On the theory of a firm:
The case of by-production of emissions.

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No 934

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

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May 2010

Acknowledgment: I am grateful to the participants of the Warwick work-shop series in Economics for very helpful comments.

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Abstract

Five attributes of emission generating technologies are identified and a concept of by-production is introduced, which implies these five attributes. Murty and Russell [2010] characterization of technologies, which requires distinguishing between intended production of firms and nature's laws of emission generation, is shown to be both necessary and sufficient for by-production. While intended production could be postulated to satisfy standard input and output free-disposability, these will necessarily be violated by nature's emission generation mechanism, which satisfies costly disposability of emission as defined in Murty [2010]. Marginal technical and economic costs of abatement are derived for technologies exhibiting by-production. The former measures the loss in intended outputs when the firm is mandated to reduce emissions, while the latter measures its loss in profits under regulation. The by-production approach reveals a rich set of abatement options available to firms. These include reductions in the use of fuel inputs, inter-fuel substitution, increase in cleaning-up efforts, and technological change. In a simple model of by-production, we show that, when faced with regulation, the firm will use all or some of these strategies. This is in contrast to the standard input-approach to modeling emission generating technologies, where we show that, under a Pigouvian tax, a firm will reduce its emissions, solely, by increasing its cleaning-up effort. The standard input-approach also allows some paths of inputs and outputs, which seem inconsistent with nature's laws of emission generation, to become technologically feasible. Our model of by-production illustrates that, while common abatement paths considered in the literature do involve a technological trade-off between emission reduction and intended production, there also almost always exist abatement paths where it is possible to have *both* greater emission reductions and greater intended outputs. Further, marginal abatement costs will usually be decreasing in the initial level of emissions of firms. Counterintuitive as these results may sound in the first instance, they are intuitively obvious in the by-production approach as it is rich enough to incorporate both standard economic assumptions, such as diminishing returns, with respect to intended production of firms and the rules of nature that govern emission generation.

Keywords: theory of a firm, technology, input and output free-disposability, diminishing returns to inputs, joint production, emission-generation, marginal abatement cost.

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1. Introduction.

What are the essential attributes of the process of emission generation by firms when they engage in the production of their intended outputs?

Firstly, in nature, there are certain goods that cause emissions under certain physical conditions. If, in the process of their intended production, firms create such conditions and employ these emission causing goods as inputs or produce these emission causing goods as their intended outputs, then they trigger-off the nature's emission generating mechanism and generate emissions as *by-products* or *incidental outputs* of their intended production.[1]

Secondly, a distinction needs to be made between inputs of emission generation and inputs of intended production. The emission causing inputs used by a firm in its intended production or the emission causing intended outputs produced by a firm are the inputs of emission generation. They have a special property: they exhibit *non-rivalness* or *jointness* in emission generation and intended production: the use of such inputs to produce intended outputs does not reduce their availability for emission generation, and the fact that some intended outputs serve also as inputs of emission generation does not reduce the amounts of such goods that the firms can offer for sale in markets.[2]

Thirdly, technologies of firms producing emissions as by-products do not satisfy free-disposability of emissions[3]: for every given vector of the emission producing inputs used by the firm or the emission producing outputs produced by the firm, there is a certain *minimal* amount of emission that will always be generated. Technical inefficiencies may imply that the firm can be generating more (but not less) than this minimum amount of emission.[4] On the other hand, a given vector of inputs is also associated with a menu of *maximum* possible combinations of intended outputs that are technologically feasible. Technical efficiencies may imply that the firm produces less (but not more) than these efficient amounts of intended outputs.

Fourthly, all the above attributes may imply systematic *correlations* between emission generation and intended production. If emissions are generated by certain inputs of

[1] For example, an emission causing input is coal. Emission of a strong odor can be caused by intended outputs such as cheese of a dairy.

[2] The same amount of coal produces both smoke and electricity. The fact that cheese produced by a dairy is an input into production of strong odor (a by-product of the dairy) does not reduce the amount of cheese that the dairy can sell.

[3] Intuitively, a technology satisfies output free disposability if a given level of output is producible from a given vector of inputs implies that any lower level of output is also producible from the same vector of inputs. A technology satisfies input free disposability if a given level of input can produce a given vector of outputs implies that any higher level of input can also produce the same vector of outputs. These definitions allow for the possibility that firms can be operating technically inefficiently.

[4] Murty [2010] calls this "costly disposability" and provides two definitions of this term, one of which is the polar opposite of the standard definition of free disposability of output employed in the literature.

intended production, then increases in the use of these inputs by firms implies a greater production of both emissions and the intended outputs.[5] If emissions are generated by certain intended outputs of the firm then, for a fixed vector of all inputs, an increase in the scale of production of these outputs (by diverting more of the inputs into their production) leads to a greater generation of emissions and lower production of the remaining intended outputs.[6]

Fifthly, by diverting some of their resources towards *cleaning-up activities* from production of intended goods, firms can mitigate emission generation.[7] But this comes at the cost of lower production of intended goods.

Many of the emissions generated by firms impose external effects (beneficial or detrimental) on the rest of the society. Efforts to reduce (increase) harmful (beneficial) emissions are often costly for the generators as they usually entail changing the profitable scale of their intended production activities. Hence, in the absence of additional incentives, the generating firms will not voluntarily internalize, in their decisions, the external effects on the rest of the society caused by by-products such as emissions that they produce.

A first step in designing policies that provide incentives to firms to regulate the generation of such by-products is a thorough understanding of the inextricable link between emission generation and intended production. In the case of harmful emissions, such an exercise delineates the various options available to firms to abate emission generation and their associated costs. The aims of this paper are to provide an understanding of this link and to identify and assess the costs of various abatement strategies available to firms.

The literature usually models emission generating technologies with the help of a single production relation between inputs, intended outputs, and emissions.[8] Such a production relation is assumed to exhibit a positive relation between intended outputs and emission generation. Such a positive relation is introduced either by bestowing the emission with properties of a standard input[9] or by treating it as an output that possesses

[5] Increases in coal used increases both the level of smoke produced and the level of electricity generated.

[6] For a given amount of milk, an increase in cheese production implies increase in the amount of odor generated and a decrease in the amount of butter produced by the dairy (as the use of the milk to produce greater amount of the cheese decreases the amount of milk for producing butter).

[7] Cleaning-up activities to reduce emissions generated include end-of pipe treatment plants, recycling, etc.

[8] See, *e.g.,* the classic text in environmental economics by Baumol and Oates [1988] and Cropper and Oates [1992].

[9] For the input approach see, *e.g.,* Baumol and Oates [1988] and Cropper and Oates [1992]. This is often justified by considering the amount of the emission generated as a proxy for the amount of the assimilative capacity of environmental resources such as air and water used to absorb it. However, a clear distinction needs to be made between environmental resources, which definitely serve as inputs of emission generation, and emission itself, which is an (incidental) output of production. A given environmental resource like air can absorb different types of emissions like CO_2, SO_2, *etc.*, and its assimilative capacity can be different for different emissions. See Murty [2010] for this distinction. In this paper we do not model the use of environmental resources as inputs of emission generation. However, see Footnote 37 for a potential way of doing so.

a radial disposability property (called weak disposability) that is weaker than standard output free-disposability.[10]

Murty and Russell [2010] show that a single production relation between inputs of intended production, intended outputs, and the emission is not rich enough to capture all the trade-offs between goods that are exhibited by an emission generating technology.[11] Early works of Frisch [1965], which have recently been applied by Førsund [2009] in the context of pollution generating technologies, show the need for multiple production relations for modeling technologies where some inputs and outputs exhibit technological non-rivalness/jointness. Murty and Russell [2010] show that all the above attributes of emission generation by firms can be captured in a model of a technology that is derived as an intersection of two technologies: (i) a technology defined by laws of nature regarding emission generation and (ii) a technology defined by the relation between inputs and outputs in intended production. Standard free disposability assumptions can be assumed for (ii), but (i) will necessarily violate free-disposability of both emission generation and the inputs that cause emissions in nature. As a result, the observed technology derived as an intersection of (i) and (ii) violates free disposability of both emissions and inputs that cause the emissions.

In Section 2, we provide a motivating example that shows the inadequacy of a single production relation in capturing all the above five features of emission generation by firms. In Section 3, we provide a definition of "by-production (BP)." Technologies that satisfy (BP) have all the five attributes of emission generation mentioned above. We show that such technologies violate standard disposability assumptions and that the Murty and Russell [2010] characterization is both a necessary and sufficient characterization of technologies that satisfy (BP). In particular, such a characterization implies that the emissions generated by a firm do not impose external effects on its own production of intended outputs.[12] On the other hand, when such external effects are present, then we show that technologies of emission generating firms may violate the fourth and fifth attributes of emission generating technologies mentioned above. Such technologies satisfy a weaker condition (WBP) than (BP). In Section 4, we give four examples to illustrate all the results in Section 3.

In Section 5, we provide an application of our by-production approach to the identification of various options open to firms for abating emissions and their associated costs. In the context of a simple model that exhibits by-production, we define the marginal technical abatement costs and the marginal economic abatement cost. Various factors that affect these marginal costs of abatement are unravelled and their properties are explained in terms of standard assumptions made in economics regarding intended production by firms. In the context of a simple model that satisfies (BP), we find that while common abatement paths considered in the literature do involve a technological trade-off between

[10] For the output approach with weak disposability, see Shephard [1953], Färe, Grosskopf, Noh, and Yaisawarng [1993], Färe, Grosskopf, and Pasurka [1986], Färe, Grosskopf, Lovell, and Pasurka [1989], Färe, Grosskopf, Noh, and Weber [2005], and Murty and Kumar [2002, 2003] among others.

[11] This is true both in the presence or absence of cleaning-up options available to firms.

[12] It may impose external effects on the rest of the society though.

emission reduction and intended production, there also almost always exist abatement paths where it is possible to have *both* greater emission reductions and greater intended outputs. We also find that marginal abatement costs are usually decreasing in the initial level of emissions of firms. Counterintuitive as these results may sound in the first instance, they are intuitively obvious when we adopt the by-production approach to modeling emission generating technologies as it is rich enough to incorporate both standard economic assumptions with respect to intended technologies of firms and the rules of nature that govern emission generation. We also compare the effects of emission regulation under the conventional input approach with the by-production approach. We conclude in Section 6. Most proofs are simple and are relegated to the appendix.

2. Notation and a motivating example.

2.1. Notation.

The commodity-space in which technologies are empirically observed is the space $\mathbf{R}_+^{n+m+2}$ formed by the coordinates $\mathbf{R}_+^{m+1}$ reserved for m intended outputs and a cleaning-up activity of the firm $\langle y, c\rangle \in \mathbf{R}_+^{m+1}$, $\mathbf{R}_+^n$ reserved for n inputs $x \in \mathbf{R}_+^n$, and $\mathbf{R}_+$ reserved for the by-product $z \in \mathbf{R}_+$.[13] We index inputs by i and intended outputs by j. We assume that, in nature, inputs $1, \ldots, n_1$ cause emissions, while the remaining $n_2 = n - n_1$ inputs do not. Similarly, intended outputs $1, \ldots, m_1$ cause emissions, while the remaining $m_2 = m - m_1$ intended outputs do not. Likewise, we partition the input and intended output vectors into $x = \langle x^1, x^2\rangle \in \mathbf{R}_+^{n_1+n_2}$ and $y = \langle y^1, y^2\rangle \in \mathbf{R}_+^{m_1+m_2}$, respectively. Let $T \subset \mathbf{R}_+^{n+m+2}$ denote a technology of the firm in the empirically observed space of commodities. Define also various restrictions of T such as $P_T(x, y, c) := \{z \geq 0 | \langle x, y, c, z\rangle \in T\}$, $P_T(x, y^{-j}, c, z) := \{y_j \geq 0 | \langle x, y^{-j}, y_j, c, z\rangle \in T\}$, and $P_T(x, y^{-j}, c) = \{\langle z, y_j\rangle \in \mathbf{R}_+^2 | \langle x, y^{-j}, y_j, c, z\rangle \in T\}$ for all $j = 1, \ldots, m$.

2.2. A motivating example.

Consider the case where a firm employs wood, water, chemicals, labor, and capital as inputs to produce paper ($y \in \mathbf{R}_+$). Wood and chemicals ($x_1 \in \mathbf{R}_+$ and $x_2 \in \mathbf{R}_+$) leave behind residuals which are the emissions ($z \in \mathbf{R}_+$) of paper production. Thus, $x^1 = \langle x_1, x_2\rangle$ and $x^2 = \langle x_3, x_4, x_5\rangle$. The larger the amount of chemicals and wood used by the firm, the greater is the maximal amount of paper it can produce and the greater is also the minimal level of emissions that it can generate. Consider any single production relation that satisfies standard assumptions with respect to the intended output and all

[13] For the sake of notational ease, we restrict our analysis to a single emission generated by any firm. We believe that the analysis can be generalized to the case of multiple emissions. It can be shown, along the lines of the analysis to follow, that in the case a firm generates more than one emission, more than two production relations may be required to characterize its technology.

inputs and aims to capture the positive correlation between emission generation and paper production. One such example is

$$y = f^y(x_1, x_2, x_3, x_4, x_5, z) = \prod_{i=1}^{5} x_i^{\frac{1}{5}} z. \tag{2.1}$$

The implied technology of the firm is[14]

$$T = \{\langle x_1, \ldots, x_5, y, z\rangle \in \mathbf{R}^7_+ |\ y \leq \prod_{i=1}^{5} x_i^{\frac{1}{5}} z\}. \tag{2.2}$$

The technically efficient production vectors are those that satisfy (2.1). Such a relation is inadequate to capture all the attributes of emission generation mentioned in Section 1 because of the following reasons.

(i) It implies that along the efficient frontier, the trade-offs between the residual and the chemical and wood inputs is negative which contradicts the fact that these inputs cause the residual:

$$\frac{\partial z}{\partial x_i} = \frac{-1}{5x_i \prod_{i' \neq i} x_{i'}^{\frac{1}{5}}} < 0 \ \forall\ i = 1, 2. \tag{2.3}$$

(ii) (2.1) implies that for a fixed vector of all inputs $\langle x_1, \ldots, x_5\rangle$, there is a whole menu of efficient paper and residual combinations (see Figure 1(a)).[15] This also contradicts the fact that the minimal amount of the residual generated changes if and only if the levels of the chemical or wood inputs change. Similarly, the maximal amount of paper produced can change if and only if inputs of paper production change.[16] This implies that if all inputs are held fixed, then there exist only one efficient combination of residual and paper that the firm can produce.

(iii) Figure 1(a) also shows that, at fixed level of all inputs, the efficient level of paper produced increases when the level of residual increase and that the menu of efficient paper-residual combinations in unbounded. This can be true only if the residual imposes a positive externality on paper production, *i.e.,* the residual is a productive input into paper production–which would seem counter-intuitive.

The literature[17] would justify production relations of the type (2.1) by assuming the existence of *implicit* cleaning-up options available to the paper producing firm, *e.g.,* an affluent treatment plant that cleans up the residual generated: holding the vector of all inputs (including chemicals and wood) fixed, the firm can be diverting more of the inputs

[14] Note, that this technology is consistent with both the input and output approaches to modeling emissions that are seen in the standard literature: it treats emission like an input and also satisfies the weak-disposability assumption of Shephard [1953].

[15] $P_T(x_1, x_2, x_3, x_4, x_5)$ in the figure is the restriction of T to the space of y and z for the given input vector.

[16] This is true assuming that the residuals do not generate external effects on paper production.

[17] See references in Footnotes 9 and 10 in Section 1.

into cleaning-up activities, so that less of both residuals and paper are produced. If such a perspective is adopted to interpret (2.1), then (2.2) is only a reduced-form description of the technology, as it does not explicitly model cleaning-up efforts. The full technology lies in the space of all goods–intended outputs, inputs, cleaning-up activities, and the emissions. Thus, the full technology would involve another production relation, in addition to (2.1), that explicitly involves cleaning-up options available to the firm. More importantly, if we require the technology to satisfy attributes one to five of emission generation in the introductory section then nature's residual generating mechanism implies that there is a minimum (and perhaps a maximum) amount of residual associated with every level of wood and chemical used by the firm, and that this minimum amount will *increase* as more and more of these inputs are used. A restriction of such a technology in the space of chemicals and the residual is shown in Figure 1(c). Clearly, the technology must violate both output free-disposability of emission and input free-disposability of chemicals. But T in (2.2) satisfies standard disposability assumptions with respect to paper and *all* its inputs (see Figure 1(b)). Thus, (2.2) may not be an appropriate specification of even the reduced-form technology underlying this example.

3. By-production and a necessary and sufficient characterization of technologies satisfying by-production.

3.1. A definition of by-production (BP) and properties of technologies satisfying (BP).

In this section we define the concept of by-production and show that technologies that satisfy by-production have all the observed properties mentioned in the introductory section.

First, we define three important projections of any observed emission generating technology T: (i) its projection into the space of all goods other than the emission– $\Omega_T := \{\langle x, y, c\rangle \in \mathbf{R}^{m+n+1} |\ \langle x, y, c, z\rangle \in T \text{ for some } z \geq 0\}$, (ii) for all $j = 1, \ldots, m$, its projection into the space of all goods other than the j^{th} intended output– $\Xi^j_T := \{\langle x, y^{-j}, c, z\rangle \in \mathbf{R}^{m+n+1} |\ \langle x, y^{-j}, y_j, c, z\rangle \in T \text{ for some } y_j \geq 0\}$, and (iii) for all $j = 1, \ldots, m$, its projection into the space of all goods other than the j^{th} intended output and the emission– $\Theta^j_T := \{\langle x, y^{-j}, c\rangle \in \mathbf{R}^{n+m}_+ |\ \langle x, y^{-j}, y_j, c, z\rangle \in T \text{ for some } \langle z, y_j\rangle \in \mathbf{R}^2_+\}$. These projections form the domains of the following mappings associated with T:[18] the functions

[18] Explicit specification of these projections is important as our examples in the next section demonstrate that these sets may be quite restricted and have structures that are different from the ones that are usually assumed in the case of non-emission generating technologies, namely, the entire non-negative orthants of associated Euclidean spaces.

$\underline{g} : \Omega_T \to \mathbf{R}_+$, $\bar{g} : \Omega_T \to \mathbf{R}_+ \cup \{\infty\}$, and $f^j : \Xi_T^j \to \mathbf{R}_+ \cup \{\infty\}$ for all $j = 1, \ldots, m$ with images[19]

$$\underline{g}(x, y, c) := \inf \{z \geq 0 | \; z \in P_T(x, y, c)\}, \tag{3.1}$$

$$\begin{aligned} f^j(x, z, c, y^{-j}) &:= \sup \{y_j \geq 0 | \; y_j \in P_T(x, y^{-j}, c, z)\}, \text{ if } P_T(x, y^{-j}, c, z) \text{ is bounded and} \\ &:= \infty, \qquad \text{otherwise,} \end{aligned} \tag{3.2}$$

$$\begin{aligned} \bar{g}(x, y, c) &:= \sup \{z \geq 0 | \; z \in P_T(x, y, c)\}, \text{ if } P_T(x, y, c) \text{ is bounded and} \\ &:= \infty, \qquad \text{otherwise,} \end{aligned} \tag{3.3}$$

and the restriction mapping $P_T : \Theta_T^j \mapsto \mathbf{R}_+^2$ with image $P_T(x, y^{-j}, c)$.

The functions $\underline{g}()$ and $\bar{g}()$ specify the minimum and maximum bounds on the level of emission a firm with technology T can generate given fixed levels of all (including emission generating) inputs, intended outputs, and its cleaning-up effort.[20] The function $f^j()$ specifies the maximum amount of the j^{th} intended output that it can produce given fixed levels of all other intended outputs, inputs, emission, and cleaning-up effort. Given the domains of these functions, the constraint sets of the optimization problems defining these functions are non-empty. The following provides a definition of an emission generating technology.

Definition: T satisfies *by-production (BP)* if

(i) T is closed,

(ii) for all $j = 1, \ldots, m$ and for all $\langle x, y^{-j}, c, z\rangle \in \Xi_T^j$, the set $P_T(x, y^{-j}, c, z)$ is bounded,

(iii) for all $\langle x, y, c\rangle \in \Omega_T$, we have $\underline{g}(x, y, c) > 0$ only if $\langle x^1, y^1\rangle \neq 0$ and there exists $\langle x, y, c\rangle \in \Omega_T$ such that $\underline{g}(x, y, c) > 0$,

(iv) for all $\langle x, y, c\rangle$ and $\langle \bar{x}, \bar{y}, \bar{c}\rangle \in \Omega_T$, we have

(a) $\underline{g}(x, y, c) \geq \underline{g}(\bar{x}, \bar{y}, \bar{c})$ if $x^1 \geq \bar{x}^1$, $y^1 \geq \bar{y}^1$, and $c \leq \bar{c}$. There exist $\langle x, y, c\rangle$ and $\langle \bar{x}, \bar{y}, \bar{c}\rangle \in \Omega_T$ such that $\langle x^1, y^1, c\rangle \neq \langle \bar{x}^1, \bar{y}^1, \bar{c}\rangle$, $x^1 \geq \bar{x}^1$, $y^1 \geq \bar{y}^1$, $c \leq \bar{c}$, and $\underline{g}(x, y, c) > \underline{g}(\bar{x}, \bar{y}, \bar{c})$,

(b) $\underline{g}(x, y, c) \neq \underline{g}(\bar{x}, \bar{y}, \bar{c})$ only if $\langle x^1, y^1, c\rangle \neq \langle \bar{x}^1, \bar{y}^1, \bar{c}\rangle$,

(c) $\bar{g}(x, y, c) \neq \bar{g}(\bar{x}, \bar{y}, \bar{c})$ only if $\langle x^1, y^1, c\rangle \neq \langle \bar{x}^1, \bar{y}^1, \bar{c}\rangle$,

(v) for all $j = m_1 + 1, \ldots, m$ and for all $\langle x, y^{-j}, c, z\rangle$ and $\langle \bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z}\rangle \in \Xi_T^j$, we have

(a) $f^j(x, y^{-j}, c, z) \geq f^j(\bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z})$ if $x \geq \bar{x}$, $y^{-j} \leq \bar{y}^{-j}$, and $c \leq \bar{c}$, and

(b) $f^j(x, y^{-j}, c, z) \neq f^j(\bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z})$ only if $\langle x, y^{-j}, c\rangle \neq \langle \bar{x}, \bar{y}^{-j}, \bar{c}\rangle$.

