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Lower semicontinuity of attractors for non-autonomous dynamical systems

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Abstract. This paper is concerned with the lower semicontinuity of attractors for semilinear non-autonomous differential equations in Banach spaces. We require the unperturbed attractor to be given as the union of unstable manifolds of time-dependent hyperbolic solutions, generalizing previous results valid only for gradient-like systems in which the hyperbolic solutions are equilibria. The tools employed are a study of the continuity of the local unstable manifolds of the hyperbolic solutions and results on the continuity of the exponential dichotomy of the linearization around each of these solutions.

1. *Introduction*

Results on the upper semicontinuity of attractors with respect to perturbations ('no explosion') are relatively easy to prove, and are now essentially classical. Results on lower semicontinuity ('no collapse') are much more difficult: generally they involve assumptions on the structure of the unperturbed attractor, and one then tries to reproduce a similar structure inside the perturbed attractor. Currently, these lower semicontinuity results are restricted to the class of autonomous dynamical systems that are gradient or 'gradient-like', i.e. systems for which the global attractor is given by the union of the unstable manifolds of a finite set of hyperbolic equilibria.

The study of lower semicontinuity of attractors under perturbation, given this gradient assumption, was set in motion by Hale and Raugel [**19**], who proved an abstract result and considered applications to partial differential equations. Their proof of this result relies on the continuity of the equilibria and lower semicontinuity of the local unstable manifolds under perturbation (see also Stuart and Humphries [**29**], and Kostin [**22**] and Elliot and Kostin [**16**], who use a similar argument based on the continuity of unstable manifolds but show that this remains applicable in certain systems with non-hyperbolic equilibria when the unperturbed problem possesses a Lyapunov functional). Several applications have been considered in [**1, 2, 5, 8, 9**] (including more complicated situations where even the continuity of the equilibria is not straightforward), which maintain the hypothesis of a gradient structure for the limit problem. In all of these papers (with the exception of [**16**] and [**22**]), a key step was to show the uniformity (with respect to the underlying parameter) of the exponential dichotomy of the linearization around each hyperbolic equilibrium.

One would expect, therefore, that any attempt to consider attractors with more general structure—and particularly attractors in non-autonomous problems, where equilibria do not, in general, exist—would have to take into account linearizations around solutions that are time-dependent, leading to the problem of understanding exponential dichotomies for non-autonomous linear equations and their robustness under perturbation. We treat this problem here, and our results in this direction enable us to consider an unperturbed attractor that is given as the union of the unstable manifolds of a (possibly infinite) number of hyperbolic time-dependent solutions under regular perturbation.

Given this assumption on the structure of the unperturbed attractor associated to a family of non-autonomous problems, we use results adapted from Carvalho and Langa [**10**] on the continuity of stable and unstable manifolds to show that the attractor behaves continuously under regular perturbation.

Before we can state our results precisely, we need to introduce some terminology which will be used throughout the paper. Let $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ be the generator of a strongly continuous semigroup $\{e^{\mathfrak{B}t} \mid t \geq 0\} \subset L(\mathcal{Z})$, and consider the semilinear problem on a Banach space $\mathcal{Z}$,

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_0(t, y), \\ y(\tau) &= y_0 \end{aligned} \tag{1}$$

and a regular perturbation of it,

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_\eta(t, y), \\ y(\tau) &= y_0. \end{aligned} \tag{2}$$

If we assume that for $\eta \in [0, 1]$, $f_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ is continuous and Lipschitz continuous in the second variable, uniformly on bounded subsets of $\mathcal{Z}$, then the problems (1) and (2) are locally well-posed. Assuming that, for each $\tau \in \mathbb{R}$ and $y_0 \in \mathcal{Z}$, the solution $t \mapsto T(t, \tau)y_0$ of (2) is defined in $[\tau, \infty)$, we write

$$T_\eta(t, \tau)y_0 = e^{\mathfrak{B}t}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(s, T_\eta(s, \tau)y_0)\, ds. \tag{3}$$

The family $\{T_\eta(t, \tau) \mid t \geq \tau\}$ defined in this way is a nonlinear process.

If $\xi_\eta^*(\cdot) : \mathbb{R} \to \mathcal{Z}$ is a solution to (2), where $\eta \in [0, 1]$, we consider the linearization of (2) around $\xi_\eta^*(\cdot)$,

$$\begin{aligned} \dot{z} &= \mathfrak{B}z + (f_\eta)_z(t, \xi_\eta^*(t))z, \\ z(\tau) &= z_0 \in \mathcal{Z} \end{aligned} \tag{4}$$

and the evolution process $\{U_\eta(t,\tau)\mid t\geq\tau\in\mathbb{R}\}$ given by

$$U_\eta(t,\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)}(f_\eta)_z(s,\xi_\eta^*(s))U_\eta(s,\tau)y_0\,ds. \tag{5}$$

Definition 1.1. We say that (4) has an *exponential dichotomy with exponent* ω *and constant* M if there exists a family of projections $\{Q_\eta(t)\mid t\in\mathbb{R}\}\subset L(\mathcal{Z})$ such that:

(i) $Q_\eta(t)U_\eta(t,s) = U_\eta(t,s)Q_\eta(s)$ for all $t\geq s$;

(ii) $U_\eta(t,s): R(Q_\eta(s))\to R(Q_\eta(t))$ is an isomorphism, with inverse denoted by $U_\eta(s,t): R(Q_\eta(t))\to R(Q_\eta(s))$ for $t\geq s$;

(iii) for some $\omega>0$,

$$\begin{aligned} \|U_\eta(t,s)(I-Q_\eta(s))\| &\leq Me^{-\omega(t-s)} \quad \text{for } t\geq s,\\ \|U_\eta(t,s)Q_\eta(s)\| &\leq Me^{\omega(t-s)} \quad \text{for } t\leq s. \end{aligned} \tag{6}$$

Definition 1.2. A global solution $\xi_\eta^*(\cdot):\mathbb{R}\to\mathcal{Z}$ of (2), $\eta\in[0,1]$, is said to be *hyperbolic* if the process $\{U_\eta(t,\tau)\mid t\geq\tau\in\mathbb{R}\}$ associated to (4) has an exponential dichotomy in the sense of Definition 1.1.

