Ω be a compact subdomain of $\mathbb{B}$. By the uniform gradient estimate there exists a constant C such that

$$\sup_{z\in\Omega} |\nabla u_i(z)|^2_{hyp} < C < \infty \text{ for all } i, \tag{2.6}$$

where C depends only on the domain Ω. Let $z_0 \in \Omega$. From the barrier argument we may assume that $|u_i(z_0)|_{hyp} < \infty$ for all i, so that by the mean value theorem and (2.6), we obtain

$$\sup_{z\in\Omega} |u_i(z)| < C < \infty \text{ for all } i, \tag{2.7}$$

where again $C = C(\Omega)$.

Next, we again define functions f_i^l on $\mathbb{B}$ by setting

$$f_i^l := 2H\sqrt{h}(u_i)h^{lm}(u_i)(D_1u_i \wedge D_2u_i)_m - \Gamma^l_{jk}(u_i)D_\alpha u_i^j D_\alpha u_i^k,\ l = 1, 2, 3.$$

We apply Schauder theory to the equation

$$\Delta u_i^l = f_i^l$$

to obtain the estimate

$$|u_i^l|_{C^{1,\alpha}(\Omega')} \leqslant C\left(\sup_\Omega |f_i^l| + \sup_\Omega |u_i^l|\right),\ \alpha \in (0, 1),$$

for some $\Omega' \Subset \Omega$. But, by definition, $|f_i^l|$ is bounded above on compact subsets by a multiple of $|\nabla u_i^l|^2$, which is in turn bounded above by some uniform constant, as described above. Therefore we obtain a uniform $C^{1,\alpha}$ bound for u_i^l, which in turn implies a uniform $C^{0,\alpha}$ bound for f_i^l. We then use the estimate

$$|u_i^l|_{C^{2,\alpha}(\Omega')} \leqslant C\left(|f_i^l|_{C^{0,\alpha}(\Omega)} + \sup_\Omega |u_i^l|\right)$$

to obtain a uniform $C^{2,\alpha}$ bound on u_i^l, and so on. Thus for every multi-index γ, the sequence of partial derivatives $\{|D_\gamma u_i|\}$ is uniformly bounded. We may now apply Ascoli-Arzela's theorem to conclude that a subsequence of $\{u_i\}$ converges to a map $u^m_{\Omega'}$ in $C^m(\Omega')$ for all m, and finally using a diagonal process on an increasing sequence of relatively compact domains in $\mathbb{B}$ we extract a further subsequence converging to a map u in $C^\infty(\mathbb{B})$ which is H-harmonic, conformal and satisfies $\partial_\infty(u(\mathbb{B})) = \Gamma$. □

Remark 2.9. The closest result of this kind in the literature is Anderson's Theorem 4.1 in [6], where he proves the existence of complete, embedded, minimal surface of the type of the disk, asymptotic to a given Jordan curve γ on the sphere at infinity of $\mathbb{H}^3$. The initial steps are similar to our own: he considers a sequence of C^2 Jordan curves on the interior of $\mathbb{H}^3$ converging to γ. Through the work of Almgren & Simon [4] he asserts the existence of a smooth, embedded minimal disk spanning each element of the sequence. By establishing appropriate bounds on the mass (in the GMT setting) of the intersection of these disks with ever increasing geodesic balls, Anderson concludes the existence of a complete integral 2-current Σ asymptotic to γ. The final (substantial) step requires again techniques from [4], and the application of Allard's regularity result to conclude that Σ is first a regularly embedded minimal surface, and then that it is of the type of the disk.

Chapter 3

Perturbation of Proper H-Harmonic Maps

The main object of study in this chapter is the linear operator J that arises as the linearisation at an H-harmonic map of the H-tension field operator τ_H. We shall apply the implicit function theorem on Banach spaces to deduce a perturbation result for H-harmonic maps between hyperbolic 2-space and hyperbolic 3-space, and for this we require J to be invertible between appropriate function spaces. J turns out to be a *uniformly degenerate* elliptic edge operator, and the analysis of its mapping properties requires the use of certain weighted function spaces, which we define in Section 3.3. A simple indicial root analysis, carried out in Section 3.4, yields a maximal interval for the weights on which we expect J to be invertible. The invertibility result is proved in Section 3.4.3 using a combination of basic L^2 estimates and techniques from Mazzeo's edge calculus [34].

3.1 The H-Harmonic Map Equation and its Linearisation

Ultimately our interest lies in the perturbation of maps from the Euclidean unit disk into hyperbolic 3-space. However for the purpose of our analysis, it will suit us to also introduce the hyperbolic metric on the domain. The H-harmonic map equation (Definition 3.1) is conformally invariant, so we are free to do this. Furthermore, the calculations become particularly straightforward if we work with the upper-half space models of hyperbolic space in both domain and target. Thus let $(\mathbb{U}^2, g)$ denote the upper-half space in $\mathbb{R}^2$, with coordinates (s,t), $t > 0$, equipped with the hyperbolic metric $g := g_e/t^2$, where g_e denotes the Euclidean metric on $\mathbb{U}^2$; $(\mathbb{U}^3, h)$ will, as usual, denote the 3-dimensional upper-half space model.

Definition 3.1. Let $H \in \mathbb{R}$. For a C^2 map $u : (\mathbb{U}^2, g) \to (\mathbb{U}^3, h)$ we define the *H-tension field* of u, denoted $\tau_H(u)$ by

$$\tau_H(u) := \tau(u) + 2H du(e_1) \wedge du(e_2), \tag{3.1}$$

where $\tau(u)$ denotes the tension field of u, and $du(e_1) \wedge du(e_2)$ denotes the cross-product of $du(e_1)$ and $du(e_2)$ with respect to the metric h, where $e_1 = t\partial_t$, $e_2 = t\partial_s$. u is said to be *H-harmonic* if $\tau_H(u) = 0$.

We will study perturbations of solutions to the equation $\tau_H(u) = 0$ by means of the implicit function theorem. We therefore study the linearisation of the H-tension field operator τ_H, linearised at an H-harmonic map u:

$$J_{H,u}(\phi) = \nabla_{\partial_t}|_{t=0}\, \tau_H(u_t) = \Delta\phi + \mathrm{tr}_g \mathcal{R}(du, \phi)du + 2H(d^\nabla \phi \wedge du)(e_1, e_2), \tag{3.2}$$

where $\{u_t\}$ is a variation of u satisfying $\nabla_{\partial_t}|_{t=0}\, u_t = \phi$, Δ denotes the rough Laplacian on the pullback bundle $u^*(T\mathbb{U}^3)$, d^∇ and $\mathcal{R}$ denote respectively the connection and the curvature tensor of $(\mathbb{U}^3, h)$, and

$$d^\nabla \phi \wedge du(e_1, e_2) = d^\nabla \phi(e_1) \wedge du(e_2) + du(e_1) \wedge d^\nabla \phi(e_2).$$

Definition 3.2. We call $J_{H,u}$ the *Jacobi operator*, and solutions to $J_{H,u}\phi = 0$ *Jacobi fields.*

Remark 3.3. Recall that when $H = 0$ our basic objects are examples of the familiar *harmonic maps* between Riemannian manifolds. The Jacobi operator in this case has been the subject of much study, though primarily for situations with compact domain and target. As an example of recent work in this direction we mention the work of Lemaire & Wood [29] on *integrable* (i.e. arising as the variation vector field through a 1-parameter family of harmonic maps) Jacobi fields along hyperspheres.

In terms of the local coordinates (s,t), the components of the H-tension field of a map $u=(a,b,c):(\mathbb{U}^2,g)\to(\mathbb{U}^3,h)$ are given by

$$\tau_H^1(u)=t^2\left\{\bar{\Delta}a-\frac{2}{c}\bar{\nabla}a\cdot\bar{\nabla}c+\frac{2H}{c}\left(b_sc_t-b_tc_s\right)\right\} \tag{3.3}$$

$$\tau_H^2(u)=t^2\left\{\bar{\Delta}b-\frac{2}{c}\bar{\nabla}b\cdot\bar{\nabla}c+\frac{2H}{c}\left(c_sa_t-c_ta_s\right)\right\} \tag{3.4}$$

$$\tau_H^3(u)=t^2\left\{\bar{\Delta}c+\frac{1}{c}\left(|\bar{\nabla}a|^2+|\bar{\nabla}b|^2-|\bar{\nabla}c|^2\right)+\frac{2H}{c}\left(a_sb_t-a_tb_s\right)\right\} \tag{3.5}$$

where $\bar{\Delta}$, $\bar{\nabla}$, $\cdot$ and $|\cdot|_e$ denote respectively the Euclidean Laplacian, gradient, inner product and norm on $\mathbb{U}^2$.

In our current set-up, the "spherical caps" described in Chapter 1 take the form of surfaces equidistant to a totally geodesic copy of $\mathbb{U}^2$ inside $(\mathbb{U}^3,h)$. Their H-harmonic parametrisations are given by

$$\Sigma_H:(s,t)\mapsto\left(tH,s,t\sqrt{1-H^2}\right),\qquad H\in(-1,1), \tag{3.6}$$

where, if (x,y,z), $z>0$, are coordinates on $\mathbb{U}^3$, the totally geodesic copy of $\mathbb{U}^2$ in $(\mathbb{U}^3,h)$ is defined by the yz-plane, and the Σ_H have the y-axis as shared ideal boundary.

3.2 Asymptotics of H-Harmonic Maps

In this section we analyse a formal series solution to the H-harmonic map equation. This analysis serves two purposes: it will allow us to construct a first approximate solution to $\tau_H=0$ (which we later perturb to an exact solution), and will also yield the asymptotic behaviour of H-harmonic maps. This latter information will be important in understanding the nature of the degeneracy of the linearised operator $J_{H,u}$.

We assume for the moment that $u:(\mathbb{U}^2,g)\to(\mathbb{U}^3,h)$ is a proper H-harmonic map that is smooth on $\overline{\mathbb{U}^2}$, and solves a Dirichlet problem at infinity. Let $u(s,t)=(a(s,t),b(s,t),c(s,t))$. Let n be some integer >4. In a neighbourhood of $s=0$ we expand the components a, b and c

as formal power series:

$$\begin{aligned} a(s,t) &= a_0(s) + a_1(s)t + a_2(s)t^2 + a_3(s)t^3 + O(t^4) + \ldots + O(t^n) \\ b(s,t) &= b_0(s) + b_1(s)t + b_2(s)t^2 + b_3(s)t^3 + O(t^4) + \ldots + O(t^n) \\ c(s,t) &= c_1(s)t + c_2(s)t^2 + c_3(s)t^3 + O(t^4) + \ldots + O(t^n) \end{aligned}$$

(we know that $c(s,t) = 0$ when $t = 0$). Denote now by $f_0 = (a_0, b_0) : \mathbb{R} \to \mathbb{R}^2$ the restriction of u to the boundary, and by $e(f_0)$ the energy density of f_0, given by

$$e(f_0) = \left(\frac{\partial a_0}{\partial s}\right)^2 + \left(\frac{\partial b_0}{\partial s}\right)^2.$$

From now on we will use $'$ to denote differentiation with respect to s. Assume that $e(f_0)$ is nowhere vanishing on $\mathbb{R}$, and that $|H| < 1$. Substitution of the above series expansion into the H-harmonic map equation yields a system of ODEs for the coefficient functions a_i, b_i and c_i. This can be done by hand, but is readily handled by a computer algebra system such as Maple. For example, the first three equations, arising from equating to 0 the coefficients of the t term in the series expansion of $\tau_H(u)$, are

$$\begin{aligned} -2a_1 + 2Hb_0' &= 0 \\ -2b_1 - 2Ha_0' &= 0 \\ \frac{1}{c_1}\left\{a_1^2 + b_1^2 + e(f_0) - c_1^2 - 2H(a_1 b_0' - a_0' b_1)\right\} &= 0 \end{aligned}$$

The resulting system of ODEs can easily be solved either by hand or again using software. It turns out that a_1, b_1, c_1, a_2, b_2 and c_2 are all formally determined by a_0 and b_0. The coefficients for the cubic term t^3 are formally undetermined by the equation, and all higher order coefficients depend on a_0, b_0, a_3, b_3 and c_3. More precisely we have the following asymptotics for proper H-harmonic maps whose boundary maps have nowhere vanishing energy density:

$$a_1 = Hb_0', \qquad b_1 = -Ha_0', \qquad c_1 = \sqrt{e(f_0)(1-H^2)} \tag{3.7}$$

$$a_2 = \tfrac{1}{2e(f_0)}\left(a_0''(b_0')^2\{1-2H^2\} + 2a_0'b_0'b_0''\{H^2-1\} - (a_o')^2 a_0''\right) \tag{3.8}$$

$$b_2 = \tfrac{1}{2e(f_0)}\left(b_0''(a_0')^2\{1-2H^2\} + 2b_0'a_0'a_0''\{H^2-1\} - (b_o')^2 b_0''\right) \tag{3.9}$$

$$c_2 = \sqrt{\tfrac{1-H^2}{e(f_0)}}(a_0'b_0'' - b_0'a_0''). \tag{3.10}$$

In particular we note that

$$\begin{aligned}
\left.\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial s}\right|_{t=0} &= a_1a_0'+b_1b_0'=0\\
\left.\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial t}\right|_{t=0} &= (a_1)^2+(b_1)^2+(c_1)^2=e(f_0)\\
\left.\frac{\partial u}{\partial s}\cdot\frac{\partial u}{\partial s}\right|_{t=0} &= (a_0)^2+(b_0)^2=e(f_0)
\end{aligned}$$

where $\cdot$ denotes the Euclidean inner product. Thus such maps are automatically "Euclidean conformal" at the boundary (cf. Section 1.4).

