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INDEPENDENT SETS OF MAXIMUM WEIGHT IN APPLE-FREE GRAPHS*

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Abstract. We present the first polynomial-time algorithm to solve the maximum weight independent set problem for apple-free graphs, which is a common generalization of several important classes where the problem can be solved efficiently, such as claw-free graphs, chordal graphs, and cographs. Our solution is based on a combination of two algorithmic techniques (modular decomposition and decomposition by clique separators) and a deep combinatorial analysis of the structure of apple-free graphs. Our algorithm is robust in the sense that it does not require the input graph G to be apple-free; the algorithm either finds an independent set of maximum weight in G or reports that G is not apple-free.

Key words. maximum independent set, clique separators, modular decomposition, polynomial-time algorithm, claw-free graphs, apple-free graphs

AMS subject classifications. 68R10, 05C69, 05C75

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1. Introduction. In 1965, Edmonds solved the MAXIMUM MATCHING problem [17] by implementing the idea of augmenting chains due to Berge [1]. Moreover, in [18] Edmonds showed how to solve the problem in case of weighted graphs. Lovász and Plummer observed in their book [24] that Edmonds' solution is "among the most involved of combinatorial algorithms." This algorithm also witnesses that the MAXIMUM WEIGHT INDEPENDENT SET (MWIS) problem, being NP-hard in general, admits a polynomial-time solution when restricted to the class of line graphs. In 1980, Minty [27] and Sbihi [30] independently generalized the idea of Edmonds and extended his solution from line graphs to claw-free graphs. With a small repair from Nakamura and Tamura [28], Minty's algorithm also works for weighted graphs. In the present paper, we further develop this fundamental line of research and extend polynomial-time solvability of the MWIS problem from claw-free graphs to apple-free graphs (see [3] for an extended abstract).

An *apple* A_k is a graph obtained from a chordless cycle C_k of length $k \geq 4$ by adding a vertex that has exactly one neighbor on the cycle (see Figure 1 for A_4 and A_5). A graph is *apple-free* if it contains no A_k, $k \geq 4$, as an induced subgraph. Odd apples were introduced by De Simone in [16], and Olariu in [29] called the apple-free graphs *pan-free*. The fact that the apple-free graphs include all claw-free graphs follows from the observation that every apple contains an induced claw centered at the unique vertex of degree 3 (see Figure 1). Along with maximum independent sets

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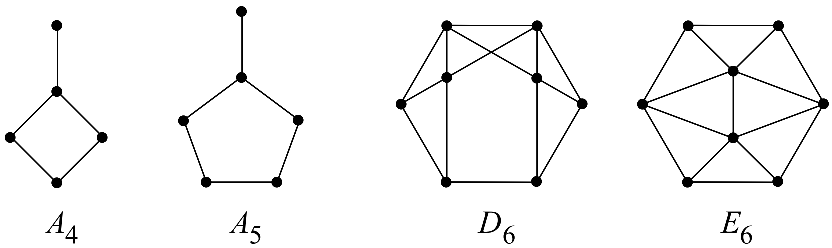

FIG. 1. *Smallest apples A_4, A_5, and the graphs D_6 and E_6.*

in claw-free graphs, our solution extends several other key results in algorithmic graph theory.

In particular, the class of apple-free graphs generalizes that of chordal graphs, since each apple contains a chordless cycle of length at least 4. Chordal graphs enjoy many attractive properties, one of which is that any noncomplete graph in this class has a separating clique, i.e., a clique deletion of which increases the number of connected components (see, e.g., [23]). This decomposability property finds applications in many algorithmic graph problems, including the problem of our interest. An efficient procedure to detect a separating clique in a graph was proposed by Tarjan [32] in 1985 (see also [33]). Recently [2], an interest to this technique was revived by combining it with another important decomposition scheme, known as *modular decomposition*. The graphs that are completely decomposable with respect to modular decomposition are called *complement reducible graphs* [15], or *cographs*, and this is another important class covered by our solution.

Our approach is based on a combination of the two decomposition techniques mentioned above and a deep combinatorial analysis of the structure of apple-free graphs, which allows us to reduce the problem to either claw-free or chordal graphs. An important feature of our solution is that it does not require the input graph G to be apple-free; it either finds an independent set of maximum weight in G or reports that G is not apple-free. Such algorithms are called *robust* in [31].

In an obvious way, our algorithm can be used to find a minimum weight vertex cover in G or a maximum weight clique in the complement of G. We also believe that the structural analysis given in this paper can be used to extend many of the combinatorial properties of claw-free graphs established in the recent line of research by Chudnovsky and Seymour [8, 7, 9, 10, 11, 12, 13, 14] to apple-free graphs.

All preliminary information related to the topic of the paper can be found in the next section. Our approach is partially based on results from [4]; in particular, Lemma 1 and all results in section 4 are contained in [4]. In order to make this paper self-contained, we repeat the proofs of the corresponding results.

2. Preliminaries. Throughout this paper, let $G = (V, E)$ be a finite undirected simple graph with $|V| = n$ and $|E| = m$. We also denote the vertex set of G as $V(G)$. For a vertex $v \in V$, let $N(v) = \{w \in V \mid vw \in E\}$ denote the *neighborhood* of v, and let $\overline{N}(v) = \{w \in V \mid w \neq v \text{ and } vw \notin E\}$ denote the *antineighborhood* of v. If $vw \in E$, then v *sees* w and vice versa, and if $vw \notin E$, then v *misses* w and vice versa. For a subset $U \subseteq V$ of vertices, let $G[U]$ denote the *induced subgraph* of G, i.e., the subgraph of G with vertex set U and two vertices of U being adjacent in $G[U]$ if and only if they are adjacent in G. We say that G is a *vertex-weighted* graph if each vertex of G is assigned a positive integer, the *weight* of the vertex. Let P_k, $k \geq 2$, denote a chordless path with k vertices and $k-1$ edges, and let C_k, $k \geq 4$, denote a chordless cycle with k vertices, say $1, 2, \ldots, k$ and k edges, say $(i, i+1)$ for $i \in \{1, 2, \ldots, k\}$ (index arithmetic modulo k). A chordless cycle with at least six vertices will be called a *long cycle*. The specific graphs D_6 and E_6 represented in Figure 1 will be needed in our solution.

A graph is *chordal* if it contains no induced subgraph C_k, $k \geq 4$. A graph is a *cograph* if it contains no induced P_4. See [6] for various properties of chordal graphs and of cographs. A *claw* K consists of four vertices, say a, b, c, d, with edges ab, ac, ad; then a is the *midpoint of* K and b, c, d are the *endpoints of* K, also denoted as $K = (a; b, c, d)$ to emphasize the difference between midpoint and endpoints. A graph is *claw-free* if it contains no induced claw.

An *independent set* in G is a subset of pairwise nonadjacent vertices. A *clique* is a set of pairwise adjacent vertices. For disjoint vertex sets $X, Y \subset V$, we say that there is a *join between* X *and* Y if each vertex in X sees each vertex in Y. A $K_{2,3}$ has five vertices, say a_1, a_2 and b_1, b_2, b_3, such that $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$ are independent vertex sets and there is a join between A and B. In an undirected graph G with vertex weight function w, the maximum total weight of an independent set in G is called the *weighted independence number* of G and is denoted by $\alpha_w(G)$. Obviously, the following identity holds:

$$\alpha_w(G) = \max_{x \in V(G)} \{w(x) + \alpha_w(G[\overline{N}(x)])\}. \tag{1}$$

An immediate consequence of (1) is the following.

PROPOSITION 1. *If for every vertex $x \in V(G)$ the MWIS problem can be solved for $G[\overline{N}(x)]$ in time T, then it can be solved for G in time $n \cdot T$, where $n = |V(G)|$.*

If, for instance, for every vertex $x \in V(G)$, $G[\overline{N}(x)]$ is chordal, then by the linear-time algorithm for the MWIS problem given in [20], this problem can be solved for G in time $\mathcal{O}(n \cdot m)$; we call such graphs *nearly chordal*. More generally, for a graph class $\mathcal{C}$, a graph G is *nearly* $\mathcal{C}$ if for all $x \in V(G)$, $G[\overline{N}(x)]$ is in $\mathcal{C}$.

