Power Indices in Large Voting Bodies

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Abstract. There is no consensus on the properties of voting power indices when there are a large number of voters in a weighted voting body. On the one hand, in some real-world cases that have been studied the power indices have been found to be nearly proportional to the weights (eg the EUCM, US Electoral College). This is true for both the Penrose-Banzhaf and the Shapley-Shubik indices. It has been suggested that this is a manifestation of a conjecture by Penrose (known subsequently as the Penrose limit theorem, that has been shown to hold under certain conditions). On the other hand, we have the older literature from cooperative game theory, due to Shapley and his collaborators, showing that, where there are a finite number of voters whose weights remain constant in relative terms, and where the quota remains constant in relative terms, while the total number of voters increases without limit - so called oceanic games - the powers of the voters with finite weight tend to limiting values that are, in general, not proportional to the weights. These results, too, are supported by empirical studies of large voting bodies (eg. the IMF/WB boards, corporate shareholder control). This paper proposes a restatement of the Penrose Limit theorem and shows that, for both the power indices, convergence occurs in general, in the limit as the Laakso-Taagepera index of political fragmentation increases. This new version reconciles the different theoretical and empirical results that have been found for large voting bodies.

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Introduction: The weighted voting effect and its asymptotic behaviour

The weighted voting effect is the fact that, in a weighted voting body - one in which different voters are assigned different numbers of votes as a means of institutionalising differences in voting power that may be required under its constitution - the resulting differences in voting power are, in general, not proportional to the differences in numbers of votes. This is well known among social choice theorists but otherwise not widely understood, although it is a relatively common occurrence since there are many weighted voting bodies: for example the US presidential electoral college in which each state casts all its votes en bloc under the winner-take-all system, corporate shareholder voting, many intergovernmental bodies such as the European Union Council of Ministers, the IMF and World Bank, and so on.

The weighted voting effect can be seen very easily in a simple example. Suppose there are three voters, with weights 49, 49 and 2 percent of the votes respectively and the decision rule is a simple majority - a decision taken by vote requires 51 votes. It is obvious that the voter with 2 votes has the same influence as one with 49. Any voting outcome has to have at least two members to achieve the majority quota of 51, and any two will do equally. If we measure voting power of any voter as his ability to swing a decision - that is to change a losing vote by the other two into one that is winning by casting his vote - then each voter can be said to have one third of the total voting power. Thus there is a big difference between weight shares and power shares. Voters 1 and 2 have 49% of the voting weight but only 33.33% of the power, while voter 3 has 2% of the weight and also 33.33% of the power. The weighted voting effect depends on all the weights and also the majority quota. If the quota is increased to 52, then only voters 1 and 2 are now powerful, and voter 3 has no influence; he cannot change an otherwise losing vote to winning with his two votes. The voting powers are closer to the weights - in the ratios 50:49, 50:49, 0:2 - but they are still not proportional.

The question addressed in this paper is: what happens to the weighted voting effect when the number of voters is large?

In general we represent a voting body which has n voters in the set $N = \{1, 2, ..., n\}$, by the notation

$$[q; w_1, w_2, \cdots, w_n] \tag{1}$$

where the weight (number of votes) of the typical voter, i, is w_i, and the voting rule is that a decision is taken if the number of votes cast in favour of it is no less than the quota q.

This is the simplest case, of a decision rule with a single quota and a single set of weights. Some voting systems in the real world involve multiple majorities, for example the current voting rule in the EU council of ministers (the system agreed as part of the Nice Treaty) is a triple-majority rule: a decision requires a majority of the member countries, possessing a majority of the weights and representing a majority of the population. Each of these three voting rules has a different quota and different voting weights. For present purposes we ignore this complication and concentrate on voting bodies which use a single decision rule with a quota weights.

I will not give a full description of the theory of voting power indices here, since there are many accounts, notably the seminal treatise by Felsenthal & Machover (1998). In this paper I assume the reader is familiar with the so-called 'classical' power indices due to Penrose, Coleman and Banzhaf (which I shall refer to here as the Penrose-Banzhaf index) and to Shapley and Shubik (the Shaply-Shubik index).

The literature contains two completely different and contradictory stories about what happens to power indices when the number of voters is large. On the one hand, very early on, in a series of Rand papers, written in the early 1960s, Shapley and his co-authors developed the limit theory, and the theory of 'oceanic games', and showed that the weighted voting effect can persist or even increase asymptotically (Shapiro & Shapley (1978); Milnor & Shapley (1978); Dubey & Shapley (1979)). They assumed a voting body with two types of voters: (1) a large number of voters (in the limit, an infinite number), each with a small weight (in the limit infinitesimal) and also (2) a fixed finite number of voters with fixed weights (which do not change asymptotically), and using a decision rule with a fixed quota. Thus in the limit, as n increases, there becomes an 'ocean' of small voters with a number of 'islands' representing the voters with finite weights. They specified this formally as a sequence of voting games and analytically derived the limiting behaviour of the power indices. They showed that, in general, in relative terms, the power indices of the finite voters tend to limits different from the weights. Some support for these theoretical results has come from applied studies of shareholder voting bodies by Leech (2002a). For many public companies, such as those listed on the London Stock Exchange studied, the shareholder voting body approximates the conditions of oceanic games quite well, having a small number of large shareholders and many thousands of very small ones.

On the other hand, a more recent strand has emerged in the literature showing asymptotic proportionality between power indices and weights as n goes to infinity. See Lindner & Machover (2004); Chang *et al.* (2006). This phenomenon, the Penrose Limit Theorem, was originally identified in a conjecture by Penrose (1952). A proof has been given by Lindner and Machover for certain special cases. It has been shown, however, not to be a general rule (Lindner & Owen (2007)). It remains an open question how general are the conditions under which it holds. This theoretical result has received empirical support from applied studies of the European Union Council of Ministers (for example, Słomczynski &

Zyczkowski (2007); Leech & Aziz (2010)) and the Electoral College (Miller (2010)).

