Király, Balázs; Simonovits, András

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BALÁZS KIRÁLY – ANDRÁS SIMONOVITS

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Authors:

Balázs Király
Ph.D. student
Institute of Physics of Budapest University of Technology, Budapest
E-mail: kiraly.balazs.graf@gmail.com

András Simonovits
research advisor
Institute of Economics
Centre for Economic and Regional Studies, Hungarian Academy of Sciences
also Mathematical Institute of Budapest University of Technology, Budapest
E-mail: simonovits.andras@krtk.mta.hu

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Abstract

Mandatory pension systems only partially replace old-age income, therefore the government also operates a voluntary pension system, where savings are matched by government grants. Accounting for the resulting tax expenditure, our models describe the income flow from shortsighted to farsighted workers. 1. In rational models, explicit results are obtained, showing the limited learning of shortsighted workers. 2. In agent-based models, this learning is improved and this raises the shortsighted workers' saving and reduces perverse income redistribution.

Keywords: life-cycle savings, overlapping generations, mandatory pensions, voluntary pensions, agent-based models

JEL classification: H55, D91

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Megtakarítás és adózás egy önkéntes nyugdíj-rendszerben: az ágensalapú modellezés felé

Király Balázs – Simonovits András

Összefoglaló

A kötelező nyugdíjrendszer gyakran csak részben helyettesíti a hiányzó időskori jövedelmet, ezért a kormányzat kiegészítésként önkéntes nyugdíjrendszert működtet, amelyben a magánmegtakarításhoz kormányzati támogatás adódik hozzá. Figyelembe véve a keletkező adóköltséget, modelljeink leírják a rövidlátóktól az előrelátókhoz áramló jövedelmeket.
1. Racionális modellekben explicite meghatározható, hogyan tanulnak meg legalább korlátozottan takarékoskodni a rövidlátók.
2. Ágensalapú modellekben a fiatalabb dolgozók idősebb társaiktól tanulva nagyobb megtakarítást érhetnek el.

Tárgyszavak: életciklus megtakarítások, együttélő nemzedékek, kötelező nyugdíjak, önkéntes nyugdíjak, ágensalapú modellek

JEL kódok: H55, D91

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Saving and taxation in a voluntary pension system: Toward an agent-based model

Balázs Király–András Simonovits

February 14, 2016

1) Institute of Physics of Budapest University of Technology, Budapest, Budafoki út 8, Hungary, 1111 e-mail: kiraly.balazs.graf@gmail.com
and
2) corresponding author: Institute of Economics, CERS, Hungarian Academy of Sciences, Budaörsi út 45, Hungary, 1112, also Mathematical Institute of Budapest University of Technology, Budapest,
e-mail: simonovits.andras@krtk.mta.hu,

Abstract

Mandatory pension systems only partially replace old-age income, therefore the government also operates a voluntary pension system, where savings are matched by government grants. Accounting for the resulting tax expenditure, our models describe the income flow from shortsighted to farsighted workers. 1. In analytical models, explicit results are obtained, showing the limited learning of shortsighted workers. 2. In agent-based models, this learning is improved and this raises the shortsighted workers' saving and reduces perverse income redistribution.

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1 Introduction

All over the developed world, governments operate *mandatory* pension systems to replace income and minimize old-age poverty. In general, the mandatory system is complemented by a *voluntary* pension system. As a rule, participants of a voluntary system can only withdraw their voluntary savings after retirement and as a compensation, their savings enjoy tax advantages.

The proponents of voluntary systems justify the subsidies as follows: a mandatory system does not and cannot ensure high enough pensions, and the mostly shortsighted workers must be made interested in raising their old-age incomes through a voluntary system (e.g. Poterba, Venti and Wise, 1996). The opponents are afraid that these subsidies are poorly targeted, mostly subsidize the well-paid savers, while worsening the burden of the others by generating the *tax expenditures* (Engen, Gale and Scholz, 1996; Duflo, Gale, Liebman, Orszag and Saez, 2007). (In a more general context, Campbell (2006) noted that "confusing financial products generate cross-subsidy from naive to sophisticated households.") Hubbard and Skinner (1996) tried to synthesize both approaches, while OECD (2005) summarized the practice of various OECD countries. Up to now those tax expenditures have generally been quite low though nonnegligible (about 0.7% in the US), but in a possible contraction of the mandatory system they may become much higher.

Since Modigliani and Brumberg (1954) and Samuelson (1958), the models of life-cycle saving and of overlapping generations have been extensively studied, respectively. A new era started with Auerbach and Kotlikoff (1987) which generalized the partial equilibrium framework into a general equilibrium one: not only savings depend on the interest rates but the interest rates also depend on savings through accumulated capital. Adding mandatory and voluntary pensions, these models have became more realistic. For example, Fehr, Habermann and Kindermann (2008) showed that reforming the voluntary pillar, existing generations lose, future generations gain. In addition, the assumption of rational expectations makes the foregoing models extremely complex (for an alternative, see Molnár and Simonovits, 1998).

A common problem of these models, however, is that they presuppose that the individuals have an extraordinary sophistication to solve them and the willpower to achieve the results. It is widely documented, however, that a large share of the population have quite limited cognitive abilities (for a survey, see Lusardi and Mitchell (2014)), quite limited information (Barr and Diamond, 2008, Box 4.2) and have weak willpower.

A class of very simple life-cycle models operate with given interest rates and wages (small open economy). In such models, various workers' ordinary life-cycle saving processes are independent, but adding government matching via a voluntary pension system introduces interdependence. Indeed, even if somebody does not participate in the scheme, he pays according to the same earmarked tax rate. This may be the reason that in voluntary systems, individual optimization is very difficult (see Appendix B) even if only young and old workers (and pensioners) are distinguished. Only by neglecting even this dimension was Simonovits (2011) able to analyze the impact of the matching rate and the cap on income redistribution in such a transfer system.

A more realistic approach to life-cycle savings is based on *behavioral economics* (started with Laibson (1997) and crowned by a recent survey by Chetty, 2015). An interesting alternative was initiated by Findley and Caliendo (2008) by assuming short planning horizons.

It is *agent-based models* (for short, ABM), which may enhance the realism of economic modeling (see Gigerenzen and Selten, 2002 and Tesfatsion, 2006). The main innovation of ABMs is that by resigning from analytical results, they are able to describe more realistic behavior of heterogeneous agents. This methodology has been used at several fields of economics. For the topic of tax evasion, related to our problem, see Pickhard and Prinz (2013) and Méder, Simonovits and Vincze (2012).

Quite recently, Varga and Vincze (2015) used an ABM to analyze a very abstract model of ordinary saving. They assumed a very long (practically infinite) horizon and excluded mandatory as well as voluntary pensions. They distinguished two types of agents: ants (who follow the prescriptions of the life-cycle model, smooth the consumption path by saving) and crickets (who spend most of their disposable income on current consumption). The main message of that paper is that notwithstanding permanent learning, different types can coexist for a very long time.

The present paper applies the ABM approach to *life-cycle savings*, especially to voluntary pension. Already Duflo and Saez (2003) emphasized the influence of colleagues' choices on participation in voluntary pension plans. We try to explain an empirically verified fact: though the share and the extent of participation in the tax favored systems are increasing functions of the wages; even controlling for wages, both indicators are heterogeneous. We take homogeneous wages, neglect the cap on the voluntary contributions, thereby eliminate unmatched savings. We consider the following dimensions: (a) We introduce age dependency into the model: the agents are young and savers in their first $R \geq 1$ periods, then old and dissavers in their remaining $D - R > 0$ periods. (b) Having a medium-size old-age mandatory pension, to smooth their consumption paths, workers have to save only a reduced amount, while credit constraints prevent them from accumulating any debt. (c) Adding tax-favored voluntary savings the weight of the mandatory pensions can be diminished but at the cost of raising the taxes financing the matching. For simplicity, we assume a stationary population without growth, inflation and interest. To make the impact of the individual decisions on the macro state negligible, we assume that there are a continuum of workers. We also distinguish two types: far- and shortsighted workers (Feldstein, 1985). The inadequacy of the shortsighted workers' savings is measured by the variance in their life-cycle consumption, while the redistribution from the shortsighted to the farsighted workers is measured by the variance between their lifetime consumptions. For the sake of brevity, the two variances are distinguished by adjectives *internal* and *external*.