We can actually derive the first order asymptotics for an H-harmonic map u assuming that u extends to be merely C^1 on $\overline{\mathbb{U}^2}$, and without resorting to a power series analysis. In the case $H=0$ this result was obtained by Li & Tam [31]. We will need the following simple lemma from [31]:

Lemma 3.4 ([31], Lemma 1.2)**.** *Let p be a C^1 function on $\overline{\mathbb{U}^2}$ that is smooth on $\mathbb{U}$. Let t denote the Euclidean distance to the boundary $\partial\mathbb{U}^2$. Then for any point $x\in\partial\mathbb{U}^2$ there exists a sequence $\{x_i\}\subset\mathbb{U}^2$ with $x_i\to x$ such that*

$$\lim_{i\to\infty} t(x_i)\bar{\Delta}p(x_i)=0.$$

We now prove:

Lemma 3.5. *Let $H\in(-1,1)$. Suppose $u=(a,b,c):(\mathbb{U}^2,g)\to(\mathbb{U}^3,h)$ is a proper H-harmonic map that extends to be a C^1 map from $\overline{\mathbb{U}^2}$ to $\overline{\mathbb{U}^3}$. Assume that the boundary map $f_0:=u|_{\partial\mathbb{U}^2}:\mathbb{R}\to\mathbb{R}^2$ has nowhere vanishing energy density $e(f_0)$. Then u satisfies the following first order compatibility conditions:*

$$\left.\frac{\partial a}{\partial t}\right|_{t=0}=H\left.\frac{\partial b}{\partial s}\right|_{t=0},\qquad \left.\frac{\partial b}{\partial t}\right|_{t=0}=-H\left.\frac{\partial a}{\partial s}\right|_{t=0},\qquad \left.\frac{\partial c}{\partial t}\right|_{t=0}=\sqrt{e(f_0)(1-H^2)}.$$

Proof. From (3.5) we have that

$$t^2\bar{\Delta}c+\frac{t^2}{c}\left(|\bar{\nabla}a|^2+|\bar{\nabla}b|^2-|\bar{\nabla}c|^2\right)=\frac{2Ht^2}{c}\left(\frac{\partial a}{\partial t}\frac{\partial b}{\partial s}-\frac{\partial a}{\partial s}\frac{\partial b}{\partial t}\right). \tag{3.11}$$

We multiply (3.11) by c/t^2 to obtain

$$\left(\frac{c}{t}\right)t\bar{\Delta}c+\left(|\bar{\nabla}a|^2+|\bar{\nabla}b|^2-|\bar{\nabla}c|^2\right)=2H\left(\frac{\partial a}{\partial t}\frac{\partial b}{\partial s}-\frac{\partial a}{\partial s}\frac{\partial b}{\partial t}\right). \tag{3.12}$$

Now let $x \in \mathbb{R}$, and let $\{x_i\}$ be the sequence of points in $\mathbb{B}$ converging to x constructed in Lemma 3.4. Evaluating (3.12) at x_i and letting $i \to \infty$ we obtain that at $t = 0$

$$\left(\frac{\partial a}{\partial t}\right)^2 + \left(\frac{\partial b}{\partial t}\right)^2 + \left(\frac{\partial a}{\partial s}\right)^2 + \left(\frac{\partial b}{\partial s}\right)^2 - \left(\frac{\partial c}{\partial t}\right)^2 = 2H\left(\frac{\partial a}{\partial t}\frac{\partial b}{\partial s} - \frac{\partial a}{\partial s}\frac{\partial b}{\partial t}\right).$$

Now,

$$\left|\frac{\partial a}{\partial t}\frac{\partial b}{\partial s}\right| \leqslant \frac{1}{2}\left(\frac{\partial a}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial b}{\partial s}\right)^2,$$

and similarly for $\frac{\partial a}{\partial s}\frac{\partial b}{\partial t}$. Therefore

$$\begin{aligned}
\left(\frac{\partial c}{\partial t}\right)^2 &= \left(\frac{\partial a}{\partial t}\right)^2 + \left(\frac{\partial b}{\partial t}\right)^2 + e(f_0) - 2H\left(\frac{\partial a}{\partial t}\frac{\partial b}{\partial s} - \frac{\partial a}{\partial s}\frac{\partial b}{\partial t}\right) &(3.13)\\
&\geqslant \left(\frac{\partial a}{\partial t}\right)^2 + \left(\frac{\partial b}{\partial t}\right)^2 + e(f_0) - H\left\{\left(\frac{\partial a}{\partial t}\right)^2 + \left(\frac{\partial b}{\partial t}\right)^2\right\} - He(f_0) &(3.14)\\
&= (1-H)\left\{\left(\frac{\partial a}{\partial t}\right)^2 + \left(\frac{\partial b}{\partial t}\right)^2\right\} + (1-H)e(f_0) &(3.15)\\
&> 0.
\end{aligned}$$

Therefore

$$\left.\frac{\partial c}{\partial t}\right|_{t=0} \neq 0. \tag{3.16}$$

Applying the same idea to the equation $\tau_H^1(u) = 0$, we obtain, at $t = 0$,

$$\frac{\partial a}{\partial t}\frac{\partial c}{\partial t} = H\frac{\partial b}{\partial s}\frac{\partial c}{\partial t}, \tag{3.17}$$

and since $\frac{\partial c}{\partial t} \neq 0$, we conclude that

$$\left.\frac{\partial a}{\partial t}\right|_{t=0} = H\left.\frac{\partial b}{\partial s}\right|_{t=0}. \tag{3.18}$$

Similarly, from $\tau_H^2(u) = 0$, we obtain

$$\left.\frac{\partial b}{\partial t}\right|_{t=0} = -H\left.\frac{\partial a}{\partial s}\right|_{t=0}. \tag{3.19}$$

Finally, substituting (3.18) and (3.19) into (3.13) we obtain

$$\left.\frac{\partial c}{\partial t}\right|_{t=0} = \sqrt{e(f_0)(1-H^2)}.$$

□

In particular we have that for such a map $u = (a, b, c)$,

$$\lim_{t\to 0} \frac{t}{c} = \frac{1}{\sqrt{e(f_0)(1-H^2)}} < \infty \tag{3.20}$$

In fact, we have the following important corollary of Lemma 3.5, entirely analogous to Corollary 1.4 in [31]:

Corollary 3.6. *Assume $H \in (-1, 1)$. Let u be an H-harmonic map from $(\mathbb{U}^2, g)$ to $(\mathbb{U}^3, h)$ that extends to be a C^1 map from $\overline{\mathbb{U}^2}$ to $\overline{\mathbb{U}^3}$. Suppose that $f_0 := u|_{\partial\mathbb{U}^2}$ has nowhere vanishing energy density as map from $\mathbb{R}$ to $\mathbb{R}^2$. Then the hyperbolic energy density $e(u)$ is bounded, and furthermore*

$$\lim_{t\to 0} e(u) = \frac{2}{1-H^2}. \tag{3.21}$$

We now use the asymptotics (3.7)-(3.10) to extend an arbitrary boundary map to a map of $\mathbb{U}^2$ whose H-tension field vanishes to second order.

The Extension Operator $\mathcal{E}$

Assume we have an H-harmonic map $u : (\mathbb{U}^2, g) \to (\mathbb{U}^3, h)$ which extends to be at least C^2 up to the boundary, and assume, as always, that the boundary map $f_0 = u|_{\mathbb{U}^2} : \mathbb{R} \to \mathbb{R}^2$ has nowhere vanishing energy density $e(f_0)$. Let $f = (f^1, f^2)$ be some given map again satisfying $e(f) \neq 0$ everywhere on $\mathbb{R}$, that lies in a $C^{2,\alpha}(\mathbb{R}, \mathbb{R}^2)$ neighbourhood of f_0. We begin by extending f to a map $\tilde{f} = (\tilde{f}^1, \tilde{f}^2, \tilde{f}^3) : \mathbb{U}^2 \to \mathbb{U}^3$, where the $\tilde{f}^i$ are polynomials in t of order 2 whose coefficients are functions of $f^1(s)$ and $f^2(s)$ determined by the asymptotics (3.7)-(3.10). In the same way we extend f_0 to $\tilde{f}_0$. Note that these extensions are C^2 only on a neighbourhood of $t = 0$. Next we pick a smooth cut-off function χ which equals 1 on a small enough neighbourhood of $t = 0$, and define

$$\mathcal{E}(f) := \chi(\tilde{f} - \tilde{f}_0) + u. \tag{3.22}$$

For a section ϕ of the pullback bundle $u^*T\mathbb{U}^3$, let $|\phi|_e$ denote the Euclidean norm of ϕ. The operator $\mathcal{E}$ has been specifically concocted to ensure that it extends an arbitrary boundary map to an interior map which is “asymptotically H-harmonic”. More specifically we have the following lemma:

Lemma 3.7. *For u, f_0 and f as above, the extension operator $\mathcal{E}$ satisfies*

(i) $\mathcal{E}(f_0) = u$,

(ii) $\mathcal{E}(f)|_{t=0} = f$,

(iii) $|\tau_H(\mathcal{E}(f))|_e = O(t^3)$.

The remainder of this chapter is devoted to showing the existence of an appropriate "correction term" ϕ_c which will make $\mathcal{E}(f) + \phi_c$ exactly H-harmonic. We will require ϕ_c not to upset the boundary values (which are taken care of by $\mathcal{E}(f)$, which means we require it to decay at infinity. The appropriate rate of decay is determined by the mapping properties of $J_{H,u}$ on certain weighted Sobolev and Hölder spaces, which we now describe.

Remark 3.8. For the harmonic, H=0 case, the extension just constructed is essentially the same as the one constructed by Li & Tam in [30], modified so as to extend the boundary values of a given harmonic map back to the given map. In their subsequent papers [31], [32], Li & Tam use a more sophisticated construction to retain more control over the regularity of the extension, which they then use to deduce regularity results for proper harmonic maps. In the current work we are not concerned with optimal regularity results, so our more basic extension construction suffices. For future work it might however prove useful to modify the Li & Tam technique (which as a first step involves extending boundary values to their *Euclidean* harmonic extensions) to deal with the $H \neq 0$ case.

3.3 Weighted Function Spaces

As remarked earlier, the Jacobi operator $J_{H,u}$ is a *degenerate* elliptic operator. The purpose of the weighted function spaces that we now describe is to provide a setting in which (i) $J_{H,u}$ is a (weight preserving) bounded linear operator, and (ii) weighted versions of the usual elliptic regularity results hold true.

We fix an H-harmonic map $u : \mathbb{U}^2 \to \mathbb{U}^3$ in $C^\infty(\mathbb{U}^2, \mathbb{U}^3)$, and set $\mathcal{P} := u^*(T\mathbb{U}^3)$. We will denote the space of sections of a vector bundle F over $\mathbb{U}^2$ by $\Gamma(F)$. We set $T^j\mathbb{U}^2 :=$

$\overbrace{T^*\mathbb{U}^2 \otimes \ldots \otimes T^*\mathbb{U}^2}^{\text{j times}}$ and $g^j := \overbrace{g^{-1} \otimes \ldots \otimes g^{-1}}^{\text{j times}}$. The natural fibre-preserving metric on $\Gamma(T^j\mathbb{U}^2 \otimes \mathcal{P})$ is then $g^j \otimes h$. In order to simplify notation we shall write ∇ for $u^*(d^\nabla)$, where d^∇ is the Levi-Civita connection on $(\mathbb{U}^3, h)$.

Let (s,t), $t > 0$ be the usual rectangular coordinates on $\mathbb{U}^2$. For $\alpha \in (0,1)$ we define the space $C^{0,\alpha}(\mathbb{U}^2)$ to be the space of functions v on $\mathbb{U}^2$ for which the norm

$$||v||_{0,\alpha} := \sup_{\mathbb{U}^2} |v| + \sup_{\mathbb{U}^2} (t+t') \frac{|v(s,t) - v(s',t')|}{|s-s'|^\alpha + |t-t'|^\alpha}$$

is finite. The space $C^{k,\alpha}(\mathbb{U}^2)$ is then defined as the space of functions v for which $(t\partial_t)^j(t\partial_s)^l v \in C^{0,\alpha}(\mathbb{U}^2)$ for all $j + l \leqslant k$, equipped with the corresponding norm

$$||v||_{k,\alpha} := \sum_{j+l \leqslant k} ||v||_{0,\alpha}.$$

Next we set $C^{k,\alpha}_\delta(\mathbb{U}^2) := t^\delta C^{k,\alpha}(\mathbb{U})^2$ and define the norm

$$||v||_{k,\alpha,\delta} := ||t^{-\delta} v||_{k,\alpha}. \tag{3.23}$$

Finally we define the space of weighted sections $C^{k,\alpha}_\delta(\mathbb{U}^2, \mathcal{P})$ to consist of those $\phi \in \Gamma(\mathcal{P})$ whose components ϕ^i, $i = 1,2,3$, in rectangular coordinates all lie in $C^{k,\alpha}_\delta(\mathbb{U}^2)$, equipped with norm

$$||\phi||_{k,\alpha,\delta} := \sum_{i=1}^{3} ||\phi^i||_{k,\alpha,\delta}.$$

We will also require the use of certain weighted Sobolev spaces. For k a non-negative integer, $1 < p < \infty$, we set

$$W^{k,p}(\mathbb{U}^2) = \{v \,|\, (t\partial_t)^j (t\partial_s)^l v \in L^p(\mathbb{U}^2, dsdt)\}, \text{for all } j + l \leq k,$$

equipped with the norm

$$||v||^p_{k,p} = \sum_{j+l \leq k} \int_{\mathbb{U}^2} |(t\partial_t)^j (t\partial_s)^l v|^p \, dsdt.$$

As for the Hölder spaces, we extend this definition component-wise to obtain the Sobolev spaces $W^{k,p}(\mathbb{U}^2, \mathcal{P})$, with associated norm $|| \cdot ||k, p$. Finally we define weighted versions by setting

$$W^{k,p}_\delta(\mathbb{U}^2, \mathcal{P}) := t^\delta W^{k,p}(\mathbb{U}^2, \mathcal{P}), \tag{3.24}$$

equipped with the norm

$$||\phi||^p_{k,p,\delta} := ||t^{-\delta}\phi||^p_{k,p}. \tag{3.25}$$

For a subdomain $\Omega \subset \mathbb{U}^2$, we denote by $||\cdot||_{k,p;\Omega}$ and $||\cdot||_{k,\alpha;\Omega}$ respectively the restriction to Ω of the norms $||\cdot||_{k,p}$ and $||\cdot||_{k,\alpha}$, and by $W^{k,p}(\Omega,\mathcal{P})$ and $C^{k,\alpha}(\Omega,\mathcal{P})$ respectively the corresponding Banach spaces of sections for which these norms are finite.