A hyperbolic global solution is the natural generalization to the non-autonomous framework of a hyperbolic equilibrium. Indeed, in Carvalho and Langa [**10**] it is shown that if we perturb an autonomous equation by the addition of a non-autonomous nonlinear term, then each hyperbolic equilibrium becomes a hyperbolic global solution which behaves continuously with respect to the perturbation. On the other hand, globally asymptotically stable solutions are also examples of this kind of hyperbolicity (see, for instance, the papers by Rodríguez-Bernal and Vidal [**28**] and Langa *et al* [**24**]).

One of the most useful tools for describing the asymptotic dynamics of non-autonomous differential equations is the pullback attractor. This concept was introduced by Chepyzhov and Vishik (see [**14**], for example), who termed the sets $A_\eta(t)$ 'kernel sections'; the term 'pullback attractor' appeared in a paper by Kloeden and Schmalfuss [**21**] as well; see also Crauel *et al* [**12**].

Definition 1.3. A family $\{A_\eta(t)\mid t\in\mathbb{R}\}$ of subsets of $\mathcal{Z}$ attracts a bounded set $B\subset\mathcal{Z}$ under $\{T_\eta(t,\tau)\mid t\geq\tau\in\mathbb{R}\}$ *in the pullback sense* if

$$\lim_{\tau\to-\infty}\operatorname{dist}(T_\eta(t,\tau)B, A_\eta(t)) = 0 \quad \text{for all } t\in\mathbb{R}.$$

A family of compact sets $\{A_\eta(t)\subset\mathcal{Z}\mid t\in\mathbb{R}\}$ is a *pullback attractor* for $\{T_\eta(t,\tau)\mid t\geq\tau\in\mathbb{R}\}$ if it is invariant, i.e. $T_\eta(t,\tau)A(\tau)=A(t)$ for all $t\geq\tau$, and attracts all bounded subsets of $\mathcal{Z}$ under $\{T_\eta(t,\tau)\mid t\geq\tau\in\mathbb{R}\}$ in the pullback sense.

In this paper we assume that $\{T_\eta(t,\tau)\mid t\geq\tau\}$ has a pullback attractor $\{A_\eta(t)\mid t\geq 0\}$ for each $\eta\geq 0$, and that (1) has a collection of hyperbolic global solutions $\xi_i^*(\cdot)$, $i\in\mathbb{N}$, such that $A_0(t)$ is the closure of the union of their unstable manifolds $W^u(\xi_i^*)$,

$$A_0(t) = \overline{\bigcup_{j=1}^{\infty} W^u(\xi_j^*(\cdot))(t)}. \tag{7}$$

Since a hyperbolic global solution is the non-autonomous generalization of a hyperbolic equilibrium, the assumption that the pullback attractor has this structure is very natural;

however, none of the previous results in the literature suffice to prove lower semicontinuity for this kind of attractor (although the papers by Carvalho and Langa [**10**], Carvalho *et al* [**11**] and Langa *et al* [**23**] contain results for non-autonomous perturbations of *autonomous* gradient systems). Furthermore, it is known (see [**11**]) that it is precisely attractors of this form that arise for small non-autonomous perturbations of gradient systems.

For the regular perturbation of the nonlinearity, we assume that

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\sup_{z\in B(0,r)}\{\|f_\eta(t,z)-f_0(t,z)\|_{\mathcal{Z}}+\|(f_\eta)_z(t,z)-(f_0)_z(t,z)\|_{L(\mathcal{Z})}\}=0 \tag{8}$$

for all $r>0$, where $(f_\eta)_z(t,z)\in L(\mathcal{Z})$ denotes the derivative of f_η with respect to the variable z at the point (t,z).

Under assumption (8), it is easy to see that for each $T>0$ and compact subset K of $\mathcal{Z}$,

$$\sup_{t\in[0,T]}\sup_{\tau\in\mathbb{R}}\sup_{z\in K}\|T_\eta(t,\tau)z-T_0(t,\tau)z\|_{\mathcal{Z}}\xrightarrow{\eta\to 0}0. \tag{9}$$

For a bounded global hyperbolic solution $\xi^*_{i,0}(\cdot)$ of (1) (i.e. a solution for which $\{U_0(t,\tau)\mid t\geq\tau\in\mathbb{R}\}$ has an exponential dichotomy), we prove that given $\epsilon>0$, there exists η_0 such that for all $\eta\leq\eta_0$ there is a unique global solution $\xi^*_{i,\eta}(\cdot)$ of (2) with

$$\sup_{t\in\mathbb{R}}\|\xi^*_{i,\eta}(t)-\xi^*_{i,0}(t)\|_{\mathcal{Z}}\leq\epsilon.$$

It follows from (8) that for each $T>0$,

$$\sup_{\tau\in\mathbb{R}}\sup_{t\in[0,T]}\|U_\eta(t+\tau,\tau)-U_0(t+\tau,\tau)\|_{L(\mathcal{Z})}\xrightarrow{\eta\to 0}0. \tag{10}$$

It also follows from Henry's monograph [**20**, Theorem 7.6.10] that for η_0 sufficiently small, $\{U_\eta(t,\tau)\mid t\geq\tau\in\mathbb{R}\}$ has an exponential dichotomy and, consequently, that $\xi^*_{i,\eta}(\cdot)$ is hyperbolic for all η.

Definition 1.4. The *unstable manifold* of a hyperbolic solution ξ^*_η to (2) is the set

$$W^u_\eta(\xi^*_\eta)=\Big\{(\tau,\zeta)\in\mathbb{R}\times\mathcal{Z}\mid \text{there is a backward solution } z(t,\tau,\zeta) \text{ of (2) satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to-\infty}\|z(t,\tau,\zeta)-\xi^*_\eta(t)\|=0\Big\}.$$