In this paper I investigate this apparent contradiction and propose a general reformulation of the limit theorem which can encompass both. I argue that the two strands of the literature can be reconciled by defining the limiting behaviour in terms of the Laakso-Taagepera index of voter fragmentation (Laakso & Taagepera, 1979) rather than the number of voters.

1 Asymptotic behaviour of power indices and 'oceanic games' as models of large voting bodies

The asymptotic properties of the two classical voting power indices were developed in three seminal papers by Shapiro & Shapley (1978), Milnor & Shapley (1978) and Dubey & Shapley (1979).

These papers all assume a sequence of voting bodies in which the number of voters increases without limit while the total voting weight remains finite. Specifically, they assume that voters can be separated into two groups: major voters, of whom there are a finite number, m, and minor voters, of whom there are a number, n, which is allowed to increase without limit while their combined vote share remains finite. They derived the limits of the indices, in the Shapley-Shubik case analytically, and in the Penrose-Banzhaf case by showing that the voting body reduces to a simpler one in the limit.

The voting body, indexed by n in the sequence, can be written (modifying their notation slightly for consistency with the rest of the paper):

$$[q; w_1, w_2, \cdots, w_m, a_1, a_2, \cdots, a_n] \tag{2}$$

such that,

$$\sum_{j=1}^{n} a_j = \alpha.$$

The number of voters is $m + n$ where it is assumed that $n \to \infty$. The weights of the large voters, $w_1, \cdots, w_m$, the quota, q, and α are positive constants.

The sequence is such that,

$$\max_j a_j \to 0 \text{ as } n \to \infty.$$

1.1 The Shapley-Shubik index

The first paper to study the Shapley-Shubik index (henceforth the SSI) in large voting games was a 1960 Rand Corporation report by Shapley and Shapiro, later published in *Mathematics of Operations Research* (Shapiro & Shapley, 1978).

Let let $M = \{1, 2, \cdots, m\}$ represent the set of large voters, $\phi_{i\infty}$ be the limiting SSI for large voter $i \in M$ as $n \to \infty$. Also let $s = |S|$ and $w(S) = \sum_{i \in S} w_i$ for any set of voters S.

The most important class of voting body is where $w(M) < q < \alpha$, what Shapiro and Shapley call the interior case, where the combined votes of the large voters are neither winning nor blocking. Shapiro and Shapley proved the following:

Theorem 1 (Shapiro and Shapley): For the sequence of voting games defined above, for each major voter, $i \in M$, his SSI converges on the limit $\phi_{i\infty}$,

$$\phi_{i\infty} = \sum_{S \subseteq M \setminus \{i\}} \int_{(q - w(S \cup \{i\}))/\alpha}^{(q - w(S))/\alpha} t^s (1-t)^{m-s-1} dt. \tag{3}$$

Proof (Shapiro & Shapley, 1978)

This holds for a voting body with an arbitrary finite number of major voters. We can obtain from it exact results for special cases. (It is of some interest that, in this case, the limiting power index does not depend on the quota.)

Case 1: One major voter: It can be shown that for the case $m = 1$, expression (3) is equal to,

$$\phi_{1\infty} = \frac{w_1}{\alpha} \tag{4}$$

Illustrative examples of (4) for a range of values of w_1 are presented in Table 1. In this example the weights of all voters, major and minor, represented by the set N, sum to 1, $w(N) = w_1 + \alpha = 1$, so $\alpha = 1 - w_1$, and weights and voting powers can be directly compared.

Thus we find that relative powers and weights for the largest voter are very different except when the weight is small, when they begin to approach proportionality. In general there are no grounds for assuming any tendency towards power to weight proportionality as the number of voters increases.

Table 1: Shapley-Shubik index with one major voter

w_1	$\phi_{1\infty}$
0.4	0.6667
0.3	0.4286
0.2	0.25
0.1	0.1111
0.05	0.0526
0.01	0.0101

Case 2: Two major voters When $m = 2$, the limiting Shapley-Shubik indices can be derived from expression (3) as:

$$\phi_{1\infty} = \frac{w_1}{\alpha}(1 - \frac{w_2}{\alpha}),\ \phi_{2\infty} = \frac{w_2}{\alpha}(1 - \frac{w_1}{\alpha}) \tag{5}$$

Some illustrative examples of (5) are presented in Table 2. Again we assume the weights sum to one so that weights and powers are fractions and can be directly compared. Here, we have $\alpha = 1 - w_1 - w_2$.

Table 2: Shapley-Shubik index with two major voters

w_1	w_2	$\phi_{1\infty}$	$\phi_{2\infty}$
0.3	0.1	0.4167	0.0833
0.2	0.2	0.2222	0.2222
0.3	0.05	0.4260	0.0414
0.2	0.1	0.2449	0.1020
0.2	0.05	0.2489	0.0489

Thus, we find, in these special cases (which appear to be fairly unexceptional), that the limiting values of the power indices are very different from the weights. There is no evidence here of a general tendency towards asymptotic proportionality.

These results assume a sequence of voting games in which the number of voters increases without limit, becoming asymptotically an 'ocean'. In a subsequent paper Milnor and Shapley proposed a game with a continuum of agents, an infinite number of players with infinitesimal weights, and a finite number of players with finite weights, an 'oceanic' game, and showed that expression (3) is obtained without the use of a sequence of games. (Milnor & Shapley, 1978)

1.2 The Penrose-Banzhaf index

The analysis of the Penrose-Banzhaf index (henceforth the PBI) case was presented by Dubey & Shapley (1979). This case is rather easier analytically than the SSI case because the coalitional model underlying it is simpler. Each voter is assumed to vote 'for' or 'against' a proposed action with equal probability and independently. It follows that in large games the votes of the minor players can be assumed to follow the law of large numbers, and tend to a constant split with equal numbers voting on each side: $\alpha/2$ 'for' and $\alpha/2$ 'against'. This means that the limiting values of the power indices can be found simply by modifying the quota in a game with just the major players.