Our results are as follows: (a) By saving in a tax-favored system, the farsighted workers simply exploit the shortsighted ones. To simplify the calculations, we assume that the farsighted workers try to smooth their consumption paths without any intertemporal substitution. (b) Turning to dynamics, we assume a special form of *global* learning: the shortsighted workers guess the amount of farsighted counterparts' saving as the ratio of the tax rate and the matching rate and they save a given share of this estimation: *relative propensity to save*. Note that if the share of the farsighted workers is very low, then they only play the role of the catalyzer but without them the system ceases working. The process converges to a steady state and the degree of exploitation is significantly reduced. (c) We disaggregate shortsighted workers according to their different relative saving propensities and assume that these subtypes also learn *locally* from each other.

We mention the following ABM-results: (i) in the basic run, some heterogeneity in savings of the short-sighted workers remains; (ii) increasing the spread between the propensities diminishes both variances but with serious oscillations; (iii) the increase of the number of types diminishes the external variance but increases the internal variance; (iv) randomly perturbing the network may homogenize the shortsighted workers' savings; (v) the rise in the number of acquaintances does not reduce the variances; (vi) reducing the density of the connections by factor 4, the convergence is much slower; and (vii) speeding up the learning process does not influence the external variance but increases the internal variance. Further work is needed to check the robustness of these results.

We add Table 0 which shows how our model relates to five selected countries' pension systems according to the strength of progressivity and the size of the mandatory (public+private) and voluntary systems. (Note that it is not easy to define the size of a pension system, because in addition to the relative size of the cap; the contribution rate and the matching rate also influence the size of the mandatory and the voluntary systems, respectively.) We see that our model goes even beyond the German and the Hungarian systems in eliminating any redistribution in the mandatory system, and it copies the US system's medium size. Concerning the voluntary system, our model is similar to the US, the Dutch and the Hungarian systems having no progressivity. Not shown in the table, but our model copies the German voluntary system of having mandatory life annuities, and approximates Hungary of having very high cap on the voluntary system. In summary, we have copied various features of various countries arbitrarily, just to make the model as simple as possible but reflecting the learning dimension.

Table 0. *Features of pension systems of selected countries*

	Mandatory		**Voluntary**	
Country	Progressive	Size	Progressive	Size
Uinted States	medium	medium	no	medium
Germany	weak	large	medium	small
Netherlands	strong	large	no	medium
Czech Republic	strong	large	strong	small
Hungary	weak	large	no	large
Model	no	medium	no	large

The structure of the remainder of the present paper is as follows: Section 2 discusses an analytical model of life-cycle saving, where the shortsighted workers do not care at all: they are passive. Section 3 makes them active. Section 4 studies the corresponding ABM. Section 5 concludes. Appendix A gives analytical results on the stability of the steady state with active shortsighted workers in the case $\mathbf{R} = 2$ and $\mathbf{D} = 3$ (fat letters refer to decades rather than years). Appendix B sketches the rational decision version allowing for intertemporal substitution and discounting.

2 Passive shortsighted workers

We shall consider a simple model of mandatory and voluntary pensions. To simplify exposition, we consider a stationary population, with overlapping cohorts. At the end of every time-period, D cohorts live together and every cohort becomes older by one period except for the old, which dies and the youngest, which just enters the labor market. There are $R > 0$ working cohorts and $D - R > 0$ retired cohorts, where R and D are positive integers. The workers earn unitary wages, pay $\tau > 0$ as a mandatory pension contribution. The retired cohorts receive universal pension benefits b. Introducing notations for the ratios of working span to total adult life span and that of working span to retirement span,

$$\rho = \frac{R}{D} < 1 \qquad \text{and} \qquad \beta = \frac{\rho}{1-\rho},$$

the benefit is

$$b = \frac{R\tau}{D-R} = \frac{\rho\tau}{1-\rho} = \beta\tau. \tag{1}$$

It is easy to see that the net wage and the pension benefit are equal if

$$\tau = \bar{\tau} = 1 - \rho.$$

Due to incentive constraints, the government keeps the contribution rate well below this *critical* value: $\tau < \bar{\tau}$ and encourages private savings in a voluntary pension system: the annual saving is denoted by $s \geq 0$. To promote participation, every euro paid into the voluntary system is matched by $\alpha > 0$ euros. In contrast to the bulk of the literature, we explicitly model the earmarked tax needed to finance such matching from wage taxes with a flat rate θ. There is no other tax in our models. A basic observation is that different types of workers—even with identical earnings—save different amounts in voluntary systems.

Until the end of this Section, we shall assume that there are only two types: shortsighted (L) and farsighted (H), with shares $f_L, f_H > 0$ and $f_L + f_H = 1$. In this Section, the shortsighted worker is passive, does not save at all: $s^L = 0$ and the farsighted worker saves s^H. Denoting worker i's and pensioner i's consumption by $c_{1,i}$ and $c_{D,i}$, respectively, we have the following tax equation:

$$\theta = \alpha f_H s_H \tag{2}$$

and consumption equations:

$$c_{1,L} = 1 - \tau - \theta, \qquad c_{D,L} = b \tag{3 − L}$$

and

$$c_{1,H} = 1 - \tau - \theta - s^H, \qquad c_{D,H} = \beta[\tau + (1+\alpha)s^H]. \tag{3 − H}$$

We assume that H saves as much as needed to smooth out her projected consumption path. Introducing $\chi = 1 - (1+\beta)\tau > 0$, we have then

$$1 - \tau - \theta - s^H = \beta[\tau + (1+\alpha)s^H], \qquad \text{i.e.} \qquad s^H = \frac{\chi - \theta}{1 + \beta(1+\alpha)}. \tag{4}$$

We display the special value of s^H at $\alpha = 0$:

$$s^H(0) = \frac{\chi}{1+\beta}. \tag{4'}$$

Substituting (4) into (2) yields an implicit equation for the balanced tax rate:

$$\theta = \alpha f_H \frac{\chi - \theta}{1 + \beta(1+\alpha)}.$$

Hence follows

Theorem 1. *In the two-type model with passive shortsighted workers, the government sets the balanced tax rate*

$$\theta^{\text{o}} = \frac{\alpha f_H \chi}{1 + \beta(1+\alpha) + \alpha f_H} > 0. \tag{5}$$

Then every farsighted worker chooses her saving $s^H = \theta^{\text{o}}/(\alpha f_H)$ *and every shortsighted worker saves nothing.*

Remarks. 1. In this model, the introduction of voluntary saving simply redistributes from the shortsighted to the farsighted workers. The higher the matching rate, the stronger the redistribution is.

2. In such a zero-sum game, the use of voluntary pensions is only justifiable if there is wage heterogeneity (w_i) and the mandatory (public) pension is progressive: $b_i = B_0 + Bw_i$, with $B_0 > 0$ and $B > 0$, but this is beyond the scope of this paper.

3. Even these formulas are quite cumbersome, therefore the impact of various parameters are far from clear. For example, what happens if the matching rate (unrealistically) goes to infinity? Dividing the numerator and the denominator of (5) by α, $\alpha \to \infty$ then implies that $\theta^{\mathrm{o}}_{\infty} = f_H\chi/(\beta + f_H)$. But $s^H_{\infty} = 0$!

To help understanding, we shall numerically illustrate our results. Let us calculate in decades: $\mathbf{R} = 4$, $\mathbf{D} = 6$, $\rho = 2/3$ and choose a contribution rate $\tau = 0.2$ far below the maximum: $\bar{\tau} = 1/3$. Table 1 displays the two-type characteristics for three matching rates: $\alpha = 0, 0.5, 1$; for population shares $f_L = 3/4$ and $f_H = 1/4$. To display perverse redistribution, we also show the shortsighted workers' *average lifetime* consumption $c^L = \rho c^L_R + (1 - \rho)c^L_{R+1}$. As the matching rate increases, so decreases the average shortsighted consumption: at $\alpha = 1$, the earmarked tax is equal to 0.019 and the average consumption of the shortsighted type drops from 0.667 to 0.654. Using the obvious formula for the expected average consumption: $c = f_L c^L + f_H c^H = \rho$, the simplest measure of perverse redistribution is the (squared) *external variance* of the average lifetime consumptions:

$$\varepsilon_E^2 = f_L(c^L - \rho)^2 + f_H(c^H - \rho)^2.$$

In addition, to measure the *internal variance*, we also introduce

$$\varepsilon_I^2 = f_L\rho(c_1^L - c^L)^2 + f_L(1 - \rho)(c_D^L - c^L)^2.$$

The former grows from zero to 0.022, while the latter diminishes from 0.163 to 0.156. To relate these values to the absolute maximum where everybody is shortsighted and there is no mandatory pension, we give the the corresponding maximum: $\bar{\varepsilon}_I = 0.47$.