The section of the unstable manifold at time τ is denoted by

$$W^u_\eta(\xi^*_\eta)(\tau)=\{z\mid(\tau,z)\in W^u_\eta(\xi^*_\eta)\}.$$

Under assumption (8), we shall prove in §2 that for each ξ^*_0 there is a $\delta>0$ and an $\eta_\delta>0$ such that if

$$V_\delta(\xi^*_0)=\{(\tau,z)\in\mathbb{R}\times\mathcal{Z}:\|\xi^*_0(\tau)-z\|_{\mathcal{Z}}<\delta,\ \tau\in\mathbb{R}\}$$

and

$$W^{u,\delta}_\eta(\xi^*_\eta):=W^u_\eta(\xi^*_\eta)\cap V_\delta(\xi^*_\eta),$$

then

$$\mathrm{dist_H}(W^{u,\delta}_\eta(\xi^*_\eta), W^{u,\delta}_0(\xi^*_0)) \to 0 \quad \text{as } \eta \to 0,$$

where $\mathrm{dist_H}$ is the symmetric Hausdorff distance defined by

$$\mathrm{dist_H}(B, C) = \max(\mathrm{dist}(B, C), \mathrm{dist}(C, B)).$$

As a consequence, we are able to prove in §3 that if (1) has an attractor $A_0(t)$ that is the closure of the union of the unstable manifolds of a (possibly infinite) number of global hyperbolic solutions, and if the equations (2) have global attractors $A_\eta(t)$ such that $\bigcup_{t\in\mathbb{R}} \bigcup_{0\le\eta\le\eta_0} A_\eta(t)$ is compact, then the family $\{A_\eta(t) \mid 0 \le \eta \le \eta_0\}$ is upper and lower semicontinuous at $\eta = 0$ uniformly in $t \in \mathbb{R}$.

We emphasize that we do not require the attractor to be the union of a *finite* number of these unstable manifolds; but finding conditions under which the number of hyperbolic global solutions would be finite is an interesting open problem (see §4). Whether such a structure (or an appropriate generalization) can be expected to be in any way typical is unclear: the recent paper by Berger and Siegmund [**4**] indicates the potential complexity of attractors in non-autonomous systems.

Finally, we note that while there is a strong connection between lower semicontinuity results and knowledge of the structure of the unperturbed attractor, here we are not able to characterize the whole attractor of the perturbed problem.

2. *Existence and continuity of hyperbolic solutions and associated unstable manifolds*

In this section we show that global hyperbolic solutions and their unstable manifolds behave continuously with respect to regular perturbations by following, step by step, a procedure similar to that used by Carvalho and Langa in [**10**], which treated regular non-autonomous perturbations of manifolds associated with hyperbolic equilibria.

We start by proving that near a global hyperbolic solution $\xi^*_0(\cdot) : \mathbb{R} \to \mathcal{Z}$ of (1) there is a unique global hyperbolic solution of (2).

If $y(t)$ is a solution to (1) and $z(t) = y(t) - \xi^*_0(t)$, then z satisfies

$$\begin{aligned} \dot{z} &= \mathcal{A}_0(t)z + h_0(t, z), \\ z(\tau) &= z_0, \end{aligned} \tag{11}$$

where $\mathcal{A}_0(t) = \mathfrak{B} + f'_0(t, \xi^*_0(t))$ and

$$h_0(t, z) = f_0(t, \xi^*_0(t) + z) - f_0(t, \xi^*_0(t)) - f'_0(t, \xi^*_0(t))z,$$

with $f'_0(t, x)$ denoting the derivative with respect to the second variable. Then $\mathcal{A}_0(t)$ generates a linear evolution process $\{U_0(t, \tau) \mid t \ge \tau \in \mathbb{R}\}$. Moreover, 0 is an equilibrium solution for (11) and $h_0(t, 0) = 0$, $(h_0)_z(t, 0) = 0 \in L(\mathcal{Z})$.

THEOREM 2.1. *Let $\xi^*_0(\cdot)$ be a bounded hyperbolic solution for (1) and assume that (8) holds. Then there exists an $\eta_0 > 0$ such that for each $0 < \eta < \eta_0$, there is a unique bounded solution $\xi^*_\eta(\cdot) : \mathbb{R} \to \mathcal{Z}$ of (2) that satisfies*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|\xi^*_\eta(t) - \xi^*_0(t)\|_{\mathcal{Z}} = 0.$$

Furthermore, $\xi^*_\eta(\cdot) : \mathbb{R} \to \mathcal{Z}$ is hyperbolic for all η sufficiently small.

Proof. The proof of this result is completely analogous to the proof of [**10**, Theorem 2.1], and we include its main steps here for completeness only. Denote by $y(t) \equiv y(t, \tau; y_0)$ the solution of the initial value problem (2). Then, if we define $\phi(t) = y(t) - \xi_0^*(t)$, we have

$$\phi(t) = U_0(t, \tau)\phi(\tau) + \int_\tau^t U_0(t, s) g_\eta(s, (\phi(s)))\, ds, \tag{12}$$

where $g_\eta(t, \phi) = f_\eta(t, \phi + \xi_0^*(t)) - f_0(t, \xi_0^*(t)) - f_0'(t, \xi_0^*(t))\phi$. Hence, if we project by $Q_0(t)$ and $I - Q_0(t)$ and take limits as $\tau \to +\infty$ and $\tau \to -\infty$, respectively, we get

$$Q_0(t)\phi(t) = \int_\infty^t U_0(t, s) Q_0(s) g_\eta(s, (\phi(s)))\, ds$$

and

$$(I - Q_0(t))\phi(t) = \int_{-\infty}^t U_0(t, s)(I - Q_0(s)) g_\eta(s, (\phi(s)))\, ds.$$

Consequently, a unique global bounded solution for (12) exists in a small neighborhood of $\phi = 0$ if and only if

$$\begin{aligned} \mathcal{T}(\phi)(t) = &\int_\infty^t U_0(t, s) Q_0(s) g_\eta(s, (\phi(s)))\, ds \\ &+ \int_{-\infty}^t U_0(t, s)(I - Q_0(s)) g_\eta(s, (\phi(s)))\, ds \end{aligned}$$

has a unique fixed point in the set

$$\left\{ \phi : \mathbb{R} \to \mathcal{Z} \,\middle|\, \sup_{t \in \mathbb{R}} \|\phi(t)\|_{\mathcal{Z}} \leq \epsilon \right\}$$

for some ϵ sufficiently small. The existence of such a fixed point follows from the hyperbolicity of $\xi_0^* : \mathbb{R} \to \mathcal{Z}$ via a standard contraction mapping argument.

It follows, in addition, that $\xi_\eta^*(\cdot)$ is uniformly close to ξ_0^* and tends to ξ_0^* as $\eta \to 0$.