We therefore have:

Theorem 2 (Dubey and Shapley): For the sequence of voting games defined above, the PBI of each of the major voters, $i \in M$,converges on the limit $\beta i\infty$ which can be found as the corresponding PBI in the modified voting body:

$$[q - \frac{\alpha}{2}; w_1, w_2, \cdots, w_m] \tag{6}$$

Proof Dubey & Shapley (1979)

Table 3: Penrose-Banzhaf indices with two major voters

w_1	w_2	α	$q - \frac{\alpha}{2}$	$\beta_{1\infty}$	$\beta_{2\infty}$
0.3	0.1	0.6	0.21	1.0	0.0
0.2	0.2	0.6	0.21	0.5	0.5
0.3	0.05	0.65	0.185	1.0	0.0
0.2	0.1	0.7	0.16	1.0	0.0
0.2	0.05	0.75	0.135	1.0	0.0

Table 3 shows the limiting values of the normalised PBI's for the same voting bodies as in Table 2, with a quota of $q = 0.51$. Table 4 shows some results for the case of 3 major voters. It is clear that, in general, the asymptotic, as $n_l \to \infty$, PBI's are far from being proportional to the weights.

It is worth noting, incidentally, that, comparing Tables 2 and 3, it is clear from these examples that, in general, the values of the PBI can be very different indeed from those of the SSI. This finding is important and interesting not least because it shows the fallacy that is often made by writers claiming that the two 'classical' power indices are very similar. It should be assumed that, generally, in large voting bodies they could be very different.

Table 4: Penrose-Banzhaf indices with three major voters

w_1	w_2	w_3	$\beta_{1\infty}$	$\beta_{2\infty}$	$\beta_{3\infty}$
0.3	0.2	0.1	0.6	0.2	0.2
0.3	0.1	0.1	1.0	0.0	0.0

2 Empirical findings for shareholder voting power and comparison with theoretical results for 'oceanic' power indices

The theoretical results discussed in the previous section are derived for limiting cases. Asymptotic results tell us nothing about the rate of convergence to the limit and therefore these results can give us no guidance on large finite voting bodies in the real world. We still have the question: how large does n have to be in order that the limiting theory can be assumed to hold as a reasonable approximation? Voting bodies encountered in the real world can never conform completely to the limiting case.

Earlier work (Leech (2002a)) on shareholder voting bodies for a sample of companies listed on the London Stock Exchange provides some insight into this question. Typically large companies in the UK have many thousands of shareholders, the great majority of whom own only a very small stake. There are a few large shareholders whose holdings will be powerful because of the weighted voting effect and who will thereby be in possession of a high degree of control over the company. The number of voters, although strictly finite, is large (tens or hundreds of thousands for the largest companies) and such voting bodies might be thought of as approximating the conditions of an 'oceanic' game. But because the number of voters is finite, we have been able to compute power indices for these shareholder voting bodies by algorithms that do not rely on the same limit theorems (Leech (2003)) and therefore we can check on the realism of the theoretical results.

Tables 5 and 6 give the results for two randomly selected companies, one with concentrated ownership, where there is at least one large shareholder, and one with dispersed ownership, where the largest shareholder is relatively small. In each case, we show the weights for the top six shareholders, as proportions, and the corresponding limiting PBI's, the indices for the finite game, the limiting SSI's and the SSI's for the finite game. In each case the data is such that only the upper tail of the distribution of shareholdings is observed: in the case of Conder Group the 18 largest shareholders are observed and in the case of Aurora, it is the largest 38 shareholders. We assume (for the sake of convenience) that the unobserved shareholdings are as large as possible, that they are all no bigger than w_{18} in the former and w_{38} in the latter cases. Therefore the finite voting bodies are assumed to have $n = 78$ and $n = 176$ voters respectively. In calculating the limiting oceanic voting power indices we assume $m = 18$ and $m = 38$ respectively for the PBI's and $m = 5$ for the SSI's.

There is a different pattern in the two cases. In the concentrated ownership case, Conder Group, the largest shareholding is 35%, the second-largest 12.5% and the third 7.9%. In comparison with companies listed on the London Stock Exchange, this is a very concentrated ownership structure, with a very large shareholder. This concentration is reflected in the value for the normalised PBI for shareholder 1, 0.9222, or a power index of over 92%. By contrast the PBI for shareholder 2 is only 0.0033, that is that a shareholder with 12.5% of the votes has only 0.3% of the voting power. The concentration of voting weight tends to lead to a greater concentration of voting power. The limiting PBI's, $\beta i\infty$, are very similar to the finite indices, suggesting that the asymptotic results are a reasonable approximation to the real power indices.

The two versions of the SSI's are also very close to each other, agreeing to two places of decimals. However they are very different from the PBI's.

What is notable is that both the PBI's and the SSI's are quite different from the weights. So although this is a large voting body, we cannot infer from these results that power and weight are proportional.

Table 5: Shareholder power indices: Conder Group - concentrated ownership

$i =$	1	2	3	4	5	6
w_i	0.3526	0.1252	0.0790	0.0364	0.0297	0.0242
$\beta_i(n = 78)$	0.9222	0.0033	0.0033	0.0032	0.0029	0.0023
$\beta i\infty(m = 18)$	0.9912	0.0009	0.0009	0.0009	0.0009	0.0004
$\phi_i(n = 78)$	0.5052	0.0887	0.0643	0.0286	0.0232	0.0188
$\phi_{i\infty}(m = 5)$	0.5066	0.0879	0.0646	0.0287	0.0232	——

On the other hand, the findings for the company with relatively dispersed share ownership, Aurora in Table 6, are quite different. Here all the indices, with the exception of the limiting PBI's, $\beta i\infty$, appear to be close to each other and the weights. This suggests that it may be the dispersion of the weights rather than the number of voters that is the important factor.