Now we recall two decomposition techniques used in our algorithm. A *clique separator* in a connected graph G is a subset Q of vertices of G which induces a complete graph, such that the graph $G[V \setminus Q]$ is disconnected. Tarjan showed in [32] that the MWIS problem can be reduced in polynomial time to graphs without clique separators which are also called *atoms*, and a clique separator decomposition of a given graph can be determined in polynomial time (see also [33]).

Let $G = (V, E)$ be a graph, $U \subset V$, and $x \in V \setminus U$. We say that x *distinguishes* U if x has both a neighbor and a nonneighbor in U. A subset $U \subset V$ is a *module* in G if no vertex outside U distinguishes U. A module U is *nontrivial* if $1 < |U| < |V|$, otherwise it is *trivial*. A graph is *prime* if all its modules are trivial.

It is well known (see, e.g., [26]) that the MWIS problem can be reduced in polynomial time from any hereditary (i.e., closed under taking induced subgraphs) class $\mathcal{C}$ to prime graphs in $\mathcal{C}$. Recently, in [2], decomposition by clique separators was combined with modular decomposition in a more general decomposition scheme.

THEOREM 1 (see [2]). *Let $\mathcal{C}$ be a hereditary class of graphs. If the MWIS problem can be solved in time T for those induced subgraphs of graphs in $\mathcal{C}$ which are prime atoms, then MWIS is solvable in time $\mathcal{O}(n^2 \cdot T)$ for graphs in $\mathcal{C}$.*

In [2, 5], some examples are given where this technique can be applied. The aim of this paper is to show that this approach leads to a polynomial-time solution for MWIS on apple-free graphs. Below we outline five major steps in our solution.

(1) Theorem 3(i) proves that a prime apple-free atom containing a chordless cycle C_k with $k \geq 7$ is claw-free. Together with polynomial-time solvability of the problem in the class of claw-free graphs, this reduces the problem from apple-free graphs to (A_4, A_5, A_6, C_7, $C_8, \ldots$)-free graphs.
(2) Lemma 7 proves that prime $(A_4, A_5, A_6, C_7, C_8, \ldots)$-free atoms are nearly D_6- and E_6-free, which reduces the problem from $(A_4, A_5, A_6, C_7, C_8, \ldots)$-free graphs to ($A_4$, A_5, A_6, D_6, E_6, C_7, $C_8, \ldots$)-free graphs.
(3) Theorem 3(ii) proves that a prime $(A_4, A_5, A_6, D_6, E_6, C_7, C_8, \ldots)$-free atom containing a C_6 is claw-free, which reduces the problem from $(A_4, A_5, A_6, D_6, E_6, C_7, C_8, \ldots)$-free graphs to $(A_4, A_5, C_6, C_7, C_8, \ldots)$-free graphs.
(4) Lemma 9 proves that prime $(A_4, A_5, C_6, C_7, C_8, \ldots)$-free atoms are nearly C_5-free, which reduces the problem to $(A_4, C_5, C_6, C_7, C_8, \ldots)$-free graphs.
(5) Lemma 8 proves that prime $(A_4, C_5, C_6, C_7, \ldots)$-free atoms are nearly C_4-free (i.e., nearly chordal), which reduces the problem to chordal graph.

Steps 4 and 5 are relatively simple and we solve them separately in section 4. The main result is Theorem 3 which deals with steps 1 and 3 of the above outline. This theorem requires a number of preparatory results given in section 3. In the same section we also prove Lemma 7, which is not used in the proof of the main result, but is a mandatory technical step between steps 1 and 3. Then in section 5 we prove the main result and present the algorithm that solves the problem.

3. Preparatory results. In this section, we prove various auxiliary results that will be needed in the course of our study. We start by proving a result which is valid for any A_4-free graph, not necessarily apple-free.

LEMMA 1 (see [4]). *Prime A_4-free graphs are $K_{2,3}$-free.*

Proof. Suppose to the contrary that a prime A_4-free graph G contains an induced $K_{2,3}$, say with vertices a_1, a_2 in one color class and b_1, b_2, b_3 in the other, i.e., the edges are $a_i b_j$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Let Q be the connected component in the complement of the graph $G[N(b_1) \cap N(b_2) \cap N(b_3)]$ that contains a_1 and a_2. Since G is prime, Q must contain two vertices a'_1, a'_2 such that $a'_1 a'_2 \notin E(G)$, which are distinguished by a vertex $z \notin Q$, say $za'_1 \in E(G)$ and $za'_2 \notin E(G)$. If $zb_1 \notin E$, then z is adjacent to at most one of the vertices b_2 and b_3, since otherwise b_2, z, b_3, a'_2, b_1 induce an A_4. Suppose that $zb_2 \notin E(G)$; then b_2, a'_2, b_1, a'_1, z induce an A_4, a contradiction. Thus $zb_1 \in E(G)$, and by an analogous argument, also $zb_2 \in E(G)$ and $zb_3 \in E(G)$, but now z is in Q, a contradiction again. $\square$

From now on, let G be an apple-free graph. According to Theorem 1 and Lemma 1, we may assume that

- G is prime and has no clique separators, i.e., G is a prime atom, and G is $K_{2,3}$-free.

Consider a chordless cycle $C = (1, 2, \ldots, k)$ of length $k \geq 4$ in G. Recall that if $k \geq 6$, then C is called a *long cycle*. Let v be a vertex of G outside C. Denote by $N_C(v)$ the set of neighbors of v in C. We say that v *is universal for* C if v is adjacent to every vertex of C. Let F_u denote the set of all universal vertices for C. Moreover, for $i \geq 0$, we say that v is *a vertex of type* i or *an* i*-vertex* for C if it has exactly i neighbors in C, and we denote by F_i the set of all vertices of type i (for C). The following facts are easy to see.

FACT 1. *Every nonuniversal vertex for C is of type* 0, 2, 3, *or* 4.

FACT 2. *Every vertex of type* 2 *has two consecutive neighbors in C, every vertex of type* 3 *has three consecutive neighbors in C, and every vertex of type* 4 *has two pairs of consecutive neighbors in C.*

FACT 3. *Every vertex v outside C adjacent to vertex i in C is also adjacent to $i-1$ or $i+1$ (i.e., v has no isolated neighbor in C).*

FACT 4. *If two distinct nonadjacent vertices $x, y \in F_2$ see vertices of the same connected component in $G[F_0]$, then $N_C(x) \neq N_C(y)$.*

FACT 5. *If C is long, then no vertex in F_0 can see a vertex in $F_3 \cup F_4$.*

FACT 6. *If C is long, then every vertex in F_u sees every vertex in $F_3 \cup F_4$.*

FACT 7. *If C is long, then the set F_u is a clique (since otherwise a $K_{2,3}$ arises).*

FACT 8. *Let C be a long cycle, and let $v \in F_2$ and $w \in F_3$. If $N_C(v) \subset N_C(w)$, then $vw \in E$. If $N_C(v) \cap N_C(w) = \emptyset$, then $vw \notin E$.*

FACT 9. *Let C be a long cycle, and let $v \in F_2$ and $w \in F_4$. If $|N_C(v) \cap N_C(w)| \leq 1$, then $vw \notin E$. If $N_C(v) \subset N_C(w)$, then $vw \in E$, unless $N_C(w) = \{i-1, i, i+1, i+2\}$ and $N_C(v) = \{i, i+1\}$ in which case $vw \notin E$.*

Now let us introduce more notations. Denote by $F_2(i, i+1)$ the set of vertices in F_2 which see exactly i and $i+1$ on C. Similarly, by $F_3(i, i+1, i+2)$ we denote the set of vertices in F_3 which see exactly $i, i+1$ and $i+2$, and by $F_4(i, i+1, j, j+1)$ we denote the set of vertices in F_4 which see exactly $i, i+1$ and $j, j+1$. We will distinguish between vertices of type 4 with *consecutive neighbors* and vertices of type 4 with *opposite neighbors* (if the neighbors are not consecutive in C). Also, for a vertex $v \in F_0$, let

- $F_0(v)$ denote the connected component in $G[F_0]$ containing v,
- $S(v) := \{x \mid x \text{ sees } F_0(v) \text{ and } C\}$.

We call $S(v)$ the set of *contact vertices of C and $F_0(v)$*. Obviously, $S(v)$ is a separator between v and C. We will frequently use the following properties of $S(v)$.