Table 1. *Paths for mandatory and voluntary systems: passive L*

		Consumption of				Variance	
Matching rate α	Tax rate θ	H-type c^H	L-worker c^L_R	L-pensioner c^L_{R+1}	L-average c^L	internal ε_I	external ε_E
0.0	0	0.667	0.800	0.4	0.667	0.163	0
0.5	0.012	0.691	0.788	0.4	0.659	0.159	0.014
1.0	0.019	0.705	0.781	0.4	0.654	0.156	0.022

3 Active shortsighted workers

In Section 2, we assumed that shortsighted workers are passive, they do not understand anything from the logic of the system. Here we assume that these workers are active, they understand something and react to exploitation. First we rely on steady state analysis, then we turn to the dynamics.

3.1 Steady state

Type L presumes that all the others are type H and knowing the tax rate θ and the matching rate α, relying on (2), he naively presumes their saving is equal to θ/α. Due to his myopia, he is ready to save only γ times this quantity, $(0 < \gamma \le 1)$, therefore

$$s^L = \frac{\gamma\theta}{\alpha}.$$

We shall refer to γ as *relative propensity to save.* Retaining (4), the modified tax balance equation (2) becomes

$$\theta = \gamma f_L \theta + \alpha f_H \frac{\chi - \theta}{1 + \beta(1 + \alpha)}. \tag{6}$$

With a simple calculation, we have obtained

Theorem 2. *The steady state with active shortsighted workers is*

$$\theta^{\mathrm{o}} = \frac{\alpha f_H \chi}{\nu}, \qquad s^H = \frac{(1-\gamma f_L)\chi}{\nu} \qquad \textit{and} \qquad s^L = \frac{\gamma f_H \chi}{\nu}, \tag{7}$$

where

$$\nu = (1-\gamma f_L)[1+\beta(1+\alpha)] + \alpha f_H > 0.$$

Remarks. 1. Looking at the steady state balanced tax rate (7) (active), note that the higher the relative propensity to save γ, the higher is the balanced tax rate, and the lower is the redistribution. For $\gamma = 1$, the shortsighted becomes farsighted and exploitation disappears.

2. Disaggregating the shortsighted workers into $n-1 > 1$ types with different γ_is, we can open the door to multitype models (to be studied in Section 4). Indeed, let f_i be the population share of the shortsighted workers with relative saving propensities $\gamma_i < 1$, $i = 1, 2, \ldots, n-1$. Then the detailed model can be aggregated as

$$f_L = \sum_{i=1}^{n-1} f_i < 1 \qquad \text{and} \qquad \gamma = \frac{\sum_{i=1}^{n-1} f_i \gamma_i}{f_L} \leq 1.$$

Then s^L in Theorem 2 can also be disaggregated:

$$s^i = \frac{\gamma_i f_H \chi}{\nu}, \qquad i = 1, \ldots, n-1. \tag{7M}$$

Table 2 displays the impact of the relative propensity to save γ with $\alpha = 1$. The first row replicates the third row of Table 1. As γ increases from 0 to 1, the earmarked tax rate rises from 0.019 to 0.067, and even the shortsighted type's consumption paths becomes smooth, i.e. age-invariant. Eventually both the internal and the external variances drop to zero.

Table 2. *Paths for mandatory and voluntary systems: active L, $\alpha = 1$*

		Consumption of				Variance	
Relative propensity to save	Tax rate	H-type	L-worker	L-pensioner	L-average	internal	external
γ	θ	c^H	c^L_R	c^L_{R+1}	c^L	ε_I	ε_E
0.00	0.019	0.705	0.781	0.400	0.654	0.156	0.022
0.25	0.023	0.701	0.771	0.423	0.655	0.142	0.020
0.50	0.030	0.696	0.756	0.459	0.657	0.121	0.017
0.75	0.041	0.687	0.728	0.523	0.660	0.084	0.012
1.00	0.067	0.667	0.667	0.667	0.667	0	0

3.2 Dynamic model

In a standard overlapping generations model, the agents differ not only by age but also by the time they start working. We shall denote age for workers by $a = 1, 2, \ldots, R$ and for pensioners by $a = R + 1, \ldots, D$. Every period t, D adult cohorts overlap: those entering the labor market in period $t, t - 1, \ldots, t - D + 1$, respectively. For technical reasons we assume that the new cohort entered the labor market in period $t+1$ rather than t. Subindex triple $(a, i, t+a)$ refers to type i of age a in time $t+a$. To have a recursive model, we assume that in every period, the government determines and announces the appropriate tax rate and then the various types calculate the corresponding age-dependent savings.

For each pair (a, i), the accumulated savings satisfy a dynamic relation:

$$S_{a,i,t+a} = S_{a-1,i,t+a-1} + s_{a,i,t+a}, \qquad t = 0, 1, 2, \ldots, \tag{8}$$

where the initial conditions are given:

$$S_{a-1,i,-1}, \qquad a = 2, \ldots, R \qquad \text{and} \qquad i = 1, 2.$$

For informational reasons we relax the previous tax equation (6) and allow the government to run temporary surpluses and deficits. Let E_t be the tax expenditure in period t and $\mathcal{D}_t$ be the stock of government debt at the end of period t. We have the following two identities:

$$E_t = \alpha \sum_{i=L}^{H} f_i \sum_{a=1}^{R} s_{a,i,t} \tag{9}$$

and

$$\mathcal{D}_t = \mathcal{D}_{t-1} + E_t - R\theta_t, \qquad t = 0, 1, 2, \ldots, \qquad \mathcal{D}_{-1} = 0. \tag{10}$$

Using a trial-and-error method, at the beginning of period t the government chooses and announces the tax rate which would have covered the expenditures in the previous period:

$$\theta_t = \frac{E_{t-1}}{R}. \tag{11}$$

Note that this leads to $\mathcal{D}_t = \mathcal{D}_{t-1} + E_t - E_{t-1} = E_t$.

By definition, we have two classes of consumption equations. Consumption at work:

$$c_{a,i,t+a} = 1 - \tau - \theta_{t+a} - s_{a,i,t+a}, \qquad a = 1, 2, \ldots, R. \tag{12}$$

Consumption at retirement:

$$c_{a,i,t+R+1} = \cdots = c_{a,i,t+D} = b + d_{i,t+R}, \qquad a = R+1, \ldots, D, \tag{13}$$

where the private life annuity is given by

$$d_{i,t+R} = \psi(1+\alpha)S_{R,i,t+R}, \qquad \text{where} \qquad \psi = \frac{1}{D-R}, \qquad i = L, H.$$

Instead of the steady state estimation of s_L, assume that the age- and time-dependent saving varies with the time-variant θ_{t+a}:

$$s_{a,L,t+a} = \frac{\gamma\theta_{t+a}}{\alpha}, \qquad a = 1, \ldots, R. \tag{14}$$

Even the farsighted workers do not know their future savings, they naively assume that they will save the same amount until retiring as they save now. Projected private life-annuity at age a:

$$d_{a,H,t+R} = \psi[(1+\alpha)S_{a-1,H,t+a-1} + (1+\alpha)(R-a+1)s_{a,H,t+a}]. \tag{15}$$

Projected consumption at retirement:

$$\tilde{c}_{a,H,t+R+1} = \cdots = \tilde{c}_{a,H,t+D} = b + d_{a,H,t+R}. \tag{16}$$