[19] A function $f^c()$, similar to $f^j()$, can also be defined for cleaning-up effort of the firm and the definition of by-production below can be extended to include properties with respect to cleaning-up effort as well. Here, in the interest of economizing on notation, we ignore this aspect. Nothing qualitatively substantive is lost by this omission in this analysis.

[20] Burning a given amount of coal always results in a certain minimum amount of smoke. Technical inefficiency may generate more smoke than the minimum possible, but it is reasonable to assume that there is also a bound on the maximum smoke that can be emitted with the fixed amount of coal.

(i) and (ii) in the definition of (BP) ensure that, for any j, $f^j()$ is well-defined. Hence, there exists a maximum amount of intended output j for every vector of inputs, cleaning-up effort, intended outputs other than j, and emission. (iii) implies that the emission is generated by the firm only because it employs inputs x^1 and outputs y^1. The other inputs and intended outputs are not emission generating. (i) and (iii) also imply that $\underline{g}()$ is well defined and that T is non-trivially an emission generating technology, *i.e.,* there is a level of its operation where it emits positive amount of the emission. (iv) captures the expected monotonicity properties of function $\underline{g}()$. Part (a) of (iv) implies that the minimum level of emission increases if the firm employs more and more of the emission producing inputs or produces more and more of the emission producing outputs or decreases its cleaning-up effort. Parts (b) and (c) of (iv) imply that the minimum and maximum level of emission depend only on x^1, y^1, and c and are unaffected by changes in inputs x^2 and intended outputs y^2. Part (a) of (v) captures the expected monotonicity properties of function $f^j()$, where j is a non-emission generating intended output: The maximum amount of intended output j is non-decreasing in all inputs, non-increasing in the remaining intended outputs and cleaning-up efforts. Part (b) of (v) says that if all inputs, cleaning-up effort, and other intended outputs are held fixed, then the maximum amount of output j is unaffected by the level of emissions.[21]

For all $j = 1, \ldots, m$, define the correspondences $\underline{P}^j : \Theta_T^j \mapsto \mathbf{R}_+^2$ and $\bar{P}^j : \Theta_T^j \mapsto \mathbf{R}_+^2$ with images

$$\begin{aligned}\underline{P}^j(x, y^{-j}, c) &= \{\langle z, y_j\rangle \in P_T(x, y^{-j}, c) |\ y_j = f^j(x, y^{-j}, c, z)\ \wedge\ z = \underline{g}(x, y^{-j}, y_j, c)\}\\ \bar{P}^j(x, y^{-j}, c) &= \{\langle z, y_j\rangle \in P_T(x, y^{-j}, c) |\ y_j = f^j(x, y^{-j}, c, z)\ \wedge\ z = \bar{g}(x, y^{-j}, y_j, c)\}.\end{aligned} \tag{3.4}$$

The correspondence $\underline{P}^j()$ identifies the menu of efficient combinations of the emission and the j^{th} intended output for given levels of all inputs, remaining intended outputs, and cleaning-up effort.[22] Theorem 1 demonstrates that for technologies that satisfy (BP) this correspondence is single-valued–there is only one minimum level of the emission and one maximum level of the non-emission causing intended output j, when all resources and all other outputs are held fixed. (Contrast this with the example in Section 2.) This theorem also demonstrates a positive correlation between the emission and any non-emission generating intended output when some inputs are the cause of the emission and a negative correlation between the emission and any non-emission generating intended output when some intended outputs are the cause of the emission.

[21] Thus, a firm's emission does not impose external effects on its own intended production activities. This is a common implicit assumption in most empirical works in the literature (see, *e.g.,* references in Footnote 10) that involve a firm-level analysis. We will consider a weaker assumption later, which allows such externalities.

[22] Similarly, we can interpret the correspondence $\bar{P}^j()$.

Theorem 1: *Suppose $m_1 < m$ and $T \subset \mathbf{R}_+^{n+m+2}$ satisfies (BP). Then, for all $j = m_1+1, \ldots, m$,*
(1.) the correspondences $\underline{P}^j()$ and $\bar{P}^j()$ are single-valued (functions) and
(2.) if $\langle z, y_j \rangle \in \underline{P}^j(x^1, x^2, y^1, y^{2-j}, c)$ and $\langle \bar{z}, \bar{y}_j \rangle \in \underline{P}^j(\bar{x}^1, x^2, \bar{y}^1, y^{2-j}, \bar{c})$ then
(a) $z \geq \bar{z}$ and $y_j \geq \bar{y}_j$ whenever $\bar{x}^1 \leq x^1$, $\bar{c} \geq c$, and $y^1 = \bar{y}^1$ and
(b) $z \geq \bar{z}$ and $y_j \leq \bar{y}_j$ whenever $\bar{x}^1 = x^1$, $\bar{c} = c$, and $y^1 \geq \bar{y}^1$.

Input and output free disposability are defined in a standard way:

Definition: T satisfies *input free disposability (IFD)* if $\langle x, y, c, z \rangle \in T$ and $\bar{x} \geq x$ implies $\langle \bar{x}, y, c, z \rangle \in T$.

Definition: T satisfies *output free disposability (OFD)* if $\langle x, y, c, z \rangle \in T$ and $\langle \bar{y}, \bar{c} \rangle \leq \langle y, c \rangle$ implies $\langle x, \bar{y}, \bar{c}, z \rangle \in T$.

Theorem 2 shows that if T satisfies (BP) then it violates (IFD) and (OFD). T violates free disposability of the emission causing inputs and cleaning-up effort of the firm. This is because $\underline{g}()$ provides the lower bound on the emission and this function is non-decreasing in the emission causing inputs and non-increasing in the cleaning-up effort of the firm.[23] If the upper bound on emissions $\bar{g}$ is also increasing in the emission causing intended outputs, then free-disposability of these outputs is also violated.

Theorem 2: *Suppose $T \subset \mathbf{R}_+^{n+m+2}$ satisfies (BP) and*
(1) $m_1 = 0$ or
(2) there exist $\langle x, y, c \rangle \in \Omega_T$ and $\langle x, \bar{y}^1, y^2, c \rangle \in \Omega_T$ such that $\bar{y}^1 \leq y^1$, $\bar{y}^1 \neq y^1$, and $\bar{g}(x, y, c) > \bar{g}(x, \bar{y}^1, y^2, c)$.
Then T does not satisfy (OFD) and (IFD).

While (BP) is inconsistent with free disposability of goods that affect emission generation, we could still assume standard free disposability in the non-emission causing goods. In particular, we define

Definition: T satisfies *restricted output free disposability (ROFD)* if $\langle x, y, c, z \rangle \in T$ and $\bar{y}^2 \leq y^2$ implies $\langle x, y^1, \bar{y}^2, c, z \rangle \in T$.[24]

We also define costly disposability of emissions as the case where all convex combinations of the upper and the lower bounds on emissions are also technologically feasible.

Definition: T satisfies *costly disposability of by-product (CDB)* if $\langle x, y, c \rangle \in \Omega_T$ implies $\underline{g}(x, y, c) < \bar{g}(x, y, c)$ and $\langle x, y, c, \lambda \underline{g}(x, y, c) + (1-\lambda)\bar{g}(x, y, c) \rangle \in T$ for all $\lambda \in [0, 1]$.

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23 Recall Figure 1(c).
24 Similarly we can define *restricted input free disposability (RIFD)*.

Define the graphs of the functions $\underline{g}()$ and $f^j()$ for all $j = 1, \ldots, m$:

$$\begin{aligned} \underline{G}_T &:= \big\{ \langle x, y, c, z \rangle \in T |\ z = \underline{g}(x, y, c) \big\} \\ F_T^j &:= \big\{ \langle x, y, c, z \rangle \in T |\ y_j = f^j(x, y^{-j}, c, z) \big\} \end{aligned}. \tag{3.5}$$

Theorem 3 shows that if T satisfies (BP) and (ROFD) or (CDB) then the graphs of the functions $\underline{g}()$ and $f^j()$ for any non-emission generating intended output j are not identical, *i.e.,* these functions are not inverses of another and provide distinct information regarding the production relations that define T. Contrast this with the example in Section 2, where $\underline{g}(x^1, x^2, y) = \frac{y}{\prod_{i=1}^{5} x_i^{\frac{1}{5}}}$ is the inverse of the production function f^y in (2.1). On the other hand Theorem 3 also shows that the functions f^j are inverses of one another for all the non-emission causing intended outputs j.[25] Thus, all such functions f^j provide the same information about T.[26] Theorem 5 in the next section will provide a functional representation of T employing the two distinct relations underlying $\underline{g}()$ and $f^j()$ for any $j = m_1 + 1, \ldots, m$.

Theorem 3: *Suppose $T \subset \mathbf{R}_+^{n+m+2}$ satisfies (BP) and $m_1 < m$.*

(1) If T also satisfies (CDB) then, for all $j = m_1 + 1, \ldots, m$, we have $\underline{G}_T \neq F_T^j$.

(2) If T also satisfies (ROFD) then, for all $j = m_1 + 1, \ldots, m$, we have $\underline{G}_T \neq F_T^j$.

(3) For all $j, j' = m_1 + 1, \ldots, m$ such that $j \neq j'$ either $F_T^j = F_T^{j'}$ or if there exist $\langle x, y, c, z \rangle \in T$ such that $y_j = f^j(x, y^{-j,j'}, y_{j'}, c, z)$ but $y_{j'} \neq f^j(x, y^{-j,j'}, y_j, c, z) =: \tilde{y}_{j'}$ then $y_j = f^j(x, y^{-j,j'}, \tilde{y}_{j'}, c, z)$.

3.2. A necessary and sufficient characterization of technologies satisfying by-production.

By-products such as emissions are outputs of firms, whose production is governed by certain laws of nature. From the point of view of these natural laws, the inputs required to produce the by-product z are x^1, y^1, and c. In fact, the cleaning-up efforts c of firms are unproductive (bad) inputs of by-product generation. On the other hand, from the point of view of intended production, y^1, y^2, and c are outputs and x^1 and x^2 are inputs. Further, if we allow for external effects that a firm's emission z can impose on the production of its own intended outputs, then z is also an input into intended production.[27] Moreover, while inputs x exhibit rivalness in the production of y and c,[28] they *jointly* produce intended

[25] Precisely, this result holds if we exclude the weakly efficient portions of T from our consideration.

[26] With respect to the non-emission generating outputs, T shows properties similar to standard multiple-output non-emission generating technologies.

[27] If it imposes a negative external effect, then it is an unproductive or a bad input used in the production of y and c.

[28] *i.e.,* a given level x of inputs is *shared* between the productions of various intended outputs and cleaning-up effort

outputs and the by-product.[29] Similarly, the use of intended outputs y^1 as inputs in the production of z does not reduce the marketable amounts of these intended outputs.

The properties of technology T defined in the empirically observed space $\mathbf{R}_+^{n+m+2}$ of goods are a result of all the above different roles that goods y, x, c, and z play in both intended production and by-product generation. In this section we will disentangle and delineate all these different roles by describing technological relations in a larger space of commodities.[30] By doing so, we will obtain another characterization of a technology T that satisfies (BP) in the empirically observed space $\mathbf{R}_+^{n+m+2}$.

$z_+ \in \mathbf{R}_+$ will denote the amount of by-product (incidental output) that is produced by a firm and $z_- \in \mathbf{R}_+$ will denote the role of the by-product produced by the firm in generating external effects on the production of the intended outputs. $y_+ \in \mathbf{R}_+^m$ will denote the vector of intended outputs produced by the firm, while $y_- \in \mathbf{R}_+^m$ will denote the vector of intended outputs that go in as inputs in the process of generation of the by-product z_+ in nature. $x \in \mathbf{R}^n$ will denote the role of the associated goods as inputs in both intended production and by-production. $c_+ \in \mathbf{R}_+$ is the cleaning-up effort produced in intended production and $c_- \in \mathbf{R}_+$ is the input of cleaning-up effort used to mitigate the by-product. We now define two types of technologies in an extended commodity space $\mathbf{R}_+^{n+2(m+2)}$.

The first type, called an intended production technology (IPT), will be denoted by $\mathcal{T}_1 \subset \mathbf{R}_+^{n+2(m+2)}$. It is concerned only with the roles as outputs of the goods denoted by y and c, the roles as inputs of the goods denoted by x, and the role of z as an input.[31] Thus, it contains vectors of the form $\langle x, y_+, y_-, c_+, c_-, z_+, z_-\rangle \in \mathbf{R}_+^{n+2(m+2)}$ with $z_+ = 0$, $c_- = 0$, and $y_- = 0$. We define the following sets induced by this technology: $\Omega_{\mathcal{T}_1} := \{\langle x, y_+, c_+\rangle \in \mathbf{R}_+^{n+m+1} | \ \langle x, y_+, 0, c_+, 0, 0, z_-\rangle \in \mathcal{T}_1 \text{ for some } z_- \in \mathbf{R}_+\}$, $\Xi^j_{\mathcal{T}_1} := \{\langle x, {y^{-j}}_+, c_+, z_-\rangle \in \mathbf{R}_+^{n+m+1} | \ \langle x, {y^{-j}}_+, y_{j+}, 0, c_+, 0, 0, z_-\rangle \in \mathcal{T}_1 \text{ for some } y_{j+} \in \mathbf{R}_+\}$, $\Theta^j_{\mathcal{T}_1} := \{\langle x, y_+^{-j}, c_+\rangle \in \mathbf{R}^{n+m} | \ \langle x, y_+^{-j}, y_{j+}, 0, c_+, 0, 0, z_-\rangle \in \mathcal{T}_1 \text{ for some } \langle y_j, z\rangle \in \mathbf{R}_+^2\}$ for $j = 1, \ldots, m$. In an obvious way, we can also define various restrictions of $\mathcal{T}_1$ such as $P_{\mathcal{T}_1}(x, y_+, c_+)$, $P_{\mathcal{T}_1}(x, {y^{-j}}_+, c_+, z_-)$, and $P_{\mathcal{T}_1}(x, {y^{-j}}_+, c_+)$ for $j = 1, \ldots, m$.[32]

The second type, called a by-product generating technology (BPT), will be denoted by $\mathcal{T}_2 \subset \mathbf{R}_+^{n+2(m+2)}$. It is concerned only with the roles of the goods denoted by y and c as inputs, the roles of the goods denoted by x as inputs, and the role of z as an output of production.[33] Thus, it contains vectors of the form $\langle x, y_+, y_-, c_+, c_-, z_+, z_-\rangle \in \mathbf{R}_+^{n+2(m+2)}$ with $z_- = 0$, $c_+ = 0$, and $y_+ = 0$. In an obvious manner, we define the sets $\Omega_{\mathcal{T}_2}$, $\Xi^j_{\mathcal{T}_2}$,

[29] *i.e.,* if generation of the by-product requires a certain level of x^1, then it does not reduce the amount of x^1 left to be shared between the production of y and c.

[30] Not dissimilar to commodity spaces found in Milleron [1972] in the context of public goods.

[31] Note, these are the roles that these goods assume in intended production activities of firms.

[32] For example, $P_{\mathcal{T}_1}(x, y_+, c_+) = \{z_- \in \mathbf{R}_+ | \ \langle x, y_+, 0, c_+, 0, 0, z_-\rangle \in \mathcal{T}_1\}$.

[33] Note, these are the roles that these goods assume in nature's by-product generation mechanism.

and $\Theta^j_{\mathcal{T}_2}$ for $j = 1, \ldots, m$ and also restrictions $P_{\mathcal{T}_2}(x, y_-, c_-)$, $P_{\mathcal{T}_2}(x, y^{-j}{}_-, c_-, z_+)$, and $P_{\mathcal{T}_2}(x, y^{-j}{}_-, c_-)$.[34]

As in the case of technologies defined in the empirically observed space $\mathbf{R}^{n+m+2}_+$, we define functions that specify the bounds on intended outputs and levels of the externality that are consistent with $\mathcal{T}_1$ holding all other goods fixed: $\underline{\mathcal{G}}_{\mathcal{T}_1} : \Omega_{\mathcal{T}_1} \to \mathbf{R}_+$, $\bar{\mathcal{G}}_{\mathcal{T}_1} : \Omega_{\mathcal{T}_1} \to \mathbf{R}_+ \cup \{\infty\}$, and $\mathcal{F}^j_{\mathcal{T}_1} : \Xi^j_{\mathcal{T}_1} \to \mathbf{R}_+ \cup \{\infty\}$ for all $j = 1, \ldots, m$ with images

$$\underline{\mathcal{G}}_{\mathcal{T}_1}(x, y_+, c_+) := \inf \{z_- \geq 0 |\ z_- \in P_{\mathcal{T}_1}(x, y_+, c_+)\}, \tag{3.6}$$

$$\begin{aligned} \mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}{}_+, c_+, z_-) &:= \sup \{y_{j+} \geq 0 |\ y_{j+} \in P_{\mathcal{T}_1}(x, y^{-j}{}_+, c_+, z_-)\}, \\ &\qquad \text{if } P_{\mathcal{T}_1}(x, y^{-j}{}_+, c_+, z_-) \text{ is bounded and} \\ &:= \infty, \qquad \text{otherwise.} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \bar{\mathcal{G}}_{\mathcal{T}_1}(x, y_+, c_+) &:= \sup \{z_- \geq 0 |\ z_- \in P_{\mathcal{T}}(x, y_+, c_+)\}, \\ &\qquad \text{if } P_{\mathcal{T}_1}(x, y_+, c_+) \text{ is bounded and} \\ &:= \infty, \qquad \text{otherwise.} \end{aligned} \tag{3.8}$$

Similarly too, we can define functions: $\underline{\mathcal{G}}_{\mathcal{T}_2} : \Omega_{\mathcal{T}_2} \to \mathbf{R}_+$, $\bar{\mathcal{G}}_{\mathcal{T}_2} : \Omega_{\mathcal{T}_2} \to \mathbf{R}_+ \cup \{\infty\}$, and $\mathcal{F}^j_{\mathcal{T}_2} : \Xi^j_{\mathcal{T}_2} \to \mathbf{R}_+ \cup \{\infty\}$ for all $j = 1, \ldots, m$ with images $\underline{\mathcal{G}}_{\mathcal{T}_2}(x, y_-, c_-)$, $\mathcal{F}^j_{\mathcal{T}_2}(x, y^{-j}{}_-, c_-, z_+)$, and $\bar{\mathcal{G}}_{\mathcal{T}_2}(x, y_-, c_-)$.

The properties of an IPT are summarized in the definition below. They include standard assumptions of boundedness of intended outputs and free disposability of inputs and outputs that are normally imposed on non-emission generating technologies (qualified only by external effects of z on intended production).

Definition: $\mathcal{T}_1 \subset \mathbf{R}^{n+2(m+2)}_+$ is an *intended production technology (IPT)* if

(i) $\mathcal{T}_1$ is closed and $\langle x, y_+, y_-, c_+, c_-, z_+, z_- \rangle \in \mathcal{T}_1$ implies that $z_+ = 0$, $c_- = 0$, and $y_- = 0$,

(ii) for all $j = 1, \ldots, m$, $P_{\mathcal{T}_1}(x, y^{-j}_+, c_+, z_-)$ is bounded for all $\langle x, y^{-j}_+, c_+, z_- \rangle \in \Xi^j_{\mathcal{T}_1}$, and

(iii) for all $j = 1, \ldots, m$ and for all $\langle x, y^{-j}_+, c_+, z_- \rangle$, $\langle \bar{x}, \bar{y}^{-j}_+, \bar{c}_+, z_- \rangle \in \Xi^j_{\mathcal{T}_1}$, we have $P_{\mathcal{T}_1}(x, y^{-j}_+, c_+, z_-) \subseteq P_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}_+, \bar{c}_+, z_-)$ if $\bar{x} \geq x$, $\bar{y}^{-j}_+ \leq y^{-j}_+$, and $\bar{c}_+ \leq c_+$.[35]

From the definition of an IPT, the following obvious remark regarding the monotonicity properties of function $\mathcal{F}^j$ with respect to inputs and outputs of intended production

[34] For example, $\Omega_{\mathcal{T}_2} := \{\langle x, y_-, c_- \rangle \in \mathbf{R}^{n+m+1}_+ |\ \langle x, 0, y_-, 0, c_-, 0z_+, 0 \rangle \in \mathcal{T}_2$ for some $z_+ \in \mathbf{R}_+\}$ and $P_{\mathcal{T}_1}(x, y^{-j}{}_-, c_-, z_+) = \{y_{j\,-} \in \mathbf{R}_+ |\ \langle x, 0, y^{-j}_-, y_{j\,-}, 0, c_-, z_+, 0 \rangle \in \mathcal{T}_2\}$.

[35] This is equivalent to $\mathcal{T}_1$ satisfying free disposability of all inputs, intended outputs, and abatement output.

(excluding the externality z_-) follow, namely, $\mathcal{F}^j$ is non-decreasing in x and non-increasing in y_+^{-j} and c_+.[36]

Remark 1: Let $\mathcal{T}_1 \subset \mathbf{R}_+^{n+2(m+2)}$ be an IPT. For all $j = 1, \ldots, m$ and for all $\langle x, y^{-j}{}_+, c_+, z_- \rangle$, $\langle \bar{x}, \bar{y}_+^{-j}, \bar{c}_+, z_- \rangle \in \Xi_{\mathcal{T}_1}$, (iii) in the definition of an IPT implies

(a) $\mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}{}_+, c_+, z_-) \geq \mathcal{F}^j_{\mathcal{T}_1}(\bar{x}, \bar{y}_+^{-j}, \bar{c}_+, z_-)$ if $\bar{x} \leq x$, $\bar{y}_+^{-j} \geq y^{-j}{}_+$, and $\bar{c}_+ \geq c_+$ and

(b) $\mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}{}_+, c_+, z_-) > \mathcal{F}^j_{\mathcal{T}_1}(\bar{x}, \bar{y}_+^{-j}, \bar{c}_+, z_-)$ only if $\langle \bar{x}, \bar{y}_+^{-j}, \bar{c}_+ \rangle \neq \langle x, y^{-j}{}_+, c_+ \rangle$ and it is not the case that $\bar{x} \geq x$, $\bar{y}_+^{-j} \leq y_+^{-j}$, and $\bar{c}_+ \leq c_+$.