Other real-world examples in which measures of voting power are not proportional to weights for at least some voters include the IMF and World Bank, bodies for which $n > 180$ (Leech, 2002b). Another example might be a hypothetical world assembly where voting weights are proportional to populations.

Table 6: Shareholder power indices: Aurora - dispersed ownership

$i =$	1	2	3	4	5	6
w_i	0.0293	0.0282	0.0225	0.0220	0.0174	0.0169
$\beta_i(n = 176)$	0.0304	0.0291	0.0229	0.0224	0.0175	0.0170
$\beta i\infty(m = 38)$	0.0742	0.0711	0.0556	0.0543	0.0425	0.0412
$\phi_i(n = 176)$	0.0300	0.0288	0.0228	0.0223	0.0176	0.0171
$\phi_{i\infty}(m = 5)$	0.0301	0.0289	0.0230	0.0224	0.0177	———

3 The Penrose Limit Theorem and recent evidence supporting it

Recent literature, beginning with Lindner & Machover (2004), has examined the question from the point of view of the Penrose limit theorem, which was a result stated by Penrose (1952). He did not actually present it as a formal theorem and gave no proof; he merely mentioned it in passing, in describing his Square Root Rule[1], so perhaps it might more accurately be described as a conjecture. We can state the theorem as follows (after Lindner and Macover).

Theorem 3 (Penrose Limit Theorem): If the number of voters increases indefinitely while existing voters always keep their old weights and the relative quota is pegged at half the total weight, then, under suitable conditions,

$$\frac{\beta_i}{\beta_j} \to \frac{w_i}{w_j}$$

for any pair of existing voters, i, j, where β_i and β_j are the normalised or non-normalised Penrose-Banzhaf indices. Lindner and Machover also give a version of the theorem in terms of the SSI.

Proof Lindner & Machover (2004)

[1] The Penrose square root rule is a rule for equalising *a priori* voting power in a federal voting body, where there are two levels of voting. At the first, local stage, if voting is by majority rule with one-person-one-vote, the voting power index of any voter is inversely proportional to the square root of the electorate. Therefore the second, higher level should adopt a rule whereby the voting power of each delegate should be proportional to the square root of his electorate.This is hard to implement in general. However, if the conditions for the Penrose limit theorem hold, an approximation can be used whereby the voting weights at the higher level are chosen proportional to the square roots of the electorates in the knowledge that they will be proportional to the powers.

Lindner and Machover show, by a counterexample, that the limit does not hold in general, hence the need for the 'under certain conditions' qualification.(See also Lindner & Owen (2007).) But they argue that such counter-examples are atypical.

Studies which find evidence of approximate proportionality in large voting bodies include the analysis of the Nice system of qualified majority voting currently used by the Council of the European Union[2] for which power and weight are close to being proportional in the present 27-member EU. This is true not only for the PBI but also the SSI. Support for the theorem has also come from a simulation study by Chang *et al.* (2006). Their overall conclusion is "Both real-life and randomly generated Weighted Voting Games with many voters provide much empirical evidence that the Penrose Limit Theorem holds in most cases, as a general rule".

Other empirical studies also appear to support the case. For example Słomczynski & Zyczkowski (2007) have proposed a weighted majority voting rule for the EU Council in which decisions are taken by a single supermajority with votes being assigned to each country in proportion to the square root of its population, a variant of the Penrose Square Root rule. The PBI's are extremely close to proportionality. Leech & Aziz (2010) have also studied the voting power properties of this proposed voting system - the 'Jagiellonian compromise' - under hypothetical scenarios for an expanding union and found it to be remarkably proportional also in much larger possible European unions. However this does not in itself constitute evidence that this is due to n being large. (And it cannot be due to the weights being population square roots.)

Other apparent empirical support comes from a recent study by Miller (2010) of the US presidential Electoral College. See Figure 1. He finds power closely proportional to weight in all states except the one with the largest weight, California, where it is slightly greater. Again it is not clear that this is due to n being large (in this study, $n = 51$) or something else. It is interesting to note that the departure from proportionality is with the state with most electoral votes by a wide margin.

There are undoubtedly many more such examples. However, as we have seen above, there are also many real-world examples that do not support this version of the Theorem. However a careful reading of Penrose can give rise to a slightly different interpretation of his conjecture. In the next section I discuss this question in detail and propose a restatement of the Theorem which can reconcile the conflicting empirical observations.

[2] Actually a triple-majority rule with supermajorities

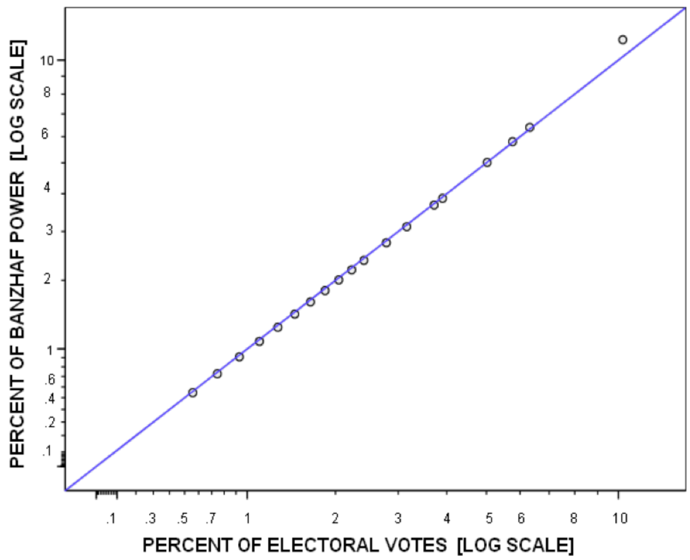

Figure 1: Share of voting power by share of electoral votes(Miller, 2009)

4 Penrose's approximation

The approximation that Penrose (1952) gives is based on the assumption of a probabilistic voting model and an appeal to the central limit theorem to enable the use of the Gaussian distribution. The accuracy of the approximation therefore depends crucially on whether the conditions for the central limit theorem actually do hold in any particular case.