LEMMA 2. *Let C be a long cycle and $v \in F_0$. Then for the set $S(v)$ of contact vertices, the following properties hold:*

(i) *$S(v) \subseteq F_2 \cup F_u$;*
(ii) *$S(v)$ is no clique;*
(iii) *if x and y are two distinct nonadjacent vertices in $S(v)$, then $x, y \in F_2$ and $N_C(x) \neq N_C(y)$.*

Proof. Condition (i) follows from Facts 1 and 5. For (ii), note that $S(v)$ is a separator between v and C and G is an atom. For (iii), let $x, y \in S(v)$ with $x \neq y$ and $xy \notin E$. Recall that F_u is a clique (Fact 7). Thus, at least one of x, y is not in F_u. Moreover, if $x \in F_u$ and $y \in F_2$, say $y \in F_2(1, 2)$, then let P_{xy} denote a shortest path in $F_0(v)$ connecting x and y. Now, $3, x, P_{xy}, y, 1$ is an apple. Thus, $x, y \in F_2$, and by Fact 4, $N_C(x) \neq N_C(y)$. ∎

The proof of our main result will be prepared by various other lemmas. We recall that throughout the paper graph G is assumed to be a prime atom.

LEMMA 3. *If the chordless cycle C is either*

(i) *a C_k with $k \geq 7$ in the apple-free graph G or*
(ii) *a C_6 in the $(A_4, A_5, A_6, D_6, C_7, C_8, \ldots)$-free graph G,*

then every universal vertex for C sees every vertex of type 2.

Proof. Suppose by contradiction that there is a universal vertex $u \in F_u$ of C which misses a vertex $v \in F_2$. Without loss of generality, let $N_C(v) = \{1, 2\}$. Denote $I := N_C(v)$ and $T := \{w \in F_2 \mid N_C(w) = I\}$. We split T into two subsets T_0 and T_1 so that every vertex of T_0 has a nonneighbor in F_u, and T_1 has a join to F_u. Note that $v \in T_0$, while T_1 may be empty. Also, denote $T_1(v) := \{w \in T_1 \mid vw \in E\}$, $F_4(v) := \{w \in F_4 \mid vw \in E \textit{ and } I \subset N_C(w)\}$ and $F_3(v) := \{w \in F_3 \mid vw \in E \textit{ and } I \subset N_C(w)\}$.

CLAIM 1. *$Q := F_u \cup I \cup T_1(v) \cup F_3(v) \cup F_4(v)$ is a clique.*

Proof of Claim 1. By Fact 7, F_u is a clique, and by Fact 6, F_u has a join to $F_3(v) \cup F_4(v)$. By definition, F_u has a join to $T_1(v)$, and I is a clique that has a join to $F_u \cup T_1(v) \cup F_3(v) \cup F_4(v)$. Thus, if Q has a pair of nonadjacent vertices x and y, then $x, y \in T_1(v) \cup F_3(v) \cup F_4(v)$. By definition, x and y see v. If there is a vertex $z \in C \setminus I$ which is adjacent neither to u nor to v, then G has an apple A_4 induced by u, v, x, y, z. The only case where such a pair $x, y \in T_1(v) \cup F_3(v) \cup F_4(v)$ can see all vertices of $C \setminus I$ is the case where C is of length 6 and, up to symmetry, $N_C(x) = \{1, 2, 3, 4\}$ and $N_C(y) = \{5, 6, 1, 2\}$. But then C together with x and y induce a D_6, contradicting the assumption. This shows Claim 1.

Since G is an atom and thus Q is no clique separator and, in particular, the clique Q does not separate v from $C \setminus I$, there is a path P in $G \setminus Q$ connecting v and $C \setminus I$, say $P = (v = v_1, v_2, \ldots, v_\ell, v_{\ell+1})$ with $v = v_1$ and $v_{\ell+1} \in C \setminus I$, with at least one internal vertex v_2, i.e., $\ell \geq 2$. Without loss of generality, let P be as short as possible among such paths between T_0 and $C \setminus I$.

CLAIM 2. *$v_2 \in F_0$ and $v_2u \notin E$.*

Proof of Claim 2. Since $v_2 \notin Q$, we know that $v_2 \notin F_u \subseteq Q$. Assume $v_2 \in F_2 \cup F_3 \cup F_4$. Then $I \not\subseteq N_C(v_2)$. Indeed, if $I \subseteq N_C(v_2)$, then $v_2 \in T = F_2(1, 2)$, since otherwise $v_2 \in F_3(v) \cup F_4(v) \subset Q$, and therefore $v_2 \in T_1$, as P has no internal vertices in T_0. By $v_2 \notin Q$ it follows that $v_2 \in T_1 \setminus T_1(v)$, which means that $vv_2 \notin E$, a contradiction showing that $I \not\subseteq N_C(v_2)$. Without loss of generality, let $v_21 \notin E$.

Assume first $v_2u \in E$. Then $v_23 \in E$, since otherwise $1, v_1, v_2, u, 3$ induce an A_4, and for similar reasons, $v_24 \in E$ and $v_25 \in E$. Consequently, $v_22 \in E$, since otherwise $v_1, v_2, 2, 3, 5$ induce an A_4. Now by Fact 1, $v_2 \in F_4$, i.e., $N_C(v_2) = \{2, 3, 4, 5\}$, but then $v_2, u, 1, v_1, 6$ induce an A_4, a contradiction.

Assume now $v_2u \notin E$. Then, by Fact 6, $v_2 \notin F_3 \cup F_4$, i.e., $v_2 \in F_2$. Therefore, $v_23 \notin E$, since otherwise $1, v_1, v_2, u, 3, 5$ induce an A_5, and thus by Fact 3, also $v_22 \notin E$. As a result, $v_25 \notin E$, since otherwise $v_2, v_1, 1, u, 5, 3$ induce an A_5, and thus by Fact 3, also $v_24 \notin E$. But then v_2 misses all vertices $t \in \{6, \ldots, k\}$, since otherwise $v_2, v_1, 2, u, t, 4$ induce an A_5.

The contradiction in both cases shows that $v_2 \notin F_2 \cup F_3 \cup F_4$. Therefore, $v_2 \in F_0$. Moreover, if $v_2u \in E$, then $2, v_1, v_2, u, 4$ induce an A_4, which completes the proof of Claim 2.

CLAIM 3. *If $\ell > 3$, then for all $t \in \{3, \ldots, \ell - 1\}$, $v_t \in F_0$.*

Proof of Claim 3. First, let $t = 3$. As $v_3 \notin Q$ and v_3 misses $C \setminus I$, v_3 is either in F_0 or a 2-vertex adjacent to 1 and 2. But if v_3 is adjacent to 2 and not to 3, then $v_1, v_2, v_3, 2, 3$ induce an A_4. Thus, $v_3 \in F_0$. Similarly, $v_t \in F_0$ for $3 < t \leq \ell - 1$, which shows Claim 3.

Now consider again the path $P = (v = v_1, v_2, \ldots, v_\ell, v_{\ell+1})$ in $G \setminus Q$ connecting $v = v_1$ and $v_{\ell+1} \in C \setminus I$. Since v_ℓ sees $C \setminus I$ but $v_\ell \notin Q$, v_ℓ is of type 2, 3, or 4, and since $v_{\ell-1}$ is of type 0, by Fact 5, v_ℓ is not of type 3 or 4. Then, without loss of generality, v_ℓ does not see 1. If v_ℓ sees u, then $v_\ell, u, 1$, the path P up to $v_{\ell-1}$ and a neighbor y of u in C which is a nonneighbor of v_1, v_ℓ, and 1 create an apple. Such a vertex y must exist, since C has at least six vertices. If v_ℓ does not see u, then let x be a neighbor of v_ℓ in C which is closest to 1. If $x = k$, then P up to v_ℓ, x, u, 2, and a neighbor y of u in C which is a nonneighbor of x, v_ℓ, and 2 create an apple. If $x < k$, then P up to v_ℓ, x, u, 1, and a neighbor y of u in C which is a nonneighbor of x, v_ℓ, and 1 create an apple. Again, such a vertex y must exist, since C has at least six vertices. This contradiction completes the proof of Lemma 3. ∎

LEMMA 4. *If the chordless cycle C is either*

(i) *a C_k with $k \geq 7$ in the apple-free graph G or*

(ii) *a C_6 in the $(A_4, A_5, A_6, D_6, C_7, C_8, \ldots)$-free graph G,*

then C has no universal vertex.