The properties of a BPT are summarized in the definition below. They specify the laws of nature regarding by-product generation. They are similar to (CDB) and properties (iii) and (iv) that were assumed in the definition of (BP) in the empirically observed commodity space.[37]

Definition: $\mathcal{T}_2 \subset \mathbf{R}_+^{n+2(m+2)}$ is *nature's by-product producing technology (BPT)* if

(i) $\mathcal{T}_2$ is closed and $\langle x, y_+, y_-, c_+, c_-, z_+, z_- \rangle \in \mathcal{T}_2$ implies that $z_- = 0$, $c_+ = 0$, and $y_+ = 0$,

(ii) $\underline{\mathcal{G}}_{\mathcal{T}_2}(x, y_-, c_-) > 0$ implies $\langle x^1, y_-^1, c_- \rangle \neq 0$ and there exists $\langle x, y_-, c_- \rangle \in \Omega_{\mathcal{T}_2}$ such that $\underline{\mathcal{G}}_{\mathcal{T}_2}(x, y_-, c_-) > 0$,

(iii) $z_+ = \lambda \underline{\mathcal{G}}_{\mathcal{T}_2}(x, y_-, c_-) + (1-\lambda)\bar{\mathcal{G}}_{\mathcal{T}_2}(x, y_-, c_-) \in P_{\mathcal{T}_2}(x, y_-, c_-)$ for all $\lambda \in [0, 1]$ and for every $\langle x, y_-, c_- \rangle \in \Omega_{\mathcal{T}_2}$, and

(iv) for all $\langle x^1, x^2, y_-^1, y_-^2, c_- \rangle \in \Omega_{\mathcal{T}_2}$ and $\langle \bar{x}^1, \bar{x}^2, \bar{y}_-^1, \bar{y}_-^2, \bar{c}_- \rangle \in \Omega_{\mathcal{T}_2}$, we have

(a) $\underline{\mathcal{G}}_{\mathcal{T}_2}(x^1, x^2, y_-^1, y_-^2, c_-) \geq \underline{\mathcal{G}}_{\mathcal{T}_2}(\bar{x}^1, \bar{x}^2, \bar{y}_-^1, \bar{y}_-^2, \bar{c}_-)$ if $\bar{y}_-^1 \leq y_-^1$, $\bar{x}^1 \leq x^1$, and $\bar{c}_- \geq c_-$ and there exist $\langle x^1, x^2, y_-^1, y_-^2, c_- \rangle \in \Omega_{\mathcal{T}_2}$ and $\langle \bar{x}^1, \bar{x}^2, \bar{y}_-^1, \bar{y}_-^2, \bar{c}_- \rangle \in \Omega_{\mathcal{T}_2}$ such that $\langle \bar{x}^1, \bar{y}_-^1, \bar{c}_- \rangle \neq \langle x^1, y_-^1, c_- \rangle$, $\bar{y}_-^1 \leq y_-^1$, $\bar{x}^1 \leq x^1$, $\bar{c}_- \geq c_-$, and $\underline{\mathcal{G}}_{\mathcal{T}_2}(x^1, x^2, y_-^1, y_-^2, c_-) > \underline{\mathcal{G}}_{\mathcal{T}_2}(\bar{x}^1, \bar{x}^2, \bar{y}_-^1, \bar{y}_-^2, \bar{c}_-)$,

(b) $\underline{\mathcal{G}}_{\mathcal{T}_2}(x^1, x^2, y_-^1, y_-^2, c_-) \neq \underline{\mathcal{G}}_{\mathcal{T}_2}(\bar{x}^1, \bar{x}^2, \bar{y}_-^1, \bar{y}_-^2, \bar{c}_-)$ only if $\langle \bar{x}^1, \bar{y}_-^1, \bar{c}_- \rangle \neq \langle x^1, y_-^1, c_- \rangle$, and

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36 The monotonicity properties with respect to z_- depends on the nature of externalities that it generates in intended production.

37 Environmental resources such as air and water are inputs of nature's emission generating mechanism, so they should form a part of the BPT. Moreover, such resources are non-subsitutable inputs in emission generation. An example of a production relation that incorporates a natural resource, denoted by $r \in \mathbf{R}_+$, as a non-substitutable input of the BPT is

$$z_+ \geq \min\{\underline{\mathcal{G}}_{\mathcal{T}_2}(x, y_-, c_-), \phi r\}, \tag{3.9}$$

where $\phi > 0$ is the assimilative capacity of the natural resource to absorb one unit of the emission. Though in our model, we abstract from including r, the demand for this input is immediately determined from (3.9) once the level of emission is known.

(c) $\bar{\mathcal{G}}_{\mathcal{T}_2}(x^1, x^2, y^1_-, y^2_-, c_-) \neq \bar{\mathcal{G}}_{\mathcal{T}_2}(\bar{x}^1, \bar{x}^2, \bar{y}^1_-, \bar{y}^2_-, \bar{c}_-)$ only if $\langle \bar{x}^1, \bar{y}^1_-, \bar{c}_- \rangle \neq \langle x^1, y^1_-, c_- \rangle$.[38]

Given any IPT $\mathcal{T}_1 \subset \mathbf{R}_+^{n+2(m+2)}$ and BPT $\mathcal{T}_2 \subset \mathbf{R}_+^{n+2(m+2)}$ we can derive a technology $T(\mathcal{T}_1, \mathcal{T}_2)$ in the empirically observed space $\mathbf{R}_+^{n+m+2}$ from them:[39]

$$T(\mathcal{T}_1, \mathcal{T}_2) := \big\{\langle x, y, c, z\rangle \in \mathbf{R}_+^{n+m+2} |\ \langle x, y, 0, c, 0, 0, z\rangle \in \mathcal{T}_1\ \wedge\ \langle x, 0, y, 0, c, z, 0\rangle \in \mathcal{T}_2\big\}. \tag{3.10}$$

It is easy to derive the relevant projections for the technology $T(\mathcal{T}_1, \mathcal{T}_2)$. These inherit their structures from both the IPT and BPT underlying $T(\mathcal{T}_1, \mathcal{T}_2)$.

$$\Omega_{T(\mathcal{T}_1,\mathcal{T}_2)} = \{\langle x, y, c\rangle \in \Omega_{\mathcal{T}_1} \cap \Omega_{\mathcal{T}_2} |\ P_{\mathcal{T}_1}(x, y, c) \cap P_{\mathcal{T}_2}(x, y, c) \neq \emptyset\}, \tag{3.11}$$

$$\Xi^j_{T(\mathcal{T}_1,\mathcal{T}_2)} = \{\langle x, y^{-j}, c, z\rangle \in \Xi^j_{\mathcal{T}_1} \cap \Xi^j_{\mathcal{T}_2} |\ P_{\mathcal{T}_1}(x, y^{-j}, c, z) \cap P_{\mathcal{T}_2}(x, y^{-j}, c, z) \neq \emptyset\}, \tag{3.12}$$

and

$$\Theta^j_{T(\mathcal{T}_1,\mathcal{T}_2)} = \{\langle x, y^{-j}, c\rangle \in \Theta^j_{\mathcal{T}_1} \cap \Theta^j_{\mathcal{T}_2} |\ P_{\mathcal{T}_1}(x, y^{-j}, c) \cap P_{\mathcal{T}_2}(x, y^{-j}, c) \neq \emptyset\} \tag{3.13}$$

for all $j = 1, \ldots, m$. In that case, for all $\langle x, y, c\rangle \in \Omega_{T(\mathcal{T}_1,\mathcal{T}_2)}$, for all $j = 1, \ldots, m$, and for all $\langle x, y^{-j}, c, z\rangle \in \Xi^j_{T(\mathcal{T}_1,\mathcal{T}_2)}$, we have

$$\begin{aligned}
\underline{g}(x, y, c) &= \min\{z \in \mathbf{R}_+ |\ z \in P_{T(\mathcal{T}_1,\mathcal{T}_2)}(x, y, c)\} \\
&= \min\{z \in \mathbf{R}_+ |\ z \in P_{\mathcal{T}_1}(x, y, c) \cap P_{\mathcal{T}_2}(x, y, c)\}, \\
\bar{g}(x, y, c) &= \max\{z \in \mathbf{R}_+ |\ z \in P_{T(\mathcal{T}_1,\mathcal{T}_2)}(x, y, c)\} \\
&= \max\{z \in \mathbf{R}_+ |\ z \in P_{\mathcal{T}_1}(x, y, c) \cap P_{\mathcal{T}_2}(x, y, c)\}, \text{ and} \\
f^j(x, y^{-j}, c, z) &= \max\{y_j \in \mathbf{R}_+ |\ y_j \in P_{T(\mathcal{T}_1,\mathcal{T}_2)}(x, y^{-j}, c, z)\} \\
&= \max\{y_j \in \mathbf{R}_+ |\ y_j \in P_{\mathcal{T}_1}(x, y^{-j}, c, z) \cap P_{\mathcal{T}_2}(x, y^{-j}, c, z)\}.
\end{aligned} \tag{3.14}$$

(3.14) suggests that the functions $\underline{g}()$, $\bar{g}$, and $f^j()$ inherit their properties from both the IPT $\mathcal{T}_1$ and the BPT $\mathcal{T}_2$.[40]

Theorem 4 provides a sufficient characterization of a technology satisfying (BP) in the observed space $\mathbf{R}_+^{n+m+2}$.[41] If the production relation underlying an IPT ($\mathcal{T}_1$) is independent of external effects generated by the by-product z then, given that the production relation underlying a BPT ($\mathcal{T}_2$) is independent of goods x^2 and y^2, we find that the technology $T(\mathcal{T}_1, \mathcal{T}_2)$ in the observed commodity space $\mathbf{R}_+^{n+m+2}$ satisfies (BP).

[38] Recall, the minimum and maximum amount of by-product is not affected by y^2 and x^2 given our primitive assumptions.

[39] Having distinguished between an IPT and BPT in terms of the roles played by different goods in intended production and the production of by-products, we conveniently, and without confusion, ignore all the subscripts + and − that were employed earlier in this section.

[40] The next section provides some examples to illustrate this.

[41] It is a formalization of the results in Murty and Russell [2010].

Theorem 4: *Let $\mathcal{T}_1 \subset \mathbf{R}_+^{n+2(m+2)}$ be an IPT and $\mathcal{T}_2 \subset \mathbf{R}_+^{n+2(m+2)}$ be a BPT and define $T := T(\mathcal{T}_1, \mathcal{T}_2)$. Suppose the following hold:*
(a) for all $j = 1, \ldots, m$ and for all $\langle x, y, c\rangle \in \Omega_{\mathcal{T}_1}$, we have

$$P_{\mathcal{T}_1}(x, y, c) = \mathbf{R}_+ \tag{3.15}$$

and
(b) for all $\langle x^1, y^1, c, z\rangle \in \mathbf{R}^{n_1+m_1+2}$ such that $P_{\mathcal{T}_2}(x^1, y^1_-, c_-, z_+) \neq \emptyset$, we have

$$P_{\mathcal{T}_2}(x^1, y^1, c, z) = \mathbf{R}_+^{n_2+m_2}. \tag{3.16}$$

Then T satisfies (BP), (CDB), (RIFD), and (ROFD).

If $T \subset \mathbf{R}_+^{n+m+2}$ satisfies (BP) then it follows that, for any $j = m_1 + 1, \ldots, m$, the function $f^j()$ will be invariant to changes in z and the functions $\underline{g}()$ and $\bar{g}()$ will be invariant to changes in y^j. In that case, it follows from Theorem 1 that, for all $j = m_1 + 1, \ldots, m$, there exist functions $\rho^j : \Theta_T^j \to \mathbf{R}_+$, $\underline{\sigma}^j : \Theta_T^j \to \mathbf{R}_+$, and $\bar{\sigma}^j : \Theta_T^j \to \mathbf{R}_+$ such that

$$\begin{aligned} &\underline{P}^j(x, y^{-j}, c) = \langle \rho^j(x, y^{-j}, c),\ \underline{\sigma}^j(x, y^{-j}, c)\rangle \text{ and} \\ &\bar{P}^j(x, y^{-j}, c) = \langle \rho^j(x, y^{-j}, c),\ \bar{\sigma}^j(x, y^{-j}, c)\rangle. \end{aligned} \tag{3.17}$$

In Theorem 5, conclusions of Theorem 1 are employed, firstly, to obtain a functional representation of a technology T that satisfies (BP) in the empirically observed space $\mathbf{R}_+^{n+m+2}$ and, secondly, to construct an IPT ($\mathcal{T}_1$), which is independent of z, and a BPT ($\mathcal{T}_2$), which is independent of y^2 and x^2, such that $T = T(\mathcal{T}_1, \mathcal{T}_2)$. This will demonstrate that the sufficient characterization in Theorem 4 of a technology satisfying (BP) in the observed space of commodities is also necessary.

Theorem 5: *If $T \subset \mathbf{R}_+^{n+m+2}$ satisfies (BP), (ROFD), and (CDB) then,*
(1) given any $j = m_1 + 1, \ldots, m$, T can be functionally represented by

$$\begin{aligned} T := \big\{\langle x, y, c, z\rangle \in \mathbf{R}_+^{n+m+2} |\ &\langle x, y^{-j}, c\rangle \in \Theta_T^j \ \wedge\ y_j \leq \rho^j(x, y^{-j}, c) \\ &\wedge\ \bar{\sigma}^j(x, y^{-j}, c) \geq z \geq \underline{\sigma}^j(x, y^{-j}, c)\big\} \text{ and} \end{aligned} \tag{3.18}$$

(2) there exist an IPT $\mathcal{T}_1 \subset \mathbf{R}_+^{n+2(m+2)}$ satisfying Assumption (a) of Theorem 4 and a BPT $\mathcal{T}_2 \subset \mathbf{R}_+^{n+2(m+2)}$ satisfying Assumption (b) of Theorem 4 such that $T = T(\mathcal{T}_1, \mathcal{T}_2)$.

We now provide a weaker definition of technologies that generate emissions than (BP). All that is required by this definition is that there exist some intended outputs or inputs of a firm that can generate an emission, a positive amount of emission is actually observed at some combination of intended outputs and inputs, and that there are maximal and minimal bounds on emission generation and the production of intended outputs. No restrictions of monotonicity are placed on the various bounds on $T \in \mathbf{R}_+^{n+m+2}$.

Definition: $T \in \mathbf{R}_+^{n+m+2}$ satisfies *weak by-production (WBP)* if
(i) T is closed,
(ii) for all $j = 1, \ldots, m$ and for all $\langle x, y^{-j}, c, z\rangle \in \Xi_T^j$, the set $P(x, y^{-j}, c, z)$ is bounded, and
(iii) there exists $\langle x, y, c\rangle \in \Omega_T$ such that $\underline{g}(x, y, c) > 0$.

Theorem 6 shows that, in general, combining any IPT in the extended commodity space with any BPT ensures only that the resulting technology in the empirically observed space satisfies (WBP). Bounds of the observed technology may violate the monotonicity properties in the definition of (BP) if the underlying IPT is not independent of external effects of the emission.[42] Employing the bounds of the underlying IPT and the BPT, it also provides a functional representation of the derived observed technology in $\mathbf{R}_+^{n+m+2}$.

Theorem 6: *Let $\mathcal{T}_1 \subset \mathbf{R}_+^{n+2(m+2)}$ be an IPT and $\mathcal{T}_2 \subset \mathbf{R}_+^{n+2(m+2)}$ be a BPT. Suppose the following hold:*
(a) there exists $\langle x, y, c\rangle \in \Omega_{T(\mathcal{T}_1 \cap \mathcal{T}_2)}$ such that $\underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c) > 0$ and
(b) for all $\langle x^1, y^1, c, z\rangle \in \mathbf{R}^{n_1+m_1+2}$ such that $P_{\mathcal{T}_2}(x^1, y^1, c, z) \neq \emptyset$, we have

$$P_{\mathcal{T}_2}(x^1, y^1, c, z) = \mathbf{R}_+^{n_2+m_2}. \tag{3.19}$$

Then $T(\mathcal{T}_1, \mathcal{T}_2)$ satisfies (WBP), (RIFD), and (ROFD) and, given any $j = m_1+1, \ldots, m$, $T(\mathcal{T}_1, \mathcal{T}_2)$ can functionally be represented by[43]

$$\begin{aligned} T(\mathcal{T}_1, \mathcal{T}_2) = \big\{\langle x, y, c, z\rangle \in \mathbf{R}_+^{n+m+2} \mid\ & \langle x, y^{-j}, c, z\rangle \in \Xi_{\mathcal{T}_1}^j \ \wedge\ \langle x, y, c\rangle \in \Omega_{\mathcal{T}_2} \\ & \wedge\ y^j \leq \mathcal{F}_{\mathcal{T}_1}^j(x, y^{-j}, c, z) \\ & \wedge\ \min\{\bar{\mathcal{G}}_{\mathcal{T}_1}(x, y, c), \bar{\mathcal{G}}_{\mathcal{T}_2}(x, y, c)\} \geq z \geq \max\{\underline{\mathcal{G}}_{\mathcal{T}_1}(x, y, c), \underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c)\}\big\}. \end{aligned} \tag{3.20}$$

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[42] Examples 3 and 4 in the next section demonstrate this.

[43] $\langle x, y^{-j}, c, z\rangle \in \Xi_{\mathcal{T}_1}^j$ and $y^j \leq \mathcal{F}_{\mathcal{T}_1}^j(x, y^{-j}, c, z)$ imply that $\langle x, y, 0, c, 0, 0, z\rangle \in \mathcal{T}_1$. Hence, $\langle x, y, c\rangle \in \Omega_{\mathcal{T}_1}$ and $\underline{\mathcal{G}}_{\mathcal{T}_1}(x, y, c)$ and $\bar{\mathcal{G}}_{\mathcal{T}_1}(x, y, c)$ are well-defined. $\langle x, y, c\rangle \in \Omega_{\mathcal{T}_2}$ and $\min\{\bar{\mathcal{G}}_{\mathcal{T}_1}(x, y, c), \bar{\mathcal{G}}_{\mathcal{T}_2}(x, y, c)\} \geq z \geq \max\{\underline{\mathcal{G}}_{\mathcal{T}_1}(x, y, c), \underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c)\}$ imply that $\langle x, 0, y, 0, c, z, 0\rangle \in \mathcal{T}_2$.

Corollary to Theorem 6: *If all conditions of Theorem 4 hold then, given any* $j = m_1 + 1, \ldots, m$, T *can be functionally represented by*

$$T = \{\langle x, y, c, z\rangle \in \mathbf{R}_+^{n+m+2} | \ \langle x, y^{-j}, c, z\rangle \in \Xi^j_{\mathcal{T}_1} \ \wedge \ \langle x, y, c\rangle \in \Omega_{\mathcal{T}_2} \ \wedge \\ y_j \leq \mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}, c, z) \ \wedge \ \underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c) \leq z \leq \bar{\mathcal{G}}_{\mathcal{T}_2}(x, y, c)\}. \tag{3.21}$$

4. Examples of by-production.

In this section, we study four examples to illustrate the by-production approach to modeling emission generating technologies. Each example specifies a particular pair of IPT and BPT, from which a technology in the space of the observed variables is derived and its properties are studied. The first two examples lead to observed technologies that satisfies (BP), while the last two examples lead to observed technologies that satisfy (WBP). In the first example, some inputs are the cause of the emission, while in the second case, an intended output is the cause of the emission. The third and fourth examples consider pairs of IPT and BPT, where the emissions also impose (negative and positive) external effects on the IPT. All our results of the previous section are illustrated through these examples. The projections and restrictions of the technologies in the observed spaces that are derived from various pairs of IPT and BPT as well as the various bounds on them exhibit non-standard structures. This is because they inherit features from both the IPTs and BPTs associated with them. In particular, standard disposability properties are violated by those goods that affect both intended production and nature's emission generation. When an emission of a firm also impose externalities on its intended production then, contrary to conclusions of Theorem 1, for given levels of all inputs, there may exist a considerable menu of efficient combinations of the emission and intended outputs.