In the next subsection I will present and restate his argument, amplifying it for clarity where necessary, following the same structure, and section numbering that he used in his Appendix, beginning on p71. I will use his notation (supplementing it with my own where necessary) with one change: he denoted the number of voters by u, whereas I will stick with n, which seems more natural to me.

4.1 Penrose's argument

Let us suppose a voting body[3] has n voters with votes $a_1, a_2, \cdots, a_n$. Penrose defines $S(a) = \sum a_i$ and $S(a^2) = \sum a_i^2$. He assumes a simple majority quota throughout. I will use his section numbering.

[3]Penrose uses the term assembly - he was presumably thinking specifically, at the time he was writing, of a then-planned United Nations General Assembly which he supposed would use weighted voting.

(i) Penrose begins by showing that there is an equivalence between a general weighted voting body and a hypothetical one in which there are a number of blocs each having the same number of votes.

Let there be a number N of equal-sized blocs, [4] each with a_N votes. Then, obviously, $Na_N = S(a)$.

Equivalence is defined in terms of the probability distribution of the total votes of the body, assuming probabilistic voting. The total number of votes cast 'for' an action must be the same in the real and the hypothetical voting bodies.

Let each voter cast his votes 'for' the action under consideration with probability $\frac{1}{2}$ independently of the other voters. Let the total number of 'for' votes be X. Then X is a random variable with mean and variance,

$$\mu_X = \frac{S(a)}{2} = \frac{Na_N}{2},$$

$$\sigma_X^2 = \frac{S(a^2)}{4} = \frac{Na_N^2}{4}.$$

We therefore have,

$$Na_N = S(a)$$

$$Na_N^2 = S(a^2)$$

and hence,

$$a_N = \frac{S(a^2)}{S(a)}$$

and,

$$N = \frac{S(a)^2}{S(a^2)}. \tag{7}$$

Note the distinction between the number of voters, n, and the equivalent number of equal-sized blocs, N.

(ii) Penrose then defines the power of a single bloc of size a_N in an assembly of N equally sized blocs. It is as well to think of this as being a separate group of voters from the one whose power we wish to measure: that is the complete voting body is of size $n+1$ and the hypothetical equal voting body of size $N+1$.

[4] This is Penrose's exact notation.

The absolute *a priori* voting power of this bloc (absolute PBI) is the probability that it will be the swing voter. We know the mean and variance of the number of 'for' votes, X. In addition, if N is large enough (or equivalently n large enough *and* the variance of the weights not too large) then X can be assumed to have a Gaussian distribution by the central limit theorem. Now we can find the swing probability. Penrose assume that the voter with a_N votes is able to swing the decision if the 'for' votes just balance the 'no' votes; that is, $X = \mu_X$.

We know the probability density function of the Gaussian distribution of X is

$$f(X) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}[\frac{X-\mu_X}{\sigma_X}]^2}$$

and therefore,

$$f(\frac{Na_N}{2}) = \frac{1}{\sqrt{2\pi}\sigma_X} = \frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{N}a_N} = \frac{1}{a_N}\sqrt{\frac{2}{\pi N}}.$$

Therefore, the absolute voting power of the bloc, P_{a_N}, is given by

$$P_{a_N} \approx a_N f(\frac{Na_N}{2}).$$

That is,

$$P_{a_N} \approx \sqrt{\frac{2}{\pi N}}.$$

(iii) Penrose then gives the approximate power of a bloc of size a_1, P_{a_1}:

$$P_{a_1} \approx a_1 f(\frac{Na_N}{2}) \approx \frac{a_1}{a_N} P_{a_N}.$$

(iv) Therefore, the power of any bloc with a_1 votes is, from (ii) and (iii),

$$P_{a_1} \approx \frac{a_1}{a_N}\sqrt{\frac{2}{\pi N}}.$$

This requires $\frac{a_1}{a_N}$ to be small as well as N large.

(v) Then, since $a_N = \frac{S(a^2)}{S(a)}, N = \frac{S(a)^2}{S(a^2)}$, we can write

$$P_{a_1} \approx a_1 \frac{S(a)}{S(a^2)} \frac{\sqrt{S(a^2)}}{S(a)} \sqrt{\frac{2}{\pi}} = a_1 \sqrt{\frac{2}{\pi S(a^2)}}.$$

(vi) Therefore, this shows that, provided the conditions for the approximation hold, if countries are allocated votes in a world assembly according to the square roots of their populations, so that $S(a^2) = \sum_{i=1}^{n} a_i^2$ is the total world population, voting powers are proportional to weights. Penrose says that this approximation is accurate for small countries and even for large countries up to $\frac{1}{5}$ of the world's population.

(vii) Penrose next shows that this square root rule has the property that it equalises voting power among all individuals in all countries, since the power of a single vote in a country with population a_1^2 is $\frac{1}{a_1}\sqrt{\frac{2}{\pi}}$, each person's power of influencing a decision in this 2-stage voting body is

$$\frac{1}{a_1}\sqrt{\frac{2}{\pi}}P_{a_1} \approx \frac{1}{a_1}\sqrt{\frac{2}{\pi}}a_1\sqrt{\frac{2}{\pi S(a^2)}} = \frac{2}{\pi\sqrt{S(a^2)}}.$$

This is the original statement of what is now known as the Penrose Square Root Rule.