Now, proceeding as before, we change variables by defining $z(t) = y(t) - \xi_\eta^*(t)$ and rewrite (2) as

$$\begin{aligned} \dot{z} = (\mathcal{A}_0(t) + B_\eta(t))z + h_\eta(t, z), \\ z(\tau) = z_0, \end{aligned} \tag{13}$$

where

$$\mathbb{R} \ni t \mapsto B_\eta(t) = f_\eta'(t, \xi_\eta^*(t)) - f_0'(t, \xi_0^*(t)) \in L(\mathcal{Z})$$

and

$$h_\eta(t, z) = f_\eta(t, \xi_\eta^*(t) + z) - f_\eta(t, \xi_\eta^*(t)) - f_\eta'(t, \xi_\eta^*(t))z.$$

Hence, 0 is an equilibrium for (13), $h_\eta(t, 0) = 0$ and $(h_\eta)_z(t, 0) = 0 \in L(\mathcal{Z})$.

The exponential dichotomy for (4) follows from [**20**, Theorem 7.6.11]. □

Now we are ready to study the unstable manifold of a hyperbolic solution $\xi_\eta^*(\cdot)$.

We will show that the unstable manifold of ξ_η^* is given by a map

$$\mathbb{R} \times \mathcal{Z} \ni (t, z) \mapsto \Sigma_\eta^u(t, Q_\eta(t)z) \in (I - Q_\eta(t))\mathcal{Z}.$$

The points in the unstable manifold will have the form

$$(t,\, Q_\eta(t)z + \Sigma^u_\eta(t,\, Q_\eta(t)z)) \in \mathbb{R} \times \mathcal{Z} \quad \text{with } (t,\, z) \in \mathbb{R} \times \mathcal{Z} \text{ and } z \text{ small.}$$

Note that the way in which we obtained (13) allows us to concentrate on the existence of invariant manifolds of global hyperbolic solutions around the zero stationary solution. Thus, if z is a solution of (13), we write $z^+(t) = Q_\eta(t)z$ and $z^-(t) = z - z^+(t)$; then we have

$$\begin{aligned} \dot{z}^+ &= \mathcal{A}^+_\eta(t)z^+ + H(t,\, z^+,\, z^-),\\ \dot{z}^- &= \mathcal{A}^-_\eta(t)z^- + G(t,\, z^+,\, z^-), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \mathcal{A}^+_\eta(t) &= (\mathcal{A}_0(t) + B_\eta(t))Q_\eta(t),\\ \mathcal{A}^-_\eta(t) &= (\mathcal{A}_0(t) + B_\eta(t))(I - Q_\eta(t)),\\ H(t,\, z^+,\, z^-) &= Q_\eta(t)h_\eta(t,\, z^+ + z^-) \end{aligned}$$

and

$$G(t,\, z^+,\, z^-) = (I - Q_\eta(t))h_\eta(t,\, z^+ + z^-).$$

Since at $(t,\, 0,\, 0)$ the functions H and G are zero with zero derivatives (with respect to z^+ and z^-), from the continuous differentiability of H and G, uniform with respect to t, we obtain that given $\rho > 0$ there exists $\delta > 0$ such that if $\|z\|_{\mathcal{Z}} = \|(z^+ + z^-)\|_{\mathcal{Z}} < \delta$, then

$$\begin{aligned} \|H(t,\, z^+,\, z^-)\|_{\mathcal{Z}} &\le \rho,\\ \|G(t,\, z^+,\, z^-)\|_{\mathcal{Z}} &\le \rho,\\ \|H(t,\, z^+,\, z^-) - H(t,\, \tilde{z}^+,\, \tilde{z}^-)\|_{\mathcal{Z}} &\le \rho(\|z^+ - \tilde{z}^+\|_{\mathcal{Z}} + \|z^- - \tilde{z}^-\|_{\mathcal{Z}}),\\ \|G(t,\, z^+,\, z^-) - G(t,\, \tilde{z}^+,\, \tilde{z}^-)\|_{\mathcal{Z}} &\le \rho(\|z^+ - \tilde{z}^+\|_{\mathcal{Z}} + \|z^- - \tilde{z}^-\|_{\mathcal{Z}}). \end{aligned} \tag{15}$$

Firstly, we can obtain the existence of the unstable manifold under the assumption that (15) holds for all $z = (z^+,\, z^-) \in \mathcal{Z}$ with some suitably small $\rho > 0$. Also, supposing that (15) holds for all $z = (z^+,\, z^-) \in \mathcal{Z}$, we get continuity of the unstable and stable manifolds. Finally, we conclude the existence and continuity of local unstable and stable manifolds for the case where h_η satisfies (15) only for $\|z\|_{\mathcal{Z}} = \|z^+ + z^-\|_{\mathcal{Z}} < \delta$ with $\delta > 0$ suitably small.

In order to show existence of the function $\Sigma^{*,u}_\eta(\tau,\, \cdot)$, we will use the Banach contraction principle. For this, let us fix $D > 0$, $L > 0$, $0 < \vartheta < 1$ and choose $\rho > 0$ such that

$$\begin{aligned} &\frac{\rho M}{\omega} \le D, \qquad \frac{\rho M}{\omega}(1 + L) \le \vartheta < 1,\\ &\frac{\rho M^2(1 + L)}{\omega - \rho M(1 + L)} \le L, \qquad \rho M + \frac{\rho^2 M^2(1 + L)(1 + M)}{2\omega - \rho M(1 + L)} < \omega. \end{aligned} \tag{16}$$

Definition 2.1. Denote by $\mathcal{LB}(D,\, L)$ the complete metric space of all bounded and globally Lipschitz continuous functions $\mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$, $(\tau,\, z) \mapsto \Sigma(\tau,\, Q_\eta(\tau)z) \in (I - Q_\eta(\tau))\mathcal{Z}$, that satisfy

$$\begin{aligned} &\sup_{(\tau,z)\in\mathbb{R}\times\mathcal{Z}} \|\Sigma(\tau,\, Q_\eta(\tau)z)\|_{\mathcal{Z}} \le D,\\ &\|\Sigma(\tau,\, Q_\eta(\tau)z) - \Sigma(\tau,\, Q_\eta(\tau)\tilde{z})\|_{\mathcal{Z}} \le L\|Q_\eta(\tau)z - Q_\eta(\tau)\tilde{z}\|_{\mathcal{Z}}\\ &\qquad\qquad \text{for all } (\tau,\, z,\, \tilde{z}) \in \mathbb{R} \times \mathcal{Z} \times \mathcal{Z}, \end{aligned} \tag{17}$$

where the distance between $\Sigma, \tilde{\Sigma} \in \mathcal{LB}(D, L)$ is defined as

$$|||\Sigma(\cdot,\cdot) - \tilde{\Sigma}(\cdot,\cdot)||| := \sup_{(\tau,z)\in\mathbb{R}\times\mathcal{Z}} \|\Sigma(\tau, Q_\eta(\tau)z) - \tilde{\Sigma}(\tau, Q_\eta(\tau)z)\|_{\mathcal{Z}}.$$

First of all, we need to prove a result on the continuity of the projections $Q_\eta(t)$ and $Q_0(t)$.