4.1. Example 1 (paper production): Inputs as sources of emission and no emission-externality on intended production.

The example of paper production in Section 2 is considered again. Cleaning-up effort of the firm is also explicitly modeled. Here, $n = 5$, $n_1 = 2$, $n_2 = 3$, $m = 1$, and $m_1 = 0$. Suppose the intended production technology and the nature's by-product generating technology have the following forms:

$$\mathcal{T}_1 = \{\langle x^1, x^2, y, 0, c, 0, 0, z\rangle \in \mathbf{R}_+^{11} | \ y \leq \prod_{i=1}^{5} x_i^{v_i} - c\} \text{ and} \\ \mathcal{T}_2 = \{\langle x^1, x^2, 0, y, 0, c, z, 0\rangle \in \mathbf{R}_+^{11} | \ \gamma + \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq z \geq \sum_{i=1}^{2} \alpha_i x_i - \beta c\}, \tag{4.1}$$

where $v_i > 0$ for all $i = 1, \ldots, 5$, $\sum_{i=1}^{5} v_i \leq 1$, $\alpha_i > 0$ for all $i = 1, 2$, $\beta > 0$, and $\gamma > 0$. It is easy to check that $\mathcal{T}_1$ and $\mathcal{T}_2$ in (4.1) satisfy the definitions of an IPT and a BPT, respectively. Let us consider various projections and restrictions of the IPT and BPT:

$$\begin{aligned} \Omega_{\mathcal{T}_1} &= \{\langle x, y, c\rangle \in \mathbf{R}^7_+ |\ y \leq \prod_{i=1}^{5} x_i^{v_i} - c\} \text{ and} \\ P_{\mathcal{T}_1}(x, y, c) &= \mathbf{R}_+ \ \forall\ \langle x, y, c\rangle \in \Omega_{\mathcal{T}_1}, \\ \Xi^y_{\mathcal{T}_1} &= \{\langle x, z, c\rangle \in \mathbf{R}^7_+ |\ \prod_{i=1}^{5} x_i^{v_i} - c \geq 0\} \text{ and} \\ P_{\mathcal{T}_1}(x, z, c) &= \{y \in \mathbf{R}_+ |\ y \leq \prod_{i=1}^{5} x_i^{v_i} - c\} \ \forall\ \langle x, z, c\rangle \in \Xi^y_{\mathcal{T}_1}. \end{aligned} \tag{4.2}$$

The structure of $P_{\mathcal{T}_1}(x, z, c)$ in (4.2) demonstrate that, from the point of view of intended production, paper production is unaffected by the presence of chemical residuals. In addition, the structure of $P_{\mathcal{T}_1}(x, y, c)$ demonstrates that Assumption (a) of Theorem 4 holds for this example. Thus, in this example, $\mathcal{T}_1$ is independent of generation of emissions. Some of the projections and restrictions induced by $\mathcal{T}_2$ are

$$\begin{aligned} \Omega_{\mathcal{T}_2} &= \{\langle x, y, c\rangle \in \mathbf{R}^5_+ |\ \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq -\gamma\} \text{ and} \\ P_{\mathcal{T}_2}(x, y, c) &= \{z \in \mathbf{R}_+ |\ \gamma + \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq z \geq \sum_{i=1}^{2} \alpha_i x_i - \beta c\} \ \forall\ \langle x, y, c\rangle \in \Omega_{\mathcal{T}_2}, \\ \Xi^y_{\mathcal{T}_2} &= \{\langle x, z, c\rangle \in \mathbf{R}^5_+ |\ \gamma + \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq z \geq \sum_{i=1}^{2} \alpha_i x_i - \beta c\} \text{ and} \\ P_{\mathcal{T}_2}(x, z, c) &= \mathbf{R}_+ \ \forall\ \langle x, z, c\rangle \in \Xi^y_{\mathcal{T}_2}. \end{aligned} \tag{4.3}$$

The structure of $P_{\mathcal{T}_2}(x, z, c)$ in (4.3) demonstrates that nature's by-product generating technology is, ceteris paribus, unaffected by the output of paper. The structure of $P_{\mathcal{T}_2}(x, z, c)$ in (4.3) shows that Assumption (b) of Theorem 4 also holds for this example. Thus, in this example, the BPT is independent of the level of the intended output.

Denote the observed technology obtained from the pair of IPT and BPT in (4.1) by $T := T(\mathcal{T}_1, \mathcal{T}_2) \subset \mathbf{R}^8_+$. For this observed technology, note that for all $\langle x, y, c\rangle \in \Omega_{\mathcal{T}_1} \cap \Omega_{\mathcal{T}_2}$, (4.2) and (4.3) imply that $P_{\mathcal{T}_1}(x, y, c) \cap P_{\mathcal{T}_2}(x, y, c) = P_{\mathcal{T}_2}(x, y, c)$. Hence, (3.13) implies that

$$\Omega_T = \Omega_{\mathcal{T}_2} \text{ and } P_T(x, y, c) = P_{\mathcal{T}_2}(x, y, c), \tag{4.4}$$

i.e., the observed technology T inherits its projection Ω_T and restriction $P_T(x, y, c)$ from the BPT of (4.1). On the other hand, it can be shown that (4.2) and (4.3) also imply that it inherits its projection Ξ^y_T and restriction $P_T(x, c, z)$ from the IPT of (4.1):

$$\Xi^y_T = \Xi^y_{\mathcal{T}_1} \text{ and } P_T(x, c, z) = P_{\mathcal{T}_1}(x, c, z). \tag{4.5}$$

The function $\underline{g} : \Omega_T \to \mathbf{R}_+$ is obtained as

$$\begin{aligned}\underline{g}(x, y, c) &:= \min\{z \in \mathbf{R}_+ | \; z \in P_T(x, y, c)\} = \min\{z \in \mathbf{R}_+ | \; z \in P_{T_2}(x, y, c)\} \\ &= \sum_{i=1}^{2} \alpha_i x_i - \beta c \quad \text{if } \sum_i \alpha_i x_i - \beta c \geq 0 \text{ and} \\ &= 0 \qquad\qquad \text{if } -\gamma \leq \sum_{i=1}^{2} \alpha_i x_i - \beta c \leq 0\end{aligned} \tag{4.6}$$

Similarly, we obtain $\bar{g} : \Omega_T \to \mathbf{R}_+$ with image

$$\bar{g}(x, y, c) = \gamma + \sum_{i=1}^{2} \alpha_i x_i - \beta c. \tag{4.7}$$

The function $f^y : \Xi_T^y \to \mathbf{R}_+$ is obtained as

$$\begin{aligned}f^y(x, z, c) &:= \max\{y \in \mathbf{R}_+ | \; y \in P_T(x, z, c)\} = \max\{y \in \mathbf{R}_+ | \; y \in P_{T_1}(x, z, c)\} \\ &= \prod_{i=1}^{5} x_i^{v_i} - c.\end{aligned} \tag{4.8}$$

We now check the properties of T. Clearly the T is closed. For all $\langle x, c, z\rangle \in \Xi_T^y$, (4.2) implies that the set $P_T(x, c, z) = P_{T_1}(x, c, z)$ is bounded. For appropriately chosen values of the parameters, it is not difficult to find a production vector $\langle x, y, c, z\rangle \in T$ such that $\underline{g}(x, y, c) > 0$.[44] Thus, T satisfies conditions (i) to (iii) in the definition of (BP).

The function $\underline{g}()$ is non-increasing in the extent of cleaning-up efforts of the firm and non-decreasing in the levels of chemicals and wood used in intended production. Ceteris paribus, it is unaffected by the output of paper y. This is because, according to the nature defined BPT in this example, the residual generated is affected only by the use of chemical and wood inputs and the cleaning-up efforts of the firm. Hence, so long as all inputs and cleaning-up efforts are held fixed, an increase in paper production (which is still possible, if there is technical inefficiency in paper production at this fixed level of inputs) has no effect on residual generation.

The function $f^y()$ is increasing in the amounts of all inputs used by the firm and decreasing in the amount of cleaning up efforts of the firm. Ceteris paribus, it is unaffected by the level of residual generated–if all inputs and cleaning-up efforts are held fixed then any change in residual generation (which is still possible if there are technical inefficiencies in residual generation) has no effect on paper production. The properties of the functions $\underline{g}()$, $\bar{g}()$, and $f^y()$ indicate that T also satisfies conditions (iv) and (v) in the definition of (BP). Therefore, T satisfies (BP).

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44 The production vector with $y = \prod_{i=1}^{5} x_i^{v_i} - c,\ x_i > \frac{\beta^2}{\alpha_i^2}$ for $i = 1, 2,\ x_i = 1$ for $i = 3, 4, 5,\ c = \sum_{i=1}^{2} \frac{\beta}{\alpha_i}$, and $z = \sum_i \alpha_i x_i - \beta c$ does the job when $\frac{\beta^{4v_1 v_2}}{\alpha_1^2 \alpha_1^2} - \frac{\beta}{\alpha_1 + \alpha_2} > 0$.

(4.4) and (4.5) imply that T satisfies (CDB) and (ROFD). It also satisfies free disposability in any non-residual generating input.[45] But it violates free-disposability of c and any emission-generating input. This is shown in Figures 2(a) and 2(b) where we draw

$$P_T(x_2, x_4, x_5, y, c, z) := P_{\mathcal{T}_1}(x_2, x_4, x_5, y, c, z) \cap P_{\mathcal{T}_2}(x_2, x_4, x_5, y, c, z)$$
$$= \{\langle x_1', x_3' \rangle \in \mathbf{R}^2_+ | \ \frac{y+c}{\prod_{i \neq 1,3} x_i^{v_i}} \leq {x_1'}^{v_1} {x_3'}^{v_3} \wedge \ \frac{z + \beta c - \alpha_2 x_2 - \gamma}{\alpha_1} \leq x_1' \leq \frac{z + \beta c - \alpha_2 x_2}{\alpha_1}\} \text{ and}$$

$$P_T(x_2, x_3, x_4, x_5, y, z) := P_{\mathcal{T}_1}(x_2, x_3, x_4, x_5, y, z) \cap P_{\mathcal{T}_2}(x_2, x_3, x_4, x_5, y, z)$$
$$= \{\langle x_1', c' \rangle \in \mathbf{R}^2_+ | \ c' \leq \prod_{i \neq 1} x_i^{v_i} {x_1'}^{v_1} - y \ \wedge \ \frac{z + \beta c' - \alpha_2 x_2 - \gamma}{\alpha_1} \leq x_1' \leq \frac{z + \beta c' - \alpha_2 x_2}{\alpha_1}\}. \tag{4.9}$$

It is clear from these figures that T satisfies input freely disposability in non-emission generating x_3, but violates input and output free disposability in x_1 and c, respectively. This is because x_3 affects only the IPT, while x_1 and c affect both IPT and BPT. While IPT has standard free disposability properties in all inputs, y, and c and is independent of z, BPT violates standard free disposability of x_1, x_2, c, and z. Thus, all conclusions of Theorem 4 hold for this example.

T satisfies all assumptions of parts (1) and (2) of Theorem 3. (4.6) and (4.8) shows that $\underline{g}()$ and $f^y()$ (which T inherits from the BPT and IPT, respecitively) are not inverses of one another–their graphs are not identical. Thus conclusions of parts 1 and 2 of Theorem 3 hold for this example. Further, in this example, it can be shown that T inherits Θ^y_T from both the IPT and BPT

$$\Theta^y_T = \Theta^y_{\mathcal{T}_1} \cap \Theta^y_{\mathcal{T}_2} = \{\langle x, c \rangle \in \mathbf{R}^2_+ | \ \prod_{i=1}^{5} x_i^{v_i} - c \geq 0 \ \wedge \ \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq -\gamma\}. \tag{4.10}$$

Figure 2(c) shows the restriction $P_T(x, c) \subset \mathbf{R}^2_+$ for $\langle x, c \rangle \in \Theta^y_T$.[46] In the figure, $z^* = \bar{g}(x, c, y^*)$, $z_* = \underline{g}(x, c, y^*)$, and $y^* = f^y(x, c, z_*)$. In particular, we note that the correspondence $\underline{P}^y : \Theta^y_T \mapsto \mathbf{R}^2_+$ has image

$$\begin{aligned} \underline{P}^y(x, c) = \langle y, z \rangle &= \langle \prod_{i=1}^{5} x_i^{v_i} - c, \ \sum_{i=1}^{2} \alpha_i x_i - \beta c \rangle \quad \text{if } \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq 0 \\ &= \langle \prod_{i=1}^{5} x_i^{v_i} - c, \ 0 \rangle \qquad\qquad \text{if } -\gamma \leq \sum_{i=1}^{2} \alpha_i x_i - \beta c \leq 0. \end{aligned} \tag{4.11}$$

[45] This is seen in Figure 2(c) that will be discussed later.

[46] Note that this figure illustrates that T satisfies output free disposability with respect to y but there is a lower bound for z. Hence, it violates output free disposability with respect to the by-product z. It satisfies (CDB).

Thus, it is single valued. It is clear (see also Figure 2(d)) that this correspondence is non-increasing in c, non-decreasing in x_1 and x_2, and non-decreasing in x_3, x_4, and x_5. In the figure $\langle y^*, z_* \rangle = \underline{P}^y(x, c)$ and $\langle y^\sim, z_\sim \rangle = \underline{P}^y(\tilde{x}^1, x^2, \tilde{c})$ with $\tilde{x}^1 > x^1$ and $\tilde{c} < c$. This demonstrates that conclusions of Theorem 1 hold for this example.

Noting that functions $f^y()$, $\underline{g}()$, and $\bar{g}()$ are independent of z, y, and y, respectively, we can obtain a functional representation of T as in part 1 of Theorem 5:

$$T = \Big\{ \langle x, y, c, z \rangle \in \mathbf{R}^8_+ |\ y \leq \prod_{i=1}^{5} x_i^{v_i} - c \wedge \\ \gamma + \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq z \geq \sum_{i=1}^{2} \alpha_i x_i - \beta c \text{ if } \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq 0 \wedge \\ \gamma + \sum_{i=1}^{2} \alpha_i x_i - \beta c \geq z \geq 0 \text{ if } -\gamma \leq \sum_{i=1}^{2} \alpha_i x_i - \beta c \leq 0 \Big\}. \tag{4.12}$$

4.2. Example 2 (a dairy): An intended output as a source of emission and no emission externality on intended production.

Consider the case of a dairy that uses milk (x) as an input to produce cheese (y_1), butter (y_2), and cream (y_3) as its intended outputs. The input milk itself is not a cause of any emission, but a strong odor (an emission) emanates from its output of cheese. Thus, $n = 1, n_2 = 1$, $m = 3$, and $m_1 = 1$. Suppose the intended production technology and the by-product generating technology have the following forms:

$$\mathcal{T}_1 = \{ \langle x, y_1, y_2, y_3, 0, 0, 0, 0, z \rangle \in \mathbf{R}^9_+ |\ y_1 \leq x^{\frac{1}{2}} - y_2 - y_3 \} \text{ and} \\ \mathcal{T}_2 = \{ \langle x, 0, 0, 0, y_1, y_2, y_3, z, 0 \rangle \in \mathbf{R}^9_+ |\ \gamma + \alpha y_1 \geq z \geq \alpha y_1 \}. \tag{4.13}$$

As in Example 1, it can be checked that in this example all the assumptions of Theorem 4 hold and so $T := T(\mathcal{T}_1, \mathcal{T}_2)$ satisfies (BP). In particular, we find that, whenever the various restrictions of T are non-empty, they take the following structures:

$$P_T(x, y_1, y_2, y_3) = \{ z \in \mathbf{R}_+ |\ \gamma + \alpha y_1 \geq z \geq \alpha y_1 \}, \\ P_T(x, y_2, y_3, z) = \{ y_1 \in \mathbf{R}_+ |\ y_1 \leq x^{\frac{1}{2}} - y_2 - y_3 \ \wedge\ \frac{z}{\alpha} \geq y_1 \geq \frac{z - \gamma}{\alpha} \}, \text{ and} \\ P_T(x, y_1, y_{j'}, z) = \{ y_j \in \mathbf{R}_+ |\ y_j \leq x^{\frac{1}{2}} - y_1 - y_{j'} \} \ \forall\ j, j' = 2, 3. \tag{4.14}$$

The structure of $P_T(x, y_2, y_3, z)$ demonstrates that T violates free disposability of cheese, the output that causes the odor. However, for $j, j' = 2, 3$, the structure of $P_T(x, y_1, y_{j'}, z)$ shows that T satisfies free disposability of outputs butter and cream and the input milk that do not cause the emission. Further, the structure of $P_T(x, y_1, y_2, y_3)$ demonstrates that T satisfies (CDB). It can be checked that, in this example,

$$\begin{aligned} &\underline{g}(x, y_1, y_2, y_3) := \alpha y_1 = \underline{\mathcal{G}}_{\mathcal{T}_2}(x, y_1, y_2, y_3), \\ &\bar{g}(x, y_1, y_2, y_3) := \gamma + \alpha y_1 = \bar{\mathcal{G}}_{\mathcal{T}_2}(x, y_1, y_2, y_3), \\ &f^1(x, y_2, y_3, z) = \min\ \{x^{\frac{1}{2}} - y_2 - y_3,\ \frac{z}{\alpha}\},\ \text{and} \\ &f^j(x, y_1, y_{j'}, z) = x^{\frac{1}{2}} - y_1 - y_{j'} = \mathcal{F}^j_{\mathcal{T}_1}(x, y_1, y_{j'}, z)\ \forall\ j, j' = 2, 3. \end{aligned} \tag{4.15}$$

The function $\underline{g}$ (the minimum level of strong odor) inherits all its properties from $\underline{\mathcal{G}}_{\mathcal{T}_2}$ of the BPT. It is increasing in cheese, the intended output causing the odor. However, it is unaffected by the levels of butter and cream produced or the amount of milk used. For any non-odorous output j, the function f^j inherits all its properties from $\mathcal{F}^j_{\mathcal{T}_1}$. It is increasing in the amount of the input milk used, it is decreasing in the levels of the other two intended outputs, and it is unaffected by the odor itself. Note also that $\underline{g}()$ and $f^j()$ are not inverses of one another. However, the two functions $f^2()$ and $f^3()$ are inverses of one another. Thus, all conclusions of Theorem 3 hold for this example non-trivially. The function f^1 corresponding to the odorous output (cheese) inherits its properties from both $\mathcal{F}^1_{\mathcal{T}_1}$ and $\mathcal{F}^1_{\mathcal{T}_2}$. It is, ceteris paribus, non-decreasing in x, non-increasing in y_2 and y_3, and is not independent of (it is non-decreasing in) z. Further, its graph is distinct from $\underline{g}()$, $\bar{g}()$, $f^2()$, and $f^3()$. Thus, in this example too, more than one production relation is needed to functionally represent the observed technology T (see Theorem 5):

$$T = \{\langle x, y_1, y_2, y_3, z\rangle \in \mathbf{R}^5_+ |\ y_1 \leq x^{\frac{1}{2}} - y_2 - y_3\ \wedge\ \alpha y_1 + \gamma \geq z \geq \alpha y_1\}. \tag{4.16}$$

4.3. Example 3 (a thermal power plant): An input as a source of emission and a negative emission externality on intended production.

Consider the case of a thermal power plant that employs inputs of coal (x_1) and labor (x_2) to produce electricity (y) and a by-product smoke (z). Thus, in this case $n = 2,\ n_1 = 1, m = 1$, and $m_1 = 0$). Suppose emission z decreases the average productivity of labor in intended production. Suppose the underlying IPT and BPT are

$$\begin{aligned} &\mathcal{T}_1 = \{\langle x_1, x_2, y, 0, 0, z\rangle \in \mathbf{R}^6_+ |\ y \leq x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - z\}\ \text{and} \\ &\mathcal{T}_2 = \{\langle x_1, x_2, 0, y, z, 0\rangle \in \mathbf{R}^6_+ |\ \gamma + \alpha x_1 \geq z \geq \alpha x_1\}. \end{aligned} \tag{4.17}$$

$\mathcal{T}_1$ and $\mathcal{T}_2$ are valid IPT and BPT, respectively. We again focus on the structures of those restrictions of the IPT and BPT that are non-empty.[47] The fact that the smoke affects the

[47] These correspond to points in the projections $\Omega_{\mathcal{T}_1}$, $\Omega_{\mathcal{T}_2}$, $\Xi^y_{\mathcal{T}_1}$, and $\Xi^y_{\mathcal{T}_2}$, which can be derived in a straightforward way.

intended production of electricity implies that Assumption (a) of Theorem 4 is violated in this example:

$$\begin{aligned} &P_{\mathcal{T}_1}(x_1, x_2, y) = \{z \in \mathbf{R}_+ |\ z \leq x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - y\} \text{ and} \\ &P_{\mathcal{T}_1}(x_1, x_2, z) = \{y \in \mathbf{R}_+ |\ y \leq x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - z\}. \end{aligned} \tag{4.18}$$

(4.18) shows that the production of electricity is not independent of the level of smoke produced. The restriction $P_{\mathcal{T}_1}(x_1, x_2, z)$ shrinks as z increases. However, the structure of $P_{\mathcal{T}_2}(x_1, x_2, y)$ shows that the production of smoke is not (directly) affected by the level of electricity produced (in nature, smoke is caused by the burning of coal):

$$\begin{aligned} &P_{\mathcal{T}_2}(x_1, x_2, y) = \{z \in \mathbf{R}_+ |\ \gamma + \alpha x_1 \geq z \geq \alpha x_1\} \text{ and} \\ &P_{\mathcal{T}_2}(x_1, x_2, z) = \mathbf{R}_+. \end{aligned} \tag{4.19}$$

(4.19) demonstrates that Assumption (b) of Theorem 4 holds for this example. Define $T := T(\mathcal{T}_1, \mathcal{T}_2)$. We now study the properties of the observed technology T.