While working, type H always tries to smooth her consumption path, $c_{a,H,t+a} = \tilde{c}_{a,H,t+R+1}$, i.e. by (15)–(16):

$$1-\tau-\theta_{t+a}-s_{a,H,t+a} = b+\psi(1+\alpha)S_{a-1,H,t+a-1}+\psi(R-a+1)(1+\alpha)s_{a,H,t+a},$$

hence her age- and time-dependent saving is given by

$$s_{a,H,t+a} = \frac{\chi - \psi(1+\alpha)S_{a-1,H,t+a-1} - \theta_{t+a}}{1+\psi(R-a+1)(1+\alpha)} = \varphi_a(\chi - \theta_{t+a}) - \sigma_a S_{a-1,H,t+a-1}. \tag{17}$$

3.3 Dynamic analysis

To use (9)–(11), we shall need the saving rules in t rather than in $t+a$, therefore we shift (14), (17), (9) and (8) back by a.
L-saving

$$s_{a,L,t} = \frac{\gamma\theta_t}{\alpha}, \qquad a = 1, \ldots, R. \tag{14'}$$

H-savings

$$s_{a,H,t} = \varphi_a(\chi - \theta_t) - \sigma_a S_{a-1,H,t-1}, \qquad a = 1, \ldots, R. \tag{17'}$$

Tax expenditure

$$E_{t-1} = \alpha \sum_{i=L}^{H} f_i \sum_{a=1}^{R} s_{a,i,t-1}. \tag{9'}$$

Accumulated H-savings

$$S_{a,H,t} = S_{a-1,H,t-1} + s_{a,H,t}, \qquad a = 1, 2, \ldots, R, \tag{8'}$$

where the initial conditions are given:

$$\theta_{-1} = 0, \quad S_{a-1,H,-1}, \quad s_{a-1,H,-1}, \quad a = 1, \ldots, R.$$

To minimize the dimension of the system, we drop the debt dynamics as a reducible component, and $(S_{a-1,L,-1})_{a=2}^{R}$ as reducible initial conditions. Substituting (14′) into (17′) and then into (11) and repeating the remaining equations of the irreducible system, namely (17′) and (8′) for $t = 1, 2, \ldots$, for $\alpha \neq 0$:

$$\theta_t = \gamma f_L \theta_{t-1} + \alpha R^{-1} f_H \sum_{a=1}^{R} s_{a,H,t-1}, \tag{18}$$

$$s_{a,H,t} = \varphi_a \chi - \varphi_a \gamma f_L \theta_{t-1} - \varphi_a R^{-1} \alpha f_H \sum_{x=1}^{R} s_{x,H,t-1} - \sigma_a S_{a-1,H,t-1}, \tag{19}$$

and (8′).

System (18)–(19)–(8′) is an inhomogeneous linear system of dimension $m = 2R - 1$.

Theorem 3. *a) In the two-type OLG model, the government sets the tax rate θ_t according to (18), the farsighted and the active shortsighted workers save according to (19)–(8′) and (14′), respectively.*

b) For any sufficiently low matching rate α, the system converges to the steady state of Theorem 2.

Remarks. 1. It is implicitly assumed that the initial values are sufficiently close to their steady state values to generate viable paths, i.e. $s_{a,i,t+a}, c_{a,i,t+a} \geq 0$ for all $(a, i, t + a)$s.

2. It is an open question how low matching rates guarantee the stability of the steady state in general but we shall give partial answers later.

3. The model can be generalized for heterogeneous shortsighted workers with different γ_is as in Theorem 2.

Proof. (a) We have proved part a) above.

(b) To prove stability, we can drop the constant terms from (19). Then we have a simple solution for $\alpha = 0 = \theta_t$:

$$s_{a,H,t} = -\sigma_a S_{a-1,H,t-1}. \tag{19′}$$

Substituting (19′) into (8′) results in

$$S_{a,H,t} = (1 - \sigma_a) S_{a-1,H,t-1} = (1 - \sigma_a) \cdots (1 - \sigma_1) S_{0,H,t-a} = 0 \quad (t \geq a). \tag{20}$$

By continuity, stability survives for sufficiently low matching rate. ∎

To deepen our understanding, we consider the simplest case, OLG 1-1.

Example 1. Let $\mathbf{R} = 1$ and $\mathbf{D} = 2$. In this case, $S_{1,H,t} = s_{1,H,t}$:

$$\theta_t = \gamma f_L \theta_{t-1} + \alpha f_H s_{1,H,t-1} \tag{18''}$$

and

$$s_{1,H,t} = \frac{\chi - \theta_t}{2+\alpha}. \tag{19''}$$

Shifting (19″) back by 1 period and inserting the shifted (19″) into (18″) yields

$$\theta_t = \alpha f_H \frac{\chi}{2+\alpha} + \left[\gamma f_L - \frac{\alpha f_H}{2+\alpha}\right] \theta_{t-1}.$$

The path generated by this first-order linear difference equation is obviously stable. For $\gamma^* = \dfrac{\alpha f_H}{(2+\alpha) f_L}$, the tax rate jumps to the steady state. For $0 < \gamma < \gamma^*$, the tax rate oscillates around the steady state, while for $\gamma^* < \gamma \leq 1$, the tax rate increasingly converges to the steady state. (Note that the second interval is empty, i.e. $\gamma^* > 1$ if and only if $(2+\alpha)/[2(1+\alpha)] < f_H \leq 1$.) In Appendix A we discuss the more realistic and more complex case of $\mathbf{R} = 2$, $\mathbf{D} = 3$.

Finally, three parts of Table 3 display the numerical illustrations for $\mathbf{R} = 4$ and $\mathbf{D} = 6$ (decades), matchin rate $\alpha = 1$ and the relative propensity to save $\gamma = 1/2$ (Table 2, middle row). We expect that the process converges to the steady state described in row 2 in Table 2. Our expectations are correct, at least for the initial values $\theta_0 = 0$, $\mathcal{D}_{-1} = 0$, furthermore we choose the initial values for L and H savings as 0 and $s^H(0)$, belonging to $\alpha = 0$. Table 3a displays the paths of the debt, of the tax rate and of the H-saving. As expected, the debt converges (to 0.12) while the tax rate converges to the steady state 0.03. As the matching system builds up, the H-savings drop to the steady state values of $s^H = 0.074$, regardless of age. The internal and external variances converge to their respective steady state values.

Table 3a. *H-saving in overlapping generations*

			Variance		Saving of H-workers			
Period	Debt	Tax rate	internal	external	youn-gest	younger	older	oldest
t	$\mathcal{D}_{\mathbf{t}}$	$\theta_{\mathbf{t}}$	$\varepsilon_{I,\mathbf{t}}$	$\varepsilon_{E,\mathbf{t}}$	$s_{\mathbf{1},H,\mathbf{t}}$	$s_{\mathbf{2},H,\mathbf{t}}$	$s_{\mathbf{3},H,\mathbf{t}}$	$s_{\mathbf{4},H,\mathbf{t}}$
0	0.048	0	0.240	0.063	0.080	0.067	0.044	0
1	0.101	0.012	0.224	0.005	0.078	0.077	0.080	0.098
2	0.111	0.025	0.196	0.027	0.075	0.074	0.073	0.070
3	0.116	0.028	0.178	0.017	0.074	0.074	0.074	0.075
4	0.118	0.029	0.174	0.018	0.074	0.074	0.074	0.074
5	0.118	0.029	0.172	0.017	0.074	0.074	0.074	0.074
6	0.118	0.030	0.171	0.017	0.074	0.074	0.074	0.074

Turning to the farsighted workers' consumption paths, note that before the introduction of the transfers, H saved much more than just after it: $0.133 > 0.08$. Therefore when the transfer system is introduced in period $t = 0$, the farsighted workers' savings drop and their consumption jumps, at least temporarily. We experience the same fast convergence to the age-invariant steady state. The life-cycle average consumption is also displayed though not for a longitudinal path but for a cross-section profile: $c_t^i = D^{-1} \sum_{a=1}^{D} c_{a,i,t}$, $i = H, L$.