The following density result is often useful:

Proposition 3.9 ([28], Lemma 3.9)**.** *If $1 < p < \infty$, $\delta \in \mathbb{R}$ and $k \geqslant 0$, the set of compactly supported smooth sections of $\mathcal{P}$, $C^\infty_c(\mathbb{U}^2,\mathcal{P})$, is dense in $W^{k,p}_\delta(\mathbb{U}^2,\mathcal{P})$.*

Finally we also define Hölder spaces of sections that are continuous on $\overline{\mathbb{U}^2}$. Set $C^{k,\alpha}_{(0)}(\overline{\mathbb{U}^2})$ to be the space of $C^{k,\alpha}$ functions on $\overline{\mathbb{U}^2}$ (in the usual, Euclidean sense), and define a subspace $C^{k,\alpha}_{(\delta)}(\overline{\mathbb{U}^2}) \subset C^{k,\alpha}_{(0)}(\overline{\mathbb{U}^2})$ by

$$C^{k,\alpha}_{(\delta)}(\overline{\mathbb{U}^2}) = \{u \in C^{k,\alpha}_{(0)}(\overline{\mathbb{U}^2}) \,|\, u = O(t^\delta)\}. \tag{3.26}$$

Consider u now as a map from $\overline{\mathbb{U}^2} \to \overline{\mathbb{U}}$, and define $\overline{\mathcal{P}} := u^*T\overline{\mathbb{U}}$. We define $C^{k,\alpha}_{(\delta)}(\overline{\mathbb{U}^2},\overline{\mathcal{P}})$ to be the space of sections $\phi \in \Gamma(\overline{\mathcal{P}})$ such that the components of ϕ in rectangular coordinates lie in $C^{k,\alpha}_{(\delta)}(\overline{\mathbb{U}^2})$.

Scaling

We now make some remarks on the scaling properties of the weighted spaces defined above. This will allow us to write down equivalent weighted norms which are more practical for actual computations. The arguments in this sections are somewhat standard and can be found, for example, in [20], [7], [28] and [34]. For a point $\xi = (\xi^1,\xi^2) \in \mathbb{U}^2$ set $R_\xi := \xi^2/2$. We define the "Whitney square" $Q_\xi \subset \mathbb{U}^2$ centered at ξ to be the set

$$Q_\xi := \{(\zeta^1,\zeta^2) \in \mathbb{U}^2 \,|\, |\zeta^1 - \xi^1| < R_\xi,\, |\zeta^2 - \xi^2| < R_\xi\}.$$

The following "Whitney decomposition" type lemma is easy to prove (see [20], [7]).

Lemma 3.10. *There exists a countable collection of points $\{\xi_i\} \subset \mathbb{U}^2$ and corresponding open sets $\{Q_{\xi_i}\}$ such that the Q_{ξ_i} cover $\mathbb{U}^2$ and are uniformly locally finite: there exists an N such that for each i, Q_{ξ_i} has nontrivial intersection with Q_{ξ_j} for at most N values of j.*

The idea is now to take a Whitney covering of the type described in Lemma 3.10, and to pull back the norm restricted to each square Q_{ξ_i} to a standard, fixed square. To this end we fix an "origin" $o = (0, 2) \in \mathbb{U}^2$, (so that $R_o = 1$) and define the affine map $\Lambda_\xi : Q_o \to Q_\xi$ by

$$\Lambda_\xi(\zeta) := \xi + R_\xi(\zeta - o).$$

Finally, assume that a countable collection of points $\{\xi_i\}$ as described in Lemma 3.10 has been chosen. For a point $\zeta \in \mathbb{U}^2$ we let $t(\zeta)$ denote the vertical height component (i.e. the distance to the boundary $\partial\mathbb{U}^2$). We make the abbreviations $t_i := t(\xi_i)$, $Q_i := Q_{\xi_i}$ and $\Lambda_i := \Lambda_{\xi_i}$. Then we make the following two important observations:

(i) for $\zeta \in Q_i$ we have $\frac{1}{2}t(\zeta) \leqslant t_i \leqslant \frac{3}{2}t(\zeta)$, for all i;

(ii) for a function v and any multi-index γ with $|\gamma| = k$ we have $\partial^\gamma v \circ \Lambda_i \approx t_i^{-k}\partial^\gamma(v \circ \Lambda_i)$.

For a section $\phi \in \Gamma(\mathcal{P})$, we let ϕ_i denote the restriction of ϕ to Q_i. The above two observations suffice to give us the desired result:

Lemma 3.11. *Let $\{\xi_i\} \subset \mathbb{U}^2$ be a countable collection of points such that the open sets $\{Q_i\}$ form a uniformly locally finite cover of $\mathbb{U}^2$. Then we have the following norm equivalences:*

$$||\phi||_{k,p,\delta} \approx \sum_i t_i^{-\delta}||\phi_i||_{k,p;Q_o} \tag{3.27}$$

$$||\phi||_{k,\alpha,\delta} \approx \sup_i t_i^{-\delta}||\phi_i||_{k,\alpha;Q_o} \tag{3.28}$$

3.4 The Jacobi Operator

We now begin our analysis of the Jacobi operator $J_{H,u}$ associated to an H-harmonic map u. We first prove a basic lemma: we show that $J_{H,u}$ is formally self-adjoint on a certain weighted L^2

space. We assume throughout that our initial H-harmonic map u is fixed and that $H \in (-1,1)$. We will often suppress reference to u and H in the symbol for the Jacobi operator and simply write J for $J_{H,u}$.

From here onwards we abbreviate the inner product induced on the fibres of $T^j\mathbb{U}^2 \otimes \mathcal{P}$ by the hyperbolic metrics g and h to $(\cdot,\cdot)$. For ϕ, $\psi \in \Gamma(\mathcal{P})$, let $\langle\cdot,\cdot\rangle$ denote the inner product

$$\langle\phi,\psi\rangle := \int_{\mathbb{U}^2} \phi\cdot\psi\, t^{-4} ds dt.$$

Note that

$$\int_{\mathbb{U}^2} |\phi|^2_{euc}\, t^{-4} ds dt = \int_{\mathbb{U}^2} |t^{-2}\phi|^2_{euc}\, ds dt \approx ||\phi||_{0,2,2}.$$

Lemma 3.12. *$J : C^\infty(\mathbb{U}^2,\mathcal{P}) \to C^\infty(\mathbb{U}^2,\mathcal{P})$ is formally self-adjoint on $L^2_2(\mathbb{U}^2,\mathcal{P})$, with respect to the inner product $\langle\cdot,\cdot\rangle$.*

Proof. Recall that $J(\phi) = \Delta\phi + \mathrm{tr}_g\mathcal{R}(du,\phi)du + 2H(d^\nabla\phi\wedge du)(e_1,e_2)$. Suppose now that ϕ and ψ are smooth sections of $\mathcal{P}$ with compact support. We have immediately that

$$\begin{aligned}\int_{\mathbb{U}^2}(\Delta\phi + \mathrm{tr}_g\mathcal{R}(du,\phi)du,\psi)\,dV_g &= -\int_{\mathbb{U}^2}(\nabla\phi,\nabla\psi) + (\mathrm{tr}_g\mathcal{R}(du,\psi)du,\phi)\,dV_g \\ &= \int_{\mathbb{U}^2}(\Delta\psi + \mathrm{tr}_g\mathcal{R}(du,\psi)du,\phi)\,dV_g,\end{aligned}$$

so we have only to deal with the cross-product term. Define a 1-form ω on $\mathbb{U}^2$ by

$$\omega : X \mapsto (\phi\wedge du(X),\psi).$$

Now, $\omega = w^1 ds + w^2 dt$, where

$$w^1 = \omega(\partial_s) = (\phi\wedge u_s,\psi) \text{ and } w^2 = \omega(\partial_t) = (\phi\wedge u_t,\psi).$$

Therefore

$$\begin{aligned}\partial_s w^2 - \partial_t w^1 &= \partial_s(\phi\wedge u_t,\psi) - \partial_t(\phi\wedge u_s,\psi) \\ &= (\nabla_x\phi\wedge u_t,\psi) + (\phi\wedge\nabla_x u_t,\psi) + (\phi\wedge u_t,\nabla_x\psi) \\ &\quad -(\nabla_y\phi\wedge u_s,\psi) - (\phi\wedge\nabla_y u_t,\psi) - (\phi\wedge u_s,\nabla_y\psi) \\ &= (\nabla_x\phi\wedge u_t + u_s\wedge\nabla_y\phi,\psi) - (\nabla_x\psi\wedge u_t + u_s\wedge\nabla_y\psi,\phi) \\ &\quad +(\phi\wedge(\nabla_x u_t - \nabla_y u_s),\psi)\end{aligned}$$

But $\nabla_x u_t - \nabla_y u_s = [u_s, u_t] = 0$. Recall that $dV_g = t^{-2} ds \wedge dt$, and that $e_1 = t\partial_s$ and $e_2 = t\partial_t$. Therefore

$$\begin{aligned}
\int_{\mathbb{U}^2} ((d^{\nabla}\phi \wedge du)(e_1, e_2), \psi) - ((d^{\nabla}\psi \wedge du)(e_1, e_2), \phi)\, dV_g &= \int_{\mathbb{U}^2} (\nabla_x \phi \wedge u_t + u_s \wedge \nabla_y \phi, \psi) \\
&\quad -(\nabla_x \psi \wedge u_t + u_s \wedge \nabla_y \psi, \phi)\, ds \wedge dt \\
&= \int_{\mathbb{U}^2} d\omega = 0.
\end{aligned}$$

Finally we note that $(\phi, \psi) = c^{-2}\phi \cdot \psi \sim t^{-2}\phi \cdot \psi$, and the result follows. □

3.4.1 Asymptotic Analysis for the Jacobi Operator

Definition 3.13. Let $L : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{P})$ be a second-order partial differential operator acting on sections of the pullback bundle $\mathcal{P}$. L is said to be *uniformly degenerate* if it can be written in coordinates (s,t) as a system of operators that are polynomials in $t\partial_t$ and $t\partial_s$ with coefficients that are at least continuous up to the boundary. For $\phi = (\phi^1, \phi^2, \phi^3) \in C^2(\mathbb{U}^2, \mathcal{P})$ we write

$$L^i\phi = (L\phi)^i = \sum_{j=1}^{3} L^i_j(z, (t\partial_t, t\partial_s))\phi^j, \quad z \in \mathbb{U}^2,\ 1 \leqslant i \leqslant 3.$$

We say L is *elliptic* (as a uniformly degenerate operator) if the homogeneous quadratic principal part $a^i_j(z, X)$ of L^i_j satisfies the usual ellipticity condition

$$\det(a^i_j(z, X)) \geqslant K|X|^6 \quad \text{for all } z \in \mathbb{U}^2,\ X \in \mathbb{R}^2,$$

for some constant $K > 0$.

We now write down the explicit form of $J_{H,u}$, in terms of the coordinates (s,t) on $\mathbb{U}^2$. Let $u(s,t) = (a(s,t), b(s,t), c(s,t))$, and let $\phi = (\phi^1, \phi^2, \phi^3)$ be a smooth section of the pullback bundle $\mathcal{P}$. For a function $X = X(s,t)$ we write X_t and X_s for $\partial X/\partial t$ and $\partial X/\partial s$ respectively. Let $J_{H,u} = (J^1_{H,u}, J^2_{H,u}, J^3_{H,u})$. Then:

$$\begin{aligned}
J^1_{H,u}(\phi) &= (t^2\phi^1_{tt}) + (t^2\phi^1_{ss}) - 2t\left(\frac{t}{c}\right)\left\{a_t\phi^3_t + \phi^1_t c_t + a_s\phi^3_s + \phi^1_s c_s\right\} + \\
&\quad 2\phi^3\left(\frac{t}{c}\right)^2\left\{a_t c_t + a_s c_s\right\} + (2Ht)\left(\frac{t}{c}\right)\left\{b_s\phi^3_t - b_t\phi^3_s + \phi^2_s c_t - \phi^2_t c_s\right\} - \\
&\quad (2H\phi^3)\left(\frac{t}{c}\right)^2\left\{b_s c_t - b_t c_s\right\}
\end{aligned}$$

$$
\begin{aligned}
J^2_{H,u}(\phi) &= (t^2\phi^2_{tt}) + (t^2\phi^2_{ss}) - 2t\left(\frac{t}{c}\right)\left\{b_t\phi^3_t + \phi^2_t c_t + b_s\phi^3_s + \phi^2_s c_s\right\} + \\
&\quad 2\phi^3\left(\frac{t}{c}\right)^2\{b_t c_t + b_s c_s\} + (2Ht)\left(\frac{t}{c}\right)\left\{c_s\phi^1_t - c_t\phi^1_s + \phi^3_s a_t - \phi^3_t a_s\right\} - \\
&\quad (2H\phi^3)\left(\frac{t}{c}\right)^2\{c_s a_t - c_t a_s\}
\end{aligned}
$$

$$
\begin{aligned}
J^3_{H,u}(\phi) &= (t^2\phi^3_{tt}) + (t^2\phi^3_{ss}) + 2t\left(\frac{t}{c}\right)\left\{a_t\phi^1_t + b_t\phi^2_t - c_t\phi^3_t + a_s\phi^1_s + b_s\phi^2_s - c_s\phi^3_s\right\} - \\
&\quad \phi^3\left(\frac{t}{c}\right)^2\left\{(a_t)^2 + (b_t)^2 - (c_t)^2 + (a_s)^2 + (b_s)^2 - (c_t)^2\right\} + \\
&\quad (2Ht)\left(\frac{t}{c}\right)\left\{a_s\phi^2_t - a_t\phi^2_s + \phi^1_s b_t - \phi^1_t b_s\right\} - (2H\phi^3)\left(\frac{t}{c}\right)^2\{a_s b_t - a_t b_s\}
\end{aligned}
$$