Note that, while in this example, $\Theta^y_{\mathcal{T}_1} = \mathbf{R}^2_+ = \Theta^y_{\mathcal{T}_2}$, Figures 3(a) and 3(b) show that, for any given $\langle x_1, x_2 \rangle \in \Theta^y_{\mathcal{T}_1} \cap \Theta^y_{\mathcal{T}_2}$,

$$\begin{aligned} P_T(x_1, x_2) &= P_{\mathcal{T}_1}(x_1, x_2) \cap P_{\mathcal{T}_2}(x_1, x_2) \\ &= \{\langle z, y \rangle \in \mathbf{R}^2_+ |\ y \leq x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - z \ \wedge\ \gamma + \alpha x_1 \geq z \geq \alpha x_1\} \end{aligned} \tag{4.20}$$

is non-empty-set if and only if $x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \geq \alpha x_1$. Hence, from (3.13) it follows that

$$\Theta^y_T = \{\langle x_1, x_2 \rangle \in \mathbf{R}^2_+ |\ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \geq \alpha x_1\}. \tag{4.21}$$

The functions $\underline{g} : \Omega_T \to \mathbf{R}_+$, $\bar{g} : \Omega_T \to \mathbf{R}_+$, and $f^y : \Xi^y_T \to \mathbf{R}_+$ are obtained as[48]

$$\begin{aligned} \underline{g}(x_1, x_2, y) &:= \min\{z \in \mathbf{R}_+ |\ z \in P_T(x_1, x_2, y)\} \\ &= \min\{z \in \mathbf{R}_+ |\ z \in P_{\mathcal{T}_1}(x_1, x_2, y) \cap P_{\mathcal{T}_2}(x_1, x_2, y)\} \\ &= \min\{z \in \mathbf{R}_+ |\ z \leq x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - y \ \wedge\ \gamma + \alpha x_1 \geq z \geq \alpha x_1\} = \alpha x_1 \\ \bar{g}(x_1, x_2, y) &= \min\{\gamma + \alpha x_1,\ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - y\} \\ f^y(x_1, x_2, z) &:= \max\{y \in \mathbf{R}_+ |\ y \in P_T(x_1, x_2, z)\} = \max\{y \in \mathbf{R}_+ |\ y \in P_{\mathcal{T}_1}(x_1, x_2, z)\} \\ &= x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - z. \end{aligned} \tag{4.22}$$

Note that although, $P_T(x_1, x_2, y)$ inherits its structure from both the IPT and the BPT, $\underline{g}()$ inherits its properties only from the BPT. This is because the lower bound on $P_{\mathcal{T}_1}(x_1, x_2, y)$ is zero for all $\langle x_1, x_2, y \rangle \in \Omega_{\mathcal{T}_1}$. Thus, $\underline{g}()$ is increasing in the usage of the emission causing

48 Note, $\Omega_T = \{\langle x_1, x_2, y \rangle \in \mathbf{R}^3_+ |\ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \geq \alpha x_1 \ \wedge\ y \leq x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - \alpha x_1\}$ and $\Xi^y_T = \{\langle x_1, x_2, z \rangle \in \mathbf{R}^3_+ |\ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \geq \alpha x_1 \ \wedge\ \gamma + \alpha x_1 \geq z \geq \alpha x_1\}$.

input of coal and it is independent of the level of electricity produced.[49] $\bar{g}$ inherits its properties from both the underlying IPT and BPT. $f^y()$ inherits its property only from the IPT, as y does not affect the nature's BPT. $f^y()$ is increasing in both the inputs. But, it is decreasing in z (see Figure 3(b)). This is because, in this example, smoke adversely affects the intended production of electricity.[50] Thus, T violates (v) in the definition of (BP). Note, however, that it satisfies (i) to (iii) in the definition of (BP), hence, it satisfies (WBP). Also, from (4.18) and (4.19) it follows that $P_T(x_1, x_2, z) = P_{\mathcal{T}_1}(x_1, x_2, z)$ and the structure of $P_{\mathcal{T}_1}(x_1, x_2, z)$ in (4.18) shows that T satisfies (ROFD)–electricity is freely disposable. Figure 3(b) also shows that T satisfies (CDB). Note that

$$P_T(y, z) = \{\langle x_1, x_2\rangle \in \mathbf{R}^2_+ | \ \frac{y^2}{z^2} \leq x_1 x_2 \ \wedge \ \frac{z-\gamma}{\alpha} \leq x_1 \leq \frac{z}{\gamma}\}, \tag{4.23}$$

which demonstrates that T violates input free-disposability with respect to coal but satisfies the same with respect to labor.

The restriction of the technology $P_T(x_1, x_2)$ for $\langle x_1, x_2\rangle \in \Theta^y_T$ is seen in Figure 3(b). It is clear from this figure that the functional representation of T is

$$\begin{aligned} T := \{\langle x_1, x_2, y, z\rangle \in \mathbf{R}^4_+ | \ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \geq \alpha x_1 \ \wedge \alpha x_1 \leq z \leq \min\{\gamma + \alpha x_1, \ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\} \ \wedge \\ y \leq x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - \alpha x_1 - z\} \end{aligned} \tag{4.24}$$

and that the correspondence $\underline{P}^y()$ is single valued with image

$$\underline{P}^y(x_1, x_2) = \langle z, y\rangle = \langle \alpha x_1, \ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - \alpha x_1\rangle, \ \forall \ \langle x_1, x_2\rangle \in \Theta^y_T. \tag{4.25}$$

Thus, all conclusions of Theorem 6 hold for this example.

4.4. Example 4 (leguminous plants in agriculture): An output as a source of emission and a positive emission externality on intended production.

Consider a farmer who produces, as his intended outputs, a leguminous crop (y_1) such as peas, beans, lentils, *etc.* and a non-leguminous crop (y_2) such as rice employing the inputs of labor (x_1) and nitrogenous fertilizer. The nitrogenous fertilizer is obtained as a by-product (z) that is produced by his leguminous crop.[51] Thus, in this example, z imposes a positive externality on intended production. $m = 2$, $m_1 = 1$, $n = 1$, $n_1 = 0$, and the IPT and BPT are

$$\begin{aligned} \mathcal{T}_1 &= \{\langle x, y_1, y_2, 0, 0, 0, z\rangle \in \mathbf{R}^7_+ | \ y_1 \leq x^{\frac{1}{2}} z - y_2\} \text{ and} \\ \mathcal{T}_2 &= \{\langle x, y_1, y_2, 0, 0, z, 0\rangle \in \mathbf{R}^7_+ | \ \gamma + \alpha y_1 \geq z \geq \alpha y_1\}, \end{aligned} \tag{4.26}$$

[49] Figure 3(b) verifies this for given values of all inputs.

[50] As noted above, this example violated Assumption (a) of Theorem 4.

[51] Leguminous plants are highly desirable crops in agriculture, as they have the ability to fix atmospheric nitrogen, due to a symbiotic relationship with bacteria (rhizobia) found in root nodules of these plants. The ability to form this symbiosis reduces fertilizer costs for farmers and gardeners who grow legumes, and allows legumes to be used in a crop rotation to replenish soil that has been depleted of nitrogen.

from which we can derive $T = T(\mathcal{T}_1, \mathcal{T}_2)$. The various projections of the IPT and BPT can be defined as before. Here we study the structures of various restrictions of the IPT and BPT, whenever these restrictions are non-empty. The restrictions

$$\begin{aligned} P_{\mathcal{T}_1}(x, y_1, y_2) &= \{z \in \mathbf{R}_+ | \ z \geq \frac{y_1 + y_2}{x^{\frac{1}{2}}}\} \text{ and} \\ P_{\mathcal{T}_2}(x, y_1, y_2) &= \{z \in \mathbf{R}_+ | \ \gamma + \alpha y_1 \geq z \geq \alpha y_1\} \end{aligned} \tag{4.27}$$

show that Assumption (a) of Theorem 4 is violated by this example. Moreover, both these restrictions impose non-trivial lower bounds on the emission.[52] Hence, the function $\underline{g}()$ inherits properties from both the IPT and the BPT. However, $P_{\mathcal{T}_1}(x, y_1, y_2)$ imposes no upper bound on the emission, and hence $\bar{g}()$ inherits properties only from the BPT:

$$\begin{aligned} \underline{g}(x, y_1, y_2) &= \max\{\frac{y_1 + y_2}{x^{\frac{1}{2}}}, \ \alpha y_1\} \\ \bar{g}(x, y_1, y_2) &= \gamma + \alpha y_1. \end{aligned} \tag{4.28}$$

The structures of the following restrictions, show that Assumption (b) of Theorem 4 holds for this example:

$$\begin{aligned} P_{\mathcal{T}_1}(x, y_1, z) &= \{y_1 \in \mathbf{R}_+ | \ y_2 \leq x^{\frac{1}{2}} z - y_1\} \text{ and} \\ P_{\mathcal{T}_2}(x, y_1, z) &= \mathbf{R}_+, \end{aligned} \tag{4.29}$$

Hence $f^2()$, the maximum amount of the non-emission generating non-leguminous crop, inherits its properties only from the IPT:

$$f^2(x, y_1, z) = x^{\frac{1}{2}} z - y_1. \tag{4.30}$$

Each of the following restrictions has a well-defined upper bound.

$$\begin{aligned} P_{\mathcal{T}_1}(x, y_2, z) &= \{y_1 \in \mathbf{R}_+ | \ y_1 \leq x^{\frac{1}{2}} z - y_2\} \text{ and} \\ P_{\mathcal{T}_2}(x, y_2, z) &= \{y_1 \in \mathbf{R}_+ | \ \frac{z - \gamma}{\alpha} \leq y_1 \leq \frac{z}{\alpha}\} \end{aligned} \tag{4.31}$$

Thus $f^1()$, the emission generating leguminous crop, inherits its properties from both the IPT and the BPT.

$$f^1(x, y_2, z) = \min\{x^{\frac{1}{2}} z - y_2, \ \frac{z}{\alpha}\}. \tag{4.32}$$

The upper bounds of both restrictions in (4.31) are increasing in z.[53] This shows that $f^1()$ is increasing in z. Hence, it violates the monotonicity properties of part (v) in the definition of (BP). Hence T violates (BP). However, it satisfies (WBP). (4.31) shows that

[52] $P_{\mathcal{T}_1}(x, y_1, y_2)$ shows that there is a minimal amount of nitrogen required to produce fixed levels of both intended crops when labor input is also held fixed.

[53] This is because z imposes a positive externality on intended production and z is caused by y_1 in nature's BPT.

T violates free disposability of the leguminous crop. But (4.29) shows that T satisfies the free disposability of the non-leguminous crop, *i.e.,* (ROFD) holds. (4.27) and (4.28) show that T satisfies (CDB). Further, since the BPT is unaffected by labor,

$$P_T(y_1, y_2, z) = \{x \in \mathbf{R}_+ | x \geq [\frac{y_1 + y_2}{z}]^2\} \tag{4.33}$$

shows that T satisfies free-disposability of labor. Thus, conclusions of Theorem 6 hold. Figure 4(a) shows that the restriction

$$\begin{aligned} P_T(x, y_2) &= P_{T_1}(x, y_2) \cap P_{T_2}(x, y_2) \\ &= \{\langle z, y_1\rangle \in \mathbf{R}^2_+ |\ y_1 \leq x^{\frac{1}{2}} z - y_2 \ \wedge\ \gamma + \alpha y_1 \geq z \geq \alpha y_1\} \end{aligned} \tag{4.34}$$

is non-empty if and only if $\gamma \geq \frac{y_2}{x^{\frac{1}{2}}}$.[54] Figure 4(b) show that the restriction

$$\begin{aligned} P_T(x, y_1) &= P_{T_1}(x, y_1) \cap P_{T_2}(x, y_1) \\ &= \{\langle z, y_2\rangle \in \mathbf{R}^2_+ |\ y_2 \leq x^{\frac{1}{2}} z - y_1 \ \wedge\ \gamma + \alpha y_1 \geq z \geq \alpha y_1\} \end{aligned} \tag{4.35}$$

is non-empty if and only if $\alpha y_1 \geq \frac{y_1}{x^{\frac{1}{2}}}$.[55] In this example, $f^1()$ and $\underline{g}()$ are inverses of one another. Figure 4(a) demonstrates this. Figure 4(b) shows that the graphs of $\underline{g}()$ and $f^2()$ can also coincide at more than one point. Thus, the correspondences

$$\begin{aligned} \underline{P}^2(x, y_1) &= \big\{\langle z, y_2\rangle \in \mathbf{R}^2_+ |\ y_2 = x^{\frac{1}{2}} z - y_1 \ \wedge\ z = \max\{\frac{y_1 + y_2}{x^{\frac{1}{2}}},\ \alpha y_1\}\big\} \\ \underline{P}^1(x, y_2) &= \big\{\langle z, y_1\rangle \in \mathbf{R}^2_+ |\ y_1 = \min\{x^{\frac{1}{2}} z - y_2,\ \frac{z}{\alpha}\}\big\} \end{aligned} \tag{4.36}$$

are not single valued.[56] A functional representations of T (see Theorem 6) is

$$\begin{aligned} T = \big\{\langle x, y_1, y_2, z\rangle \in \mathbf{R}^4_+ |\ &\langle x, y_1, y_2\rangle \in \Omega_T \ \wedge\ \ \langle x, y_1, z\rangle \in \Xi^2_T \ \wedge\ y_2 \leq x^{\frac{1}{2}} z - y_1 \ \wedge \\ &\max\{\frac{y_1 + y_2}{x^{\frac{1}{2}}},\ \alpha y_1\} \leq z \leq \gamma + \alpha y_1\big\}. \end{aligned} \tag{4.37}$$

54 This inequality defines Θ^1_T.
55 This inequality defines Θ^2_T.
56 This is unlike the case where T satisfies (BP).

5. The marginal abatement cost of emissions: the by-production approach.

In this section we focus on technologies that satisfy (BP) and show that the by-production approach to modeling technologies which generate emissions allows us to clearly distinguish between and delineate all options that are available to firms for reducing/ abating the generation of emissions. In particular, for a fixed technology, generation of emissions can be reduced by any emission generating firm if it (i) increases the amounts of resources that it diverts towards cleaning up (emission-mitigation) activities, *i.e.,* increases c, or (ii) reduces the use of inputs that cause emissions, *i.e.,* reduces x^1, or (iii) reduces the production of intended outputs that cause pollution, *i.e.,* reduces y^1, or (iv) substitutes away from inputs that are more intensive in producing the emissions to relatively cleaner (or perfectly clean) inputs. It is usually felt that all such changes in production strategies are costly in terms of the firm's resources. Here, we model two types of cost–technical cost of abatement and the economic cost of abatement. The former is defined as the reduction in the intended output of the firm when the firm reduces its emission, while the latter refers to the loss in profits of the firm when it reduces its emission. Emission reduction can also be achieved by technological changes that increase the productivity of inputs (especially, the cleaner inputs) of intended production or the effectiveness of cleaning-up options of firms.

5.1. Nature's emission generating industry and a simple model of by-production.

The nature's BPT (EGT)[57] is governed by certain universal laws, namely, the physical and chemical reactions which relate the level of emissions to the amounts of inputs and intended outputs that produce these emissions.[58] Hence, it is perhaps reasonable to assume that these laws are common to all firms. In other words, it is as if there is an emission generating industry defined by nature where (a) each firm in this industry has the same EGT, (b) entry of new firms into this industry does not change the aggregate technology of this industry, *i.e.,* the laws of nature governing emission generation apply at both firm-specific and aggregate levels, and (c) every firm necessarily enters this industry the moment it uses or produces goods in its intended production that trigger off the laws of nature that govern emission generation.

If the emission generating mechanism is viewed in this light, then it implies that the underlying EGT is additive: if there are L firms indexed by l then (i) the EGT is common to all firms, *i.e.,*

$$\mathcal{T}_2^l = \mathcal{T}_2 \subset \mathbf{R}_+^{n+2(m+2)}, \ \forall \, l = 1, \ldots, L, \tag{5.1}$$

[57] Evocatively, in this section, we will call a by-product producing technology (BPT) an emission generating technology (EGT).

[58] For example, the extent to which a given volume of coal can produce smoke can be thought of as a relation determined by nature.

and (ii) the aggregate technology of nature's emission generating industry is the same as the common technology of firms in this industry, *i.e.,*

$$\sum_{l}^{L} \mathcal{T}_2^l = \mathcal{T}_2. \tag{5.2}$$

One candidate for such an EGT is an EGT that exhibits constant returns to scale. In the following sections we assume a linear structure for the EGT. This ensures that the EGT is additive. However, the linear structure rules out all second-order effects of changes in emission causing inputs and outputs on emission generation–we are not sure of these properties of nature's laws of emission generation and hence suppress them in the analysis below. We rely only on the standard assumptions made in the literature regarding intended production to unravel the properties of the abatement costs of emission generating firms.

We study a simple model where the firm produces a single output y and a single emission z. It also undertakes cleaning-up efforts c. It employs three inputs, x_1, x_2, and x_3. In nature, emissions are potentially caused by x_1 and x_2.[59] The IPT and the EGT have the following forms:

$$\begin{aligned}\mathcal{T}_1 &= \{\langle x_1, x_2, x_3, y, 0, c, 0, 0, z\rangle \in \mathbf{R}_+^9 \mid y \leq F(x_1, x_2, x_3, c)\}\\ \mathcal{T}_2 &= \{\langle x_1, x_2, x_3, 0, y, 0, c, z, 0\rangle \in \mathbf{R}_+^9 \mid z \geq \alpha_1 x_1 + \alpha_2 x_2 - \beta c\}.\end{aligned} \tag{5.3}$$

Let T denote $T(\mathcal{T}_1, \mathcal{T}_2) \subset \mathbf{R}_+^{n+m+2}$ that is derived from (5.3). We assume that $F()$ is concave and smooth in the interior of its domain. The derivatives of $F()$ satisfy $F_c() < 0$, $F_i() > 0$ for $i = 1, \ldots, 3$, and $F_{i,c}() = 0$ for $i = 1, 2$. These sign restrictions capture the fact that the marginal productivities of all inputs are positive in intended production and that production of cleaning-up activities is not intensive in inputs x_1 and x_2.[60] The production of cleaning-up activities is, however, costly in terms of a firm's resources: x_3 is a common input that is shared between the production of y and the production of c. It is not the cause of emissions in nature. A greater amount of x_3 facilitates both an increase in production of the intended output and a reduction in emissions. In our analysis, however, we wish to focus purely on those abatement strategies of firms that potentially involve a trade-off between intended production and emission reduction. For this purpose we restrict our analysis to abatement strategies of firms which involve no change in the level of the third input, *i.e.,* the level of this input will be held fixed. This implies that different abatement strategies of firms will generally involve different distributions of the fixed amount of input three between the production of y and the production of c. We will assume throughout

[59] In the next section, we interpret these as two fuel inputs used by the firm which produce energy.

[60] The simple models we study are only illustrative examples to study the issue of abatement costs employing the by-production approach. The assumption $F_{i,c}() = 0$ for $i = 1, 2$ can of course be generalized if this assumption is empirically false. However, in that case, many of the derivatives of the marginal abatement cost function that we obtain below will have ambiguous signs. Assuming $F_{i,c}() = 0$ is convenient and leads to definitive signs for these derivatives which are consistent with views in the literature or are intuitively meaningful.

the next two sections that the firm always operates in a technically efficient manner, that is, its choice of the production vectors $\langle x_1, x_2, x_3, y, c, z\rangle \in \mathbf{R}^6_+$ are such that

$$\begin{aligned} y &= F(x_1, x_2, x_3, c) \\ z &= \alpha_1 x_1 + \alpha_2 x_2 - \beta c. \end{aligned} \tag{5.4}$$

5.2. The marginal technical cost of abatement (MTCA) and its properties.

In this section, we interpret the simple model above to be the model of a technology that is energy intensive. Inputs one and two are two sources of energy. These two have different abilities to produce the energy input, in addition to the differences in their propensities to generate emissions. Thus, in this section, the function $F()$ is assumed to have the following form:

$$F(x_1, x_2, x_3, c) = f(e(x_1, x_2), x_3, c). \tag{5.5}$$

Here, we interpret $e()$ as the aggregator function that specifies the total energy input obtained from employing x_1 and x_2 amounts, respectively, of inputs one and two. Let us make a further simplifying assumption:

$$e(x_1, x_2) = \theta_1 x_1 + \theta_2 x_2 \tag{5.6}$$

with $\theta_1 > \theta_2$, *i.e.,* input one is more productive in energy production than input two. At the same time, we also assume that $\alpha_1 > \alpha_2$, *i.e.,* input 1 has a greater propensity to generate emissions that input two. β measures the effectiveness of the cleaning-up efforts of the firm in reducing emissions. We assume $\beta > 0$, $\alpha_1 > 0$, and $\alpha_2 \geq 0$. If $\alpha_2 = 0$ then input 2 is a perfectly clean input (has no impurities) and hence is not a source of emission even though it is a source of energy input for the firm. Thus, in this case $n_2 = 2$ and $n_1 = 1$.[61] On the other hand, if $\alpha_2 > 0$, then $n_1 = 2$ and $n_2 = 1$.[62] Technological developments in cleaner methods for producing intended outputs increase the value of θ_2, that is, make the clean(er) input more productive in producing energy for intended production. Technological development can also improve the effectiveness of cleaning-up methods used by the firm, *i.e.,* increase the value of β. We assume, however, that α_1 and α_2 are defined by nature. The derivatives of $f()$ satisfy $f_e() > 0$, $f_3() > 0$, $f_c() < 0$, and $f_{e,c}() = 0$.

In this simple model, the different abatement options open to a firm include cleaning-up activities, reduction in the usage of inputs (fuel-inputs) that cause emissions, and switching between inputs (inter-fuel substitution) that vary in terms of their emission generating propensities. Starting at an initial efficient production vector $\langle \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}, \bar{c}, \bar{z}\rangle \in$

[61] Examples of such inputs include wind, solar, or hydro power to generate energy.
[62] Example of such an input is a cleaner variety of coal that has less carbon content, and hence, is less productive in energy generation, while at the same time has a smaller propensity to generate emissions.

T, an abatement strategy of the firm is assumed to be a linear path of inputs and cleaning-up activities of the firm that is parametrized by the mapping $\delta : \mathbf{R}_+ \to \mathbf{R}^3_+$ with image

$$\delta(t) = \begin{bmatrix} x_1(t) = \bar{x}_1 + \dot{x}_1 t \\ x_2(t) = \bar{x}_2 + \dot{x}_2 t \\ c(t) = \bar{c} + \dot{c}t \end{bmatrix} \tag{5.7}$$

Denote the gradient of δ with respect to t by $\nabla_t \delta$. Then, (5.7) indicates that

$$\nabla_t \delta(t) = [\, \dot{x}_1 \quad \dot{x}_2 \quad \dot{c} \,] =: \dot{\delta}. \tag{5.8}$$

Each linear path $\delta(t)$ involves a particular mix of rates of change in x_1, x_2, and c. Note, $\delta(0) = \langle \bar{x}_1, \bar{x}_2, \bar{c} \rangle$ and, for a given technology, different values of $\dot{x}_1$, $\dot{x}_2$, and $\dot{c}$ coincide with different paths (different strategies) of abatement undertaken by the firm. We restrict our analysis to linear paths $\delta(t)$ for which $\| \dot{\delta} \| = 1$.[63] $\dot{\delta}$ is thus a direction of change in all inputs and cleaning-up efforts of a firm, starting from an initial production vector.