Proof. Suppose by contradiction that F_u is nonempty. Since G is prime, it must contain a vertex v which is nonadjacent to some $u \in F_u$. By Facts 6 and 7, $v \in F_0 \cup F_2$, and by Lemma 3, $v \in F_0$. Without loss of generality, we assume that C and $v \in F_0$ are chosen so that the distance between them is as small as possible.

By Lemma 2, there are $x, y \in S(v) \cap F_2$ with $xy \notin E$ and $N_C(x) \neq N_C(y)$. Let $N_C(x) = \{1, 2\}$ and $N_C(y) = \{j, j+1\}$, and assume that $C[j+1, \ldots, k, 1]$ is not shorter than $C[2, \ldots, j]$. Let P_{xy} be a shortest path in $F_0(v)$ connecting x and y. Then let C_{xy} be the cycle consisting of $x, P_{xy}, y, C[j+1, \ldots, k, 1]$. Note that if the length of C is at least 7, then the length of C_{xy} is at least 7 too, and if the length of C is 6, then the length of C_{xy} is 6 too. Since u sees at least five vertices in C_{xy}, u is universal for C_{xy}. Therefore, since u misses v, vertex v cannot be in C_{xy} (by definition) and cannot be of type 3 or 4 for C_{xy} (by Fact 6). In addition, by Lemma 3(i) (if $k \geq 7$) or Lemma 3(ii) (if $k = 6$), v cannot be of type 2 for C_{xy}. Thus, v is of type 0 for C_{xy}. If x or y is on a shortest path between v and C, then v is a vertex of type 0 for C_{xy} which is closer to C_{xy} than to C, contradicting the choice of C and v as a pair of minimum distance. Thus, assume that $z \in S(v)$ is a contact vertex on a shortest path between v and C, and w is its predecessor on the path. Since $N_C(x) \neq N_C(y)$, we can assume that, without loss of generality, $N_C(z) \neq N_C(x)$. If $xz \notin E$, then in the pair (x, z) at least one of them, namely z, is on a shortest path which leads to the same contradiction as above. Thus, $xz \in E$, and since x is not on a shortest path between v and C, $xw \notin E$, but now w, x, z, and a part of C form an apple. This contradiction completes the proof of Lemma 4. ∎

LEMMA 5. *If the chordless cycle C is either*

(i) *a C_k with $k \geq 7$ in the apple-free graph G or*

(ii) *a C_6 in the $(A_4, A_5, A_6, D_6, C_7, C_8, \ldots)$-free graph G,*

then every set $F_2(i, i+1)$, $i \in \{1, 2, \ldots, k\}$, is a clique.

Proof. Suppose, without loss of generality, that there are two nonadjacent vertices $x, y \in F_2(1, 2)$. Denote by $M(x, y)$ the connected component in the complement of $G[F_2(1, 2)]$ containing x and y. Since G is prime, there must exist two vertices $x', y' \in M(x, y)$ such that $x'y' \notin E(G)$, and there is a vertex $z \notin F_2(1, 2)$ distinguishing x' and y', say $zx' \in E(G)$ and $zy' \notin E(G)$.

Assume $z \in F_2 \cup F_3 \cup F_4$. If $z2 \notin E(G)$, then also $zi \notin E(G)$ for all $i \in \{3, 4, \ldots, k\}$, since otherwise for the smallest neighbor $j \in \{3, 4, \ldots, k\}$ of z, the vertices $z, x', 2, 3, \ldots, j, y'$ induce an apple. Thus $z2 \in E(G)$. By symmetry, $z1 \in E(G)$. Since $z \notin F_2(1, 2)$, we conclude that $z \notin F_2$. Since $zy' \notin E(G)$, by Facts 8 and 9, z sees k and 3, but now $zx' \in E(G)$ is a contradiction to Fact 9, which proves $z \notin F_3 \cup F_4$. Also, from Lemma 4 we know that $z \notin F_u$. Therefore, $z \in F_0$.

As before, let $F_0(z)$ denote the connected component in $G[F_0]$ containing z. By Fact 4, we can assume that y' sees no vertex in $F_0(z)$. By Lemma 2(ii) and (iii), there are $w_1, w_2 \in S(z) \cap F_2$ with $w_1 w_2 \notin E(G)$ and $N_C(w_1) \neq N_C(w_2)$. Without loss of generality, let $N_C(x') \neq N_C(w_2)$, and let $N_C(w_2) = \{i, i+1\}$.

Case 1. $x'w_2 \notin E$. Let $P_{x'w_2}$ denote a chordless path in $F_0(z)$ connecting x' and w_2.

Case 1.1. $y'w_2 \notin E$. If $N_C(w_2) = \{k, 1\}$, then in the chordless cycle formed by w_2, $P_{x'w_2}$, x', $C[2, \ldots, k]$, vertex y' has the isolated neighbor 2. Otherwise, in the chordless cycle formed by $x', P_{x'w_2}$, w_2, $C[i+1, \ldots, k, 1]$, vertex y' has the isolated neighbor 1. In either case, we have a contradiction to Fact 3.

Case 1.2. $y'w_2 \in E$. If $N_C(w_2) = \{k, 1\}$ or $N_C(w_2) = \{i, i+1\}$ with $i > 3$, then in the chordless cycle formed by $w_2, P_{x'w_2}, x', 2, y'$, vertex 3 has the isolated neighbor

2. Otherwise, in the chordless cycle formed by $1, x', P_{x'w_2}, w_2, y'$, vertex $i+1$ has the isolated neighbor w_2. In either case, we have a contradiction to Fact 3.

Case 2. $x'w_2 \in E$. This case is similar to Case 1, where the path $P_{x'w_2}$ is replaced by the edge $x'w_2$. □

LEMMA 6. *Let G be either*

(i) *an apple-free graph containing a claw K and a chordless cycle C of length ≥ 7 or*

(ii) *an $(A_4, A_5, A_6, D_6, C_7, C_8, \ldots)$-free graph containing a claw K and a chordless cycle C of length* 6.

If K and C are chosen so that the distance between them is as small as possible and $K \cap C = \emptyset$, then C sees the midpoint of K.

Proof. Suppose by contradiction that the midpoint a of K has no neighbors on C, i.e., $a \in F_0$. As before, let $F_0(a)$ denote the connected component in F_0 containing a, and let $S(a)$ be as in Lemma 2, i.e., there are $x, y \in S(a)$ with $xy \notin E$ and $N_C(x) \neq N_C(y)$. Observe that by Lemma 4 there are no universal vertices for C. Then a long part of C together with a shortest path P_{xy} in $F_0(a)$ connecting x and y create a chordless cycle C' of length at least 7 (in case (i)) or of length 6 (in case (ii)). If x or y belong to a *shortest* path from a to C, then a is closer to C' than to C, a contradiction. Otherwise, consider any shortest path P from a to C, and let z be a vertex of type 2 in P. Also, let w be a vertex of P preceding z and assume, without loss of generality, that $N_C(z) \neq N_C(x)$. Then x is adjacent to z, since otherwise we can replace y by z in the above arguments. Also, x is not adjacent to w, since otherwise x belongs to a shortest path connecting a to C, but now G contains an induced apple with w, z, x and a long part of C. □

We complete this section with the following technical result, which is not used in the proof of the main theorem (Theorem 3), but which is a mandatory step in our solution.

LEMMA 7. *Prime $(A_4, A_5, A_6, C_7, C_8, \ldots)$-free atoms are nearly D_6- and E_6-free.*

Proof. Let G be a prime $(A_4, A_5, A_6, C_7, C_8, \ldots)$-free atom. Suppose that v is a vertex with a cycle C of length 6 in $\overline{N}(v)$. By Lemma 2, there are $w_1, w_2 \in S(v) \cap F_2$ with $w_1w_2 \notin E$. Since G is $(C_7, C_8, \ldots)$-free, w_1 and w_2 see opposite edges of C, i.e., if w_1 sees i and $i+1$, then w_2 sees $i+3$ and $i+4$, and we can assume that w_1 and w_2 have a common neighbor v' in $F_0(v)$ (otherwise, there is a C_k with $k \geq 7$, in G).