Table 3b. *H-consumption profiles in overlapping generations*

	Consumption					
	H-workers				of type H	
Period	youngest	younger	older	oldest	pensioner	average
t	$c_{\mathbf{1},H,\mathbf{t}}$	$c_{\mathbf{2},H,\mathbf{t}}$	$c_{\mathbf{3},H,\mathbf{t}}$	$c_{\mathbf{4},H,\mathbf{t}}$	$c_{\mathbf{5},H,\mathbf{t}}$	$c_{\mathbf{t}}^{H}$
0	0.720	0.733	0.756	0.800	0.933	0.813
1	0.710	0.711	0.708	0.690	0.591	0.667
2	0.700	0.700	0.701	0.705	0.734	0.712
3	0.698	0.698	0.698	0.697	0.692	0.696
4	0.697	0.697	0.697	0.697	0.698	0.697
5	0.696	0.696	0.697	0.696	0.696	0.696
6	0.696	0.696	0.696	0.696	0.696	0.696

Ending the presentation with the shortsighted workers' consumption paths, similar overconsumption can be observed which stabilizes quite fast at 0.76, while pensioner's consumption rises from 0.4 to 0.453. Comparing the two life-consumptions at $\mathbf{t} = 6$, we see the difference: $c_6^L = 0.656 < 0.698 = c_6^H$.

Table 3c. *L-consumption profiles in overlapping generations*

	Consumption					
	L-workers				of type L	
Period $\mathbf{t}$	youngest $c_{\mathbf{1},L,\mathbf{t}}$	younger $c_{\mathbf{2},L,\mathbf{t}}$	older $c_{\mathbf{3},L,\mathbf{t}}$	oldest $c_{\mathbf{4},L,\mathbf{t}}$	pensioner $c_{\mathbf{5},L,\mathbf{t}}$	average $c_{\mathbf{t}}^L$
0	0.800	0.800	0.800	0.800	0.400	0.667
1	0.782	0.782	0.782	0.782	0.400	0.655
2	0.762	0.762	0.762	0.762	0.424	0.649
3	0.758	0.758	0.758	0.758	0.451	0.656
4	0.756	0.756	0.756	0.756	0.456	0.656
5	0.756	0.756	0.756	0.756	0.458	0.657
6	0.756	0.756	0.756	0.756	0.459	0.657

It is evident that adding a given part of the past debt to the past expenditure in (11), will eventually eliminate the debt. Formally,

$$\theta_t = \frac{E_{t-1}}{R} + \zeta \mathcal{D}_{t-1}, \tag{11'}$$

where $\zeta > 0$ is an adjustment coefficient, for example, $\zeta = 1/R$. Then (18) modifies to

$$\theta_t = \gamma f_L \theta_{t-1} + \alpha R^{-1} f_H \sum_{a=1}^{R} s_{a,H,t-1} + \zeta \mathcal{D}_{t-1}.$$

4 An ABM for life-cycle savings

In the previous model, even the active shortsighted workers learn relatively little. An important feature of the ABM is that everybody learns from others, therefore we shall apply ABM to make shortsighted workers to learn not only globally but also locally.

We assume that there are n types, where $n > 1$ is a relatively small integer. Type i is characterized by relative propensity to save $\gamma_i = i/n$,

$i = 1, 2, \ldots n-1$, with rational frequencies $f_i > 0$, $\sum_{i=1}^{n-1} f_i = f_L$ and $\gamma = f_L^{-1} \sum_{i=1}^{n-1} f_i \gamma_i$. Type n with frequency $f_H = 1 - f_L$ is farsighted. (Without assuming the existence of farsighted agents, the steady state tax rate would not be defined in (25) below.) We assume that there are a finite but large number of workers ($M = RN$) with the total mass of each cohort normalized to unity. Therefore at the start of each cohort, the number of type i workers aged 1 is about Nf_i, indexed by $k = N_{i-1}+1, \ldots, N_i$, where $N_{i+1} = N_i + Nf_i$, $N_0 = 0$.

First we modify (9) as

$$E_t = \alpha \sum_{i=1}^{n} f_i \sum_{k=N_{i-1}+1}^{N_i} \sum_{a=1}^{R} s_{a,k,t}, \qquad t = 0, 1, 2, \ldots. \tag{25}$$

Before outlining the ABM framework, we have to touch on the issue of the lifetime utility function, which is maximized in standard economics. Let $u(\cdot)$ be a strictly concave increasing per-period utility function, and then the lifetime utility function is equal to the discounted sum:

$$U(c_{1,k,t+1}, \ldots, c_{D,k,t+D}) = \sum_{a=1}^{D} \delta_k^{a-1} u(c_{a,k,t+a}), \quad (0 \leq \delta_k \leq 1). \tag{26}$$

For our type H, $U(c_{1,k,t+1}, \ldots, c_{D,k,t+D}) = \min_{a=1}^{D} c_{a,k,t+a}$. In Section 2, $\delta_L = 0$ and $\delta_H = 1$.

Unfortunately, the worker at age $a < R$ does not know her future saving path. As before, we assume that she extrapolates her current saving for the future, yielding the *projected* private annuity and consumption path:

$$\tilde{c}_{a,k,t+y} = 1 - \tau - \theta_{t+a} - s_{a,k,t+a}, \qquad y = a, a+1, \ldots, R \tag{27}$$

and

$$d_{a,k,t+R} = \psi[(1+\alpha)S_{a-1,k,t+a-1} + (R-a+1)s_{a,k,t+a}], \quad \tilde{c}_{a,k,t+R+1} = b + d_{a,k,t+R}. \tag{28}$$

We are now able to formulate the projected lifetime utility function as well:

$$\tilde{U}_{a,k,t+a}(c_{1,k,t+1}, \ldots, c_{a,k,t+a}, \tilde{c}_{a,k,t+a+1} \ldots, \tilde{c}_{D,k,t+D})$$
$$= \sum_{x=1}^{a-1} \delta_k^{x-1} u(c_{x,k,t+x}) + \sum_{y=a}^{R} \delta_k^{y-1} u(c_{a,k,t+a}) + \sum_{z=R+1}^{D} \delta_k^{z-1} u(\tilde{c}_{R+1,k,t+R+1}). \tag{29}$$

Evidently, the middle term can be further simplified:

$$\sum_{y=a}^{R} \delta_k^{y-1} u(c_{a,k,t+a}) = \delta_k^a \frac{1-\delta_k^{R-a+1}}{1-\delta_k} u(c_{a,k,t+a}).$$

By (28)–(29), in period $t+a$, this function only depends on $s_{a,k,t+a}$, but we cannot expect that every ordinary worker can evaluate such a function even with the simplest parameterization, when $u(c) = \log c$. In this case we cannot pretend that our worker chooses her decision by copying the fittest decision of another type.

We try with a simpler indicator, the average lifetime consumption mentioned in Section 2. In our new setting, its projected value at $(a, k, t+a)$ is given by

$$c_{a,t+a}^k = \frac{1}{D}[C_{a-1,k,t+a-1} + (R-a+1)c_{a,k,t+a} + (D-R)\tilde{c}_{a,k,R+1}], \qquad (30)$$

where

$$C_{a,k,t+a} = \sum_{x=1}^{a} c_{x,k,t+x} = C_{a-1,t+a-1} + c_{a,k,t+a}.$$

Again, by (30), in period $t+a$, $c_{a,t+a}^k$ is a simple linear function of a single variable $s_{a,k,t+a}$. Finally, at retirement, the projected value is crystalized into $c_t^k = c_{R,t+R}^k$. This indicator is far from ideal, because it does not reflect the main aim of the pension system: consumption smoothing. Nevertheless, in the world of bounded rationality, together with ε_I and ε_E [(31) below] it reflects the undersaving of L and the exploitation of L by H in Sections 2 and 3.

We generalize the variances of the average consumption along the M paths. Let the expected average consumption be denoted by c_t, therefore the squared variances are respectively

$$\varepsilon_{t,E}^2 = \tfrac{1}{N}\sum_{k=1}^{N}(c_t^k - c_t)^2 \qquad \text{and} \qquad \varepsilon_{t,I}^2 = \tfrac{1}{NR}\sum_{k=1}^{N}\sum_{a=1}^{D}(c_{a,t}^k - c_t^k)^2. \qquad (31)$$

We assume that every shortsighted worker k knows a small number of other shortsighted workers, indexed as $l \in L_k$. (For the sake of simplicity, we forbid shortsighted workers to lean from farsighted workers.) We also assume that no set of acquaintances changes in time; the number of acquaintances is denoted by $|L_k| > 0$, $k = 1, 2, \dots, N$. For simplicity, the acquaintances are just one period older than the foregoing worker. Except for Hs, every agent k at every age and time signals his current type $i(a, k, t + a)$.