PROPOSITION 2.1. *Let $Q_\eta(\cdot)$, $Q_0(\cdot)$ be the projections associated to the linear dichotomies introduced in §2. Then, for any $\tau \in \mathbb{R}$,*

$$\lim_{\eta\to 0} \sup_{s\le\tau} \|Q_\eta(s) - Q_0(s)\|_{L(\mathcal{Z})} \to 0 \tag{18}$$

and

$$\lim_{\eta\to 0} \sup_{s\ge\tau} \|Q_\eta(s) - Q_0(s)\|_{L(\mathcal{Z})} \to 0. \tag{19}$$

Proof. The proof of this result is completely analogous to the proof of [**10**, Lemma 5.1], with the obvious changes needed to cope with the fact that ξ_0 need not be a stationary solution. We include an abridged version of the proof here for completeness. For $\zeta \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto z_\eta(t) \in \mathcal{Z}$ be the unique solution of

$$\dot{z} = (\mathcal{A}_0(t) + B_\eta(t))z,$$
$$z(\tau) = Q_\eta(\tau)\zeta.$$

Upon writing out the variation-of-constants formula, projecting with $(I - Q_0(t))$, and noting that this solution is bounded in $(-\infty, \tau]$, we obtain

$$(I - Q_0(\tau))Q_\eta(\tau)\zeta = \int_{-\infty}^{\tau} U_0(\tau, s)(I - Q_0(s))B_\eta(s)z(s)\, ds \quad \text{for all } \zeta \in \mathcal{Z}.$$

From this it follows that

$$\lim_{\eta\to 0} \sup_{s\le\tau} \|(I - Q_0(s))Q_\eta(s)\|_{L(\mathcal{Z})} \to 0. \tag{20}$$

Consider $\xi \in \mathcal{Z}$, and let $\mathbb{R} \ni t \mapsto x(t) \in \mathcal{Z}$ be the unique solution of

$$\dot{x} = \mathcal{A}_0(t)x = (\mathcal{A}_0(t) + B_\eta(t))x - B_\eta(t)x,$$
$$x(\tau) = Q_0(\tau)\xi.$$

Proceeding in a similar manner, we obtain

$$(I - Q_\eta(\tau))Q_0(\tau)\xi = -\int_{-\infty}^{\tau} U_\eta(\tau, s)(I - Q_\eta(s))B_\eta(s)z(s)\, ds \quad \text{for all } \xi \in \mathcal{Z}.$$

Consequently,

$$\lim_{\eta\to 0} \sup_{s\le\tau} \|(I - Q_\eta(s))Q_0(s)\|_{L(\mathcal{Z})} \to 0. \tag{21}$$

The proof of (18) now follows from the identity

$$Q_\eta(s) - Q_0(s) = (I - Q_0(s))Q_\eta(s) - Q_0(s)(I - Q_\eta(s)). \tag{22}$$

To prove (19) we proceed in a similar way. For $\zeta \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto z_\eta(t) \in \mathcal{Z}$ be the unique solution of

$$\dot{z} = (\mathcal{A}_0(t) + B_\eta(t))z,$$
$$z(\tau) = (I - Q_\eta(\tau))\zeta.$$

By using the variation-of-constants formula, projecting with $Q_0(t)$, and noting that $z_\eta(t)$ is bounded in $[\tau, \infty)$, we obtain

$$Q_0(\tau)(I - Q_\eta(\tau))\zeta = \int_\infty^\tau U_0(t, s)Q_0(s)B_\eta(s)z(s)\,ds \quad \text{for all } \zeta \in \mathcal{Z}$$

and hence

$$\lim_{\eta\to 0} \sup_{s\geq\tau} \|Q_0(s)(I - Q_\eta(s))\|_{L(\mathcal{Z})} \to 0. \tag{23}$$

Similarly, for $\xi \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto x(t) \in \mathcal{Z}$ be the unique solution of

$$\dot{x} = \mathcal{A}_0(t)x = (\mathcal{A}_0(t) + B_\eta(t))x - B_\eta(t)x,$$
$$x(\tau) = (I - Q_0(\tau))\xi.$$

We then obtain

$$Q_\eta(\tau)(I - Q_0(\tau))\xi = -\int_\infty^\tau U_\eta(\tau, s)Q_\eta(s)B_\eta(s)x(s)\,ds \quad \text{for all } \xi \in \mathcal{Z}$$

and

$$\lim_{\eta\to 0} \sup_{s\geq\tau} \|Q_\eta(s)(I - Q_0(s))\|_{L(\mathcal{Z})} \to 0. \tag{24}$$

Thus (19) follows from (22), (23) and (24). □

With the above results and notation, the existence and continuity of unstable manifolds is now a consequence of some results of Carvalho and Langa proved in [**10**].

THEOREM 2.2. (Theorems 3.1 and 3.5 in [**10**]) *Suppose that all the conditions in (16) are satisfied. Then there exists a function $\Sigma_\eta^{*,u}(\tau, \cdot) \in \mathcal{LB}(D, L)$ such that the unstable manifold $W_\eta^u(0, 0)$ for (14) is given by*

$$W_\eta^u(0, 0) = \{(\tau, w) \in \mathbb{R} \times \mathcal{Z} \mid w = (Q_\eta(\tau)w, \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w))\}. \tag{25}$$

If, in addition,

$$\left[\frac{\rho M}{\omega} + \frac{\rho^2 M^2(1+L)}{\omega(2\omega - \rho M(1+L))}\right] \leq \frac{1}{2},$$

then for any $r > 0$,

$$\sup_{t\leq\tau} \sup_{\substack{z\in\mathcal{Z} \\ \|z\|_{\mathcal{Z}}\leq r}} \Big\{\|Q_\eta(t)(z) - Q_0(t)(z)\|_{\mathcal{Z}} + \|\Sigma_\eta^{*,u}(t, Q_\eta(t)(z)) - \Sigma_0^{*,u}(t, Q_0(t)(z))\|_{\mathcal{Z}}\Big\} \xrightarrow{\eta\to 0} 0.$$

From the previous results, we can deduce the existence and continuity of local unstable manifolds when h, h_η satisfy (15) only for $\|z\|_{\mathcal{Z}} = \|z^+ + z^-\|_{\mathcal{Z}} < \delta$ with $\delta > 0$ sufficiently small.