Now assume that x is a vertex of type 4 with $x \in F_4(1,2,3,4)$. Then if w_1 sees 2 and 3, and w_2 sees 5 and 6, then by Fact 3, $w_1x \notin E$, since otherwise w_1 has the isolated neighbor x in the $C_5 = (1, x, 4, 5, 6)$, and by Fact 9, $w_2x \notin E$, but then $v', w_1, 2, x, 4, 5, w_2$ induce a C_7 in G, a contradiction.

Thus, up to symmetry, the only possibility for contact vertices to a C_6 with a 4-vertex $x \in F_4(1,2,3,4)$ is $w_1 \in F_2(3,4)$ and $w_2 \in F_2(6,1)$, in which case $w_1x \in E$ by Fact 3 with respect to the $C_5 = (1, x, 4, 5, 6)$ and w_1, and $w_2x \notin E$ by Fact 3 with respect to the $C_4 = (v', w_1, x, w_2)$ and vertex 2.

For a D_6 in $\overline{N}(v)$ with 4-vertices $x \in F_4(1,2,3,4)$ and $y \in F_4(3,4,5,6)$, $xy \notin E$, the above discussion implies that $w_1 \in F_2(2,3)$ and $w_2 \in F_2(5,6)$ is impossible, and also $w_1 \in F_2(1,2)$ and $w_2 \in F_2(4,5)$ is impossible. Thus, $w_1 \in F_2(3,4)$ and $w_2 \in F_2(6,1)$. Then w_1 sees x and y, but now $v', w_1, y, 6, 1, x$ induce an A_5, a contradiction.

For an E_6 in $\overline{N}(v)$ with 4-vertices $x \in F_4(1,2,3,4)$ and $y \in F_4(4,5,6,1)$ with $xy \in E$, the above discussion implies, without loss of generality, that $w_1 \in F_2(3,4)$ and $w_2 \in F_2(6,1)$. Then w_1 sees x but not y, while w_2 sees y but not x. But now $v', w_1, x, y, w_2, 5$ induce an A_5, a contradiction. □

4. MWIS on $(A_4, A_5, C_6, C_7, \ldots)$-free graphs. This section collects some preparatory steps contained in [4].

LEMMA 8. *Prime $(A_4, C_5, C_6, C_7, \ldots)$-free atoms are nearly chordal.*

Proof. Suppose that a prime $(A_4, C_5, C_6, C_7, \ldots)$-free atom G is not nearly chordal. Then there is a vertex v in G such that $G[\overline{N}(v)]$ contains an induced cycle C of length 4. Since G is A_4-free, the set $S(v)$ of contact vertices contains only 2-vertices and universal vertices. We claim that $S(v)$ is a clique:

By Lemma 1 and since G is A_4-free, $S(v) \cap F_u$ is a clique: If $u_1, u_2 \in S(v) \cap F_u$ with $u_1u_2 \notin E$ have a common neighbor in $F_0(v)$, then there is a $K_{2,3}$, and if not, then there is an A_4 in G, a contradiction.

Moreover, by Lemma 1, 2-vertices have consecutive neighbors in C, and since G is $(A_4, C_5, C_6, C_7, \ldots)$-free, every set $S(v) \cap F_2(i, i+1)$ of 2-vertices in $S(v)$ is a clique.

Finally, since G is $(A_4, C_5, C_6, C_7, \ldots)$-free, there is a join between $S(v) \cap F_u$ and $S(v) \cap F_2(i, i+1)$ for each $i \in \{1, \ldots, 4\}$, and there is a join between $S(v) \cap F_2(i, i+1)$ and $S(v) \cap F_2(j, j+1)$ for $i \neq j$ (note that if $S(v) \cap F_2(i, i+1) \neq \emptyset$, then $S(v) \cap F_2(i+1, i+2) = \emptyset$).

Now $S(v)$ is a clique separator between v and C, a contradiction. Thus, G is nearly chordal. $\square$

LEMMA 9. *Prime $(A_4, A_5, C_6, C_7, \ldots)$-free atoms are nearly C_5-free.*

Proof. Suppose that a prime $(A_4, A_5, C_6, C_7, \ldots)$-free atom G is not nearly C_5-free. Then there is a vertex v such that $G[\overline{N}(v)]$ contains an induced cycle C of length 5. Clearly, C has no 1-vertex. We first claim that $S(v)$ contains only 2-vertices and universal vertices: If $x \in S(v)$ is a 3-vertex for C, it must have consecutive neighbors $i, i+1, i+2$ in C, but then a neighbor y of x in $F_0(v)$, x, and $i, i+2, i+3, i-1$ would induce an A_5 in G. A similar argument holds for 4-vertices of C.

Next, we claim that $S(v)$ is a clique: By Lemma 1 and since G is A_4-free, $S(v) \cap F_u$ is a clique by similar arguments as in the proof of Lemma 8. Moreover, 2-vertices have consecutive neighbors in C, and every set $S(v) \cap F_2(i, i+1)$ is a clique, since G is $(A_4, A_5, C_6, C_7, \ldots)$-free. Finally, since G is $(A_4, A_5, C_6, C_7, \ldots)$-free, there is a join between $S(v) \cap F_u$ and $S(v) \cap F_2(i, i+1)$ for each $i \in \{1, 2, \ldots, 5\}$, and there is a join between $S(v) \cap F_2(i, i+1)$ and $S(v) \cap F_2(j, j+1)$ for $i \neq j$ (note that if $S(v) \cap F_2(i, i+1) \neq \emptyset$, then $S(v) \cap F_2(i+1, i+2) = \emptyset$). Now $S(v)$ is a clique separator between v and C, a contradiction. Thus, G is nearly C_5-free. $\square$

COROLLARY 1 (see [4]). *In a prime $(A_4, A_5, C_6, C_7, \ldots)$-free atom G, for every vertex $v \in V(G)$, the prime atoms of $G[\overline{N}(v)]$ are nearly chordal.*

Together with Theorem 1, Corollary 1 implies polynomial-time solvability of the MWIS problem in the class of $(A_4, A_5, C_6, C_7, \ldots)$-free graphs. We formally state this conclusion in Theorem 2 and describe the solution in Algorithm 1. To simplify the description, we assume, by Theorem 1, that the input graph is a prime atom.

Algorithm 1.

Input: A vertex-weighted prime atom G.

Output: A MWIS in G or the output "G is not $(A_4, A_5, C_6, C_7, \ldots)$-free."

(a) Check whether for every vertex $v \in V(G)$, the prime atoms of $G[\overline{N}(v)]$ are nearly chordal (see, e.g., [23] for linear-time recognition of chordal graphs).

(b) If yes, apply a polynomial-time algorithm for the MWIS problem on chordal graphs [20, 21] and combine the partial results according to (1) and Theorem 1.

(c) Otherwise, G is not $(A_4, A_5, C_6, C_7, \ldots)$-free.

The following theorem is a direct consequence of Lemmas 8 and 9.

Theorem 2. *Algorithm* 1 *solves the MWIS problem on* $(A_4, A_5, C_6, C_7, \ldots)$*-free graphs (and an even larger class) in polynomial time in a robust way.*

5. Main result and algorithm. Theorem 3 is the main result of this paper; the algorithmic consequences are based on it.

Theorem 3. *Let* G *be a prime atom.*

(i) *If* G *is apple-free, then* G *is either* $(C_7, C_8, C_9, \ldots)$*-free or claw-free.*

(ii) *If* G *is* $(A_4, A_5, A_6, D_6, E_6, C_7, C_8, \ldots)$*-free, then* G *is either* C_6*-free or claw-free.*

Proof. We simultaneously prove (i) and (ii) of Theorem 3, i.e., we assume that G satisfies either condition (i) or condition (ii). By contradiction, we assume that G contains both a chordless cycle $C = (1, 2, \ldots, k)$ (with $k \geq 7$ in case (i) or with $k = 6$ in case (ii)) and an induced claw $K = (a; b, c, d)$ with midpoint a. Without loss of generality, we assume that C and K are chosen so that the distance between them is as small as possible. Then by Lemma 6, either $a \in C$ or C sees a. Below we analyze all (up to symmetry) possible dispositions of the claw and the cycle with respect to each other.

By Lemma 4, we can assume that $F_u = \emptyset$. In particular, by Facts 1 and 3, for any claw $K = (a; b, c, d)$, $K \cap C = \{b, c, d\}$ is impossible, and $K \cap C = \{a, b, c\}$ is impossible.