5 The Laakso-Taagepera index

Penrose's parameter N defined in the last section is a numbers-equivalent. It is the hypothetical number of voters who, having equal weights, would be equivalent, under probabilistic voting, to the actual voters in the sense that the distribution of the total number of votes they cast 'for' an action would have the same expectation and variance. Actually it turns out that this N is identical to the Laakso-Taagepera index (Laakso & Taagepera, 1979) of party fragmentation (hanceforth the L-T index), as we now show.

Definition: The Laakso-Taagepera index. If the weights of the members of a voting body are $w_1, w_2, ..., w_n$ then the number L defined by

$$L = \frac{1}{\sum\left(\frac{w_i}{\sum w_i}\right)^2} = \frac{\left(\sum w_i\right)^2}{\sum w_i^2}. \tag{8}$$

is the Laakso-Taagepara index.

Alternatively, if w_i is a vote share, such that $\sum w_i = 1$, we have

$$L = \frac{1}{\sum w_i^2}.$$

(Laakso & Taagepera, 1979)

It is apparent that L is the same as Penrose's N defined in equation (7) above because $S(a) \equiv \sum w_i$ and $S(a^2) \equiv \sum w_i^2$.

The index L is widely used in political science to measure party fragmentation in parliaments. It is a numbers-equivalent for the 'effective number of parties' taking into account both the actual number of parties and their comparative sizes. In a weighted voting body its value depends on both the number of voters and the dispersion of the weights.[5]

Feld & Grofman (2007) show that the index can be written in terms of its mean and variance, as:

$$L = \frac{1}{\bar{w} + ns^2}$$

where $\bar{w}$ is the mean and s^2 the variance of the weights, $s^2 = \frac{1}{n} \sum \left(\frac{w_i}{\sum w_i} \right)^2 - \bar{w}^2$.

If the $w_i s$ are vote shares, then $\bar{w} = \frac{1}{n}$ and

$$L = \frac{1}{\frac{1}{n} + ns^2}.$$

The L-T index combines the number of voters with the dispersion of their weighted votes.

6 The Penrose Limit Theorem restated

We can now give a restatement and proof of the theorem. We do not restrict the analysis to simple-majority voting rules.

6.1 The Banzhaf index

Theorem 4 (Penrose Limit Theorem Restated) Consider a sequence of voting bodies with n voters,

$$[q; w_1, w_2, \cdots, w_n],$$

where $\sum w_i$ is constant. The quota q is not restricted to half the total weight. Let the Absolute PBI of a voter a, having w_a votes, be β'_a and the L-T index be L. Then, for any two voters, a, b, with constant finite weights,

$$\lim_{L \to \infty} \frac{\beta'_a}{\beta'_b} = \frac{w_a}{w_b} \tag{9}$$

[5]It is related to, and mathematically formally the same as, the Hirschman-Herfindahl index of industrial concentration used in industrial organisation economics and anti-trust policy.

Proof

Assume each voter casts his votes 'for' or 'against' with equal probability, independently of the others. Let the number of votes 'for' cast by voter i be u_i, a dichotomous random variable equal to 0 or w_i with equal probability. Then, $E(u_i) = \frac{w_i}{2}, var(u_i) = \frac{w_i^2}{4}$. Let the total number of votes cast 'for' be $y = \sum_{i=1}^n u_i$.

Now we make use of the Lindeberg-Feller theorem to determine the asymptotic behaviour of y.

The Lindeberg-Feller theorem (see Feller (1971)) states that, if a sequence of independent random variables $x_1, x_2, \cdots, x_n$ with $E(x_i) = 0$ and $E(x_i^2) = \sigma_i^2 < \infty$ for all i such that,

$$\frac{\sigma_j^2}{\sum_{i=1}^n \sigma_i^2} < \epsilon, \tag{10}$$

for all j, and some $\epsilon > 0$, then,

$$\lim_{n\to\infty} P\left[b < \frac{\sum_{i=1}^n x_i}{\sqrt{\sum_{i=1}^n \sigma_i^2}} < c\right] = \Phi(c) - \Phi(b) \tag{11}$$

for constants b, c, where $\Phi(x)$ is the standard Gaussian cdf.

Let us define $x_i = u_i - \frac{w_i}{2}$. ($u_i$ was defined above.) Then x_i has mean zero and variance $\sigma_i^2 = \frac{w_i^2}{4}$. (Also, note that $y = \sum u_i = \sum x_i + \frac{\sum w_i}{2}$.)

Therefore condition (10) requires that

$$\frac{w_j^2}{\sum_{i=1}^n w_i^2} \to 0$$

as $n \to \infty$, for all j, which is obviously not true in general.

However, we can find another condition in terms of the equivalent voting body with equal weights that satisfies the analogous version of (10).

The total 'for' votes, y has a distribution with expectation and variance

$$E(y) = \frac{Lw_L}{2} = \frac{\sum_{i=1}^n w_i}{2}, V(y) = \frac{Lw_L^2}{4} = \frac{\sum_{i=1}^n w_i^2}{4}.$$

Then, let us redefine x_i in terms of equivalently distributed random variables with constant variance.

Write

$$x_i = v_i - \frac{w_L}{2},$$

where i= 1, 2, ..., L and the random variables v_i all have the same dichotomous distribution: $v_i = 0$ with probability 1/2, or $v_i = w_L$ with probability 1/2. So, $E(v_i) = \frac{w_L}{2}, V(v_i) = \frac{w_L^2}{4}$. Also $y = \sum_{i=1}^{L} v_i$.