THEOREM 2.3. (Theorem 6.1 in [**10**]) *For* $\eta \in [0, 1]$, *suppose that* $h_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ *is differentiable and consider the initial value problem*

$$\dot{z} = \mathcal{A}_0(t)z + h_\eta(t, z), \quad z(\tau) = z_0 \in \mathcal{Z}. \tag{26}$$

Assume that $h_0 : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ *satisfies* $h_0(t, 0) = 0$, $(h_0)_z(t, 0) = 0 \in L(\mathcal{Z})$ *and that* $\dot{z} = \mathcal{A}_0(t)z$ *has an exponential dichotomy with projections* $\{Q_0(t) \mid t \in \mathbb{R}\}$. *Suppose also that*

$$\lim_{\eta\to 0} \sup_{z\in B(0,r)} \|h_\eta(t, z) - h_0(t, z)\|_{\mathcal{Z}} + \|(h_\eta)_z(t, z) - (h_0)_z(t, z)\|_{L(\mathcal{Z})} = 0 \tag{27}$$

for some $r > 0$. *Then the following properties hold.*

(1) *For each* η *sufficiently small, there exists a globally defined solution of* (26), $\xi^*_\eta : \mathbb{R} \to \mathcal{Z}$, *with* $\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|\xi^*_\eta(t)\|_{L(\mathcal{Z})} = 0$ *and such that*

$$\dot{z} = (\mathcal{A}_0(t) + (h_\eta)_z(t, \xi^*_\eta(t)))z \tag{28}$$

has an exponential dichotomy; that is, there is a family of projections $\{Q_\eta(t) \mid t \in \mathbb{R}\}$ *such that the conditions in Definition 1.1 are satisfied and* $U_\eta(t, \tau)$ *is the solution operator associated to (28).*

(2) *For any* $T > 0$,

$$\lim_{\eta\to 0} \sup_{t\in[0,T]} \sup_{\tau\in\mathbb{R}} \|U_\eta(t + \tau, \tau) - U_0(t + \tau, \tau)\|_{L(\mathcal{Z})} \to 0.$$

(3) *The family of projections* $\{Q_\eta(t) \mid t \in \mathbb{R}\}$ *satisfies*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|Q_\eta(t) - Q_0(t)\|_{L(\mathcal{Z})} = 0.$$

(4) *There exist an* $\eta_0 > 0$, *a neighborhood* V *of* $z = 0$ *in* $\mathcal{Z}$ *(independent of* η*) that has* $\xi^*_\eta(t) \in V$ *for all* $t \in \mathbb{R}$ *and* $\eta \in [0, \eta_0]$ *and, for each* $0 \le \eta \le \eta_0$, *a function* $(\tau, z) \mapsto \Sigma^{*,u}_\eta(\tau, Q(\tau)z) : \mathbb{R} \times V \to \mathcal{Z}$ *such that the local unstable manifold* $W^u_{\mathrm{loc},\eta}(\xi^*_\eta) = W^u_\eta(\xi^*_\eta) \cap V$ *for (26) is given by*

$$W^u_{\mathrm{loc},\eta}(\xi^*_\eta) = \{(\tau, w) \in V \mid w = Q_\eta(\tau)w + \Sigma^{*,u}_\eta(\tau, Q(\tau)w)\}.$$

(5) *The unstable manifolds behave continuously at* $\eta = 0$ *in the sense that*

$$\sup_{t\le\tau} \sup_{z\in V} \{\|Q_\eta(t)z - Q_0(t)z\|_{\mathcal{Z}} + \|\Sigma^{*,u}_\eta(t, Q_\eta(t)z) - \Sigma^{*,u}_0(t, Q_0(t)z)\|_{\mathcal{Z}}\} \overset{\eta\to 0}{\longrightarrow} 0.$$

Note that it would be possible to prove a similar result concerning the existence and stability of local stable manifolds (as in [**10**]), but we do not need such a result here.

3. *Lower semicontinuity of global attractors*

Establishing the upper semicontinuity of attractors for autonomous dynamical systems is a relatively simple matter which depends only on obtaining uniform bounds on the attractors and proving continuity of the nonlinear semigroups (see the paper by Hale *et al* [**17**] and the books by Hale [**18**], Robinson [**27**] or Temam [**30**] for the autonomous case; for results derived in a non-autonomous framework, see Caraballo *et al* [**6, 7**]). On the other hand, results on the lower semicontinuity of attractors are much more difficult. Such results, in general, rely on reproducing the structures present in the limiting attractor inside the perturbed attractors. Because of this, a gradient-like structure for the attractors of the limiting problem has always been taken as the starting point (for autonomous problems by Hale and Raugel [**19**], Stuart and Humphries [**29**] and various other authors [**1, 2, 5, 8, 9**]; for non-autonomous perturbations of autonomous systems by Langa *et al* [**23**] and Carvalho *et al* [**11**]). The following result, however, requires only that the limiting attractor be the closure of the union of a (possibly infinite) number of unstable manifolds of hyperbolic global solutions.