Having dropped the age index, as a starting point, we shall experiment with the simplest network, described as follows. Let e be a positive integer, called *radius*, $0 < e \ll N$:

$$L_k^e = \{l, |l - k| \le e\},$$

where the N agents of the same cohort are allocated *randomly* on the circle with N points, and artificial types $N + 1$, $N + 2$, etc. stand for 1, 2, etc. For example, the set of 1's acquaintances is

$$L_1^e = \{1, 2, 3, \dots, e, e + 1; N - e, N - e + 1, \dots, N - 1, N\}.$$

We assume that worker (a, k) in period $t + a$ adopts that type $i_{a,k,t-1+a}$'s γ which produced the highest average *projected* consumption among her acquaintances:

$$c^l_{a,k,t+a} \le c^{i_{a,k,t-1+a}}_{a,k,t+a}, \qquad l \in L_k. \tag{32}$$

If there is more than one optimal decision, he will pick one with the minimal index i or randomize.

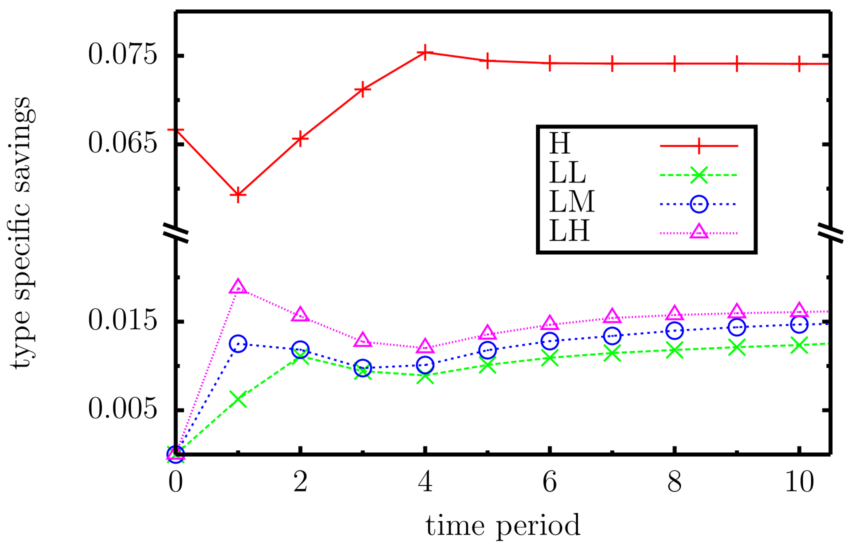
To start the dynamic system at period 0, we have to define the initial conditions. For comparability with Table 3a, we assume that all the previous shortsighted savings were zero and those of the farsighted were $s^H(0)$ in (4-0). Hoping that the process converges fast enough, we choose $T = 10$, which represents ten decades, i.e. 100 years.

First the number of types is $n = 4$: $f_1 = f_2 = f_3 = f_4 = 1/4$ with $\gamma_1 = 1/4$, $\gamma_2 = 1/2$ and $\gamma_3 = 3/4$, and type 4 is H. Note that the average relative saving propensity, $\gamma = 0.5$ is as in Table 3, therefore the two cases are comparable. Adding debt servicing, we choose $\zeta = 1/R$ in (11′). The number of workers at each cohort is $N = 120$ and $e = 1$, i.e. everybody has 3 acquaintances. The parameter values are as follows: $\mathbf{D} = 6$, $\mathbf{R} = 4$, strategies: H, LL, LM, LH, no learning from H types.

(i) Figure 1 shows the four types' average saving paths. Note that on average, the middle shortsighted workers catch up with the higher ones, but the lower ones lag behind. The H types create "walls" that separate different types of behavior. Agents within such a domain can only become as smart as the initially smartest agent was inside the domain. The average

L-learning remains below 0.015, slightly less than previously. (Note that even extending the learning from farsighted workers would not significantly improve learning.)

Figure 1: Rise of savings in time



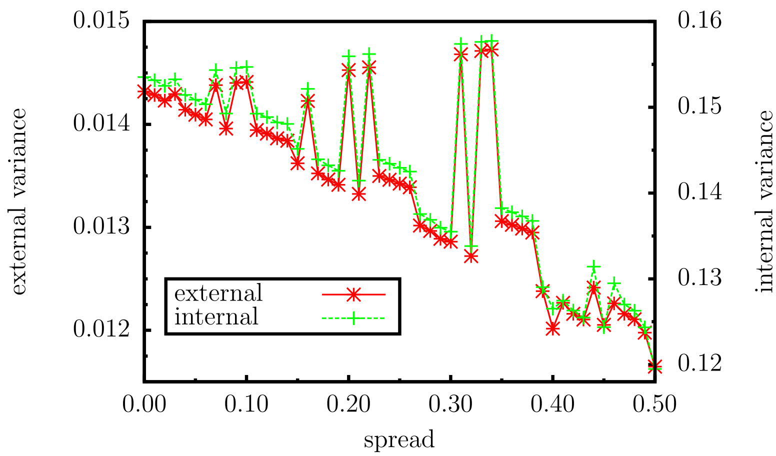
(ii) Next we continuously change the distance between the extreme γs, the *spread* ξ, while fixing the middle at $\gamma_2 = 1/2$: $\gamma_1 = 1/2 - \xi$ and $\gamma_3 = 1/2 + \xi$, $\xi \in (0, 1/2)$. Figure 2 (left-hand scale) shows the degree of perverse redistribution ε_E as a function of ξ, for a fixed $T = 10$. We expected ε_E is decreasing with ξ and we see now its extent. But we see peaks around $\xi = 0.2, 0.3$ and 0.4, which is unexpected. This is probably due to the induced change in the tax rate dynamics. Even in the case $\xi = 0.5$, variance ε_T cannot become zero, because of two reasons. Firstly, LL types do not save in their first working period. Secondly, the domain structure can prevent some LL type agents from becoming LH types. We get a similar curve for the internal variance (right scale).

(iii) In the third round, we increase the number of types from $n = 4$ to 6, 8 and 10. Retaining symmetry, we have $n = 2m$,

$$\gamma_i = \frac{i}{2m}, \qquad i = 1, 2, \ldots, m, \ldots, 2m - 1.$$

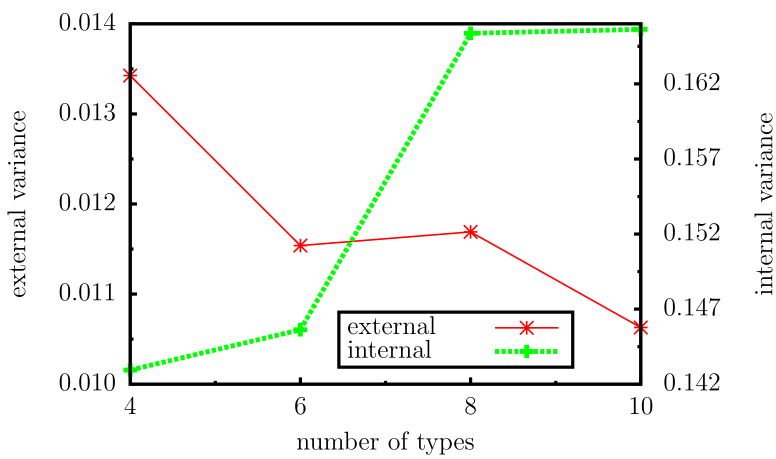
We expect that ε_E is decreasing with m and Figure 3 shows its strength because the learning process eventually turns most L types into LH types

Figure 2: Wider spread, typically smaller variance of lifetime average consumption: $n = 4$



(with maximal γ), which become more and more similar to H types as m increases. It is of interest that the internal variances (right scale) rises rather than sinks.