Case 1. $K \cap C = \{a, d\}$, say $a = i$ and $d = i + 1$. According to Fact 3, both b and c see $i - 1$. By Lemma 5, b and c cannot both belong to $F_2(i - 1, i)$, and if b and c both see $i - 2$, then $b, c, i - 2, i, i + 1$ induce an A_4. Thus, at most one of b, c is in F_3, and at least one of b, c is not in F_2.

First suppose that $b \in F_3(i-2, i-1, i)$, and let C_b denote the cycle resulting from C by replacing $i - 1$ with b. Then c has isolated neighbor a on C_b, which contradicts Fact 3. The same argument works if $b \in F_4(i - 3, i - 2, i - 1, i)$.

Thus, b and c are neither of type 3 nor of type 4 with consecutive neighbors in C. If one of them is of type 2 and the other of type 4 with opposite neighbors, say $b \in F_4(i-1, i, j, j+1)$, then obviously there is an apple with a, b, c, d and $C[i+2, \ldots, j]$. Thus both b and c are of type 4 with opposite neighbors. If both see $i + 2$, then $a, b, c, d, i + 2$ induce a $K_{2,3}$, a contradiction to Lemma 1. If exactly one, say b, sees $i + 2$, then $a, b, c, d, i + 2$ induce an A_4. If b and c see j in $C[i + 3, i + 4, \ldots, i - 2]$, then a, b, c, d, j induce an A_4. If b and c have no common neighbors other than i and $i - 1$, say b sees $j, j + 1$ and c sees $j', j' + 1$ with $j + 1 < j'$, then d, a, b, c and $C[j + 1, j + 2, \ldots, j']$ induce an apple. This settles Case 1. Observe that the proof of this case does not require the graph to be D_6- or E_6-free.

Case 2. $K \cap C = \{b, c\}$. By Fact 3 and Lemma 4, a is of type 3 or 4. If $a \in F_3$, say $a \in F_3(1, 2, 3)$ with $b = 1$ and $c = 3$, then the chordless cycle $C_a = C - 2 + a$ is as long as C, and d has the isolated neighbor a on C_a, which contradicts Fact 3. Thus $a \in F_4$. If $a \in F_4(1, 2, 3, 4)$ with $b = 1$ and $c = 4$, then again d has the isolated neighbor a on $C_a = C - \{2, 3\} + a$, which contradicts Fact 3, since the length of C_a is $|C| - 1$. If $a \in F_4(1, 2, 3, 4)$ with $b = 1$ and $c = 3$, then, without loss of generality, d sees 4 by Fact 3 with respect to the cycle $C_a = C - \{2, 3\} + a$ and d sees 5 by Fact 3 with respect to the cycle C. But then, denoting by j the neighbor of d on C which is closest to b between 5 and b, we conclude that a, b, c, d and $C[j, \ldots, k]$ induce an apple. Now assume that $a \in F_4(1, 2, i, i + 1)$ with $3 < i < k - 1$. Then, if $b = 1$ and $c = i + 1$ (or $b = 2$ and $c = i$, respectively), then d contradicts Fact 3. Thus, without loss of generality, $b = 1$ and $c = i$, but then, by Fact 3 with respect to the cycles $(a, C[i+1, \ldots, k, 1])$ and C, d sees $i+1$ and $i+2$, and as before, a, b, c, d and $C[j, \ldots, k]$ induce an apple. This settles Case 2.

Case 3. $K \cap C = \{d\}$. We use the following obvious fact (which holds for A_4-free graphs).

FACT 10. *Every nonneighbor of a and d sees at most one of b, c.*

CLAIM 4. *Midpoint a cannot be of type* 3.

Proof. Without loss of generality, assume to the contrary that $a \in F_3(1, 2, 3)$ or $a \in F_3(k, 1, 2)$ with $d = 2$. If $a \in F_3(k, 1, 2)$, then for the long cycle $C_a := C - 1 + a$, K is a claw as in Case 1. Thus, we have to discuss the case $a \in F_3(1, 2, 3)$. To avoid Case 1 for $C_a := C - 2 + a$, b, or c sees 1, and b or c sees 3. To avoid Case 1 for C with claw $(3; d, c, b)$, at most one of b, c sees 3, and similarly for 1 instead of 3. Without loss of generality, assume that c sees 3 but not 1 and b sees 1 but not 3. Then by Fact 3 with respect to C, c sees 4. Then, by Fact 10, $b4 \notin E$, and moreover, none of the vertices in $C[4, \ldots, k]$ sees both b and c. Since b sees 1, by Fact 3, b sees k, and there is an apple with K and a part of C between the last neighbor of c and the first neighbor of b in $C[4, \ldots, k]$. Thus, $a \notin F_3(1, 2, 3)$ and a is no vertex of type 3, which shows Claim 4.

CLAIM 5. *Midpoint a cannot be of type* 4.

Proof. Assume to the contrary that a is of type 4. We have to analyze the following cases.

Case A. $a \in F_4(1, 2, 3, 4)$ and $d = 1$ or $d = 2$. (The other cases for $d = 3$ and $d = 4$ are similar.)

Case A.1. $d = 1$. If the length of C is at least 7, then K is a claw as in Case 1 for the long cycle $C_a := C - \{2, 3\} + a$. Remember that Case 1 holds for any long cycle, regardless of D_6- or E_6-freeness of the input graph. Thus, let C be a cycle of length 6. Since $(a; d, 3, b)$ is no claw as in Case 2, b sees 3. Similarly, c sees 3, and since $(a; d, 4, b)$, $(a; d, 4, c)$ is no claw as in Case 2, b and c see 4. Since $(4; b, c, 5)$ is no claw as in Case 1, 5 sees b or c, and by Fact 10, 5 sees exactly one of b, c, say $5b \in E$ and $5c \notin E$. If $b \in F_4(3, 4, 5, 6)$, then $a, b, c, d, 6$ form an A_4. Thus, $b6 \notin E$. Since $a, b, 5, 6, d, c$ is no A_5, c sees 6, but now $a, c, 6, d, b$ induce an A_4, a contradiction showing that $a \in F_4(1, 2, 3, 4)$ and $d = 1$ is impossible.

Case A.2. $d = 2$. If the length of C is at least 7, then 1 sees b or c, since otherwise the claw $(a; 1, b, c)$ and the long cycle $C_a := C - \{2, 3\} + a$ are as in Case 1 (which is valid for any long cycle, regardless of D_6- or E_6-freeness of the input graph). Since $(1; b, c, d)$ is no claw as in Case 1 for C, 1 sees at most one of b, c. Thus 1 sees exactly one of b, c, say $b1 \in E$ and $c1 \notin E$. Then by Fact 3, b sees also k in C. Since $(a; d, 4, b)$ is no claw as in Case 2 for C, b sees 4. Thus, b is of type 4 with opposite neighbors, i.e., b sees 3 or 5. If b sees 5, then a, b and $C[5, \ldots, k]$ induce an apple. Thus, $b5 \notin E$ and $b3 \in E$. Since $(3; d, b, c)$ is no claw as in Case 1 for C, $c3 \notin E$, and recall that $c1 \notin E$, but now $(a; 1, 3, c)$ is a claw as in Case 2 for C, a contradiction.

If the length of C is 6, then, since $(3; d, b, c)$ is no claw as in Case 1 for C, 3 sees at most one of b, c, and similarly, 1 sees at most one of b, c. If $1b \notin E$ and $3b \notin E$, then $(a; 1, 3, b)$ is a claw as in Case 2 for C and, similarly, for c instead of b. Thus, without loss of generality, $1b \in E$, $1c \notin E$, and $3c \in E$, $3b \notin E$. Then by Fact 3, b sees 6 and c sees 4. Since $d, a, 4, 5, 6, b$ is neither an A_5 nor contains an A_4, b sees 4, but then b also sees 5, and now a, b, and C form an E_6, a contradiction.

Case B. $a \in F_4(1, 2, i, i+1)$ for $i \notin \{3, k-1\}$, and let $d = 1$. If the length of C is 6, then since $(a; d, b, 4)$ is no claw as in Case 2 for C, we have $b4 \in E$ and, similarly, $c4 \in E$, $b5 \in E$, and $c5 \in E$. Since $(5; 6, b, c)$ is no claw as in Case 1, 6 sees b or

c. Since $(6; b, c, d)$ is no claw as in Case 1, 6 sees exactly one of b, c, say $b6 \in E$ and $c6 \notin E$, but then $1, a, b, 6, c$ induce an A_4, a contradiction.