Therefore,

$$E(x_i) = 0,\ V(x_i) = \frac{w_L^2}{4},\ i = 1, 2, ..., L.$$

Then,

$$\beta'_a = Pr\left[q - w_a < y \leq q\right] = Pr\left[q - \frac{Lw_L}{2} - w_a < \sum_{1=1}^{L} x_i \leq q - \frac{Lw_L}{2}\right].$$

Letting $c = q - \frac{Lw_L}{2} = q - \frac{\sum_{i=1}^{n} w_i}{2}$, this can be written as,

$$\beta'_a = Pr\left[c - w_a < \sum_{1=1}^{L} x_i \leq c\right]. \tag{12}$$

In this case, we have that $\sigma_i^2 = V(x_i) = \frac{w_L^2}{4}$ for all $i = 1, 2, ..., L$. Therefore, condition (10) becomes now (replacing n by L etc):

$$\frac{w_L^2}{\sum_{i=1}^{L} w_L^2} = \frac{w_L^2}{Lw_L^2} = \frac{1}{L} \to 0 \tag{13}$$

as $L \to \infty$, which is always satisfied.

Therefore, for large L, by the Lindeberg-Feller theorem,

$$\beta'_a = Pr\left[c - w_a < \sum_{1=1}^{L} x_i \leq c\right] \to \Phi(c) - \Phi(c - w_a).$$

The first order Taylor approximation of $\Phi(c - w_a)$ is

$$\Phi(c - w_a) = \Phi(c) - w_a\phi(c).$$

Hence,

$$\beta'_a \to w_a\phi(c) \text{ as } L \to \infty.$$

Therefore, for two voters, a and b, with constant finite weights,

$$\frac{\beta'_a}{\beta'_b} \to \frac{w_a}{w_b} \text{ as } L \to \infty.$$

6.2 The Shapley-Shubik index

The same property is possessed also by the SSI. We now state and prove this in Theorem 5.

Theorem 5 (Limiting behavior of the SSI)

For the sequence of voting bodies defined in Theorem 4 above, and L the L-T index, let the SSI for voters a, b be ϕ_a, ϕ_b. Then,

$$\lim_{L \to \infty} \frac{\phi_a}{\phi_b} = \frac{w_a}{w_b}.$$

Proof

The proof uses a generalisation of that of Theorem 4 together with the approach of (Owen, 1972).

The SSI for voter a (where we assume $a \notin N$) is defined as,

$$\phi_a = \sum_{T \in S} \frac{t!(n-t)!}{n!}, \tag{14}$$

where $S, T \subseteq N$ and $S = \{T \subset N | q - w_a < \sum_{i \in T} w_i \leq q\}$. S is the set of swings for voter a and $T \in S$ is a particular swing.

The term inside the summation in (14) is a beta function, and so,

$$\frac{t!(n-t)!}{n!} = \int_0^1 p^t (1-p)^{n-t} dp,$$

and therefore, substituting,

$$\phi_a = \sum_{T \in S} \int_0^1 p^t (1-p)^{n-t} dp = \int_0^1 \left[\sum_{T \in S} p^t (1-p)^{n-t} \right] dp. \tag{15}$$

Assume voter i votes 'for' with probability p, and 'against' with probability $1-p$. Then we can interpret the term inside the square brackets in (15) as the probability of a swing.

The number of votes 'for' cast by i are u_i, a random variable with expectation $w_i p$ and variance $w_i^2 p(1-p)$. Therefore the total votes 'for' cast by all voters is $y = \sum_{i=1}^n u_i$. We can redefine the set of swings S in terms of the values of y: $S = \{y | q - w_a < y \leq q\}$.

We can now equate the two expressions for the probability of a swing,

$$\sum_{T \in S} p^t (1-p)^{n-t} = Pr\left[q - w_a < y \leq q\right] \tag{16}$$

Now let $x_i = v_i - pw_L$, where v_i is a random variable taking the values 0 and w_L with probabilities p and $1-p$. Then, $E(x_i) = 0$, $Var(x_i) = w_i^2 p(1-p)$ and $y = \sum_{i=1}^{L} x_i - pLw_L$.

We can rewrite the RHS of (16) as

$$Pr\left[q - w_a < y \leq q\right] = Pr\left[c(p) - w_a < \sum_{i=1}^{L} x_i \leq c(p)\right] \tag{17}$$

where $c(p) = q + pw_L)$.

The condition for the Lindeberg-Feller theorem holds because $\sigma_j^2 = w_L^2 p(1-p), \forall j = 1, ..., L$, and so $\frac{\sigma_j^2}{\sum_{i=1}^{L} \sigma_i^2} = \frac{1}{L} \to \infty$ as $L \to \infty$. Therefore (17) becomes, in the limit,

$$\lim_{L \to \infty} Pr\left[c(p) - w_a < \sum_{i=1}^{L} x_i \leq c(p)\right] = \left[\Phi(c(p)) - \Phi(c(p) - w_a)\right] = w_a \phi(c(p)).$$

We therefore have the limiting SSI,

$$\lim_{L \to \infty} \phi_a = w_a \int_0^1 \phi(c(p)) dp,$$

and

$$\frac{\phi_a}{\phi_b} \to \frac{w_a}{w_b}.$$

6.3 Approximation

Theorem 4 has been stated and proved in terms of the limiting power index of voter $a \notin N$ and the L-T index for the weights of voters $N = \{1, 2, ..., n\}$. This is the same approach as used by Penrose who ignored complications arising when $a \in N$ and essentially treated a as an extra voter.

If we want to apply it to voter $i \in N$, then the restated theorem is in terms of the limiting behavior of the L-T index for the set of voters not including i, $N \backslash \{i\}$. It is of interest to use the L-T index for the whole set of voters, N. We can find the relation as an approximation. Consider the voting power index for voter number 1.