The following Lemma is immediate:

Lemma 3.14. *Let $H \in (-1, 1)$. Suppose $u : (\mathbb{U}^2, g) \to (\mathbb{U}^3, h)$ is a proper H-harmonic map that extends to be a C^1 map from $\overline{\mathbb{U}^2}$ to $\overline{\mathbb{U}^3}$. Assume that the boundary map $f_0 := u|_{\partial\mathbb{U}^2} : \mathbb{R} \to \mathbb{R}^2$ has nowhere vanishing energy density. Then $J_{H,u}$ is a uniformly degenerate elliptic partial differential operator.*

Proof. By inspection, using the compatibility conditions listed in Lemma 3.5. Recall that if $e(f_0)$ denotes the energy density of the boundary map, then $\frac{t}{c}\big|_{t=0} = \frac{1}{\sqrt{e(f_0)(1-H^2)}} < \infty$. ☐

Indicial Root Analysis

The first stage in the analysis of a uniformly degenerate operator is the determination of its *indicial roots*, which govern the asymptotic behaviour of the objects in its kernel; i.e. in our situation, the behaviour of the Jacobi fields as $t \to 0$. For $\sigma \in \mathbb{R}$, define the *indicial operator* $I_\sigma : C^\infty(\overline{\mathbb{U}^2}, \overline{\mathcal{P}}\big|_{\mathbb{R}}) \to C^\infty(\overline{\mathbb{U}^2}, \overline{\mathcal{P}}\big|_{\mathbb{R}})$ associated to J by

$$I_\sigma(\phi) := \lim_{t\to 0}(t^{-\sigma} J(t^\sigma\tilde{\phi})),$$

where $\tilde{\phi}$ is an arbitrary extension of ϕ to a smooth section of $\mathcal{P}$ in a neighbourhood of $\partial\mathbb{U}^2$. σ is said to be an *indicial root* of J if $I_\sigma \equiv 0$. In the present situation, the indicial operator takes a particularly simple form:

Lemma 3.15. *J has indicial operator equal to*

$$I_\sigma : \phi \mapsto (\sigma^2 - 3\sigma)\phi,$$

and indicial roots 0 and 3.

Proof. We note that for any function $X = X(s,t)$, $\sigma \in \mathbb{R}$,

$$\begin{aligned} t^{-\sigma} t (t^\sigma X)_t &= t^{-\sigma} t (\sigma t^{\sigma-1} X + t^\sigma X_t) \\ &= sX + tX_t, \end{aligned}$$

and

$$\begin{aligned} t^{-\sigma} t^2 (t^\sigma X)_{tt} &= t^{-\sigma} t^2 (\sigma(\sigma-1) t^{\sigma-2} X + 2\sigma t^{\sigma-1} X_t + t^\sigma X_{tt}) \\ &= \sigma(\sigma-1)X + 2\sigma t X_t + t^2 X_{tt}. \end{aligned}$$

Also

$$\begin{aligned} t^{-\sigma} t (t^\sigma X)_s &= tX_s \\ t^{-\sigma} t^2 (t^\sigma X)_{ss} &= t^2 X_{ss}. \end{aligned}$$

So that

$$\begin{aligned} \lim_{t\to 0} t^{-\sigma} t (t^\sigma X)_t &= \sigma X, & \lim_{t\to 0} t^{-\sigma} t (t^\sigma X)_s &= 0, \\ \lim_{t\to 0} t^{-\sigma} t^2 (t^\sigma X)_{tt} &= \sigma(\sigma-1)X, & \lim_{t\to 0} t^{-\sigma} t^2 (t^\sigma X)_{ss} &= 0. \end{aligned}$$

Now let $\phi \in \Gamma(\overline{\mathcal{P}}|_{\mathbb{R}})$, and let $\tilde{\phi}$ be an arbitrary extension of ϕ to a neighbourhood of $\partial\mathbb{U}^2$. Employing the above observations we obtain:

$$\begin{aligned} \lim_{t\to 0} t^{-\sigma} J^1_{H,u}(t^\sigma \tilde{\phi}) = \sigma(\sigma-1)\phi^1 &- 2\left(\frac{t}{c}\right)_{t=0} \left\{ \left.\frac{\partial c}{\partial t}\right|_{t=0} \sigma\phi^1 \right\} \\ &+ 2\phi^3 \left(\frac{t}{c}\right)^2 \left\{ \left.\frac{\partial a}{\partial t}\right|_{t=0} \left.\frac{\partial c}{\partial t}\right|_{t=0} + \left.\frac{\partial a}{\partial s}\right|_{t=0} \left.\frac{\partial c}{\partial s}\right|_{t=0} \right\} \\ &+ 2H\phi^3 \left(\frac{t}{c}\right)^2 \left\{ \left.\frac{\partial b}{\partial t}\right|_{t=0} \left.\frac{\partial c}{\partial s}\right|_{t=0} - \left.\frac{\partial b}{\partial s}\right|_{t=0} \left.\frac{\partial c}{\partial t}\right|_{t=0} \right\} \end{aligned}$$

We now make use of the compatibility conditions (3.7)-(3.10). We have that

$$\begin{aligned}
\left(\frac{t}{c}\right)_{t=0} &= \frac{1}{\sqrt{e(f_0)(1-H^2)}}, \\
\left.\frac{\partial c}{\partial t}\right|_{t=0} &= \sqrt{e(f_0)(1-H^2)}, \\
\left.\frac{\partial a}{\partial t}\right|_{t=0} &= H\left.\frac{\partial b}{\partial s}\right|_{t=0} \\
\left.\frac{\partial c}{\partial s}\right|_{t=0} &= 0.
\end{aligned}$$

Plugging in these equalities we see that the ϕ^3 terms cancel, and we are left with

$$\lim_{t\to 0} t^{-\sigma} J^1_{H,u}(t^\sigma \tilde{\phi}) = (\sigma^2 - 3\sigma)\phi^1.$$

Similar computations for $J^2_{H,u}$ and $J^3_{H,u}$ give us that $J_{H,u}$ has indicial operator given by

$$I_\sigma : \phi \to (\sigma^2 - 3\sigma)\phi.$$

□

Let $\phi \in \Gamma(\overline{\mathcal{P}})$, and let $|\phi|_e$ denote the Euclidean norm of ϕ. The following result illustrates the role of the indicial roots:

Corollary 3.16. *Suppose ϕ is an Jacobi field that is smooth on $\overline{\mathbb{U}^2}$. Then either $|\phi|_e = O(1)$ or $|\phi|_e = O(t^3)$ as $t \to 0$*

Proof. Since ϕ is smooth up to the boundary, $|\phi|_e = O(t^p)$ for some $p \geqslant 0$. Set $\psi := t^{-p}\phi$. ψ is again smooth on $\overline{\mathbb{U}^2}$, and since ϕ is an Jacobi field, $J(t^p\psi) = 0$. In particular,

$$\lim_{t\to 0} t^{-p} J(t^p \psi) = 0 = I_p(\psi|_{t=0}).$$

Thus, if ϕ (and therefore ψ) is non-trivial, $p = 0$ or 3. □

The above indicial root analysis is equivalent to the fact that for $\phi \in C^2(\overline{\mathbb{U}^2}, \overline{\mathcal{P}})$, we have

$$J(t^\sigma \phi) = \sigma(\sigma - 3)\phi + O(t^{\sigma+1}).$$

This in turn implies that if $\sigma \neq 0$ or 3, then we can find a solution in $\phi \in C^2(\overline{\mathbb{U}^2}, \overline{\mathcal{P}})$ to the equation

$$J(t^\sigma \phi) = t^\sigma \psi + O(t^{\sigma+1}),$$

for any $\psi \in C^2(\overline{\mathbb{U}^2}, \overline{\mathcal{P}})$, simply by setting $\phi := (\sigma(\sigma-3))^{-1}\psi$. Heuristically speaking, this simple observation suggests that the maximal interval of weights for which we can expect to invert is $(0,3)$.

3.4.2 Invertibility on $L^2_2(\mathbb{U}^2, \mathcal{P})$

The following two basic results are proven using the method of scaling described above.

Proposition 3.17. *J extends naturally as a bounded mapping between the following weighted spaces:*

(a) $J : W^{k,p}_\delta(\mathbb{U}^2, \mathcal{P}) \to W^{k-2,p}_\delta(\mathbb{U}^2, \mathcal{P})$ for all $\delta \in \mathbb{R}$, $1 < p < \infty$, $k \geqslant 2$

(b) $J : C^{k,\alpha}_\delta(\mathbb{U}^2, \mathcal{P}) \to C^{k-2,\alpha}_\delta(\mathbb{U}^2, \mathcal{P})$ for all $\delta \in \mathbb{R}$, $0 \leqslant \alpha < 1$, $k + \alpha \geqslant 2$

We also have weighted versions of standard elliptic regularity results:

Lemma 3.18 ([28], Lemma 4.8 and [20], Proposition 3.4)**.** *(a) Let $\delta \in \mathbb{R}$, $\alpha \in (0,1)$ and $2 \leqslant k \in \mathbb{Z}$. Let $\phi \in C^{0,\alpha}_\delta(\mathbb{U}^2, \mathcal{P})$ and suppose that $J\phi \in C^{k-2,\alpha}_\delta$. Then $\phi \in C^{k,\alpha}_\delta(\mathbb{U}^2, \mathcal{P})$ and*

$$||\phi||_{k,\alpha,\delta} \leqslant C\,(\,||J\phi||_{k-2,\alpha,\delta} + ||\phi||_{0,\alpha,\delta})\,. \tag{3.29}$$

(b) Let $\delta \in \mathbb{R}$, $1 < p < \infty$ and $2 \leqslant k \in \mathbb{Z}$. Let $\phi \in W^{0,p}_\delta(\mathbb{U}^2, \mathcal{P})$ and suppose that $J\phi \in W^{k-2,p}_\delta$. Then $\phi \in W^{k,p}_\delta(\mathbb{U}^2, \mathcal{P})$ and

$$||\phi||_{k,p,\delta} \leqslant C\,(\,||J\phi||_{k-2,p,\delta} + ||\phi||_{0,p,\delta})\,. \tag{3.30}$$

Proof. We give the proof of (b) to illustrate the method of scaling. Let $\phi \in W^{0,p}_\delta(\mathbb{U}^2, \mathcal{P})$ and suppose that $J\phi \in W^{k-2,p}_\delta$. Then

$$||\phi||_{k,p,\delta} \leqslant C \sum_i t_i^{-\delta} ||\phi_i||_{k,p;Q_o} \tag{3.31}$$

$$\leqslant C' \sum_i t_i^{-\delta}\,(\,||J\phi_i||_{k-2,p;Q_o} + ||\phi_i||_{0,p;Q_o}) \tag{3.32}$$

$$\leqslant C''\,(\,||J\phi||_{k-2,p,\delta} + ||\phi||_{0,p,\delta})\,. \tag{3.33}$$

In the first and third line we have used the norm equivalence (3.27), and in the second line the standard elliptic estimate on Sobolev spaces (as described, for example, in [18]). ☐

We now show that J is an isomorphism as an operator $J : W_2^{2,2} \to L_2^2$.

Lemma 3.19 ([28], Lemma 4.7, Lemma 4.9)**.** *For all* $\phi, \psi \in L_2^2(\mathbb{U}^2, \mathcal{P})$*,* $\langle J\phi, \psi\rangle = \langle \phi, J\psi\rangle$*. Furthermore,* J *is self-adjoint as an unbounded operator on* $L_2^2(\mathbb{U}^2, \mathcal{P})$*.*

Proof. The first claim follows from the fact that J is formally self-adjoint, and a density argument. By Lemma 3.18, part (b), the domain of J is $W_2^{2,2}(\mathbb{U}^2, \mathcal{P})$, and this space is dense in $L_2^2(\mathbb{U}^2, \mathcal{P})$ by the density of $C_c^\infty(\mathbb{U}^2, \mathcal{P})$ in $W_2^{2,2}(\mathbb{U}^2, \mathcal{P})$. Let J^* denote the Hilbert space adjoint of J. Then the domain of J^* certainly contains $W_2^{2,2}(\mathbb{U}^2, \mathcal{P})$. On the other hand, if ϕ is in $Dom(J^*)$, then there exists $\psi \in L_2^2(\mathbb{U}^2, \mathcal{P})$ such that $\langle \phi, Jv\rangle = \langle \psi, v\rangle$ for all $v \in L_2^2(\mathbb{U}^2, \mathcal{P})$. This means in particular that $J\phi = \psi$ as distributions, and by Lemma 3.18 this implies that $\phi \in W_2^{2,2}(\mathbb{U}^2, \mathcal{P})$. Thus $Dom(J^*) = Dom(J)$ and we are done. □

Lemma 3.20. *Let* $H \in (-1, 1)$*. Suppose* $u : (\mathbb{U}^2, g) \to (\mathbb{U}^3, h)$ *is a proper* H*-harmonic map that extends to be a* C^1 *map from* $\overline{\mathbb{U}^2}$ *to* $\overline{\mathbb{U}^3}$*. Assume that the boundary map* $f_0 := u|_{\partial\mathbb{U}^2} : \mathbb{R} \to \mathbb{R}^2$ *has nowhere vanishing energy density. Then* $J_{H,u}$ *extends to be an isomorphism* $J_{H,u} : W_2^{2,2}(\mathbb{U}^2, \mathcal{P}) \to L_2^2(\mathbb{U}^2, \mathcal{P})$*.*