Now let C be of length at least 7. Then $C' := (a, C[2, \ldots, i])$ and $C'' := (a, C[i+1, \ldots, k, 1])$ are two cycles, at least one of which has length at least 5. Assume, without loss of generality, that C'' is of length at least 5.

Since $(a; d, i, b)$ is no claw as in Case 2 for C, we conclude that b sees i, and similarly, b sees $i+1$ and c sees i and $i+1$. Since $(i+1; b, c, i+2)$ is no claw as in Case 1 for C, $i+2$ sees b or c, say, $i+2$ sees b, i.e., either $b \in F_3(i, i+1, i+2)$ or $b \in F_4(i, i+1, i+2, i+3)$. (Note that $b \in F_4(i-1, i, i+1, i+2)$ is impossible by Fact 9, since $ab \in E$.) Then, by Fact 10, $i+2$ is nonadjacent to c, and moreover, by Fact 10, none of the vertices $j \in \{i+2, \ldots, k-1\}$ sees both b and c. Since $(1; k, b, c)$ is no claw as in Case 1, k also cannot see both b and c. If c does not see any of the vertices in $C[i+2, \ldots, k]$ (respectively, $C[i+3, \ldots, k]$), a, b, c, d and $C[i+2, \ldots, k]$ (respectively, $C[i+3, \ldots, k]$) form an apple. If c sees two vertices $j, j+1$ in $C[i+2, \ldots, k]$ (respectively, $C[i+3, \ldots, k]$), then a, b, c, d and $C[j+1, \ldots, k]$ form an apple; note that b and c cannot be of type 4 with the same neighborhood on C. The situation for $d = 2$ is similar. This shows Claim 5.

Thus, assume that a is of type 2, say $N_C(a) = \{1, 2\}$, and let $d = 1$.

CLAIM 6. *Endpoints b and c are of type* 0.

Proof. If b sees k, then $ck \notin E$, since $(k; b, c, d)$ is no claw as in Case 1, but then d, a, b, k, c induce an A_4. Thus neither b nor c sees k. If j denotes the largest neighbor of b or c on C which does not see a and d, then, by Fact 10, j sees exactly one of b or c, but then a, b, c, d and $C[j, \ldots, k]$ form an apple. Thus, none of $3, 4, \ldots, k$ sees b or c, and 2 cannot be the only neighbor of b or c, since no vertex is of type 1. This shows Claim 6.

By Lemma 2, there are $x, y \in S(b) \cap F_2$ with $xy \notin E$ and $N_C(x) \neq N_C(y)$.

Assume first that $a \in \{x, y\}$, say $a = x$. Then $N_C(y) = \{i, i+1\} \neq \{1, 2\}$. Let P_{by} denote a shortest path in $F_0(b)$ connecting b and y. If a misses all internal vertices of P_{by}, then we obtain a contradiction to Fact 3 for c and the chordless cycle $(1, a, b, P_{by}, y, C[i+1, \ldots, k])$ or $(1, y, P_{by}, b, a)$ (if $N_C(y) = \{k, 1\}$).

Thus, a must see internal vertices in P_{by}; in particular, P_{by} has more than one edge. If $P_{by} = (b, z_1, z_2, \ldots, y)$ and a sees z_j for some $j \geq 2$, then let j' be the largest index such that a sees $z_{j'}$; then $a, P[z_{j'}, \ldots, y], C[i, i-1, \ldots, 2], b$ is an apple. Thus, a sees only z_1 in P_{by}, but then either $(a, P_{by}[z_1, \ldots, y], C[i+1, \ldots, k, 1])$ is a long cycle with the claw K as in Case 1 or $(a, P_{by}[z_1, \ldots, y], C[2, 3, \ldots, i])$ is a long cycle with the claw $(a; b, c, 2)$ as in Case 1.

Thus, $a \in \{x, y\}$ is impossible. In particular, for every pair x, y as in Lemma 2, a sees x and y. Since $N_C(x) \neq N_C(y)$, we can assume, without loss of generality, that $N_C(x) \neq N_C(a)$. Let $x \in F_2(i, i+1)$. Then let C' be the longer of the two cycles $(a, x, C[i, i-1, \ldots, 2])$ and $(a, x, C[i+1, i+2, \ldots, 1])$. Since, by Fact 3, a cannot be the only neighbor of b or c on C', it follows that b and c see x. Then replace a by x in the arguments above; $(x; i, b, c)$ is a claw with b and c of type 0. This settles Case 3.

Case 4. $K \cap C = \{a\}$. Let C be a cycle of length at least 6, and assume that $a = 1$. Then, to avoid a claw as in Case 1, vertices 2 and k, respectively, have at least two neighbors in $\{b, c, d\}$. Since G is $K_{2,3}$-free, at least one of 2 and k has a nonneighbor among b, c, d, say k sees b and c and misses d. Also, for any $(K_{2,3}, A_4)$-free graph we have the following.

FACT 11. *Every nonneighbor of a sees at most one vertex in $\{b, c, d\}$.*

Now, since k sees b and c, and since $(k; b, c, k-1)$ is no claw as in Case 1, $k-1$ sees b or c, say $k-1$ sees b and thus, by Fact 11, misses c and d. By Fact 3, d sees 2, since k misses d.

Suppose that 2 sees b. Then $b \in F_4(k-1, k, 1, 2)$ and in the chordless cycle $C_b := C - \{k, 1\} + b$, vertex 2 is not the only neighbor of d, which implies that d sees 3 (and then by Fact 11, 3 misses b and c). In the sequence $3, \ldots, k-1$, there is a pair $i < j$ closest to each other which see different neighbors in b, c, d. Then $C[i, \ldots, j]$ together with a, b, c, d induce an apple.

If 2 misses b, then, by similar reasons as before, 2 sees c and d and 3 sees one of d or c. Then a similar argument as before shows that there is an apple, a contradiction which settles Case 4.

Case 5. $K \cap C = \emptyset$ and a sees C. To analyze this case we first derive a number of helpful facts.

FACT 12. *No vertex of K is of type* 3.

Indeed, if K has a vertex of type 3, then G contains a long cycle intersecting K.

FACT 13. *Every neighbor of a on C sees exactly two vertices in $\{b, c, d\}$.*

Indeed, to avoid Case 3, every neighbor of a on C must be adjacent to at least two vertices in $\{b, c, d\}$, and to avoid Case 4, every neighbor of a on C must be adjacent to at most two vertices in $\{b, c, d\}$.

FACT 14. *If $x \in V(C)$ misses a, but sees a neighbor y of a on C, then x sees exactly one vertex in $\{b, c, d\} \cap N(y)$.*

Indeed, by Fact 13 we have, without loss of generality, $\{b, c, d\} \cap N(y) = \{b, c\}$. By Fact 11, x cannot see both b and c, and if x sees neither b nor c, then $(y; x, b, c)$ is a claw as in Case 1.

By Fact 12, vertex a is of type 2 or 4. We analyze these cases separately.

Case 5.1. a is a vertex of type 2 with $N_C(a) = \{1, 2\}$. Denote $P = C - \{1, 2\}$. If at least two vertices in $\{b, c, d\}$, say b and c, have neighbors on P, then a, b, c together with a part of P create a cycle C' of length at least 4, and d has an isolated neighbor on C', a contradiction to Fact 3. Thus, at most one of b, c, d sees P.

By Fact 13, vertex 1 sees exactly two vertices $x, y \in \{b, c, d\}$, and by Fact 14, k sees exactly one of x, y. Analogously, 2 sees exactly two vertices $x', y' \in \{b, c, d\}$, and by Fact 14, 3 sees exactly one of x', y'. Since at most one of b, c, d sees P, let, without loss of generality, b see k and 3, but then $(b; a, k, 3)$ is a claw as in Case 2 with respect to C, a contradiction. Thus a cannot be of type 2.

Case 5.2. a is a vertex of type 4. We further split this case depending on the length k of C.