The path of production of the intended output and emissions induced by any path $\delta(t)$ of inputs and cleaning-up effort is

$$\begin{aligned} y = f(\theta_1 x_1(t) + \theta_2 x_2(t), \bar{x}_3, c(t)) \\ \bar{z} - \Delta = \bar{z} - \Delta(t) := \alpha_1 x_1(t) + \alpha_2 x_2(t) - \beta c(t) \end{aligned} \tag{5.9}$$

so that $z(0) = \bar{z}$, $\Delta(0) = 0$, and $y(0) = \bar{y}$. $\Delta(t)$ is the total abatement (the change/ reduction) in the emissions from the initial level $\bar{z}$. From (5.9) it follows that

$$\begin{aligned} &\bar{z} - \Delta(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) - \beta c(t) \\ \Rightarrow &\bar{z} - \Delta(t) = \alpha_1[\bar{x}_1 + \dot{x}_1 t] + \alpha_2[\bar{x}_2 + \dot{x}_2 t] - \beta[\bar{c} + \dot{c}t] \\ \Rightarrow &\bar{z} - \Delta(t) = \bar{z} + [\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 - \beta \dot{c}]t \\ \Rightarrow &- \Delta(t) = [\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 - \beta \dot{c}]t. \end{aligned} \tag{5.10}$$

From (5.10) it follows that starting from an initial production vector, abatement is increasing along the path $\delta(t)$ if

$$\frac{\partial \Delta(t)}{\partial t} = -[\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 - \beta \dot{c}] > 0. \tag{5.11}$$

In that case, the function $\Delta(t)$ is invertible and (5.10) implies

$$t = \frac{-\Delta}{\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 - \beta \dot{c}}. \tag{5.12}$$

Employing (5.12) and substituting for t in the first equation of (5.9), we obtain

$$\begin{aligned} &y = \\ &f\Big(\sum_{i=1}^{2} \theta_i[\bar{x}_i + \frac{-\Delta \dot{x}_i}{\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 - \beta \dot{c}}],\ \bar{x}_3,\ \bar{c} + \frac{-\Delta \dot{c}}{\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 - \beta \dot{c}}\Big) \end{aligned} \tag{5.13}$$

[63] $\| \dot{\delta} \| = 1$ denotes the Euclidean norm of $\dot{\delta} \in \mathbf{R}^3$.

We define the technical cost of abatement function $C^T : \mathbf{R}^8_+ \to \mathbf{R}_+$ with image

$$C^T(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{c}) = f\big(\theta_1\bar{x}_1 + \theta_2\bar{x}_2,\ \bar{x}_3,\ \bar{c}\big) - f\Big(\sum_{i=1}^{2}\theta_i[\bar{x}_i + \frac{-\Delta\dot{x}_i}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}}],\ \bar{x}_3,\ \bar{c} + \frac{-\Delta\dot{c}}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}}\Big). \tag{5.14}$$

$C^T()$ is the change in intended output from the initial level $\bar{y}$ due to any linear path $\delta(t)$ of inputs and cleaning-up effort chosen by the firm, that results in a path $\Delta(t)$ of total abatement.

In the context of the model described above, for any given path of abatement chosen by a firm, we derive the marginal technical cost of abatement (MTCA) and study its properties. Our aim is two-fold: (i) to study the properties of MTCA along any path of abatement adopted by the firm and (ii) given any path of abatement adopted by a firm, to study the effect of changes in technology and the differences in the initial levels of usage of inputs and the initial cleaning-up efforts on MTCA. Along any such path, the marginal technical cost of abatement is given by the derivative:

$$\frac{\partial C^T(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{c})}{\partial \Delta} \equiv C^T_\Delta(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{c}) = \frac{f_e()[\theta_1\dot{x}_1 + \theta_2\dot{x}_2] + f_c()\ \dot{c}}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}}. \tag{5.15}$$

To see how the marginal technical cost of abatement varies with the abatement level along any path of abatement chosen by a firm, we consider the derivative of $C^T_\Delta()$ with respect to Δ[64]:

$$\frac{\partial^2 C^T()}{\partial \Delta^2} =: C^T_{\Delta,\Delta}() = -\ \frac{f_{e,e}()\ [\theta_1\dot{x}_1 + \theta_2\dot{x}_2]^2 + f_{c,c}()\dot{c}^2}{[\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}]^2}, \tag{5.16}$$

Similarly, we can also study how the marginal technical cost of abatement varies with the initial levels of inputs one and two, the initial level of cleaning-up efforts of the firm, and the productivities θ_1 and θ_2 of inputs one and two:

$$\begin{aligned}
\frac{\partial^2 C^T()}{\partial\Delta\partial\bar{x}_1} &=: C^T_{\Delta,\bar{x}_1}() = \frac{\theta_1\ f_{e,e}()\ [\theta_1\dot{x}_1 + \theta_2\dot{x}_2]}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}},\\
\frac{\partial^2 C^T()}{\partial\Delta\partial\bar{x}_2} &=: C^T_{\Delta,\bar{x}_2}() = \frac{\theta_2\ f_{e,e}()\ [\theta_1\dot{x}_1 + \theta_2\dot{x}_2]}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}},\\
\frac{\partial^2 C^T()}{\partial\Delta\partial\bar{c}} &=: C^T_{\Delta,\bar{c}}() = \frac{f_{c,c}()\dot{c}}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}},\\
\frac{\partial^2 C^T()}{\partial\Delta\partial\beta} &=: C^T_{\Delta,\beta}() = -\frac{[\theta_1\dot{x}_1 + \theta_2\dot{x}_2]^2\dot{c}f_{e,e}()\Delta + \dot{c}^3 f_{c,c}()\Delta - \dot{c}[\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}]^2\nabla_\Delta C^T()}{[\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}]^3},
\end{aligned} \tag{5.17}$$

[64] Recall, $f_{e,c}() = 0$.

$$\frac{\partial^2 C^T()}{\partial\Delta\partial\theta_2} =: C^T_{\Delta,\theta_2}() = \frac{x_2()f_{e,e}()[\theta_1\dot{x}_1 + \theta_2\dot{x}_2] + f_e()\dot{x}_2}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}}, \text{ and}$$
$$\frac{\partial^2 C^T()}{\partial\Delta\partial\theta_1} =: C^T_{\Delta,\theta_1}() = \frac{x_1()f_{e,e}()[\theta_1\dot{x}_1 + \theta_2\dot{x}_2] + f_e()\dot{x}_1}{\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c}}. \quad (5.18)$$

Consider, first, three obvious paths of abatement reductions that are available to a firm:

(1) ceteris-paribus, reduction in the energy intensive inputs: $\delta(t)$ such that $\dot{c} = 0$, $\dot{x}_1 \leq 0$, $\dot{x}_2 \leq 0$, and $\langle \dot{x}_1, \dot{x}_2 \rangle \neq 0$,

(2) ceteris paribus, a switch from the more emission generating to the less emission generating input (inter-fuel substitution): $\delta(t)$ such that $\dot{c} = 0$ and $\dot{x}_1 = -\dot{x}_2$ and $\dot{x}_2 > 0$, and

(3) ceteris paribus, an increase in its cleaning up efforts: $\delta(t)$ such that $\dot{c} > 0$, $\dot{x}_1 = \dot{x}_2 = 0$.

From (5.11) it follows that, along all the three paths above, abatement is increasing:

$$\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2 - \beta\dot{c} < 0. \quad (5.19)$$

Theorem 7: *: Along paths 1 to 3 of abatement,*

(i) $C^T_\Delta(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c}) > 0$,

(ii) $C^T_{\Delta,\Delta}(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c}) > 0$,

(iii) $C^T_{\Delta,\bar{x}_i}(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c}) < 0, \ \forall\ i = 1, 2$,

(iv) $C^T_{\Delta,\bar{c}}(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c}) > 0$, *and*

(v) $C^T_{\Delta,\beta}(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c}) < 0$.

Along path 2 of abatement,

(vi) $C^T_{\Delta,\theta_2}(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c}) < 0$ *and*

(vii) the sign of $C^T_{\Delta,\theta_1}(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c})$ *is ambiguous.*

Along path 1 of abatement,

(viii) the sign of $C^T_{\Delta,\theta_i}(\Delta, \theta_1, \theta_2, \beta, \bar{x}_1, \bar{x}_2, \bar{c})$ *for* $i = 1, 2$ *is ambiguous.*

Proof: Follows from (5.19), (5.15), (5.16), (5.18), and our (standard) curvature and monotonicity assumptions on the intended production function $f()$. ■

This result is quite intuitive: Along all the three paths of abatement above, the MTCA is positive ((i) of Theorem 7) because along all these paths, abatement comes at the cost of reduction of the intended output–this is either because it entails, ceteris paribus, a reduction in some inputs or because, ceteris paribus, a greater diversion of the fixed amount of input three from the production of y to the production of c or because, ceteris paribus, a substitution from an input that is more productive in intended production to a relatively less productive input.

Along all the three paths of abatement above, MTCA is increasing in the level of abatement ((ii) of Theorem 7) because of diminishing returns to resources in intended production. Abatement path 1 implies that, ceteris paribus, as the fuel intensive inputs are reduced more and more, the loss in intended output increases due to the phenomenon of diminishing returns to these inputs ($f_{e,e}() < 0$). Abatement path 2 implies that, ceteris paribus, as more and more inter-fuel substitution takes place towards the fuel that is less energy intensive, the energy input into intended input falls, diminishing returns to energy kicks in, and the loss in intended output increases; abatement path 3 implies that, ceteris paribus, as the firm undertakes more and more cleaning-up activities c, more and more of the fixed input x_3 is diverted away from production of y. Diminishing returns to this input in the production of c implies that more and more of the intended output has to be given up to produce more and more of c ($f_{c,c}() < 0$)).

Along paths 1 and 2 of abatement above, MTCA is (non-trivially) decreasing in the initial amounts of each of the energy-generating inputs ((iii) of Theorem 7): marginal costs of firms employing higher amounts of the emission generating inputs is lower. This is again due to the phenomenon of diminishing returns in intended production: at higher levels of usage of fuel inputs, the marginal productivities of these inputs is lower. Hence, for these firms, ceteris paribus, a reduction in the usage of these inputs or a switch to the relatively less energy intensive input implies a smaller reduction in the intended output than for firms which are employing lower amounts of these inputs,

Along path 3 of abatement above, MTCA is increasing in the initial amount of cleaning-up efforts ((iv) of Theorem 7): MTCAs of firms engaging in higher amounts of the cleaning-up activities is higher. This is because firms for which $\bar{c}$ is higher, the marginal productivity of inputs in producing cleaning-up output is lower due to diminishing returns, and hence, ceteris paribus, an increase in c requires a greater diversion of input three from y to c for these firms than for firms whose initial levels of cleaning-up is lower.

Along all the three paths of abatement above, MTCA is decreasing in the effectiveness β of cleaning-up effort of a firm in reducing emissions ((v) of Theorem 7). As β increases, smaller reductions in fuel inputs or lesser switching to the less productive fuel input or smaller increases in the amounts of cleaning-up efforts are required to generate a given amount of abatement Δ. Hence, diminishing returns implies lower reductions in intended outputs for firms with higher values of β.

Along path 2 of abatement above, MTCA is decreasing in the productivity of energy input x_2: marginal abatement costs of firms with higher θ_2 is lower or as θ_2 increases with technological development, MTCAs fall. There are two effects on MTCA of an increase in θ_2; (a) holding the marginal product of energy input fixed, an increase in θ_2 implies that the decrease in the energy input due to inter-fuel substitution towards the less fuel intensive input is lower and (b) holding the decrease in energy input due to inter-fuel substitution fixed, an increase in θ_2, ceteris paribus, implies a higher amount of the energy input, and hence, decreases the marginal productivity of the energy input due to diminishing returns. On the other hand, along path 1 ($\dot{x}_2 < 0$) of abatement above, nothing can be said about how the MTCA will change due to an increase in θ_2: an argument similar to (b) above holds but (a) may not along this path of abatement. Similarly, we can explain the remaining parts of Theorem 7.

Starting from any initial production vector, the by-production approach to modeling technologies that generate emissions also shows the existence of abatement strategies which lead to both higher abatement levels and lower reductions in output. This is possible, for example, when at the initial production vectors differences exist in the marginal rates of technical substitutions between energy generating inputs in emission reductions and intended production, *i.e.,* $\frac{\theta_1}{\theta_2} \neq \frac{\alpha_1}{\alpha_2}$.

Theorem 8: *Let* $\langle \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}, \bar{c}, \bar{z} \rangle \in \mathbf{R}^6_+$ *be an initial production vector such that*

$$\begin{aligned} \bar{y} &= f(\theta_1 \bar{x}_1 + \theta_2 \bar{x}_2, \bar{x}_3, \bar{c}) \\ \bar{z} &= \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 - \beta \bar{c}. \end{aligned} \tag{5.20}$$

Suppose at least one of the following holds:
(i) $\frac{\theta_1}{\theta_2} \neq \frac{\alpha_1}{\alpha_2}$ *or*
(ii) $\frac{\theta_1 \; f_e(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{c}),}{f_c(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{c})} \neq \frac{\alpha_1}{-\beta}$.
Then there exist paths of abatement $\delta(t)$ *associated with directions of change* $\dot{\delta} = \langle \dot{x}_1, \dot{x}_2, \dot{c} \rangle$ *such that MTCA is negative, that is,* $\nabla_\Delta C() < 0$.

Proof: (i) If $\frac{\theta_1}{\theta_2} > \frac{\alpha_1}{\alpha_2}$, then choose $\dot{c} = 0$, $\dot{x}_1 = 1$ and $\dot{x}_2 < 0$ such that $-\frac{\theta_1}{\theta_2} < \dot{x}_2 < -\frac{\alpha_1}{\alpha_2}$. If $\frac{\theta_1}{\theta_2} < \frac{\alpha_1}{\alpha_2}$, then choose $\dot{c} = 0$, $\dot{x}_1 = -1$ and $\dot{x}_2 > 0$ such that $\frac{\theta_1}{\theta_2} < \dot{x}_2 < \frac{\alpha_1}{\alpha_2}$. In both cases, we find that $\theta_1 \dot{x}_1 + \theta_2 \dot{x}_2 > 0$ and $\alpha_1 \dot{x}_1 + \alpha_2 + \dot{x}_2 < 0$ and, hence, $\nabla_\Delta C() < 0$.
(ii) Similar to proof of part (i). ■

This result is also intuitive. When $\frac{\theta_1}{\theta_2} > \frac{\alpha_1}{\alpha_2}$, then for a given increase in the usage of fuel input x_1, the maximum that the firm is *willing to give up* of fuel input x_2 to ensure at least the original level of intended output y is greater than the minimum amount of fuel input x_2 that it *needs to give up* to ensure at least the original level of abatement Δ. Thus, any decrease in usage of fuel input x_2 that lies between these two bounds implies both an increase in intended output and a greater reduction of emissions.[65]

5.3. The marginal economic cost of abatement (MECA).

Assuming that a firm maximizes profits, what is the loss in its profits when it is subject to an environmental regulation that requires it to reduce its emission? We define this loss as the firm's economic cost of abatement. We employ the technological specifications in (5.3). Let $\langle p_y, p_1, p_2, p_3 \rangle \in \mathbf{R}^4_+$ be a vector of output and input prices faced by the

[65] The derivates in (5.18) can be used also to understand how the MTAC will vary along a path of firm characteristics, starting from an initial configuration of firm characteristics $\langle \bar{x}_1, \bar{x}_2, \bar{c}, \beta, \theta_1, \theta_2 \rangle \in \mathbf{R}^6_{++}$.

firm. The profit function of a firm with technology $T(\mathcal{T}_1, \mathcal{T}_2) \subset \mathbf{R}_+^{n+m+2}$ is derived as the mapping $\Pi : \mathbf{R}_+^7 \to \mathbf{R}_+$ with image

$$\begin{aligned} \Pi(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3) := \max p_y y - \sum_{i=1}^{2} p_i x_i - p_3 \bar{x}_3 \\ \text{subject to} \\ y \leq F(x_1, x_2, \bar{x}_3, c) \text{ and} \\ \bar{z} - \Delta \geq \sum_{i=1}^{2} \alpha_i x_i - \beta c. \end{aligned} \tag{5.21}$$

The Lagrangian of this problem is

$$L() = p_y y - \sum_{i=1}^{2} p_i x_i - p_3 \bar{x}_3 - \mu[y - F(x_1, x_2, \bar{x}_3, c)] - \rho[\sum_{i=1}^{2} \alpha_i x_i - \beta c - \bar{z} + \Delta], \tag{5.22}$$

where $\mu \in \mathbf{R}$ and $\rho \in \mathbf{R}$ are the Lagrange multipliers on the constraints imposed by the IPT and the BPT, respectively. Note, given the inequalities that characterize the constraints in (5.21), at the optimum, they will necessarily be non-negative.

The initial level of profits prior is defined as the value $\Pi(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3)$ when $\Delta = 0$. We denote this by $\bar{\pi}$. We define the economic cost of abatement as the reduction in the initial level of profits due to abatement constraints. The economic cost of abatement function is the mapping: $C^E : \mathbf{R}_+^7 \to \mathbf{R}_+$ with image

$$C^E(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3) := \bar{\pi} - \Pi(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3). \tag{5.23}$$

The marginal economic cost of abatement (MECA) is thus the derivative of $C^E()$ with respect to Δ. Employing the envelope theorem, from (5.22), it follows that it takes the form

$$\begin{aligned} \nabla_\Delta C^E() \equiv \frac{\partial C^E(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3)}{\partial \Delta} &= -\frac{\partial \Pi(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3)}{\partial \Delta} \\ &= \rho(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3) \geq 0. \end{aligned} \tag{5.24}$$

Noting from (5.22) that $\frac{\partial \Pi(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3)}{\partial \Delta} = -\frac{\partial \Pi(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3)}{\partial \bar{z}}$, it follows that

$$\begin{aligned} \nabla_{\Delta,\Delta} C^E() \equiv \frac{\partial^2 C^E()}{\partial \Delta^2} &= \frac{\partial \rho(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3)}{\partial \Delta} \text{ and} \\ \nabla_{\bar{z},\Delta} C^E() \equiv \frac{\partial^2 C^E()}{\partial \bar{z} \partial \Delta} &= \frac{\partial \rho(\bar{z}, \Delta, p_y, p_1, p_2, p_3, \bar{x}_3)}{\partial \bar{z}} = -\frac{\partial^2 C^E()}{\partial \Delta^2}. \end{aligned} \tag{5.25}$$

We will now study how MECA varies with the level of abatement and the initial level of emission. To do so, we employ standard comparative static methods. The first-order conditions of the problem (5.21) are

$$\begin{aligned} & p_y - \mu = 0, \\ & -p_1 + \mu F_1() - \rho\alpha_1 = 0, \\ & -p_2 + \mu F_2() - \rho\alpha_2 = 0, \\ & \mu F_c() + \rho\beta = 0, \\ & y - F(x_1, x_2, \bar{x}_3, c) = 0, \text{ and} \\ & \sum_{i=1}^{2} \alpha_i x_i - \beta c - \bar{z} + \Delta = 0. \end{aligned} \tag{5.26}$$

Let the first-order conditions in (5.26) be represented by the vector-valued implicit function

$$G(y, x_1, x_2, c, \mu, \rho, \bar{z}, \Delta, p_1, p_2, p_3, \bar{x}_3) = 0, \tag{5.27}$$

where $\langle y, x_1, x_2, c, \mu, \rho\rangle \in \mathbf{R}^6_+$ are the endogenous variables and $\langle \bar{z}, \Delta, p_1, p_2, p_3, \bar{x}_3\rangle \in \mathbf{R}^6_+$ are the exogenous variables. Focusing only on the comparative statics of the choice variables with respect to Δ and differentiating (5.27) we obtain

$$\nabla_{y,x_1,x_2,c,\mu,\rho} G() \cdot \begin{bmatrix} dy \\ dx_1 \\ dx_2 \\ dc \\ d\mu \\ d\rho \end{bmatrix} = -\nabla_\Delta G() \cdot d\Delta, \tag{5.28}$$

where $\nabla_{y,x_1,x_2,c,\mu,\rho} G()$ and $\nabla_\Delta G()$ are the Jacobians

$$\nabla_{y,x_1,x_2,c,\mu,\rho} G() = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & \mu F_{1,1}() & \mu F_{1,2}() & 0 & F_1() & -\alpha_1 \\ 0 & \mu F_{1,2}() & \mu F_{2,2}() & 0 & F_2() & -\alpha_2 \\ 0 & 0 & 0 & F_{c,c}() & F_c() & \beta \\ 1 & -F_1() & -F_2() & -F_c() & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & -\beta & 0 & 0 \end{bmatrix} \text{ and } \nabla_\Delta G() = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.29}$$

Employing the implicit function theorem, from (5.28) we can solve for the following local derivatives:

$$\begin{bmatrix} \frac{\partial y()}{\partial \Delta} \\ \frac{\partial x_1()}{\partial \Delta} \\ \frac{\partial x_2()}{\partial \Delta} \\ \frac{\partial c()}{\partial \Delta} \\ \frac{\partial \mu()}{\partial \Delta} \\ \frac{\partial \rho()}{\partial \Delta} \end{bmatrix} = [\nabla_{y,x_1,x_2,c,\mu,\rho} G()]^{-1} \nabla_\Delta G() \tag{5.30}$$

Straightforward application of the Crammer's rule allows us to solve (5.30) and we obtain the following:

$$\frac{\partial y()}{\partial \Delta} = \frac{\mu^2 F_{c,c}[-F_{1,1}F_2\alpha_2 + F_{1,2}[F_1\alpha_2 + F_2\alpha_1] - F_{2,2}F_1\alpha_1] + \mu^2 F_c[F_{1,1}F_{2,2} - F_{1,2}^2]}{\mu^2 F_{c,c}[F_{1,1}\alpha_2^2 + F_{2,2}\alpha_1^2 - 2\alpha_1\alpha_2 F_{1,2}] + \beta^2\mu^2[F_{1,1}F_{2,2} - F_{1,2}^2]}, \tag{5.31}$$

$$\frac{\partial x_1()}{\partial \Delta} = \frac{\mu^2 F_{c,c}[\alpha_2 F_{1,2} - \alpha_1 F_{2,2}]}{\mu^2 F_{c,c}[F_{1,1}\alpha_2^2 + F_{2,2}\alpha_1^2 - 2\alpha_1\alpha_2 F_{1,2}] + \beta^2\mu^2[F_{1,1}F_{2,2} - F_{1,2}^2]}, \tag{5.32}$$

$$\frac{\partial x_2()}{\partial \Delta} = \frac{\mu^2 F_{c,c}[\alpha_1 F_{1,2} - \alpha_2 F_{1,1}]}{\mu^2 F_{c,c}[F_{1,1}\alpha_2^2 + F_{2,2}\alpha_1^2 - 2\alpha_1\alpha_2 F_{1,2}] + \beta^2\mu^2[F_{1,1}F_{2,2} - F_{1,2}^2]}, \tag{5.33}$$

$$\frac{\partial c()}{\partial \Delta} = \frac{\beta\mu^2[F_{1,1}F_{2,2} - F_{1,2}^2]}{\mu^2 F_{c,c}[F_{1,1}\alpha_2^2 + F_{2,2}\alpha_1^2 - 2\alpha_1\alpha_2 F_{1,2}] + \beta^2\mu^2[F_{1,1}F_{2,2} - F_{1,2}^2]}, \text{ and} \tag{5.34}$$

$$\frac{\partial \rho()}{\partial \Delta} = \frac{-\mu^3 F_{c,c}[F_{1,1}F_{2,2} - F_{1,2}^2]}{\mu^2 F_{c,c}[F_{1,1}\alpha_2^2 + F_{2,2}\alpha_1^2 - 2\alpha_1\alpha_2 F_{1,2}] + \beta^2\mu^2[F_{1,1}F_{2,2} - F_{1,2}^2]}. \tag{5.35}$$