Proof. First we show injectivity. Let $\phi \in W_2^{2,2}(\mathbb{U}^2, \mathcal{P})$ satisfy $J\phi = 0$. We wish to show that this implies $\phi = 0$, and by the density result Lemma 3.9 it suffices to show that this holds for all $\phi \in C_c^\infty(\mathbb{U}^2, \mathcal{P})$. Assume therefore that $\phi \in C_c^\infty(\mathbb{U}^2, \mathcal{P})$. We use the abbreviation $|du \wedge \phi|^2$ to denote $|du(e_1) \wedge \phi|^2 + |du(e_2) \wedge \phi|^2$. Once again we work with the intrinsic hyperbolic metrics g and h, which, as observed before, is equivalent to working on $L^2(\mathbb{U}^2)$, with density $t^{-4} ds dt$. We estimate

$$\begin{aligned}
\int_{\mathbb{U}^2} (J\phi, \phi)\, dV_g &= \int_{\mathbb{U}^2} (\Delta\phi, \phi) + \mathrm{tr}_g \mathcal{R}(du, \phi, du, \phi) + 2H(\phi, d^\nabla\phi(e_1) \wedge du(e_2) + du(e_1) \wedge d^\nabla\phi(e_2))\, dV_g \\
&= \int_{\mathbb{U}^2} -|d^\nabla\phi|^2 - |du \wedge \phi|^2 + 2H\left\{(d^\nabla\phi(e_1), du(e_2) \wedge \phi) + (d^\nabla\phi(e_2), \phi \wedge du(e_1))\right\} dV_g \\
&\leqslant \int_{\mathbb{U}^2} -|d^\nabla\phi|^2 - |du \wedge \phi|^2 + |H||d^\nabla\phi|^2 + |H||du \wedge \phi|^2\, dV_g \\
&\leqslant -\int_{\mathbb{U}^2} (1 - |H|)|d^\nabla\phi|^2\, dV_g \\
&\leqslant \frac{-(1 - |H|)}{4} \int_{\mathbb{U}^2} |\phi|^2\, dV_g,
\end{aligned}$$

where in the final line we have made use of the inequality

$$\int |d^{\nabla}\phi|^2 \geq \int |\nabla|\phi||^2,$$

(sometimes referred to as Kato's inequality), and the estimate for the scalar Laplacian on the hyperbolic plane

$$\int |\nabla f|^2 = -\int f\Delta f \geq \frac{1}{4}\int |f|^2.$$

Therefore J satisfies the following estimate:

$$||\phi||_{0,2,2} \leqslant \frac{4}{(1-|H|)}||J\phi||_{0,2,2}.$$

Therefore $J\phi = 0$ implies $\phi = 0$. Thus the L^2_2 kernel is trivial, and J is injective. But by Lemma 3.19, J is self-adjoint as an unbounded operator on L^2_2. Therefore its index is 0, which means that it is also surjective, and thus an isomorphism. ☐

3.4.3 Invertibility on Hölder Spaces

In this section we prove the following major result:

Theorem B. *Let $H \in (-1,1)$. Suppose $u : (\mathbb{U}^2, g) \to (\mathbb{U}^3, h)$ is a proper H-harmonic map that extends to be a $C^{k,\alpha}$ map from $\overline{\mathbb{U}^2}$ to $\overline{\mathbb{U}^3}$, for some $k \geqslant 2$ and $\alpha \in (0,1)$. Assume that the boundary map $f_0 := u|_{\partial\mathbb{U}^2} : \mathbb{R} \to \mathbb{R}^2$ has nowhere vanishing energy density. Then the Jacobi operator associated to u extends to be an isomorphism $J : C^{k,\alpha}_\delta(\mathbb{U}^2, \mathcal{P}) \to C^{k-2,\alpha}_\delta(\mathbb{U}^2, \mathcal{P})$ for all δ satisfying $0 < \delta < 3$.*

Remark 3.21. Note that this problem is *unobstructed*, in the sense that the appropriate linearised operator is actually invertible, not merely Fredholm of index zero. The linearised problem associated to the perturbation of CMC surfaces (c.f. Chapter 4) will turn out to be obstructed, and we can expect to perturb only at a nondegenerate solution - i.e. one for which the associated linearised (conformality) operator has trivial kernel. This observation is in keeping with the fact that we have Li & Tam's uniqueness theorem for (C^1) harmonic maps [31], whereas we know there exist curves of non-uniqueness for the asymptotic Plateau problem [6]. Linking the nondegeneracy condition just described to a geometric condition on the boundary curve is an interesting, non-trivial problem.

The transition from the *a priori* L^2 estimates obtained in the previous section to the desired Hölder-space results requires the introduction of certain techniques from the theory of microlocal analysis in the context of differential edge operators; in particular, our approach from now on will involve the explicit construction of a parametrix for the Jacobi operator J associated to an H-harmonic map. Our primary reference for this material is [34]. We will give an outline of the relevant constructions and proofs, and refer to [34] for the full details.

Remark 3.22. It is possible to push the "*a priori* estimate" approach further to obtain invertibility results on L^2_δ for δ other than 2, but except in cases where we have a high degree of symmetry (as in the case for the spherical caps, for example) it is often unclear how to pass from the L^2 estimates to Hölder space estimates. The approach involving the edge calculus, described below, requires a substantial amount of technical preparation, but the pay-off is considerable: the generalised inverses and orthogonal projectors constructed in the L^2 setting are automatically bounded operators on the weighted Hölder spaces, so the "transition" is trivial.

The Normal Operator and the Edge Calculus

The final ingredient that we need to prove our full invertibility result, Theorem B, is a linear operator obtained from the Jacobi operator by "freezing" the coefficients at the boundary; this is the so-called *normal operator* associated to J, that we now define:

Definition 3.23. Let L be a degree 2 uniformly degenerate elliptic operator acting on sections of a (rank 3) vector bundle F over $\mathbb{U}^2$, with components

$$L^{jk} = \sum_{\alpha+\beta\leqslant 2} A^{jk}_{\alpha\beta}(s,t)(t\partial_s)^\alpha(t\partial_t)^\beta,\ j,k=1,2,3.$$

We define the *normal operator* associated to L at a boundary point $p=(\tilde{s},0)\in\partial M$, denoted $N_p(L)$, to be the elliptic operator given in component form by

$$N_p(L)^{jk} = \sum_{\alpha+\beta\leqslant 2} A^{jk}_{\alpha\beta}(\tilde{s},0)(t\partial_s)^\alpha(t\partial_t)^\beta,\ j,k=1,2,3, \tag{3.34}$$

where the dependence on p is purely parametric. (We shall often omit reference to the base point p.)

Set

$$A^i = \frac{(\partial_s)u^i(p)\sqrt{1-H^2}}{e(f)(p)}.$$

Then the 3×3 matrix form of the normal operator $N(J_{H,u})$ calculated at p is

$$\begin{pmatrix} \Delta_{\mathbb{H}} - 2t\partial_t & 2Ht\partial_s & -2A^1 t\partial_s \\ -2Ht\partial_s & \Delta_{\mathbb{H}} - 2t\partial_t & -2A^2 t\partial_s \\ 2A^1 t\partial_s & 2A^2 t\partial_s & \Delta_{\mathbb{H}} - 2t\partial_t \end{pmatrix}.$$

Definition 3.23 appears to involve simply a symbolic substitution, but there is a genuine geometric meaning to the operator N, which we shall now outline. We change approach slightly, and from now on work exclusively with the manifold with boundary $\overline{\mathbb{U}^2}$. So as to simplify notation we set $M := \overline{\mathbb{U}^2}$. Our linearised operator J is a classical elliptic PDO on the interior of M, and from the pioneering work of Hörmander [24] we know that a parametrix for J can therefore be constructed there. The Schwartz kernel for such a parametrix is a distribution on the manifold with corner $M \times M$, with a well understood (conormal) singularity along the diagonal $\triangle$. The additional complication arises from the fact that this kernel must necessarily have additional singularities at the intersection of the diagonal with the corner $\partial M \times \partial M$. In order to analyse these additional singularities we perform a standard blow-up of the product $M \times M$ to obtain the so-called *edge double product* manifold $M \times_o M$ defined by

$$M \times_o M := ((M \times M) \setminus S) \sqcup (N^+(S)/\mathbb{R}),$$

where $S = (\partial M \times \partial M) \cap \triangle$, and $(N^+(S)/\mathbb{R})$ denotes the interior spherical normal bundle of S in $M \times M$. We denote by $b : M \times_o M \to M \times M$ the blow-down map which is the identity on the interior of $M \times M$, and collapses $(N^+(S)/\mathbb{R})$ to S. $M \times_o M$ has three boundary surfaces: the "left" and "right" boundaries B_{10} and B_{01}, corresponding to the boundary surfaces of $M \times M$, and the "front face" of $M \times_o M$, B_{11}, which corresponds to the interior spherical normal bundle of S. The normal operator $N(J)$ can then be defined as being the restriction to the front face (in a suitable choice of coordinate system) of the lift of J to $M \times_o M$. Denote now by G the interior parametrix obtained via classical methods mentioned above, and write

$$JG = I - R_1 \tag{3.35}$$

$$GJ = I - R_2, \tag{3.36}$$

where I is the identity operator, and consider the lift of these operators to $M \times_o M$. The (lifted) remainder terms R_1 and R_2 are not compact because they do not vanish at the front face. The invertibility of the normal operator is thus precisely what we require in order to "solve away" the boundary values of R_1 and R_2 to obtain compact error terms. We now briefly describe the appropriate operator spaces and basic results that we will need to make the above sketch argument more precise.

We begin by defining spaces of polyhomogeneous conormal distributions. The exposition here is a very brief summary of the presentation in [34], Appendix 2A; further details can also be found in [38]. Let X be a manifold with corners and let k be a nonnegative integer such that any boundary point p is contained in a corner of maximal codimension k. We fix coordinates $x^1, \ldots, x^k, y$ near p, where the x^i are boundary defining functions for the boundary hypersurfaces intersecting the corner at p and y is a set of coordinates along this corner. Let $\mathcal{V}_b$ denote the space of smooth vector fields on X which are tangent to all boundaries. We set

$$\mathcal{A}^0(X) = \{v \,|\, V^1 \ldots V^l v \in L^\infty(X), \, \forall V^i \in \mathcal{V}_b, \, \forall l\}$$

to be the basic conormal space. Let $s = (s_1, \ldots, s_J)$ be a multi-index of complex numbers, and set $\mathcal{A}^s(X) = x^s \mathcal{A}^0(X)$. Then the general conormal space of functions is defined to be $\mathcal{A}^*(X) = \cup_s \mathcal{A}^s(X)$. Suppose now that X has only one boundary hypersurface. The space of polyhomogeneous distributions on X, denoted $\mathcal{A}^*_{phg}(X)$, is defined to be the space of those conormal distributions which have the following asymptotic expansion:

$$v \sim \sum_{\mathrm{Re}\ s_j \to \infty} \sum_{p=0}^{p_j} x^{s_j} (\log x)^p a_{j,p}(x, y), \tag{3.37}$$

where the $a_{j,p}$ are functions which are smooth up to the boundary of X. Now let E denote an index set, that is, a discrete subset of $\mathbb{C} \times (\mathbb{N} \cup \{0\}$ satisfying $(s_j, p_j) \in E$, $|(s_j, p_j)| \to \infty \implies \mathrm{Re}\ s_j \to \infty$. Then $\mathcal{A}^E_{phg}(X)$ denotes the space of distributions with polyhomogeneous expansions of the form (3.37), with $(s_j, p_j) \in E$. This definition can then be extended to allow for X having many possibly intersecting codimension one boundary components. In this case we specify an index set E_i corresponding to each boundary face M_i, $i = 1, \dots, J$, and set $\mathcal{E}$ to be the J-tuple of index sets $(E_1, \dots, E_J)$. Then $\mathcal{A}^{\mathcal{E}}_{phg}(X)$ will denote the set of distributions with expansions of the form (3.37) at the interior of the face M_i with index set E_i, and with product type expansions at the corners. Now let $Y \subset X$ be an embedded submanifold, and let $\mathcal{E}$ be an index set for the boundary of X. We set

$$\mathcal{A}^{\mathcal{E}}_{phg} I^m(X, Y)$$

to be the space of distributions that have a conormal singularity of order m along Y on the interior (see [24]) and at all boundary faces have expansions of the form (3.37) with coefficients conormal to the intersection of Y with each boundary face. As always, $\mathcal{A}^{\mathcal{E}}_{phg} I^*(X, Y)$ will denote the union of these spaces over all m. Finally these constructions can of course be extended to polyhomogeneous sections of vector bundles over a manifold with corners.

Recall now the definition of the edge double product $M \times_o M$ (with boundary faces B_{10}, B_{01}, B_{11}), and let $\triangle_e$ denote the lifted diagonal $b^*\triangle$. We can now define the *small calculus*:

Definition 3.24.

$$\Psi^*_e(M) := \mathcal{A}^{\mathcal{E}_0}_{phg} I^*(M \times_o M, \triangle_e),$$

where $\mathcal{E}_0 = (\emptyset, \emptyset, (0, 0))$.

Thus, an operator $A \in \Psi^*_e$ corresponds to a kernel κ_A which is conormal along the lifted diagonal $\triangle_e$, vanishes to infinite order at the side boundaries B_{10}, B_{01} and is smooth across the front face B_{11}. Ψ^*_e is filtered by the spaces Ψ^m_e consisting of elements of order m. We define the *large calculus* by adding to the small calculus elements which are smooth on the interior on $M \times_o M$ and polyhomogeneous conormal at all boundary faces:

Definition 3.25.

$$\Psi^{m,\mathcal{E}}_e(M) := \{C = A + B \,|\, A \in \Psi^m_e(M),\ B \leftrightarrow \kappa_B \in \mathcal{A}^{\mathcal{E}}_{phg}(M \times_o M)\}.$$

Finally, if $\mathcal{F} = (F_{10}, F_{01})$ is a pair of index sets for the two boundary faces in $M \times M$, we set

$$\Psi^{-\infty,\mathcal{F}}(M) := \mathcal{A}^{\mathcal{F}}_{phg}(M \times M).$$

Remark 3.26. In the original treatment of this material, [34], the operators in question are viewed as acting on half-densities, and the various spaces we defined so far are actually constructed with this viewpoint built in. We have omitted this aspect so as to simplify the current exposition.