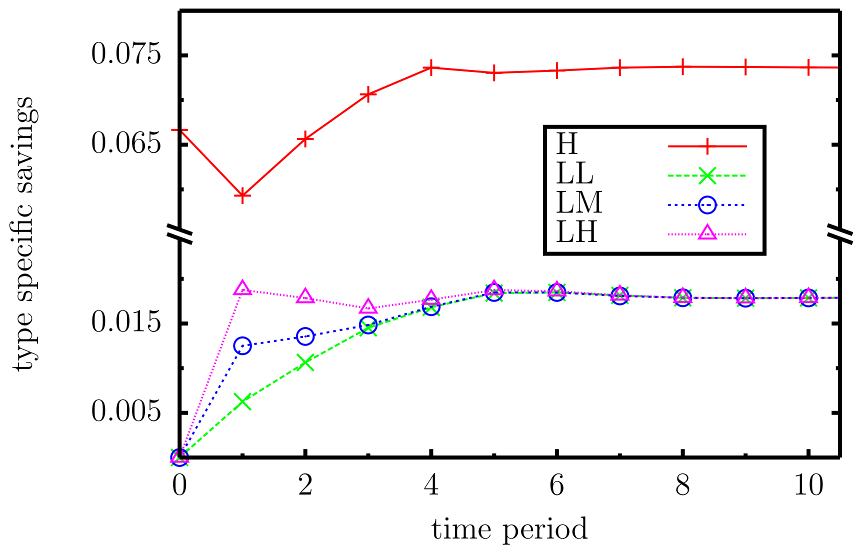
Figure 3: More types, lower external variance and higher internal variance



(iv) Next we return to Figure 1: $n = 4$ strategies: H, LL, LM and LH, but change the network's structure as well. We experiment with a random graph (Erdős and Rényi, 1959): we replace the original connections with randomly chosen ones. Links were created with probability $p = \frac{2}{119}$, thus we expect to have the same amount of edges as in the previous cases. This ensures that the expected value of connections remains invariant. As Figure 4 shows, the introduction of the random graph not only homogenizes the savings but raises their average with respect to Figure 1.

Figure 4

The impact of different structure on the saving dynamics



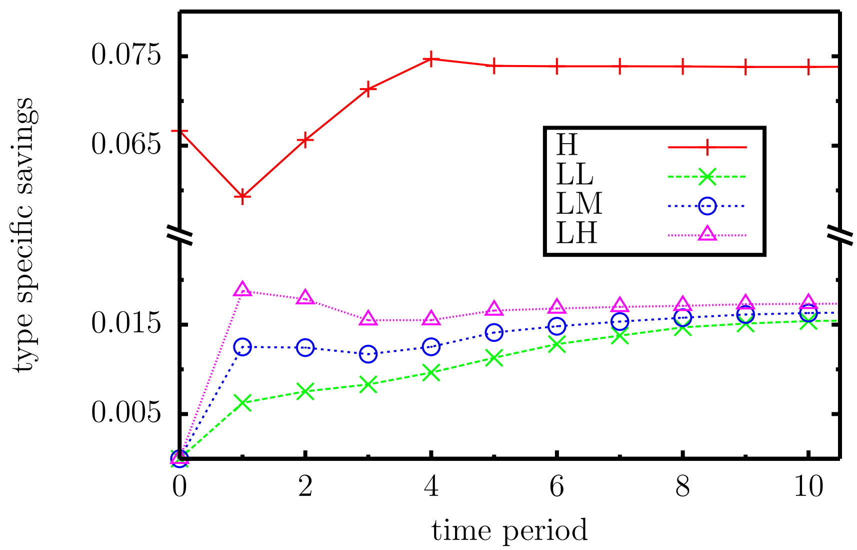
(v) What happens if we retain the better configured network of Figure 4, the density of the network is drastically reduced, diminishing the probability of being connected from 2/119 to 1/238? Figure 6 displays persistent heterogeneity and the stabilized average saving is also lower.

(vi) We also change the characteristics of the network, while fixing $m = 5$, $n = 10$. We increase the radius e from 1 to 2, 3 and 4, and expect that ε_T to increase with radius e. The parameter e controls the domain structure of the system, and therefore defines who can learn from whom. As e increases, more agents should become able to learn from L types with higher γ values, but this also changes the tax dynamics, so increasing e does not necessarily lead to a decrease in ε_E. Indeed, our calculation (omitted) show practically constant external variance.

(vii) Until now we have fixed the length of the period at 10 years. At the end, we change this important parameter as well. We expect that the shorter the length of period, the faster the shortsighted agents learn, diminishing both *relative* variances. (We turn to relative variances to neutralize the

Figure 5

The impact of less connections on the saving dynamics



impact of changing time units.) Note, however, that our modification change the debt dynamics as well, which may have unexpected repercussions as well. To check our conjecture we diminish $h = 10$ first to 5 then to 2 and end with 1. Table 4 displays the internal and external variances at $T[1] = 100$ to be denoted as $\varepsilon_I(h)$ and $\varepsilon_E(h)$. Contrary to our conjecture, the speeding up of the learning process does not decrease the variances, moreover, the internal variance slightly increases.

Table 4. *Variances and length of period*

Length of period h	Internal variance $\varepsilon_I(h)$	External variance $\varepsilon_E(h)$
10	0.143	0.013
5	0.147	0.014
2	0.160	0.014
1	0.166	0.014

5 Conclusions

We have studied a family of utterly simple life-cycle/overlapping generations models with mandatory and voluntary pensions. In the first model, (passive) short- and farsighted workers lived together, and the former did not learn anything. In the second model, the (active) shortsighted workers learned the use of participation in the voluntary system, and diminished exploitation. The third model added local learning to the global learning: in the arising ABM, the variance of heterogeneity in the relative propensities of saving decreases the variance of lifetime average consumptions, while the rise in the number of acquaintances is indifferent.

At the end we recall our qualifications (also referring to their significance). (a) The system is stationary, no rules changed after the transfer system started its operation in period 0. (This is only a technical assumption but the expected and the unexpected changes obviously affect the workers' behavior.) (b) The earnings are age-, type- and time-invariant. (c) There is no cap on private savings. (Simonovits (2011) emphasized the impact of the negative dependence of discounting on wages and the role of the cap.) (d) Savings do not earn interest. (This is a technical assumption but taking into account the riskiness of savings, it is not so bad an assumption.) (e) The private savings are used as private life annuities. (Though it would be desirable, very few countries operate such a voluntary system, and just in 2015, a most important semi-voluntary system, that of Great Britain, annulled mandatory annuatization even in the mandatory private pillar.) (f) The government revises its calculation as infrequently as the agents. If we relaxed these qualifications, the quantitative features would be changed. We can only hope that the qualitative results survive: local learning adds to a global one.

Appendix A: The case of $R = 2$ and $D = 3$

In the main text we proved stability for the trivial Example 1. In this Appendix, we shall discuss the second simplest case. Recall notations

$$\chi = 1 - 3\tau > 0, \qquad \psi = 1,$$

$$\varphi_1 = \frac{1}{3+2\alpha}, \quad \sigma_1 = (1+\alpha)\varphi_1 \qquad \text{and} \qquad \varphi_2 = \frac{1}{2+\alpha}, \quad \sigma_2 = (1+\alpha)\varphi_2.$$

Dropping the constant terms, equations (18′)–(8″) simplify to

$$\theta_t = \gamma f_L \theta_{t-1} + \frac{\alpha f_H}{2}(s_{1,H,t-1} + s_{2,H,t-1}), \tag{A.1}$$

$$s_{1,H,t} = -\varphi_1 \gamma f_L \theta_{t-1} - \frac{\varphi_1 \alpha f_H}{2} s_{1,H,t-1} - \frac{\varphi_1 \alpha f_H}{2} s_{2,H,t-1} \tag{A.2}$$

and

$$s_{2,H,t} = -\varphi_2 \gamma f_L \theta_{t-1} - \left(\frac{\varphi_2 \alpha f_H}{2} + \sigma_2\right) s_{1,H,t-1} - \frac{\varphi_2 \alpha f_H}{2} s_{2,H,t-1} \tag{A.3}$$

We have the following limited analytical stability result.