Theorem 9: *If, in addition to the maintained assumptions,*
(1) $F()$ is strictly concave and $F_{1,2}() \geq 0$ then

$$\frac{\partial y()}{\partial \Delta} < 0,\ \frac{\partial x_1()}{\partial \Delta} < 0,\ \frac{\partial x_2()}{\partial \Delta} < 0,\ \frac{\partial c()}{\partial \Delta} > 0,\ \frac{\partial \rho()}{\partial \Delta} > 0, \text{ and} \tag{5.36}$$

(2) $F()$ takes the form in (5.5), (5.6) is true, $\langle p_1, p_2\rangle$ is proportional to $\langle \theta_1, \theta_2\rangle$ then[66]

$$\frac{\partial y()}{\partial \Delta} = 0,\ \frac{\partial x_1()}{\partial \Delta} = -\frac{\partial x_2()}{\partial \Delta}\frac{\theta_2}{\theta_1},\ \frac{\partial c()}{\partial \Delta} = 0, \text{ and } \frac{\partial \rho()}{\partial \Delta} = 0. \tag{5.37}$$

Proof: (1) follows in a straightforward manner given all the assumptions on $F()$. (2) follows from the fact that for $i, i' = 1, 2$, $F_i() = \theta_i f_e()$ and $F_{i,i'}() = \theta_i\theta_{i'} f_{e,e}()$. In particular, note that, in this case, we have $F_{1,2}() < 0$, $F_{1,1}()F_{2,2}() - F_{1,2}^2() = 0$, and $\frac{\partial x_1()}{\partial \Delta} < 0$ if and only if $\frac{\alpha_1}{\alpha_2} > \frac{\theta_1}{\theta_2}$. ■

Corollary to Theorem 9: *If, in addition to the maintained assumptions, $F()$ satisfies conditions in (1) of Theorem 9 then $\frac{\partial^2 C^E()}{\partial \Delta^2} > 0$ and $\frac{\partial^2 C^E()}{\partial \bar{z}\partial \Delta} < 0$. If, in addition to the maintained assumptions, $F()$ satisfies conditions in (2) of Theorem 9 then $\frac{\partial^2 C^E()}{\partial \Delta^2} = \frac{\partial^2 C^E()}{\partial \bar{z}\partial \Delta} = 0$.*

66 This corresponds to an initial (optimal) production vector where $\rho = 0$, *i.e.*, when the firm is unregulated ($\Delta = 0$).

Proof: Follows from (5.25). ■

If the firm is a price taker and is mandated to reduce its initial level of emission, then it will (endogenously) choose that path of abatement that maximizes its profits. Along such a path, in general, the signs of the derivatives in (5.31) to (5.35) depend on the sign of $F_{1,2}()$. If this is negligible or non-negative then part (1) of Theorem 9 says that, along the optimal path of abatement chosen by the firm, the firm meets its mandated abatement requirement by reducing the use of both its fuel inputs and increasing its cleaning-up efforts. Hence, along this path, its output will fall. The corollary to Theorem 9 says that, along this path, MECA of the firm increases with increase in abatement, while it decreases with increase in the initial level of emissions, a phenomenon which again invokes the economic law of diminishing returns. If, however, $F()$ assumes the form in (5.5) and (5.6), then part (2) and the corollary of Theorem 9 say that, along the optimal path chosen by the firm, MECA is constant and the firm meets its abatement requirements purely by adopting the inter-fuel substitution strategy. Reduction in emission is achieved by no change in its output or cleaning-up effort.

5.4. Marginal abatement costs: The input approach and the by-production approach.

The single-equation input approach treats the emission as a standard input of production. It explains the positive correlation between emissions and intended outputs *solely* in terms of cleaning-up effort of the firm and not in terms of commodities x^1 and y^1: the more the resources are diverted towards cleaning up the less is the emission produced and the less is the intended output.[67] However, cleaning-up effort of the firm is not explicitly modeled. Thus, it considers a reduced-form production relation:

$$y = h(x_1, x_2, x_3, z), \; h_i() > 0 \; \forall \; i = 1, \ldots, 3, \; \wedge \; h_z() > 0. \tag{5.38}$$

In the framework of our simple model in (5.3), it is as if this reduced form technology has been derived from the following two production relations corresponding to an IPT and an EGT, respectively:

$$\begin{aligned} y &= F(x_1, x_2, x_3, c) \\ z &= \bar{z} - \beta c \end{aligned} \tag{5.39}$$

where $\bar{z} \in \mathbf{R}_+$ is the initial level of the emission.[68] This implies that

$$h(x_1, x_2, x_3, z) = F(x_1, x_2, x_3, \frac{\bar{z} - z}{\beta}) \tag{5.40}$$

[67] See Baumol and Oats [1988] and Cropper and Oates [1992].
[68] Note, the EGT is independent of x_1 and y_1.

Consider the behavior of a firm with technology (5.38) when a Pigouvian tax t is imposed on it.

$$\begin{aligned}\Pi^t(\bar{z}, t, p_y, p_1, p_2, p_3, \bar{x}_3) := \max p_y y - \sum_{i=1}^{2} p_i x_i - p_3\bar{x}_3 - tz \\ \text{subject to} \\ y \leq h(x_1, x_2, \bar{x}_3, z).\end{aligned} \tag{5.41}$$

Standard comparative static exercise demonstrates to us how the firm will respond to changes in the Pigouvian tax:[69]

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\hat{\mu}^2[h_{1,1}h_{2,2} - h_{1,2}^2]}{\hat{\mu}|H|} < 0 \\ \frac{\partial x_1}{\partial t} &= \frac{\hat{\mu}[h_{1,2}h_{2,z} - h_{1,z}h_{2,2}]}{\hat{\mu}|H|} = 0 \\ \frac{\partial x_2}{\partial t} &= \frac{\hat{\mu}[h_{1,2}h_{1,z} - h_{2,z}h_{1,1}]}{\hat{\mu}|H|} = 0 \\ \frac{\partial c}{\partial t} &= -\frac{1}{\beta}\frac{\partial z}{\partial t} > 0.\end{aligned} \tag{5.42}$$

Thus, the optimal abatement path of the firm subject to a Pigouvian tax is one where the firm meets the associated reductions in emission solely by increasing its cleaning-up effort. It does not change its use of the fuel inputs or undertake inter-fuel substitution *etc.*. Compare this with the case in the previous section where the firm's EGT explicitly modeled all goods that affect emission generation in nature:

$$z = \alpha_1 x_1 + \alpha_2 x_2 - \beta c. \tag{5.43}$$

In that case, the firm's optimal abatement strategy to meet the mandated reductions in emission Δ (which is dual to t) depended on the sign of $F_{1,2}()$. The optimal abatement strategies often involved combinations of options such as reducing x^1, inter-fuel substitution, and increasing c. See Theorem 9.

Further, some implausible paths of changes in inputs, intended output, and emission become technically feasible when the reduced-form technology is represented by (5.40), with the underlying IPT and EGT in (5.39). Consider, *e.g.,* $\delta(t)$ with $\dot{c} = 0$, $\dot{x}_1 > 0$, $\dot{x}_2 > 0$. This results in $\frac{\partial z(t)}{\partial t} = 0$. Thus, an increase in fuel inputs with no change in cleaning-up effort has no effect on emission generation, which would seem contrary to nature.

These results stem from the fact that the single-equation input approach ignores the role that commodities x^1 and y^1 play in emission generation. It implicitly attributes changes in emissions to only the cleaning-up efforts of the firm. Thus, this model also restricts the set of abatement options that the firm has to only changes in its cleaning-up effort.

[69] Recall our maintained assumption that production of c only requires x_3, so that $h_{i,z}() = 0$ for $i = 1, 2$.

6. Conclusions.

Emissions generated by firms are by-products of their intended production. In the introductory section we presented five acceptable attributes of the process of emission generation by firms. A concept of by-production was defined. We showed that technologies of firms that satisfy this condition possess all the five attributes and that any technology that satisfies the Murty and Russell [2010] characterization–*i.e.,* is derived as an intersection of two *distinct* technologies, one capturing the production relations between inputs and outputs in intended production and other capturing the production relations underlying nature's emission generating mechanism–satisfies by-production. While intended production could be postulated to satisfy standard input and output free-disposability, these will necessarily be violated by nature's emission generation mechanism, which satisfies costly disposability of emission as defined in Murty [2010]. We showed that the Murty and Russell [2010] characterization is also necessary for by-production. This is because, when by-production holds, the lower bound on emissions and the upper bounds on intended outputs provide two production relations that describe such technologies. It was shown that, under by-production, these two relations are distinct: the upper-bounds are defined by the relations that govern intended production and the lower bounds are defined by the relations in nature regarding emission generation. These two relations can be used to recover the underlying intended production technology and nature's by-product generating technology. By-production however precludes external effects that emissions generated by a firm can impose on its own intended production. When such external effects are allowed then our examples showed that the technologies may violate some of the expected correlations between emission generation and intended production.

By allowing us to distinguish between and model details about production relations that underlie intended production and the nature's laws regarding emission generation, the by-production approach provides a very rich framework for studying firms' costs of abatement when they are mandated to reduce emissions. It is not possible to obtain such details regarding abatement costs under existing formulations of emission generating technologies, which are mostly in reduced-form. The importance of marginal abatement costs for environmental policy cannot be over-stated.

The by-production approach allows us to delineate all options that are available to firms for reducing emissions. In the simple model that we studied, such options include reductions in fuel inputs, inter-fuel substitution, cleaning-up efforts such as affluent treatment plants, or technological progress that improves productivity of inputs (especially clean inputs) or improves the effectiveness of firms' cleaning-up efforts. We distinguished between technical costs and economic costs of abatement. While, the former was expressed purely in terms of the technology and considered the loss in intended outputs due to mandated reductions in emissions, the latter was defined in the context of economic behavior of firms. It considered loss in maximum profits of firms when they are faced with environmental regulation. We showed that the properties of marginal abatement costs can be explained purely in terms of known economic laws such as "diminishing returns" in intended production.

In contrast, we showed that, under the standard input-approach to modeling emission generating technologies, a firm will reduce its emissions under a Pigouvian tax solely by increasing its cleaning-up effort. We also showed that the input approach also allows for paths involving increases in emission causing inputs and outputs with no change in emissions or cleaning-up efforts to be technologically feasible, which would seems inconsistent with nature's laws of emission generation.

Though some intuitively obvious abatement strategies implied that marginal technical abatement costs are increasing in abatement, it was also seen that, as long as there are differences in the marginal rates of technical substitutions between goods in intended production and nature's emission producing mechanism, there also always exist strategies that involve *both* greater reductions in emissions and greater production of intended outputs. This is true, for example, when the marginal rate of technical substitution between two fuel inputs in producing an intended output is different from the marginal rate of technical substitution between these inputs in producing emissions. If such abatement strategies are not adopted by firms, then it must purely be due to the fact that they are not profitable–presumably because of the input costs underlying them. When mandated to reduce emissions, profit maximizing firms internalize the nature's by-production technology and endogenously choose their optimal abatement strategy.

Marginal abatement costs of firms also vary depending on the characteristics of a firm. Our model illustrated cases where marginal economic cost of abatement decreases with increase in the initial level of emission–firms with higher initial levels of emissions have lower marginal abatement costs. This is because of their higher use of emission causing inputs or lower cleaning-up efforts. As a result, the law of diminishing returns implies that their loss in intended outputs due to increase in abatement will be lower than for firms with lower initial levels of emissions.

Our simple model for studying costs of abatement is only an illustrative example. More complex models capturing more detailed aspects of both intended production and nature's laws of emission generation can be studied by employing the by-production approach. Properties of abatement cost may differ across different technological specifications.

The firm-level analysis here can perhaps be carried over to a global level with countries as the units of analysis. Inferences can be made about the division of reductions in emissions between countries based on efficiency considerations, which usually involve differences in marginal abatement costs between countries based on their initial levels of emissions, use of fuel inputs, and cleaning-up efforts such as carbon sequestration efforts *etc.*

APPENDIX

Proof of Theorem 1: (1) We prove the result for $\underline{P}^j()$. $\bar{P}^j()$ can be similarly proved. Suppose $\langle y_j, z\rangle \in \underline{P}^j(x, y^{-j}, c)$ and $\langle \bar{y}_j, \bar{z}\rangle \in \underline{P}^j(x, y^{-j}, c)$ with $\langle y_j, z\rangle \neq \langle \bar{y}_j, \bar{z}\rangle$. Thus

$$\begin{aligned} y_j = f^j(x, y^{-j}, c, z) \ \wedge \ \bar{y}^j = f^j(x, y^{-j}, c, \bar{z}) \\ z = \underline{g}(x, y^{-j}, c, z) \ \wedge \ \bar{z} = \underline{g}(x, y^{-j}, c, z) \end{aligned} \tag{A.1}$$

$\langle y_j, z\rangle \neq \langle \bar{y}_j, \bar{z}\rangle$ implies $y_j \neq \bar{y}_j$ or $z \neq \bar{z}$. Suppose $y_j \neq \bar{y}_j$. Then the first part of (A.1) is in contradiction with part (b) of (v) in the definition of (BP). If $\bar{z} \neq z$ then the second part of (A.1) is in contradiction with part (b) of (iv) in the definition of (BP).

(2) follows from part (a) of both (iv) and (v) in the definition of (BP). ■

Proof of Theorem 2:

(1) Suppose T satisfies (BP), (OFD), (IFD), and $m_1 = 0$. Part (a) of (iv) in the definition of (BP) implies that there exist $\langle x, y, c, z\rangle \in T$, $\bar{x}^1 \geq x^1$ and $\bar{c} \leq c$ such that $\langle \bar{x}^1, \bar{c}\rangle \neq \langle x^1, c\rangle$ and $\underline{g}(x, y, c) < \underline{g}(\bar{x}^1, x^2, y, \bar{c})$. T satisfies (OFD) and (IFD) implies $\langle \bar{x}^1, x^2, y, \bar{c}, z\rangle \in T$. Let $\bar{z} = \underline{g}(\bar{x}^1, x^2, y, \bar{c})$ and $\hat{z} = \underline{g}(x, y, c)$. Then $\bar{z} > \hat{z}$ and $\langle x, y, c, \hat{z}\rangle \in T$. Therefore, $\langle \bar{x}^1, x^2, y, \bar{c}, \hat{z}\rangle \notin T$. A contradiction to T satisfies (OFD) and (IFD).

(2) Suppose T satisfies (BP) and (2) in the statement of the theorem holds. Then $z := \bar{g}(x, y, c) > \bar{g}(x, \bar{y}^1, y^2, c) =: \bar{z}$ and $\langle x, y, c, z\rangle \in T$. However, $z \notin P_T(x, \bar{y}^1, y^2, c)$ since this set is bounded above by $\bar{z}$ which is less than z. Hence, $\langle x, \bar{y}^1, y^2, c, z\rangle \notin T$. Therefore, T does not satisfy (OFD). ■

Proof of Theorem 3: (1) Suppose T satisfies (BP) and (CDB) and suppose $\underline{G}_T = F_T^j$ for some $j = m_1 + 1, \ldots, m$. Let $y_j = f^j(x, y^{-j}, c, z)$. Then $z = \underline{g}(x, y, c)$. (CDB) implies there exists $\hat{z} > z$ such that $\langle x, y^{-j}, y_j, c, \hat{z}\rangle \in T$. Part (b) of (v) in the definition of (BP) implies $y_j = f^j(x, y^{-j}, c, \hat{z})$ so that $\langle x, y, c, \hat{z}\rangle \in F_T^j$. But $\underline{g}(x, y, c) = z < \hat{z}$. Hence, $\langle x, y, c, \hat{z}\rangle \notin \underline{G}_T$. This is a contradiction to $\underline{G}_T = F_T^j$.

(2) Suppose T satisfies (BP) and (ROFD) and suppose $\underline{G}_T = F_T^j$ for some $j = m_1 + 1, \ldots, m$. Let $y_j = f^j(x, y^{-j}, c, z)$. Then $z = \underline{g}(x, y, c)$. (ROFD) implies there exists $\hat{y}_j < y_j$ such that $\langle x, y^{-j}, \hat{y}_j, c, z\rangle \in T$. Part (b) of (iv) in the definition of (BP) implies $\underline{g}(x, y^{-j}, \hat{y}^j, c) = z$ so that $\langle x, y^{-j}, \hat{y}_j, c, z\rangle \in \underline{G}_T$. But $f^j(x, y^{-j}, c, z) = y_j > \hat{y}_j$. Hence, $\langle x, y^{-j}, \hat{y}_j, c, z\rangle \notin F_T^j$. This is a contradiction to $\underline{G}_T = F_T^j$.

(3) Suppose T satisfies (BP). Pick $j, j' = m_1 + 1, \ldots, m$ such that $j \neq j'$. Then either $F_T^j = F_T^{j'}$ or $F_T^j \neq F_T^{j'}$. If $F_T^j \neq F_T^{j'}$ then there exists $\langle x, y, c, z\rangle \in T$ such that

$$y_j = f^j(x, y^{-j,j'}, y_{j'}, c, z) \tag{A.2}$$

but

$$y_{j'} \neq f^{j'}(x, y^{-j,j'}, y_j, c, z) =: \tilde{y}_{j'}. \tag{A.3}$$

(A.2) implies that

$$y_j \in P_T(x, y^{-j,j'}, y_{j'}, c, z) \tag{A.4}$$

and (A.3) implies that

$$\begin{aligned} \tilde{y}_{j'} > y_{j'} \text{ and} \\ y_j \in P_T(x, y^{-j,j'}, \tilde{y}_{j'}, c, z). \end{aligned} \tag{A.5}$$

(A.2), (A.5), and part (a) of (v) in the definition of (BP) implies that

$$y_j = f^j(x, y^{-j,j'}, y_{j'}, c, z) \geq f^j(x, y^{-j,j'}, \tilde{y}_{j'}, c, z). \tag{A.6}$$

If $y_j > f^j(x, y^{-j,j'}, \tilde{y}_{j'}, c, z)$ then a contradiction to the second part of (A.5) arises. Hence, (A.6) holds as

$$y_j = f^j(x, y^{-j,j'}, \tilde{y}_{j'}, c, z). \blacksquare \tag{A.7}$$

Proof of Theorem 4: $\mathcal{T}_1$ and $\mathcal{T}_2$ are closed implies that T is closed so (i) in the definition of (BP) is satisfied for T.

For all $j = 1, \ldots, m$ and for all $\langle x, y^{-j}, c, z\rangle \in \Xi^j_T$

$$\begin{aligned} P_T(x, y^{-j}, c, z) :=& \{y_j \in \mathbf{R}_+ |\ \langle x, y^{-j}, y_j, c, z\rangle \in T\} \\ =& \{y_j \in \mathbf{R}_+ |\ y_j \in P_{\mathcal{T}_1}(x, y^{-j}, c, z) \cap P_{\mathcal{T}_2}(x, y^{-j}, c, z)\} \\ \subseteq& \{y_j \in \mathbf{R}_+ |\ y_j \in P_{\mathcal{T}_1}(x, y^{-j}, c, z)\}, \end{aligned} \tag{A.8}$$

which is bounded following (ii) in the definition of an IPT. So (ii) in the definition of (BP) holds for T.

Assumption (a) of the theorem and (3.14) imply

$$\begin{aligned} \underline{g}(x, y, c) :=& \min\{z \in \mathbf{R}_+ |\ z \in P_T(x, y, c)\} \\ =& \min\{z \in \mathbf{R}_+ |\ z \in \mathbf{R}_+ \cap P_{\mathcal{T}_2}(x, y, c)\} \\ =& \underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c). \end{aligned} \tag{A.9}$$

Part (a) of (iv) in the definition of a BPT implies that there exists $\langle x, y, c\rangle \in \Omega_T$ such that $\underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c) > 0$. (A.9) hence imply that (iii) in the definition of (BP) holds for T.

Let $\langle x, y, c\rangle,\ \langle \bar{x}, \bar{y}, \bar{c}\rangle \in \Omega_T$ such that $x^1 \geq \bar{x}^1,\ y^1 \geq \bar{y}^1,\ c \leq \bar{c}$ and $\langle x^1, y^1, c\rangle \neq \langle \bar{x}^1, \bar{y}^1, \bar{c}\rangle$. Then part (a) of (iv) in the definition of a BPT and (A.9) imply

$$\underline{g}(x, y, c) = \underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c) \geq \underline{\mathcal{G}}_{\mathcal{T}_2}(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{c}) = \underline{g}(\bar{x}, \bar{y}, \bar{c}) \tag{A.10}$$

Hence, part (a) of (iv) in the definition of (BP) holds for T.