We now list some basic results from [34] which we will need later on. By using standard symbol calculus techniques one can prove

Theorem 3.27. *If $A \in \Psi^m_e$ is elliptic then there exists $B \in \Psi^{-m}_e$ such that $R_1 := AB - I \in \Psi^{-\infty}_e$ and $R_2 := BA - I \in \Psi^{-\infty}_e$.*

Note that R_1 and R_2 are smoothing only on the interior $M \times M$, and are therefore not compact. Theorem 3.27 provides us with our first approximation for a "good" parametrix. The following two results show us boundedness of elements of the large calculus when acting between weighted Sobolev and Hölder spaces:

Lemma 3.28. *Let $A \in \Psi^{-\infty,\mathcal{E}}_e$ for some collection of index sets $\mathcal{E} = (E_{10}, E_{01}, E_{11})$. Then $A : W^{k,2}_\delta(M, \mathcal{P}) \to W^{k-2,2}_\delta(M\mathcal{P})$ is bounded for $k \geq 2$, provided $Re(E_{10}) + \frac{1}{2} > \delta > -Re(E_{01}) - \frac{1}{2}$, and $Re(E_{11}) \geq 0$.*

Lemma 3.29. *Let $A \in \Psi^{m,\mathcal{E}}_e$. Then $A : A : C^{k,\alpha}_\delta(M, \mathcal{P}) \to C^{k-2,\alpha}_\delta(M\mathcal{P})$ is bounded for all $k \geq 2$, provided $Re(E_{10}) > \delta > -Re(E_{01}) - 1$, $Re(E_{11}) \geq 0$.*

The following result gives us a criterion for compactness:

Proposition 3.30. *Suppose $A \in \Psi^{m,\mathcal{E}}_e$, where $m < 0$, $Re(E_{10}) > -\frac{1}{2}$, $Re(E_{01}) > -\frac{1}{2}$ and $Re(E_{11}) > 0$. Then A is compact as a mapping on $W^{k,2}_\delta(M, \mathcal{P})$ and on $C^{k,\alpha}_\delta(M, \mathcal{P})$.*

And finally the following is a form of "regularity up to the boundary" result for elliptic edge operators:

Theorem 3.31. *Let L be an elliptic edge operator with (constant) indicial roots $\{\sigma_j\}$. Suppose $\phi \in L^2_\delta(X, dxdy)$ satisfies $L\phi = 0$, where x is a defining function for X and y restricts to coordinates on ∂X. Then ϕ admits an asymptotic expansion of the form*

$$\phi \sim \sum_{Re\ \sigma_j \to \infty} \sum_{l=0}^{\infty} \sum_{p=0}^{p_j} x^{\sigma_j + l} (\log x)^p \phi_{j,l,p}(y), \tag{3.38}$$

with $Re(\sigma_j) > \delta - \frac{1}{2}$ for all j.

Analysis of $N(L)$ and Proof of Theorem B

We begin by making the important observation that $N(L)$ is invariant under dilations $(s,t) \mapsto (\lambda s, \lambda t)$ and linear translations in the s direction. We start our analysis by conjugating $N(L)$ by the Fourier transform in the tangential direction s to reduce to the following ordinary differential operator:

$$\widehat{N(L)}^{jk} = \sum_{\alpha+\beta \leqslant 2} A^{jk}_{\alpha\beta}(\tilde{s}, 0)(it\eta)^\alpha (t\partial_t)^\beta. \tag{3.39}$$

We next exploit the scale invariance property by making a change of variables $\lambda = t|\eta|$ and $\hat{\eta} = \eta/|\eta|$ in (3.39) to obtain

$$B(L)^{jk} = \sum_{\alpha+\beta \leqslant 2} A^{jk}_{\alpha\beta}(\tilde{s}, 0)(i\lambda\hat{\eta})^\alpha (\lambda\partial_\lambda)^\beta. \tag{3.40}$$

$B(L)$ is said to be an (elliptic), *totally characteristic* differential edge operator of *Bessel type*. It's mapping properties are closely related to those of $N(L)$. Define weighted spaces of functions on $\mathbb{R}^+$ by

$$\mathcal{H}^{k,\delta,l} = \{u \,|\, \varphi u \in \lambda^\delta W^{k,2}(\mathbb{R}^+, d\lambda),\ (1-\varphi)u \in \lambda^{-l} W^{k,2}(\mathbb{R}^+, d\lambda)\},$$

where $\varphi \in C^\infty_c(\mathbb{R}^+)$ equals 1 near zero. The basic result proven in [34] is the following

Theorem 3.32 ([34], Theorem 4.4, Lemma 5.5)**.** *$B(L) : \mathcal{H}^{2,\delta,-\delta} \to \mathcal{H}^{0,\delta,-\delta-2}$ is Fredholm for all $\delta \notin \{\delta_1 + \frac{1}{2}, \delta_2 + \frac{1}{2}\}$.*

Sketch proof: We construct parametrices separately near $\lambda = 0$ and near $\lambda = \infty$. Near $\lambda = 0$ a right parametrix is constructed by means of the general procedure outlined above: First, a first approximation parametrix A_0 is constructed using Theorem 3.27. The error term $R_1 = B(L)A_0 - I \in \Psi_e^{-\infty}$ associated to this construction is not compact since it does not vanish at the front face, so a correction term A_1 must be added. This is accomplished by solving the equation

$$(B(L)A_1)|_{B_{11}} = R_0|_{B_{11}}. \tag{3.41}$$

At this stage the value of the weight parameter δ becomes crucial: by taking the Mellin transform of the equation (3.41) one can readily ascertain that for $\delta \neq (\delta_1 + 1/2), (\delta_2 + 1/2)$ there exists a solution to (3.41). We thus obtain a right parametrix for $B(L)$ (near $\lambda = 0$) which is bounded and is a right inverse up to an error which vanishes on the front face. The parametrix near $\lambda = \infty$ is constructed directly by taking the inverse of the symbol of $B(L)$, and is not sensitive to the choice of δ. Patching these two parametrices together we obtain a right parametrix for $B(L)$ with compact associated error term. Finally a duality argument also gives us a left parametrix, and we are done. □

The basic result we need in order to prove Theorem B is the following

Lemma 3.33. *Let L be a degree 2 uniformly degenerate elliptic operator acting on sections of a vector bundle F over M, and suppose L has exactly 2 indicial roots δ_1 and δ_2. If $N(L) : W^{2,2}_{\delta+1/2}(M,F) \to L^2_{\delta+1/2}(M,F)$ is an isomorphism for one value of $\delta \in (\delta_1, \delta_2)$ then it is an isomorphism for* every *$\delta \in (\delta_1, \delta_2)$, and for all such δ, L is a Fredholm map in either of the two cases*
(i) $L : W^{2,2}_{\delta+1/2}(M,F) \to L^2_{\delta+1/2}(M,F)$ and
(ii) $L : C^{k,\alpha}_{\delta}(M,F) \to C^{k-2,\alpha}_{\delta}(M,F)$.

Proof. Assume that $N(L) : W^{2,2}_{\delta'+1/2}(M,F) \to L^2_{\delta'+1/2}(M,F)$ is invertible for some fixed $\delta' \in (\delta_1, \delta_2)$. Then $B(L) : \mathcal{H}^{2,\delta',-\delta'} \to \mathcal{H}^{0,\delta',-\delta'-2}$ is also invertible. We now show that the invertibility of $B(L)$ for a single value of δ implies its invertibility on the entire interval (δ_1, δ_2). We start by defining a 1-parameter family of Fredholm operators $B_\delta(L) : \mathcal{H}^{2,0,0} \to \mathcal{H}^{0,0,-2}$ by setting

$$B_\delta(L)\phi := t^{-\delta} B(L) t^{\delta} \phi.$$

By Theorem 3.32 above we have that for $\delta \in (\delta_1 + 1/2, \delta_2 + 1/2)$, the $B_\delta(L)$ constitute a continuous family of Fredholm operators. Thus the index is constant across the elements of the family. This in turn implies that $B(L) : \mathcal{H}^{2,\delta,-\delta} \to \mathcal{H}^{0,\delta,-\delta-2}$ itself has constant index. Now, if

$B(L)$ is invertible for $\delta = \delta'$ this means it has index zero there, and therefore has index zero for all $\delta \in (\delta_1 + 1/2, \delta_2 + 1/2)$. To show that $B(L)$ is actually invertible on this range of weights it therefore suffices to show that $B(L)$ has no kernel. For this we use Theorem 3.31: suppose we have $B(L)v = 0$ for some $v \in \mathcal{H}^{0,\delta'',-\delta''} = W^{2,2}_{\delta''}(\mathbb{R}^+, d\lambda)$. Then v has an expansion of the form (3.38), with first term $O(\lambda^{\delta_2})$. But then v must also lie in $L^2_{\delta'}(\mathbb{R}^+, d\lambda)$, and therefore $v = 0$. Thus $B(L)$, and consequently $N(L)$, is invertible for all $\delta \in (\delta_1 + 1/2, \delta_2 + 1/2)$.

We now turn to showing the Fredholm properties of L. Fix a $\delta \in (\delta_1, \delta_2)$, and assume that $N(L) : W^{2,2}_{\delta+1/2} \to L^2_{\delta+1/2}$ is invertible. Denote by $N(G) \in \Psi_e^{-2,\mathcal{H}}$ the inverse for $N(L)$, where $\mathcal{H} = (H_{10}, H_{01}, H_{11})$ is some collection of index sets whose precise form we shall not go into. We can construct parametrices for $L : W^{2,2}_{\delta+1/2} \to L^2_{\delta+1/2}$, essentially using the same procedure employed above for $B(L)$, but switching the first two steps. We thus begin by taking $A_0 \in \Psi_e^{-2,\mathcal{H}}$ to be an extension off the front face of $N(G)$. Then $LA_0 = I - R_0$, where R_0 vanishes to first order on the front face but still has a conormal singularity along the lifted diagonal $\triangle_e$, but which also vanishes on approach to the front face B_{11}. Using the small calculus (Theorem 3.27) we then pick an A_1 supported near $\triangle_e$ such that LA_1 cancels off this conormal singularity. This gives us that $L(A_0 + A_1) = I - R_1$ where $R_1 \in \Psi^{-\infty,H_{10},H_{01},1}$ and is, in particular, compact. Finally we again use a duality argument to obtain a left parametrix for L. □

For the situation at hand, $L = J_{H,u}$, we have in fact already shown that $N(J_{H,u}) : W^{2,2}_{\delta} \to L^2_{\delta}$ is invertible for $\delta = 2$. Recall (3.6) that by $\Sigma_H : \mathbb{U}^2 \to \mathbb{U}^3$ we mean the canonical H-harmonic map which defines a surface in $(\mathbb{U}^3, h)$ equidistant to a totally geodesic copy of $\mathbb{U}^2 \subset \mathbb{U}^3$.

Lemma 3.34. *Let $p = (\tilde{s}, 0) \in \partial\mathbb{U}^2$, and assume that $u = (a, b, c) : (\mathbb{U}^2, g) \to (\mathbb{U}^3, g)$ is a proper H-harmonic map with non-vanishing energy density $e(f_0)$ on the boundary, satisfying $(\partial_s a)(p) = 0$. Then the normal operator at the point p associated to the Jacobi operator $J_{H,u}$ is equal to the Jacobi operator associated to Σ_H, i.e.*

$$N_p(J_{H,u}) = J_{H,\Sigma_H}.$$

Proof. By direct computation: consider, for example, the expression for $J^1_{H,u}$:

$$\begin{aligned} J^1_{H,u}(\phi) &= (t^2\phi^1_{tt}) + (t^2\phi^1_{ss}) - 2t\left(\frac{t}{c}\right)\left\{a_t\phi^3_t + \phi^1_t c_t + a_s\phi^3_s + \phi^1_s c_s\right\} + \\ &\quad 2\phi^3\left(\frac{t}{c}\right)^2\left\{a_t c_t + a_s c_s\right\} + (2Ht)\left(\frac{t}{c}\right)\left\{b_s\phi^3_t - b_t\phi^3_s + \phi^2_s c_t - \phi^2_t c_s\right\} - \\ &\quad (2H\phi^3)\left(\frac{t}{c}\right)^2\left\{b_s c_t - b_t c_s\right\}. \end{aligned}$$

Then, since $e(f_0)(p) = b_s(p)$, we have

$$\begin{aligned} N_p(J_{H,u})^1(\phi) &= (t^2\phi^1_{tt}) + (t^2\phi^1_{ss}) - 2t\left(\frac{1}{b_s(p)\sqrt{1-H^2}}\right)\left\{Hb_s\phi^3_t + b_s(p)\sqrt{1-H^2}\phi^1_t\right\} + \\ &\quad 2\phi^3\left(\frac{1}{(b_s(p))^2(1-H^2)}\right)\left\{H(b_s(p))^2\sqrt{1-H^2}\right\} + \\ &\quad (2Ht)\left(\frac{1}{b_s(p)\sqrt{1-H^2}}\right)\left\{b_s(p)\phi^3_t + b_s(p)\sqrt{1-H^2}\phi^2_s\right\} - \\ &\quad (2H\phi^3)\left(\frac{1}{(b_s(p))^2(1-H^2)}\right)^2\left\{(b_s(p))^2\sqrt{1-H^2}\right\}. \end{aligned}$$

Thus the $b_s(p)$ terms cancel and we obtain $N_p(J_{H,u})^1(\phi) = J^1_{H,\Sigma_H}(\phi)$. $\square$

We can now proceed with the

Proof of Theorem B. By Lemma 3.34 and Lemma 3.20 we have that $N(J_{H,u})$ is invertible as a map $N(J_{H,u}) : W^{2,2}_\delta(\mathbb{U}^2, \mathcal{P}) \to L^2_\delta(\mathbb{U}^2, \mathcal{P})$ for all $\delta \in (1/2, 7/2)$, and by Lemma 3.33, for all such δ, $J_{H,u} : W^{2,2}_\delta(\mathbb{U}^2, \mathcal{P}) \to L^2_\delta(\mathbb{U}^2, \mathcal{P})$ is a Fredholm map. We pick a generalised inverse G for J and orthogonal projectors Π_1 and Π_2 onto the kernel and cokernel respectively of J in L^2_2:

$$GJ = I - \Pi_1 \tag{3.42}$$

$$JG = I - \Pi_2. \tag{3.43}$$

Again using the invertibility on L^2_2 result, Lemma 3.20, we conclude that Π_1 and Π_2 vanish. But employing the same method used for $B(L)$ above, we can conclude that Π_1 and Π_2 vanish for all $\delta \in (1/2, 7/2)$. Finally by the boundedness of these operators on weighted Hölder spaces we automatically conclude that $L : C^{k,\alpha}_\delta(\mathbb{U}^2, \mathcal{P}) \to C^{k-2,\alpha}_\delta(\mathbb{U}^2, \mathcal{P})$ is invertible for all $\delta \in (0, 3)$. $\square$

3.5 The Perturbation Result

We will need the following implicit function theorem for Banach spaces:

Theorem 3.35. *Let X_1, X_2 and Y be Banach spaces and let L be a mapping from an open set $O \subset X_1 \times Y$ into X_2. Let (x_0, y_0) be a point in O satisfying*

(i) $L(x_0, y_0) = 0$,

(ii) L is k times continuously differentiable at (x_0, y_0),

(iii) the partial Fréchet derivative $D_y L(x_0, y_0)$ is invertible.