Case 5.2.1. $k = 6$. Up to symmetry, we have to analyze the following two options: $a \in F_4(1, 2, 3, 4)$ and $a \in F_4(1, 2, 4, 5)$. Assume first that $a \in F_4(1, 2, 3, 4)$. By Fact 14, vertex 6 has exactly one neighbor in b, c, d, and similarly, vertex 5 has exactly one neighbor in b, c, d. If 5 and 6 see different vertices in b, c, d, say, 5 sees c and 6 sees b, then $a, b, c, d, 5, 6$ induce an A_5. Thus, 5 and 6 have the same neighbor in b, c, d, say 5 and 6 see b and do not see c, d. Then, since $(1; c, d, 6)$ is no claw as in Case 1, vertex 1 misses one of c and d, i.e., by Fact 13, 1 sees b, and similarly, for 4, i.e., 4 sees b, but now a and b together with C form an E_6, a contradiction.

Now suppose $a \in F_4(1, 2, 4, 5)$. Again, by Fact 14, vertex 3 must see exactly one of b, c, d, and similarly for vertex 6. If 3 and 6 have a common neighbor in b, c, d, say 3 and 6 see c then, since $G[3, 2, a, c, 6]$ is no A_4, vertex 2 sees c, and since $G[4, 3, a, c, 6]$

is no A_4, vertex 4 sees c, which is a contradiction for the neighborhood of the vertex c of type 4.

Thus, 3 and 6 have different neighbors in b, c, d, say 3 sees b but misses c and d, and 6 sees c but misses b and d. Then, since by Fact 3, b has no isolated neighbor on C, b sees 2 or 4, and similarly, c sees 5 or 1.

Since $G[2,3,4,c,6]$ is no A_4, $2c \notin E$ or $4c \notin E$. Similarly, since $G[1,6,5,b,3]$ is no A_4, $1b \notin E$ or $5b \notin E$. Since $G[2,3,4,a,c]$ is no A_4, $2c \in E$ or $4c \in E$ holds. Thus, exactly one of 2, 4 sees c, say $2c \notin E$ and $4c \in E$. Since by Fact 13 vertex 2, as a neighbor of a, sees exactly two of b, c, d, it sees b and d. Since by Fact 3, 4 is no isolated neighbor of c, c sees 5, which implies $c \in F_4(4,5,6,1)$ since by Fact 12 c is not of type 3, i.e., c sees also 1. Since $(2; b, d, 1)$ is no claw as in Case 1, vertex 1 sees b or d. Since $G[1,a,5,6,b]$ is no A_4, 1 sees b or 5 sees b. If $5b \in E$, then also $4b \in E$ and thus, $b \in F_4(2,3,4,5)$, and now there is a D_6 with b and c, a contradiction. Thus, $5b \notin E$ and $1b \in E$. Then 1 sees b and c but not d by Fact 13, but now d has the isolated neighbor 2 on C, a contradiction by Fact 3.

Case 5.2.2. $k \geq 7$. If a has consecutive neighbors in C, say $a \in F_4(1,2,3,4)$, then for the long cycle $C_a := C - \{2,3\} + a$, K is a claw as in Case 4. Thus, a is of type 4 with opposite neighbors. Obviously, if the length of C is at least 9, then there is a long cycle containing a and part of C such that K is a claw as in Case 4. The same is true if the length of C is 8 and $a \in F_4(i, i+1, i+3, i+4)$. Therefore, up to symmetry, the only uncovered case is the following one: $7 \leq k \leq 8$ and $a \in F_4(1, 2, k-3, k-2)$. By Fact 14, each of the vertices 3, k, and $k-1$ sees exactly one vertex in $\{b, c, d\}$, and to avoid an A_5, k and $k-1$ have the same neighbor in $\{b, c, d\}$. If all three vertices see b, then $(b; 3, k-1, a)$ is a claw as in Case 2 for C. Therefore, we assume, without loss of generality, that $N(3) \cap \{b, c, d\} = \{b\}$ and $N(k-1) \cap \{b, c, d\} = N(k) \cap \{b, c, d\} = \{c\}$. Suppose $1b \notin E$. Then, since $(a; 1, k-3, b)$ is no claw as in Case 2, b sees $k-3$, and similarly, b sees $k-2$, but now the vertices $1, 2, 3, b, k-2, k-1, k$ form a long chordless cycle, which together with the claw K create a pair as in Case 3. This shows that b sees 1.

Consequently, by Facts 1 and 2 and the fact that b misses k (i.e., b cannot be a vertex of type 4 with opposite neighbors 3,4,k,1), b sees 2.

Analogously, we conclude that b sees $k-2$ and $k-3$, but now b has more than four neighbors on the cycle C. This contradiction completes the proof of Case 5.2.2.

Thus, Case 5 is also impossible, which finishes the proof of Theorem 3. ∎

Now we summarize the above discussion as follows: Given a vertex-weighted graph G, Algorithm 2 either solves the MWIS problem for G or reports that G is not apple-free. Algorithm 2 is based on repeated applications of Proposition 1 and the decomposition scheme of Theorem 1 that reduces the problem from general graphs to prime atoms. According to our main results, if the input graph is apple-free, then the problem reduces to claw-free and chordal graphs. To simplify the description, we assume that the input graph is a prime atom.

Algorithm 2.

Input: A vertex-weighted prime atom G.

Output: A maximum weight independent set in G or the output "G is not apple-free."

(a) If G is claw-free, then apply a polynomial-time algorithm for MWIS on claw-free graphs [27, 28], and output the solution.

(b) For every vertex $v \in V(G)$, apply the decomposition scheme of Theorem 1 to $G[\overline{N}(v)]$, and let $G_1, \ldots, G_k$ be the list of all prime atoms obtained in this way.

(c) If for each $i = 1, \dots, k$,
 – either G_i is claw-free
 – or for each vertex $u \in V(G_i)$, the prime atoms of $G_i[\overline{N}(u)]$ are nearly chordal,

then solve the problem for G_i and use the obtained solutions to compose a solution S for G, and output S.

(d) Otherwise, output "G is not apple-free."

THEOREM 4. *Algorithm 2 is correct and solves the MWIS problem in polynomial time for apple-free graphs.*

Proof. Let G be an input graph (not necessarily apple-free). If Algorithm 2 outputs an independent set S, then obviously S is a solution for the MWIS problem in G. Therefore, to prove the correctness, we have to show that if G is apple-free, then the output of the algorithm is an independent set.

Let G be a prime apple-free atom. If the algorithm does not return an independent set after step (a), then G contains a claw, and hence by Theorem 3(i), G is $(A_4, A_5, A_6, C_7, C_8, \dots)$-free. In step (b), we apply the decomposition scheme of Theorem 1 to $G[\overline{N}(v)]$ for each vertex $v \in V(G)$, and reduce the problem to prime atoms $G_1, \dots, G_k$. Let G_i be an arbitrary graph in this list. If G_i is claw-free, we can solve the problem for it. Otherwise, by Theorem 3(ii) and Lemma 7, G_i is C_6-free, and hence the problem can be solved for G_i by Corollary 1 and Algorithm 1. This completes the proof of the correctness.

A polynomial-time bound follows from polynomial-time solvability of the problem on claw-free graphs and chordal graphs, and polynomial-time recognition algorithms for claw-free and for chordal graphs (see, e.g., [23]). ∎

6. Conclusion. The class of apple-free graphs is a natural generalization of claw-free graphs, chordal graphs, cographs, and various other classes (such as (A_4, P_5)-free graphs [25, 5], (A_4, C_5, $C_6, \dots$)-free graphs [22]), which have been extensively studied in the literature. For each of these classes, the maximum weight independent set problem is efficiently solvable in completely different ways; for cographs, it uses the cotree in a bottom-up way, for claw-free graphs, it is based on the matching algorithm, and for chordal graphs, it uses perfect elimination orderings and perfection or clique separator decomposition.

In this paper, we have shown that the maximum weight independent set problem can be solved in polynomial time on apple-free graphs. Our approach is based on a combination of clique separator decomposition and modular decomposition, and our algorithm does not require recognizing whether the input graph is apple-free. It solves the MWIS problem on a larger class (which is recognizable in polynomial time) given by the conditions in Algorithm 2. Some other important problems such as maximum clique and chromatic number are known to be NP-complete on claw-free graphs (see, e.g., [19]) and thus remain hard on apple-free graphs.

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