Let $\langle x, y, c\rangle,\ \langle \bar{x}, \bar{y}, \bar{c}\rangle \in \Omega_T$ such that $\underline{g}(x, y, c) > \underline{g}(\bar{x}, \bar{y}, \bar{c})$. Then (A.9)implies that

$$\underline{\mathcal{G}}_{\mathcal{T}_2}(x, y, c) > \underline{\mathcal{G}}_{\mathcal{T}_2}(\bar{x}, \bar{y}, \bar{c}). \tag{A.11}$$

(A.11) and part (b) of (iv) in the definition of a BPT imply that $\langle x^1, y^1, c\rangle \neq \langle \bar{x}^1, \bar{y}^1, \bar{c}\rangle$ and it is not the case that $x^1 \leq \bar{x}^1$, $y^1 \leq \bar{y}^1$, and $c \geq \bar{c}$. This proves that part (b) of (iv) in the definition of (BP) holds for T and similarly, we can prove that part (c) of (iv) in the definition of a BPT implies part (c) of (iv) in the definition of (BP).

For $j = m_1 + 1, \ldots, m$, it follows from (3.14) and Assumption (b) of the theorem that

$$\begin{aligned} f^j(x, y^{-j}, c, z) &:= \max\{y_j \in \mathbf{R}_+ |\ y_j \in P(x, y^{-j}, c, z)\} \\ &= \max\{y_j \in \mathbf{R}_+ |\ y_j \in P_{\mathcal{T}_1}(x, y^{-j}, c, z) \cap \mathbf{R}_+\} \\ &= \mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}, c, z). \end{aligned} \tag{A.12}$$

Let $\langle x, y^{-j}, c, z\rangle, \langle \bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z}\rangle \in \Xi^j_T$ with $x \geq \bar{x},\ y^{-j} \leq \bar{y}^{-j}$, and $c \leq \bar{c}$. Then (A.12) and part (a) of Remark 1 imply that

$$f^j(x, y^{-j}, c, z) = \mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}, c, z) \geq \mathcal{F}^j_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}, \bar{c}, z). \tag{A.13}$$

Assumption (a) of the theorem implies that

$$P_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}, \bar{c}, z) = P_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z}) \tag{A.14}$$

(A.12), (A.13), and (A.14) imply that

$$f^j(x, y^{-j}, c, z) = \mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}, c, z) \geq \mathcal{F}^j_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}, \bar{c}, z) = \mathcal{F}^j_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z}) = f^j(\bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z}). \tag{A.15}$$

This shows that part (a) of (v) in the definition of (BP) holds for T.
For $j = m_1+1, \ldots, m$, let $\langle x, y^{-j}, c, z\rangle, \langle \bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z}\rangle \in \Xi^j_T$ and $f^j(x, y^{-j}, c, z) > f^j(\bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z})$. Then (A.12) and (A.14) imply that

$$\begin{aligned} \mathcal{F}^j_{\mathcal{T}_1}(x, y^{-j}, c, z) &> \mathcal{F}^j_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}, \bar{c}, \bar{z}) \\ &= \mathcal{F}^j_{\mathcal{T}_1}(\bar{x}, \bar{y}^{-j}, \bar{c}, z). \end{aligned} \tag{A.16}$$

(A.16) and part (b) of Remark 1 imply that $\langle x, y^{-j}, c\rangle \neq \langle \bar{x}, \bar{y}^{-j}, \bar{c}\rangle$ and it is not the case that $x \leq \bar{x},\ y^{-j} \geq \bar{y}^{-j}$, and $c \geq \bar{c}$. This shows that part (b) of (v) in the definition of (BP) holds for T.
T satisfies (CDB) follows from Assumption (a) of the theorem and the facts that $\mathcal{T}_2$ satisfies (iii) in the definition of a BPT, $\underline{g}() = \underline{\mathcal{G}}_{\mathcal{T}_2}()$, and $\bar{g}() = \bar{\mathcal{G}}_{\mathcal{T}_2}()$. T satisfies (RIFD) and (ROFD) follow from Assumption (b) of the theorem and the fact that $\mathcal{T}_1$ satisfies (iii) in the definition of a IPT. ■

Proof of Theorem 5: (1) follows in an obvious manner and (2) follows once, given any $j = m_1 + 1, \ldots, m$, we define the sets

$$\begin{aligned} \mathcal{T}_1 &:= \big\{\langle x, y, 0, c, 0, 0, z\rangle \in \mathbf{R}^{n+2(m+2)}_+ |\ \exists\ \langle \bar{x}, \bar{y}^{-j}, \bar{c}\rangle \in \Theta^j_T \text{ such that } x \geq \bar{x}\ \wedge\ y^{-j} \leq \bar{y}^{-j} \\ &\qquad\qquad \wedge\ c \leq \bar{c}\ \wedge\ y_j \leq \rho^j(\bar{x}, \bar{y}^{-j}, \bar{c})\big\} \text{ and} \\ \mathcal{T}_2 &:= \big\{\langle x, 0, y, 0, c, z, 0\rangle \in \mathbf{R}^{n+2(m+2)}_+ |\ \langle x, y^{-j}, c\rangle \in \Theta^j_T\ \wedge\ \bar{\sigma}^j(x, y^{-j}, c) \geq z \geq \underline{\sigma}^j(x, y^{-j}, c)\big\}. \end{aligned} \tag{A.17}$$

$\mathcal{T}_1 \subset \mathbf{R}^{n+2(m+2)}_+$ and $\mathcal{T}_2 \subset \mathbf{R}^{n+2(m+2)}_+$ satisfy the definitions of an IPT and a BPT, respectively and $T = T(\mathcal{T}_1, \mathcal{T}_2)$. ■

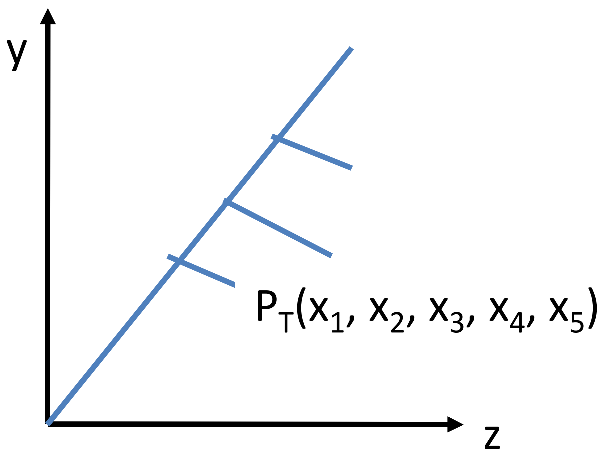

Figure 1(a)

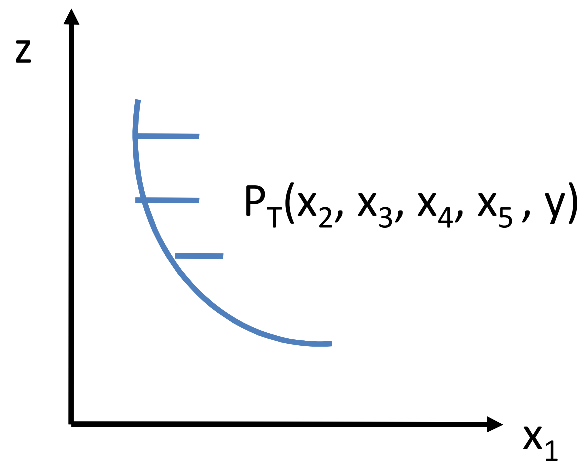

Figure 1(b)

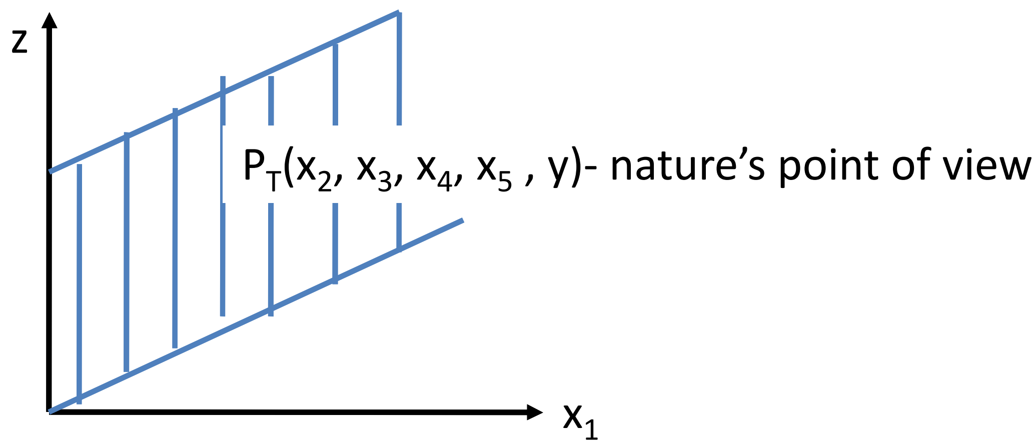

Figure 1(c)

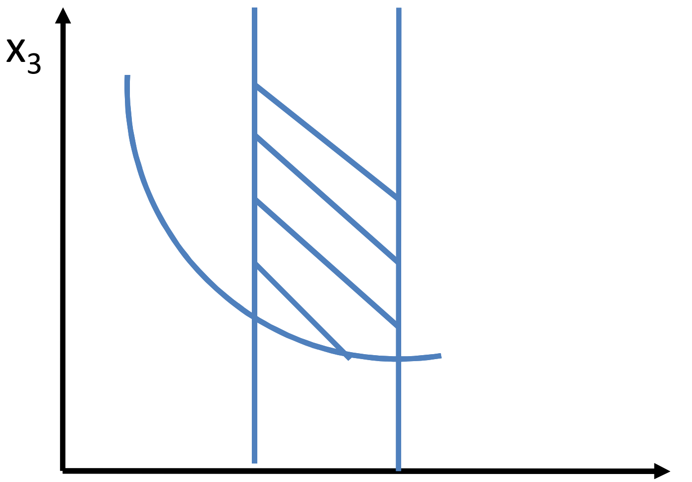

Figure 2(a)

The shaded region is $P_T(x_2, x_4, x_5, y, c, z)$

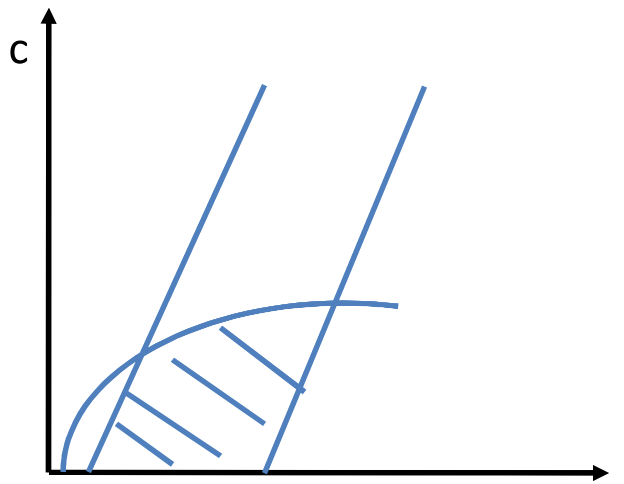

Figure 2(b)

The shaded region is $P_T(x_2, x_3, x_4, x_5, y, z)$

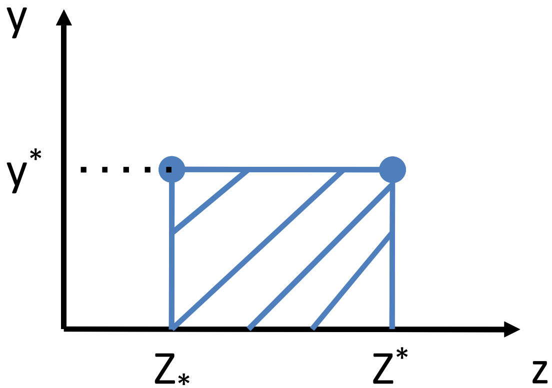

Figure 2(c)

The shaded region is $P_T(x_1, x_2, x_3, x_4, x_5, c)$

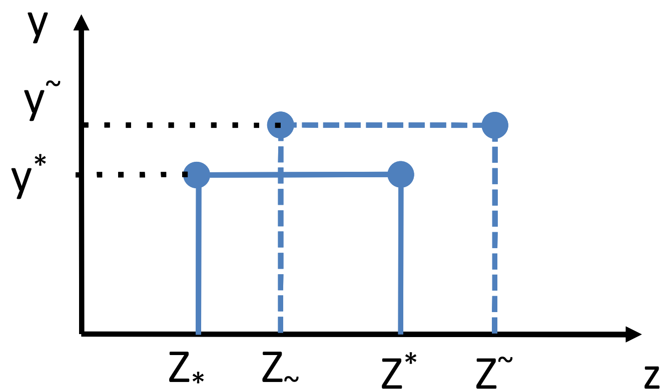

Figure 2(d)

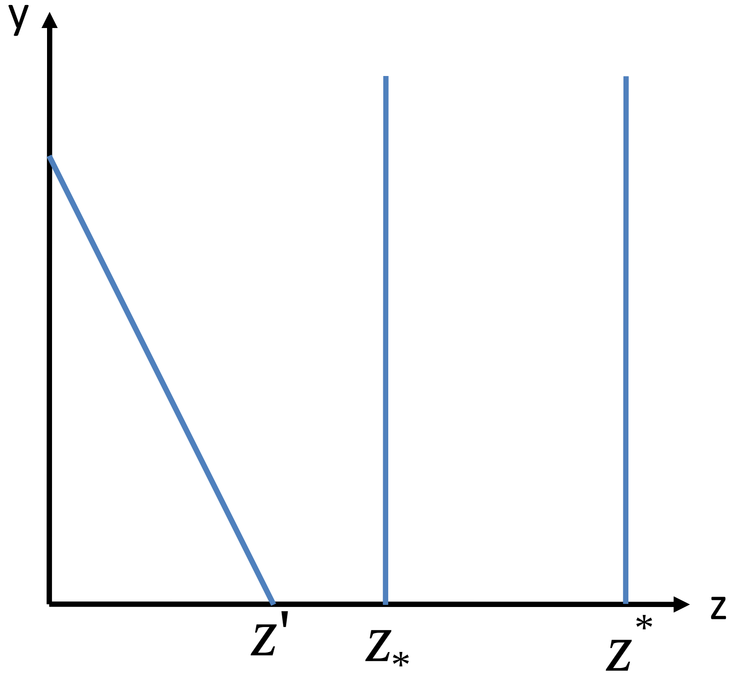

Figure 3(a)

$$P_T(x_1, x_2) = \varnothing$$

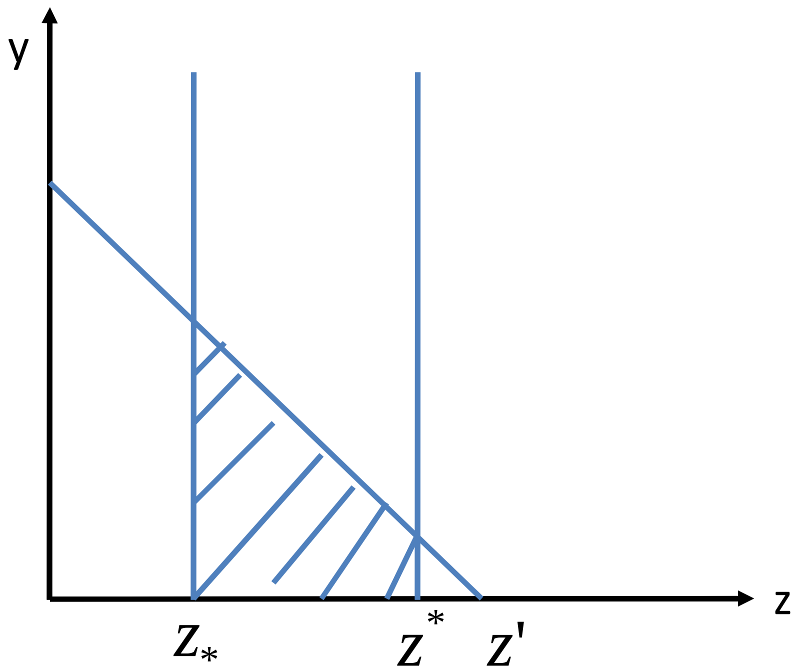

Figure 3(b)

$$P_T(x_1, x_2) \neq \varnothing$$

$z_* = \alpha x_1$, $z^* = \gamma + \alpha x_1$, and $z' = x_1^{0.5} x_2^{0.5}$

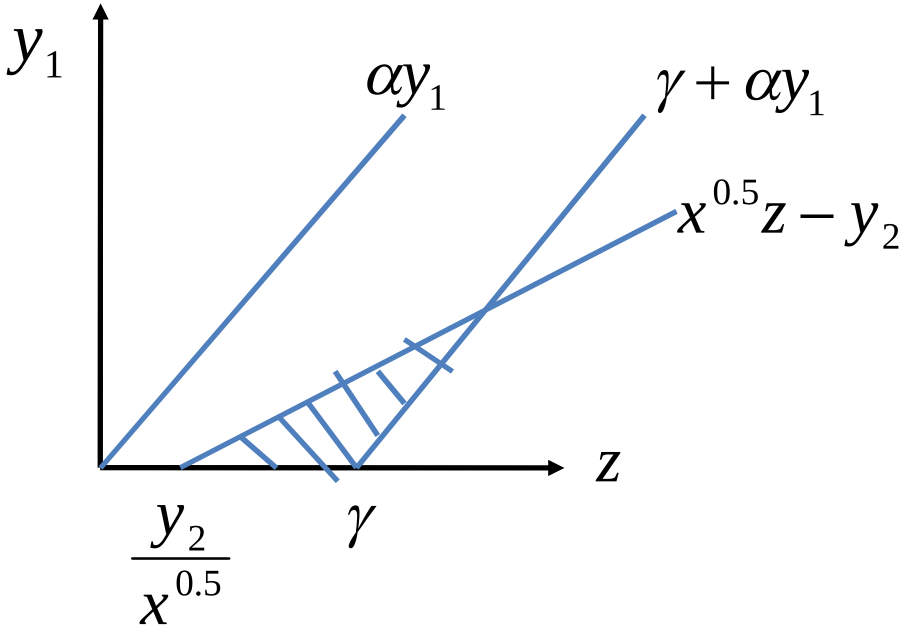

Figure 4(a)

The shaded area is $P_T(x, y_2)$

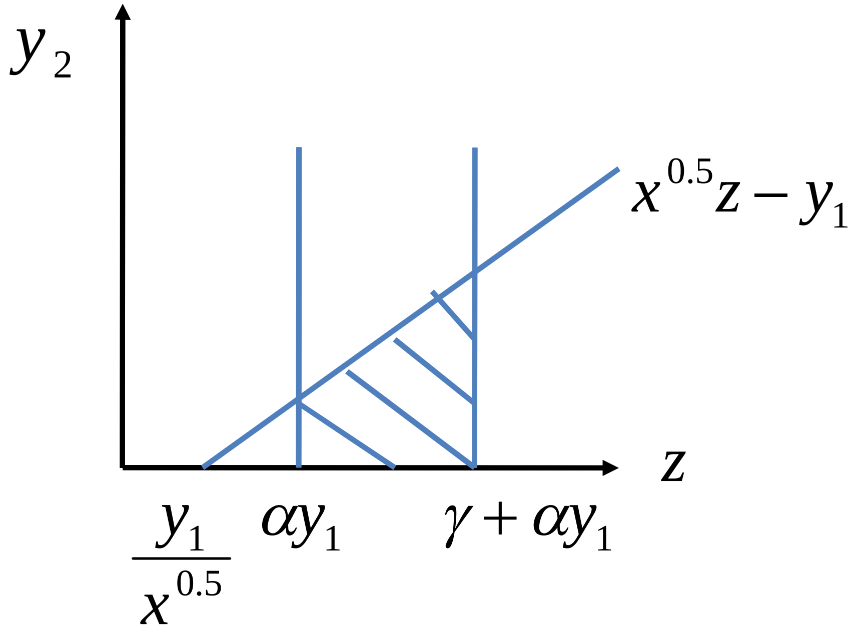

Figure 4(b)

The shaded area is $P_T(x, y_1)$

REFERENCES

Baumol, W. J., and W. E. Oates [1988], *The Theory of Environmental Policy*, Second Edition, Cambridge: Cambridge University Press.

Cropper, M. L., and W. E. Oates [1992], "Environmental Economics: A Survey," *Journal of Economic Literature* 30: 675–740.

Färe, R., S. Grosskopf, C. A. K. Lovell, and C. Pasurka [1989], "Multilateral Productivity Comparisons When Some Outputs are Undesirable: A Nonparametric Approach," *The Review of Economics and Statistics* 71: 90–98.

Färe, R., S. Grosskopf, C. A. K. Lovell, and S. Yaisawarng [1993], "Derivation of Shadow Prices for Undesirable Outputs: A Distance Function Approach," *The Review of Economics and Statistics* 75: 374–380.

Färe, R., S. Grosskopf, D. W. Noh, and W. Weber [2005], "Characteristics of a Polluting Technology: Theory and Practice," *Journal of Econometrics* 126: 469–492.

R. Färe, S. Grosskopf, and C. Pasurka [1986], "Effects of Relative Efficiency in Electric Power Generation Due to Environmental Controls," *Resources and Energy* 8: 167–184.

Førsund, F. [2009], "Good Modelling of Bad Outputs: Pollution and Multiple-Output Production," *International Review of Environmental and Resource Economics* 3: 1–38.

Frisch, R. [1965], *Theory of Production*, Dordrecht: D. Reidel Publishing Company.

Milleron, J. C. [1972], "Theory of Value with Public Goods," *Journal of Economic Theory* 5: 419–477.

Murty, M. N., and S. Kumar [2002], "Measuring Cost of Environmentally Sustainable Industrial Development in India: A Distance Function Approach," *Environment and Development Economics* 7: 467–86.

Murty, M. N., and S. Kumar [2003], "Win-Win Opportunities and Environmental Regulation: Testing of Porter Hypothesis for Indian Manufacturing Industries," *Journal of Environmental Management* 67: 139–44.

Murty, S. [2010], "Externalities and Fundamental Nonconvexities: A Reconciliation of Approaches to General Equilibrium Externality Modeling and Implications for Decentralization," *Journal of Economic Theory* 145: 331-53.

Murty, S., and R. R. Russell [2010], "On Modeling Pollution-Generating Technologies," The Warwick Economics Research Papers Series (TWERPS), No. 931.

Shephard, R. W. [1953], *Cost and Production Functions*, Princeton: Princeton University Press.