Then there exists a neighbourhood $\mathcal{N}$ of y_0 in Y such that for each $y \in Y$ there exists an $x_y \in X_1$ such that $L(x_y, y) = 0$, and furthermore the mapping $y \mapsto x_y$ is C^k smooth.

We are now ready to prove the main result of this chapter, a perturbation theorem for proper H-harmonic maps which solve a Dirichlet problem at infinity.

Theorem C. *Let $H \in (-1, 1)$. Suppose $u : (\mathbb{U}^2, g) \to (\mathbb{U}^3, h)$ is a proper H-harmonic immersion that extends to be a $C^{k,\alpha}$ map from $\overline{\mathbb{U}^2}$ to $\overline{\mathbb{U}^3}$, for $0 < \alpha < 1$. Assume that the boundary map $f_0 := u|_{\partial\mathbb{U}^2} : \mathbb{R} \to \mathbb{R}^2$ has nowhere vanishing energy density. Then there exists a neighborhood $\mathcal{N}$ of f_0 in $C^{k,\alpha}(\mathbb{R}, \mathbb{R}^2)$ such that for every $f \in \mathcal{N}$, there exists a proper H-harmonic extension of f, $u_f \in C^{k,\gamma}(\overline{\mathbb{U}^2}, \overline{\mathbb{U}^3})$, and furthermore the map $f \mapsto u_f$ is $C^{k,\gamma}$ smooth.*

Proof. Let $f \in C^{2,\alpha}(\mathbb{R}, \mathbb{R}^2)$. We begin by extending f to the "asymptotically harmonic" map $\mathcal{E}(f) : \mathbb{U}^2 \to \mathbb{U}^3$ defined by (3.22). By construction we have that $\mathcal{E}(f) \in C^{2,\gamma}(\overline{\mathbb{U}^2}, \overline{\mathbb{U}^3})$, for some $0 < \gamma < \alpha$, and $\tau(\mathcal{E}(f)) \in C^{0,\gamma}_{\gamma}(\mathbb{U}^2, \mathcal{P})$. We now perturb $\mathcal{E}(f)$ using a $\psi \in C^{2,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P})$. Namely, we define a map $u_{f,\psi} : \mathbb{U}^2 \to \mathbb{U}^3$ by

$$u_{f,\psi}(\xi) = \exp_{\mathcal{E}(f)(\xi)} \psi(\xi),$$

where exp denotes the exponential map associated to the hyperbolic metric h, and ψ is chosen to be small enough so that $u_{f,\psi}$ is again a proper immersion. We note that $u_{f,\psi} \in C^{2,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathbb{U}^3) \hookrightarrow C^{2,\gamma}(\overline{\mathbb{U}^2}, \overline{\mathbb{U}^3})$ (see, for example, [28], Chapter 3).

Note that $u_{f_0,0} \equiv u$, and furthermore, since $1 + \gamma > 0$, $u_{f,\psi}|_{\mathbb{R}} = \mathcal{E}(f)|_{\mathbb{R}} = f$ for all $\psi \in C^{2,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P})$. We wish to study a neighborhood of u via the Implicit Function Theorem, and to that end we define the map $\mathcal{T}_H : C^{2,\alpha}(\mathbb{R}, \mathbb{R}^2) \times C^{2,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P}) \to C^{0,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P})$ by

$$\mathcal{T}_H : (f, \psi) \mapsto \tau_H(u_{f,\psi}),$$

where τ_H is the H-tension field operator. Let $\mathcal{U}$ denote a small neighborhood of $(f_0, 0)$ in $C^{2,\alpha}(\mathbb{R}, \mathbb{R}^2) \times C^{2,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P})$. Then $\mathcal{T}_H : \mathcal{U} \to C^{0,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P})$ is a smooth map. We note that $\mathcal{T}_H(f_0, 0) = \tau_H(u) = 0$, and linearising $\mathcal{T}_H$ at the point $(f_0, 0)$ we obtain

$$D_{(f_0,0)}\mathcal{T}_H : (\hat{f}, \hat{\psi}) \mapsto J_{H,u} D\mathcal{E}(\hat{f}) + J_{H,u}\hat{\psi}.$$

Thus the Fréchet partial derivative $D_\psi \mathcal{T}$ at the point $(f_0, 0)$ is given by $J_{H,u} : C^{2,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P}) \to C^{0,\gamma}_{1+\gamma}(\mathbb{U}^2, \mathcal{P})$, and is therefore invertible by Theorem B, since $0 < 1 + \gamma < 3$. Applying the IFT for Banach spaces we conclude that there exists a neighborhood $\mathcal{N}$ of f_0 in $C^{2,\alpha}(\mathbb{R}, \mathbb{R})$ and a $\theta : \mathcal{N} \to C^{2,\gamma}_{\delta}(\mathbb{U}^2, \mathcal{P})$ such that for every $f \in \mathcal{N}$, $(f, \theta(f))$ satisfies $\mathcal{T}(f, \theta(f)) = 0$, i.e., $u_{f,\theta(f)}$ is H-harmonic, and furthermore $u_{f,\theta(f)}|_{\mathbb{R}} = f$. □

Remark 3.36. We make the somewhat obvious remark that we have proved a *perturbation* result for H-harmonic maps without actually having an *existence* result. Since we are ultimately interested in applying Theorem C to a conformal H-harmonic map with prescribed asymptotic boundary (for which we do have the existence result, Theorem A) we do not require a general existence result for H-harmonic maps which solve a Dirichlet problem at infinity. Nevertheless it should not be too hard to extend the methods of Li & Tam to deal with the H-harmonic case, and this problem will form part of the author's future work.

Chapter 4

The Conformality Operator and Perturbation of Spherical Caps

In this final chapter we prove a perturbation theorem for the spherical caps - the totally umbilic H-surfaces with common ideal boundary given by the unit circle in $\mathbb{R}^2$. This is achieved by studying the linearisation of the so-called *conformality operator*, the zero-set of which consists precisely of (the boundary values of) the conformal, H-harmonic maps of the unit disk $\mathbb{B}$ into hyperbolic 3-space.

4.1 The Conformality Operator

In this chapter we work exclusively with maps from the unit disk $\mathbb{B}$ into $\mathbb{U}^3$. We begin by defining the relevant function spaces. For $k \in \mathbb{Z}$, we use the abbreviation $H^k := W^{k,2}$. Define

$$\mathcal{A} := \{f \in H^5(S^1, \mathbb{R}^2) \,|\, f \text{ is an immersion}\}$$
$$\mathcal{D} := \{\gamma \in H^2(S^1, S^1) \,|\, \gamma \text{ is a diffeomorphism and satisfies a 3-point condition}\}$$
$$\mathcal{M} := \{u : \mathbb{B} \to (\mathbb{U}^3, h) \,|\, u(\mathbb{B}) \text{ is an } H\text{-surface and } u|_{S^1} \in \mathcal{A}\}$$
$$\mathcal{N} := \mathcal{A} \times \mathcal{D}, \qquad \mathcal{N}_f = \{f \circ \gamma \,|\, \gamma \in \mathcal{D}\} \quad (f \in \mathcal{A}).$$

We note that $\mathcal{N}$ has the structure of a smooth Banach manifold. Let $\Psi_H : \mathcal{A} \to \mathcal{M}$ be the map that sends $f \in \mathcal{A}$ to the H-harmonic map $u \in \mathcal{M}$ that satisfies $u|_{S^1} = f$. Let (x, y)

be Cartesian coordinates on $\mathbb{B}$. We will use a complex formulation to define the conformality operator. Thus let $z = x + iy$ denote the complex coordinate on $\mathbb{B}$, where $i = \sqrt{-1}$. We define the complex coordinate vector fields ∂_z and $\partial_{\bar{z}}$ by

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y) \qquad \text{and} \qquad \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y).$$

To simplify notation, we denote the hyperbolic metric on $\mathbb{U}^3$ by $\langle , \rangle$. For a surface $\Sigma \in \mathbb{U}^3$ we consider the complexified tangent bundle $T_{\mathbb{C}}\Sigma = T\Sigma \otimes \mathbb{C}$, and extend the metric $\langle , \rangle$ complex bi-linearly to $T_{\mathbb{C}}\Sigma$. The Levi-Civita connection ∇ of $(\mathbb{U}^3, \langle , \rangle)$ gets similarly extended. For a map $u : \mathbb{B} \to \mathbb{U}^3$ we use the abbreviations

$$u_z := \partial_z u = \frac{1}{2}(\partial_x u - i\partial_y u) \qquad \text{and} \qquad u_{\bar{z}} := \partial_{\bar{z}} u := \frac{1}{2}(\partial_x u + i\partial_y u).$$

Lemma 4.1. *In complex notation, the H-tension field of u is given by*

$$\tau_H(u) = 4\left(\nabla_{u_{\bar{z}}} u_z - iH u_z \wedge u_{\bar{z}}\right). \tag{4.1}$$

Proof. The tension field of u, is given by $\tau(u) = 4\nabla_{u_{\bar{z}}} u_z$ (see, for example, [25], Chapter 8), and

$$u_z \wedge u_{\bar{z}} = \frac{i}{2} u_x \wedge u_y.$$

□

We note also that for a map $u : \mathbb{B} \to \mathbb{U}^3$,

$$\langle u_z, u_z \rangle = \langle u_x, u_x \rangle - \langle u_y, u_y \rangle - 2i \langle u_x, u_y \rangle = \overline{\langle u_{\bar{z}}, u_{\bar{z}} \rangle},$$

Thus u is conformal if, and only if, $\langle u_z, u_z \rangle = 0$ (or equivalently, $\langle u_{\bar{z}}, u_{\bar{z}} \rangle = 0$). We therefore define the conformality operator k acting on $\mathcal{N}$ by

$$k : (f, \gamma) \mapsto \langle \partial_z \cdot \Psi_H(f \circ \gamma), \partial_z \cdot \Psi_H(f \circ \gamma) \rangle .$$

By the above we have that

Lemma 4.2. *The zero set of k in $\mathcal{N}$ consists of those (f, γ) such that $\Psi_H(f \circ \gamma)$ is a conformal, H-harmonic map.*

Now let $\mathcal{H}$ denote the space of bounded holomorphic functions on the unit disk $\mathbb{B}$. Our first crucial observation is the following:

Lemma 4.3. *Let Z denote the range of k. Then $Z \subset \mathcal{H}$.*

Proof. Let $(f, \gamma) \in \mathcal{N}$, and set $u := \Psi_H(f \circ \gamma)$, so that u is an H-harmonic map. First we show that the asymptotics for an H-harmonic map imply that the complex function $z \mapsto \langle u_z, u_z \rangle$ is bounded on $\mathbb{B}$. Let the H-harmonic map $u : \mathbb{B} \to \mathbb{U}^3$ have rectangular components $a, b, c : \mathbb{B} \to \mathbb{R}$, and let (r, θ) denote the usual polar coordinates on $\mathbb{B}$. We have that

$$\partial_z = \frac{1}{2}(\cos\theta - i\sin\theta)(\partial_r - \frac{i}{r}\partial_\theta).$$

Define a boundary defining function ρ on $\mathbb{B}$ by

$$\rho = \frac{1 - r^2}{1 + r^2},$$

so that $\partial_r = -\sqrt{1 - \rho^2}(1 + \rho)\partial_\rho$. Therefore

$$\partial_z = \frac{1}{2}(\cos\theta - i\sin\theta)(-\sqrt{1 - \rho^2}(1 + \rho)\partial_\rho - i\sqrt{\frac{1 + \rho}{1 - \rho}}\partial_\theta),$$

and

$$\langle u_z, u_z \rangle = \frac{1}{4}(\cos 2\theta - i\sin 2\theta)\left((1 - \rho)(1 + \rho)^3 \langle u_\rho, u_\rho \rangle - \left(\frac{1 + \rho}{1 - \rho}\right)\langle u_\theta, u_\theta \rangle + 2i(1 + \rho^2)\langle u_\rho, u_\theta \rangle\right).$$

We begin by evaluating $\lim_{\rho \to 0}\langle u_\rho, u_\rho \rangle$, $\lim_{\rho \to 0}\langle u_\theta, u_\theta \rangle$ and $\lim_{\rho \to 0}\langle u_\rho, u_\theta \rangle$. We shall make use of the asymptotic analysis of Section 3.2; it is easy to see that we may simply substitute the variable ρ for t there, and θ for s in the derived asymptotics (3.7)-(3.10). We thus have that as $\rho \to 0$, $c(\rho, \theta) \sim \rho$, and it therefore suffices to evaluate

$$\lim_{\rho \to 0} \frac{1}{\rho^2} u_\rho \cdot u_\rho, \ \lim_{\rho \to 0} \frac{1}{\rho^2} u_\theta \cdot u_\theta \quad \text{and} \quad \lim_{\rho \to 0} \frac{1}{\rho^2} u_\rho \cdot u_\theta.$$