THEOREM 3.1. *Consider the family* $\{T_\eta(t,\tau) \mid t \geq \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, *of nonlinear processes and assume that (9) is satisfied. Suppose that for each* $\eta \in [0,1]$ *the nonlinear process* $\{T_\eta(t,\tau) \mid t \geq \tau\}$ *has a global non-autonomous attractor* $\{A_\eta(t)\}_{t\in\mathbb{R}}$, *that* $\overline{\bigcup_{t\in\mathbb{R}} \bigcup_{\eta\in[0,\eta_0]} A_\eta(t)}$ *is compact, and that* $A_0(t)$ *is given by the closure of the union of the unstable manifolds of a collection of global hyperbolic solutions* $\{\xi_j^*(\cdot)\}_{j=1}^\infty$, *i.e.*

$$A_0(t) = \overline{\bigcup_{j=1}^{\infty} W^u(\xi_j^*(\cdot))(t)}. \tag{29}$$

Further, assume that for each $j \in \mathbb{N}$:

- *given* $\epsilon > 0$, *there exists an* $\eta_{j,\epsilon}$ *such that for all* $0 < \eta < \eta_{j,\epsilon}$ *there is a global hyperbolic solution* $\xi_{j,\eta}^*(\cdot)$ *of (2) that satisfies*

$$\sup_{t\in\mathbb{R}} \|\xi_{j,\eta}^*(t) - \xi_j^*(t)\|_{\mathcal{Z}} < \epsilon;$$

- *the local unstable manifold of* $\xi_{j,\eta}^*$ *behaves continuously as* $\eta \to 0$; *that is, there exists a* $\delta_j > 0$ *such that*

$$\mathrm{dist}_{\mathrm{H}}(W_\eta^{u,\delta_j}(\xi_{j,\eta}^*), W_0^{u,\delta_j}(\xi_j^*)) \xrightarrow{\eta\to 0} 0.$$

Then the family $\{A_\eta(t) \mid 0 \leq \eta \leq \eta_0\}$ *is upper and lower semicontinuous at* $\eta = 0$, *i.e.*

$$\sup_{t\in\mathbb{R}} \mathrm{dist}_{\mathrm{H}}(A_\eta(t), A_0(t)) \xrightarrow{\eta\to 0} 0. \tag{30}$$

Proof. First, note that upper semicontinuity is a direct consequence of the continuity of the nonlinear processes $\{T_\eta(t,\tau) \mid t \geq \tau\}$ and the compactness of $\overline{\bigcup_{t\in\mathbb{R}} \bigcup_{\eta\in[0,\eta_0]} A_\eta(t)}$; for the standard argument, see the books by Hale [**18**], Robinson [**27**] and Temam [**30**] or the paper of Caraballo *et al* [**6**]. Here we prove that the attractor is also lower semicontinuous.

To prove that $\mathrm{dist}(A_0(t), A_\eta(t)) \xrightarrow{\eta\to 0} 0$, it suffices to show that for each $t \in \mathbb{R}$ and $x \in A_0(t)$ there are sequences $\eta_k \to 0^+$ and $x_k \in A_{\eta_k}(t)$ such that $x_k \to x$.

Given $x \in A_0(t)$ and $\epsilon > 0$, there exists $x_\epsilon \in \bigcup_{j=1}^\infty W_0^u(\xi_j^*)(t)$ such that

$$\|x - x_\epsilon\|_{\mathcal{Z}} < \frac{\epsilon}{2}. \tag{31}$$

Take $j \in \mathbb{N}$ so that $x_\epsilon \in W_0^u(\xi_j^*)(t)$, and choose $\tau > 0$ such that

$$T_0(t-\tau, t)x_\epsilon \in W_0^{u,\delta_j}(\xi_j^*)(t-\tau).$$

Choose $r > 0$ so that

$$\sup_{t\in\mathbb{R}} \sup_{\|h\|_{\mathcal{Z}}\le r} \sup_{0\le\eta\le\eta_0} \sup\left\{ \|T_\eta(t+\tau,t)(z+h) - T_\eta(t+\tau,t)z\|_{\mathcal{Z}} : z \in \bigcup_{0\le\eta\le\eta_0} A_\eta(t) \right\} < \frac{\epsilon}{4}. \tag{32}$$

Let $\eta_\epsilon < \eta_0$ be such that

$$\sup_{0\le\eta\le\eta_\epsilon} \operatorname{dist}(W_0^{u,\delta_j}(\xi_j^*), W_\eta^{u,\delta_j}(\xi_{j,\eta}^*)) < r \tag{33}$$

and (by using the convergence of T_η to T_0 in (9))

$$\sup_{t\in\mathbb{R}} \sup\left\{ \|T_\eta(t+\tau,t)z - T_0(t+\tau,t)z\|_{\mathcal{Z}} : z \in \bigcup_{0\le\eta\le\eta_\epsilon} A_\eta(t) \right\} < \frac{\epsilon}{4}. \tag{34}$$

From (33), we choose $x_{\epsilon,\eta} \in W_\eta^{u,>\delta_j}(\xi_{j,\eta}^*)(t-\tau)$ such that

$$\|T_0(t-\tau,t)x_\epsilon - x_{\epsilon,\eta}\|_{\mathcal{Z}} < r$$

and, from (34) and (32), it follows that if $y_{\epsilon,\eta} = T_\eta(t,t-\tau)x_{\epsilon,\eta}$, then

$$T_\eta(t,t-\tau)x_{\epsilon,\eta} \in W_\eta^u(\xi_{j,\eta}^*)(t) \subset A_\eta(t)$$

and

$$\begin{aligned}\|x_\epsilon - y_{\epsilon,\eta}\|_{\mathcal{Z}} &= \|T_0(t,t-\tau)T_0(t-\tau,t)x_\epsilon - T_\eta(t,t-\tau)x_{\epsilon,\eta}\|_{\mathcal{Z}} \\ &\le \|T_0(t,t-\tau)T_0(t-\tau,t)x_\epsilon - T_\eta(t,t-\tau)T_0(t-\tau,t)x_\epsilon\|_{\mathcal{Z}} \\ &\quad + \|T_\eta(t,t-\tau)T_0(t-\tau,t)x_\epsilon - T_\eta(t,t-\tau)x_{\epsilon,\eta}\|_{\mathcal{Z}} < \frac{\epsilon}{2}.\end{aligned}$$

Using (31), we then obtain $y_{\epsilon,\eta} \in A_\eta(t)$ and $\|x - y_{\epsilon,\eta}\|_{\mathcal{Z}} < \epsilon$. This concludes the proof. □

Remarks.

(a) Note that our proof is also valid in an autonomous framework in which the global attractor is given by the union of a countable number of unstable manifolds of global solutions. In particular, we cover the case, appearing in Hale and Raugel [**19**], of global attractors described by the unstable manifolds of equilibria. However, because no periodic solution of an autonomous differential equation can be hyperbolic in our sense, to allow the attractor to contain the unstable manifolds of periodic orbits we would need results on the continuity of unstable manifolds of normally hyperbolic global solutions, for which a detailed study of the robustness of trichotomies under regular perturbations would be also necessary (cf. Bates *et al* [**3**] and Pliss and Sell [**25**]).