Let L_N and $L_{N\setminus\{1\}}$ be the L-T indices for the two sets of voters. Let

$$L_N = \frac{(\sum_{i=1}^n w_i)^2}{\sum_{i=1}^n w_i^2}$$

and

$$L_{N\setminus\{1\}} = \frac{(\sum_{i=1}^n w_i - w_1)^2}{\sum_{i=1}^n w_i^2 - w_1^2}.$$

Therefore,

$$\sum_{i=1}^n w_i^2 = \frac{(\sum_{i=1}^n w_i)^2}{L_N} = w_1^2 + \frac{(\sum_{i=1}^n w_i - w_1)^2}{L_{N\setminus\{1\}}},$$

and hence

$$L_N = \frac{(\sum_{i=1}^n w_i)^2 L_{N\setminus\{i\}}}{w_1^2 L_{N\setminus\{i\}} + (\sum_{i=1}^n w_i - w_1)^2.}$$

Let $\frac{w_1}{\sum_{i=1}^n w_i} = k$: the voting weight of voter number 1 remains a constant fraction of the total weight. Then,

$$L_N = \frac{L_{N\setminus\{1\}}}{k^2 L_{N\setminus\{1\}} + (1-k)^2}.$$

Therefore, as $L_{N\setminus\{1\}} \to \infty$ then $L_N \to \frac{1}{k^2}$, and we can see that the theorem holds when the L-T index is high provided that k is small.

Notice that the L-T index depends on both the number of voters and the variance of the weights. It is not sufficient that the number of voters is large to ensure approximate convergence of relative powers and weights. It depends also on what happens to the variance.

6.4 Application

There are two types of voters for whom this approximation does not hold.

1. Those with small weight in voting bodies with a few large voters. In this case we would have that $\frac{w_a}{\sum w_i}$ is small but L is also small. In this case the Gaussian approximation is not good. An example of this case might be a small shareholder in a corporation which has one (or a small number of) very large blockholders who effectively control the company.

2. Those with a relatively large weight. In this case L might be large enough but the approximation involved in going from the limit in terms of cdfs to pdfs is poor because $\frac{w_a}{\sum w_i}$ is too large. An example might be the power index of a large shareholder in a company with an otherwise very dispersed ownership structure.

7 Conclusion

I have examined the properties of the 'classical' voting power indices, those due to Penrose and Banzhaf and to Shapley and Shubik, in large voting bodies. Applied studies of actual voting bodies with large numbers of voters have found conflicting evidence of the relationship between the voting weights and the power indices when the number of voters is large. In some cases there is close to proportionality but in others they are far from proportional. The Penrose limit theorem has been taken to suggest that in most cases there should be proportionality.

This paper shows that it is false to assume a general tendency towards proportionality as the number of voters increases. Instead it proposes a restatement of the Penrose limit theorem in terms of the asymptotic behaviour of the Laakso-Taagapera index of fragmentation, rather than the number of voters. It has that the result holds for both the power indices considered, reconciling the apparently conflicting results obtained in different research studies, both empirical and theoretical.

References

Chang, P.L., Chua, V.C.H., & Machover, M. 2006. LS Penrose's limit theorem: Tests by simulation star, open. *Mathematical Social Sciences*, **51**(1), 90–106.

Dubey, P., & Shapley, L.S. 1979. Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research*, 99–131.

Feld, S.L., & Grofman, B. 2007. The Laakso-Taagepera index in a mean and variance framework. *Journal of Theoretical Politics*, **19**(1), 101.

Feller, W. 1971. *Probability Theory and its Applications, vol. II.*

Felsenthal, Dan, & Machover, Moshé. 1998. *The Measurement of Voting Power.* Cheltenham: Edward Elgar.

Laakso, M., & Taagepera, R. 1979. Effective number of parties: A measure with application to West Europe. *Comparative Political Studies*, **12**(1), 3–27.

Leech, Dennis. 2002a. An empirical comparison of the performance of classical power indices. *Political Studies*, **50**(1), 1–22.

Leech, Dennis. 2002b. Voting Power in the Governance of the International Monetary Fund. *Annals of Operations Research*, **109**, 373–95.

Leech, Dennis. 2003. Computing power indices for large voting games. *Management Science*, 831–838.

Leech, Dennis, & Aziz, Haris. 2010. The Double Majority Voting Rule of the EU Reform Treaty as a Democratic Ideal for an Enlarging Union: an Appraisal Using Voting Power Analysis. *In:* Cichocki, Marek, & Zyczkowski, Karol (eds), *Institutional Design and Voting Power in the European Union*. Ashgate.

Lindner, I., & Machover, M. 2004. LS Penrose's limit theorem: proof of some special cases. *Mathematical Social Sciences*, **47**(1), 37–49.

Lindner, I., & Owen, G. 2007. Cases where the Penrose limit theorem does not hold. *Mathematical Social Sciences*, **53**(3), 232–238.

Miller, Nicholas. 2010. A priori voting power and the US Electoral College. *In:* Holler, Manfred J, & Widgren, Mika (eds), *Essays in Honour of Hannu Nurmi:Volume 1*.

Milnor, JW, & Shapley, LS. 1978. Values of large games II: Oceanic games. *Mathematics of Operations Research*, 290–307.

Owen, G. 1972. Multilinear extensions of games. *Management Science*, **18**(5), 64–79.

Penrose, L.S. 1952. *On the objective study of crowd behavior*. HK Lewis.

Shapiro, NZ, & Shapley, LS. 1978. Values of large games, I: a limit theorem. *Mathematics of Operations Research*, 1–9.

Słomczynski, Wojciech, & Zyczkowski, Karol. 2007. *The Jagellonian Compromise: An Alternative Voting System for the Council of the European Union*. Tech. rept. Institute of Mathematics, Jagiellonian University, Krakow.