We retain the setup from Section 3.2, and write

$$a(\rho, \theta) = a_0(\theta) + a_1(\theta)\rho + a_2(\theta)\rho^2 + O(\rho^3)$$

$$b(\rho, \theta) = b_0(\theta) + b_1(\theta)\rho + b_2(\theta)\rho^2 + O(\rho^3)$$

$$c(\rho, \theta) = c_1(\theta)\rho + c_2(\theta)\rho^2 + O(\rho^3).$$

A simple calculation gives us that

$$\begin{aligned}
\frac{1}{\rho^2} u_\rho \cdot u_\rho &= \frac{(a_1')^2 + (b_1')^2 + (c_1')^2}{\rho^2} + \frac{4(a_1 a_2 + b_1 b_2 + c_1 c_2)}{\rho} + O(1) \\
\frac{1}{\rho^2} u_\theta \cdot u_\theta &= \frac{(a_0')^2 + (b_0')^2}{\rho^2} + \frac{2(a_0' a_1' + b_0' b_1')}{\rho} + O(1) \\
\frac{1}{\rho^2} u_\rho \cdot u_\theta &= \frac{a_1 a_0' + b_1 b_0'}{\rho^2} + \frac{a_1 a_1' + b_1 b_1' + 2b_0' b_2 + 2a_0' a_2 + c_1 c_1'}{\rho} + O(1)
\end{aligned}$$

Let $f_0 = u|_{S^1}$. Again using (3.7)-(3.10) we have that $(a_1')^2 + (b_1')^2 + (c_1')^2 = (a_0')^2 + (b_0')^2 = e(f_0)$, and that $4(a_1 a_2 + b_1 b_2 + c_1 c_2) = 2(a_0' a_1' + b_0' b_1') = 2H(a_0 b_0'' - b_0 a_0'')$. Finally we note that

$$(1-\rho)(1+\rho)^3 - \frac{1+\rho}{1-\rho} = O(\rho^2),$$

and conclude that $\lim_{\rho \to 0} \left\{ (1-\rho)(1+\rho)^3 \langle u_\rho, u_\rho \rangle - \left(\frac{1+\rho}{1-\rho} \right) \langle u_\theta, u_\theta \rangle \right\}$ exists. Also from the compatibility conditions we have that $a_1 a_0' + b_1 b_0' = 0$, and that $a_1 a_1' + b_1 b_1' + 2b_0' b_2 + 2a_0' a_2 + c_1 c_1' = 0$, so that $\lim_{\rho \to 0} \langle u_\rho, u_\theta \rangle$ also exists.

Finally we need to show that $z \mapsto \langle u_z, u_z \rangle$ is holomorphic. From (4.1),

$$\partial_{\bar{z}} \langle u_z, u_z \rangle = 2 \langle \nabla_{u_{\bar{z}}} u_z, u_z \rangle = 2iH \langle u_z \wedge u_{\bar{z}}, u_z \rangle = 0,$$

and we are done. ☐

Once again, we shall apply the implicit function theorem for Banach spaces. To that end we shall study the linearisation of k at a point $(f, \gamma) \in \mathcal{N}$ satisfying $k((f, \gamma)) = 0$. At the present we are only able to deal with the case $(f, \gamma) = (\mathrm{id}, \mathrm{id})$, where $\Psi_H(f \circ \gamma) = \Sigma_H$, the spherical cap with constant mean curvature H. Whenever possible we will nevertheless do the calculations for general f and γ. Assume that $(f, \gamma) \in \mathcal{N}$ satisfies $k((f, \gamma)) = 0$. Let $\{\gamma_t\}$ be a variation of γ satisfying $\gamma_0 = \gamma$, $\left.\frac{\partial \gamma_t}{\partial t}\right|_{t=0} = v$. Then

$$D_\gamma k(v) = \left.\frac{\partial}{\partial t}\right|_{t=0} k((f, \gamma_t)) = 2 \langle \nabla_{u_z} \tilde{v}, \partial_z \cdot \Psi_H(f \circ \gamma) \rangle,$$

where $\tilde{v} = D\Psi_H \cdot (f' \circ \gamma) \cdot v$. The map $v \mapsto \tilde{v}$ is an isomorphism between $T_\gamma \mathcal{D}$ and $T_{\Psi_H(f \circ \gamma)} \mathcal{M}$. Also, since $\tau_H(\Psi_H(f \circ \gamma_t)) = 0$ for every t, $\tilde{v}$ satisfies $J_{H, \Psi_H(f \circ \gamma)}(\tilde{v}) = 0$. Thus $\tilde{v}$ is the Jacobi field along $\Psi_H(f \circ \gamma)$ that satisfies $\tilde{v}|_{S^1} = (f' \circ \gamma) \cdot v$.

Lemma 4.4. *In complex notation the Jacobi operator associated to an H-harmonic map u is given by*

$$J_{H,u}(\phi) = 4\left(\nabla_{u_{\bar{z}}}\nabla_{u_z}\phi + \mathcal{R}(\phi, u_{\bar{z}})u_z - iH\left(\nabla_{u_z}\phi \wedge u_{\bar{z}} + u_z \wedge \nabla_{u_{\bar{z}}}\phi\right)\right). \tag{4.2}$$

Analogously to Lemma 4.3 we have

Lemma 4.5. *Suppose $(f, \gamma) \in \mathcal{N}$ satisfies $k((f, \gamma)) = 0$. Let $\tilde{Z}$ denote the range of $D_\gamma k$. Then $\tilde{Z} \subset \mathcal{H}$.*

Proof. Let $u = \Psi_H(f \circ \gamma)$, $v \in T_\gamma\mathcal{D}$, and set $\tilde{v} = D\Psi_H \cdot (f' \circ \gamma) \cdot v$. Then

$$\begin{aligned}
\partial_{\bar{z}}\langle\nabla_{u_z}\tilde{v}, u_z\rangle &= \langle\nabla_{u_{\bar{z}}}\nabla_{u_z}\tilde{v}, u_z\rangle + \langle\nabla_{u_z}\tilde{v}, \nabla_{u_{\bar{z}}}u_z\rangle \\
&= \langle-\mathcal{R}(\tilde{v}, u_{\bar{z}})u_z + iH\left(\nabla_{u_z}\tilde{v} \wedge u_{\bar{z}} + u_z \wedge \nabla_{u_{\bar{z}}}\tilde{v}\right), u_z\rangle \\
&\qquad + \langle\nabla_{u_z}\tilde{v}, -iHu_{\bar{z}} \wedge u_z\rangle \\
&= \langle iHu_{\bar{z}} \wedge u_z, \nabla_{u_z}\tilde{v}\rangle + \langle\nabla_{u_z}\tilde{v}, -iHu_{\bar{z}} \wedge u_z\rangle = 0.
\end{aligned}$$

□

At this point we specialise to the case $(f, \gamma) = (id, id)$. The crucial simplification is given in the following lemma:

Lemma 4.6. *Assume $(f, \gamma) = (id, id)$. Let $v \in T_{id}\mathcal{D}$, and set $\tilde{v} = D\Psi_H \cdot (f' \circ \gamma) \cdot v$. Then $\tilde{v}$ is always tangent to $\Psi_H(f \circ \gamma)$.*

Proof. Let $u = \Psi_H(f \circ \gamma) = \Psi_H(id) = \Sigma_H$, where Σ_H is the spherical cap with constant mean curvature H. Let $\{\gamma_t\}$ be a variation of $id \in \mathcal{D}$ satisfying $\left.\frac{\partial\gamma_t}{\partial t}\right|_{t=0} = v$, and let $u_t := \Psi_H(f \circ \gamma_t)$. Then $\tilde{v} = \left.\frac{\partial u_t}{\partial t}\right|_{t=0}$. We claim that for all $t \in (-\epsilon, \epsilon)$, with $\epsilon > 0$ small enough, $u_t(\mathbb{B}) = u(\mathbb{B}) = \Sigma_H(\mathbb{B})$. This is a consequence of the maximum principle for H-harmonic maps, using the spherical caps as barriers. □

We can now prove the main result of this section:

Theorem 4.7. *$D_\gamma(id, id) : T_\gamma\mathcal{D} \to \tilde{Z}$ is an isomorphism.*

Proof. We switch momentarily to using real (Cartesian) coordinates $(x, y) \in \mathbb{B}$, $r^2 = x^2+y^2$. It is easy to check that in this setup the spherical caps Σ_H, $H \in (-1, 1)$ are described parametrically by

$$\Sigma_H : (x, y) \mapsto \left(\frac{2x}{(1-H) + r^2(1+H)}, \frac{2y}{(1-H) + r^2(1+H)}, \frac{1-r^2}{(1-H) + r^2(1+H)} \right). \tag{4.3}$$

Denote the third (vertical) component by Σ_H^3. Then when $x^2 + y^2 = 1$ we have

$$\frac{\partial \Sigma_H^3}{\partial x} = -x\sqrt{1-H^2}, \qquad \frac{\partial \Sigma_H^3}{\partial y} = -y\sqrt{1-H^2}.$$

Now, suppose $v \in T_{id}\mathcal{D}$, and set $\tilde{v} = D\Psi_H \cdot (f' \circ \gamma) \cdot v$. Let $\tilde{v} = au_z + \bar{a}u_{\bar{z}} = \mathrm{Re}(a)u_x + \mathrm{Im}(a)u_y$. Since $v \in T_{id}\mathcal{D}$, $\tilde{v}$ can have no vertical component at the boundary. By the above this means that $-\mathrm{Re(a)}x\sqrt{1-H^2} - \mathrm{Im}(a)y\sqrt{1-H^2} = 0$ on S^1. Equivalently,

$$\mathrm{Re}(a\bar{z}) = 0, \text{ when } |z| = 1.$$

We now show that $D_\gamma(id, id)$ is injective. As usual, let $u = \Psi_H(f \circ \gamma)(= \Sigma_H)$. Suppose $D_\gamma(id, id)(v) = 0$, thus

$$\langle \nabla_{u_z}\tilde{v}, u_z \rangle = 0.$$

But

$$\nabla_{u_z}\tilde{v} = a_z u_z + a\nabla_{u_z}u_z + \bar{a}_z u_{\bar{z}} + \bar{a}\nabla_{u_z}u_{\bar{z}},$$

and since $\nabla_{u_z}u_{\bar{z}}$ is normal to $u(\mathbb{B})$ by the H-harmonic map equation, and $\nabla_{u_z}u_z = D_{u_z}u_z + (\nabla_{u_z}u_z)^\perp$, we have that $\bar{a}_z|u_z|^2) = 0$, where $|u_z|^2 = \langle u_z, u_{\bar{z}} \rangle$. Thus a is a holomorphic function on $\mathbb{B}$. We can now proceed using a simple argument of Tomi & Tromba ([44], Lemma 1.7): From the discussion above we have that the function

$$w := \frac{a}{iz}$$

is real valued on S^1. By the Schwartz reflection principle, w can therefore be extended to a meromorphic function on $\overline{\mathbb{C}}$ with simple poles at 0 and ∞. We deduce that w must be of the form

$$w(z) = \alpha + \beta z + \frac{\bar{\beta}}{z}, \quad \alpha \in \mathbb{R}, \beta \in \mathbb{C},$$

so that $a(z) = i(\bar{\beta} + az + \beta z^2)$. Finally, the 3-point condition implies that a vanishes at three points on S^1, and therefore $\alpha = 0 = \beta$. Thus $a = 0$ and $D_\gamma k$ is injective.

For proof of the surjectivity of $D_\gamma k$ we again follow the argument in [44]: let $h \in \mathcal{H}$ be given. Define

$$a := \bar{F} + G,$$

where $F_z = h|u_z|^{-2}$, and G is holomorphic. We want to choose G to ensure that $v \in T_{id}\mathcal{D}$. We require that $\mathrm{Re}(a\bar{z}) = 0$, equivalently that $\mathrm{Re}(G/z) = -\mathrm{Re}(Fz)$ when $|z| = 1$. We define a real valued function on S^1, $\eta := -\mathrm{Re}(zF)$. Let ξ_η be the harmonic extension of η to $\mathbb{B}$, and denote by ξ_η^* the harmonic conjugate of ξ_η, so that $\xi_\eta + i\xi_\eta^*$ is holomorphic. Define $\tilde{G} := z(\xi_\eta + i\xi_\eta^*)$. $\tilde{G}$ satisfies $\mathrm{Re}(G/z) = -\mathrm{Re}(Fz)$ on S^1, but doesn't necessarily satisfy the 3-point condition. We correct for this by setting

$$G := \tilde{G} + i(\frac{\bar{\beta}}{z} + a + \beta z),$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ are chosen appropriately. Thus $D_\gamma k$ is surjective and we are done. □

4.2 The Perturbation Theorem

Theorem D. *Let $H \in (-1, 1)$. There exists a neighbourhood η in $\mathcal{A}$ of the identity map $id : S^1 \to \mathbb{R}^2$ such that for every $f \in \mathcal{N}$ there exists a conformal, H-harmonic extension u_f satisfying $u_f|_{S^1} = f$.*

Proof. Recall the conformality operator $k : \mathcal{A} \times \mathcal{D} \to \mathcal{H}$ given by

$$k((f, \gamma)) = \langle \partial_z \cdot \Psi_H(f \circ \gamma), \partial_z \cdot \Psi_H(f \circ \gamma) \rangle .$$

By Theorem 4.7 above, $D_\gamma k(id, id)$ is an isomorphism. Thus, by the implicit function theorem for Banach spaces, there exists a neighbourhood η of $id \in \mathcal{A}$ such that for every $f \in \eta$, there exists a $\gamma_f \in \mathcal{D}$ such that $k((f, \gamma_f)) = 0$. For each such f, $u_f := \Psi_H(f \circ \gamma_f)$ is therefore a conformal, H-harmonic map that satisfies $u_f|_{S^1} = f$, and furthermore the IFT automatically yields that the map $f \to u_f$ is $C^{3,\alpha}$ smooth. □

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