(b) On the other hand, observe that nowhere do we need the pullback attraction property of the family $\{A_\eta(t)\}_{t\in\mathbb{R}}$; so our results are also applicable to any kind of non-autonomous family of attractors $\{A_\eta(t)\}_{t\in\mathbb{R}}$, described as in the above theorem. Currently, there is much active research on the relation between pullback, forward and uniform attraction for non-autonomous dynamical systems (see, for example, Cheban *et al* [**13**], Efendiev *et al* [**15**], Rodríguez-Bernal and Vidal-López [**28**], Carvalho *et al* [**11**] or Langa *et al* [**24**]). Our result would cover all of these different kinds of attraction.

(c) Finally, it is worth remarking that if we actually have a pullback attractor, then in place of (8) we could have considered the following weaker hypothesis: there exists $\tau \in \mathbb{R}$ such that

$$\lim_{\eta\to 0} \sup_{t\in(-\infty,\tau]} \sup_{z\in B(0,r)} \{\|f_\eta(t,z) - f_0(t,z)\|_{\mathcal{Z}} + \|(f_\eta)_z(t,z) - f_0'(t,z)\|_{L(\mathcal{Z})}\} = 0 \tag{35}$$

for all $r > 0$. This would imply the existence of global solutions close to $\xi_i(t)$ that are hyperbolic in $(-\infty, \tau]$, and would therefore allow for lower semicontinuity of a family of pullback attractors that are not bounded as $t \to +\infty$ (i.e. $\bigcup_{[T,+\infty)} A_\eta(t)$ is unbounded in $\mathcal{Z}$). This reinforces the idea that in non-autonomous differential equations the pullback behaviour determines crucial properties of the future dynamics of the system.

Although we have been dealing with attractors containing a possibly infinite number of global hyperbolic solutions, it is of interest to consider situations in which we can guarantee that the attractor is given as the union of the unstable manifolds of a finite number of such solutions. Note that this is true in the autonomous case, where only a finite number of hyperbolic equilibria can exist (see Hale [**18**]). In this direction, we have the following lemma.

LEMMA 3.1. *Let $\xi_0^*(\cdot)$ be an hyperbolic solution of problem (1). Then $\xi_0^*(\cdot)$ is isolated (uniformly in t), i.e. there exists an $\epsilon > 0$ such that there is no other complete trajectory $\xi(t)$ with*

$$\|\xi(t) - \xi_0^*(t)\| \le \epsilon \quad \textit{for all } t \in \mathbb{R}.$$

Proof. The proof follows closely the argument used for Theorem 2.1 since, if we define $\phi(t) = y(t) - \xi_0^*(t)$, then

$$\phi(t) = U_0(t,\tau)\phi(\tau) + \int_\tau^t U_0(t,s) g_0(s, (\phi(s)))\, ds$$

with $g_0(t,\phi) = f_0(t, \phi + \xi_0^*(t)) - f_0(t, \xi_0^*(t)) - f_0'(t, \xi_0^*(t))\phi$, and

$$\mathcal{T}(\phi)(t) = \int_\infty^t U_0(t,s) Q_0(s) g_\eta(s, (\phi(s)))\, ds + \int_{-\infty}^t U_0(t,s)(I - Q_0(s)) g_\eta(s, (\phi(s)))\, ds$$

has a unique fixed point in the set

$$\left\{ \phi : \mathbb{R} \to \mathcal{Z} \;\middle|\; \sup_{t\in\mathbb{R}} \|\phi(t)\|_{\mathcal{Z}} \le \epsilon \right\}$$

for ϵ sufficiently small. This follows from the hyperbolicity of $\xi_0^* : \mathbb{R} \to \mathcal{Z}$ via a contraction mapping argument. □

One situation for which we can ensure that there are only a finite number of global hyperbolic solutions is given in the following theorem.

THEOREM 3.2. *Assume that (1) has a pullback attractor* $\{A_0(t) \mid t \in \mathbb{R}\}$ *that is the union of the unstable manifolds of its global hyperbolic solutions. Denote by* $C_B(\mathbb{R}, \{A_0(t) \mid t \in \mathbb{R}\})$ *the space of continuous and bounded functions* $\mathbb{R} \ni t \mapsto \xi(t) \in A_0(t)$*, and assume that the set*

$$\mathfrak{A} := \{\xi \in C_B(\mathbb{R}, A_0) \mid \xi \text{ is a global hyperbolic solution of (1)}\}$$

is a compact subset of $C_B(\mathbb{R}, A_0)$*; then* $\mathfrak{A}$ *is finite.*

Proof. Suppose that $\xi_n^* : \mathbb{R} \to A_0$, $n \in \mathbb{N}$, are a countably infinite number of distinct hyperbolic solutions. From the compactness of $\mathfrak{A}$ we may assume that there is a $\xi \in C_B(\mathbb{R}, A_0)$ such that $\xi_n^* \to \xi$, i.e. $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|\xi_n^*(t) - \xi(t)\| = 0$. Clearly, ξ is a global bounded solution of (1). Since a hyperbolic solution is necessarily isolated, we arrive at a contradiction. □

4. *Final remarks and conclusions*

We have proved the lower semicontinuity of attractors that are given as the union of unstable manifolds of hyperbolic global solutions, by appealing to the recent results of Carvalho and Langa [**10**] that guarantee the existence and continuity of hyperbolic global solutions and their associated unstable manifolds under regular perturbations of non-autonomous dynamical systems in Banach spaces.

Our results suggest a new set of problems for further study. It should be possible to establish lower semicontinuity under singular perturbations of dynamical systems, although results on the robustness of exponential dichotomies would need to be proved first (see the papers [**1, 2, 5, 8, 9**] for examples of lower semicontinuity results in singularly perturbed gradient systems). It would also be very interesting to describe the geometrical structure of the perturbed attractors; note that the results of Langa *et al* [**23**] or Carvalho *et al* [**11**] are not applicable here. Moreover, the applicability of our results in the presence of normally hyperbolic structures (which, for instance, would allow one to prove lower semicontinuity of autonomous global attractors including periodic orbits) is currently one of the most challenging problems in this field (see, for example, Bates *et al* [**3**] or Pliss and Sell [**25, 26**] for some work this direction). We intend to pursue these lines of inquiry in the near future.

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