Theorem A.1. *The 3-dimensional system (A.1)–(A.3) is stable if the following sufficient conditions hold:*

$$\gamma f_L \left(1 + \frac{1}{3+2\alpha} + \frac{1}{2+\alpha}\right) < 1, \tag{A.4}$$

and

$$\frac{\alpha}{2} f_H \left(1 + \frac{1}{3+2\alpha} + \frac{1}{2+\alpha}\right) + \frac{1+\alpha}{2+\alpha} < 1, \tag{A.5}$$

Proof. Take the absolute values of entries and calculate the column sums of the matrix implicitly defined by system (A.1)–(A.3). If they are all less than 1, then stability is proved. (A.4) and (A.5) are respectively the conditions for the first and the third columns, and the second condition is implied by (A.5) ∎

It is easy to see that a lot of triples (γ, α, f_H) satisfy the two conditions. For example, for the maximal matching rate $\alpha = 1$, the pair of inequalities $\gamma f_L < 0.546$ and $f_H < 0.79$ provide a wide range. Since $\gamma \leq 1$, $f_L < 0.546$ or $f_H > 0.454$ is also sufficient. One has only to calculate the dominant characteristic root of matrix M for two intervals: $[0, 0.454]$ and $[0.8, 1]$. The numerical calculations put "all" the dominant characteristic roots inside the open unit disk, suggesting (asymptotic) stability. (We used quotations marks for all, because we only calculated a sufficiently dense grid of the parameters.)

Appendix B. Intertemporal substitution and discounting: $R = 2$ and $D = 3$

We claimed in the main text (Section 4) that apart from its dubious realism, it is technically quite cumbersome to model workers who discount the future and save while intertemporally substitute future for present consumption in a voluntary pension system. To substantiate our claim, we confine now our attention to the steady state analysis (cf. Theorem 2) of $R = 2$ and $D = 3$, distinguishing types with various discount factors. First we determine the optimal savings for a given tax rate, second we determine the balanced tax rate and third we illustrate our results numerically.

Conditional saving functions

We shall need the following notations: $\hat{t} = 1 - \tau - \theta$ and $\bar{\alpha} = 1 + \alpha$. Then Consumption–saving:

$$c_1 = \hat{t} - s_1, \qquad c_2 = \hat{t} - s_2 \qquad \text{and} \qquad c_3 = 2\tau + \bar{\alpha}(s_1 + s_2). \qquad (B.1)$$

Lifetime utility function:

$$U(c_1, c_2, c_3) = \log c_1 + \delta \log c_2 + \delta^2 \log c_3. \qquad (B.2)$$

Inserting (B.1) into (B.2) yields the reduced lifetime utility function:

$$U[s_1, s_2] = \log(\hat{t} - s_1) + \delta \log(\hat{t} - s_2) + \delta^2 \log(2\tau + \bar{\alpha}(s_1 + s_2)). \qquad (B.3)$$

Equating the partial derivatives to zero yields the first-order optimality conditions:

$$U_1'[s_1, s_2] = \frac{-1}{\hat{t} - s_1} + \frac{\delta^2 \bar{\alpha}}{2\tau + \bar{\alpha}(s_1 + s_2)} = 0 \qquad (B.4-1)$$

and

$$U_2'[s_1, s_2] = \frac{-\delta}{\hat{t} - s_2} + \frac{\delta^2 \bar{\alpha}}{2\tau + \bar{\alpha}(s_1 + s_2)} = 0. \qquad (B.4-2)$$

Comparing (B.4–1) and (B.4–2) implies

$$\frac{1}{\hat{t} - s_1} = \frac{\delta}{\hat{t} - s_2}, \qquad \text{hence} \qquad s_2 = (1 - \delta)\hat{t} + \delta s_1. \qquad (B.5)$$

Rearranging (B.4–1) and inserting (B.5) imply

$$\delta^2\bar{\alpha}(\hat{t}-s_1) = 2\tau + \bar{\alpha}(s_1+s_2) = 2\tau + \bar{\alpha}[(1-\delta)\hat{t} + (1+\delta)s_1].$$

Using $\hat{t} = \hat{\tau} - \theta$, s_1 can be expressed as a linear function of θ:

$$s_1 = \frac{\bar{\alpha}(\delta^2+\delta-1)(\hat{\tau}-\theta) - 2\tau}{\bar{\alpha}(\delta^2+\delta+1)}. \qquad (B.6)$$

To make the formula shorter, we introduce notations

$$A = \frac{\bar{\alpha}(\delta^2+\delta-1)\hat{\tau} - 2\tau}{\bar{\alpha}(\delta^2+\delta+1)} \qquad \text{and} \qquad B = \frac{\delta^2+\delta-1}{\delta^2+\delta+1}$$

resulting in $s_1 = A - B\theta$. By (B.5),

$$s_2 = (1-\delta)(\hat{\tau}-\theta) + \delta(A - B\theta) = (1-\delta)\hat{\tau} + \delta A - (1-\delta+\delta B)\theta.$$

Introducing $F = (1-\delta)\hat{\tau} + \delta A$ and $G = 1 - \delta + \delta B$, $s_2 = F - G\theta$.

Until now we have fixed the discount factor δ. From now on we differentiate a low and a high factor δ_L and δ_H and add subscript $i = L, H$ to s_1 and s_2. In summary:

$$s_{1,i} = A_i - B_i\theta \qquad \text{and} \qquad s_{2,i} = F_i - G_i\theta, \qquad i = L, H. \qquad (B.7)$$

We have not yet considered the credit constraints, namely that life-cycle savings of workers cannot be negative. Mathematical convenience would dictate neglecting the issue and make workers with negative saving $s_{a,i}$ pay a fee $\alpha s_{a,i}$. We rather exclude this case.

Balanced tax rate

Having determined the conditional savings as functions of the tax rate, we are now able to determine the balanced tax rate. Inserting (B.7) into the tax equation

$$2\theta = \alpha \sum_{i=L}^{H} f_i(s_{1,i} + s_{2,i})$$

yields

$$2\theta = \alpha \sum_{i=L}^{H} f_i[A_i + F_i - (B_i + G_i)\theta].$$

Solving for θ:

$$\theta^{o} = \frac{\alpha \sum_{i=L}^{H} f_i(A_i + F_i)}{2 + \alpha \sum_{i=L}^{H} f_i(B_i + G_i)}. \qquad (B.8)$$

Indeed, the solution is quite involved even in the simplest case.

Numerical illustration

To obtain numerical experiences we keep $f_L = 3/4$ and $f_H = 1/4$ with $\delta_H = 1$ and $\tau = 0.2$. The upper discount factor is maximal: $\delta_H = 1$. Furthermore, to avoid negative savings, we run δ_L and α from 1 to 0.75, having four runs. Note that the lower discount factor calculated annually is equal to 0.986, quite a weak discounting! Tables B.1a and b show the following.

Rows 1: If both types are farsighted, i.e. $\delta_L = 1$ and the matching rate is maximal: $\alpha = 1$, then the savings are uniformly 0.15, equal to the tax rate. The working-age consumption is just half the total wage but the old-age consumption is the double of the working age consumption and is equal to the total wage. (This is clearly wasteful.)

Rows 2: Retaining farsighted L-types, just lowering the matching rate to $\alpha = 0.75$ leads to a reduced tax rate $\theta^{o} = 0.107$ without hardly raising the saving rates and changing the consumption path.

Rows 3: If the shortsighted workers' period discount factor is only $\delta_L = 0.75$, while $\alpha = 1$ again, then they hardly save at all when they are young workers but save a little bit more than the farsighted workers at middle age: $0.178 > 0.163$. The balanced tax rate is slightly higher: $\theta = 0.11$. The farsighted workers' and old-age consumption are slightly higher than in the ideal case but the shortsighted workers' middle-age consumption is only 0.512.

Rows 4: The reduced lower discount factor $\delta_L = 0.75$ and the reduced matching rate $\alpha = 0.75$ diminishes the otherwise excessive old-age consumption of the shortsighted workers from $c_{3,L} = 0.769$ to 0.713 while raising middle age consumption from $c_{2,L} = 0.512$ to 0.544.

Table B.1a. *Neoclassical life-cycle saving paths*

			Saving of L-workers		Saving of H-workers	
L discount factor δ_L	Matching rate α	Tax rate θ	younger $s_{1,H}$	older $s_{2,H}$	younger $s_{1,L}$	older $s_{2,L}$
1.00	1.00	0.150	0.150	0.150	0.150	0.150
1.00	0.75	0.107	0.155	0.155	0.155	0.155
0.75	1.00	0.110	0.163	0.163	0.007	0.178
0.75	0.75	0.076	0.165	0.165	–0.001	0.180

Table B.1b. *Neoclassical life-cycle consumption paths*

		L-Consumption			H-Consumption		
L-discount factor δ_L	Matching rate α	younger $c_{1,H}$	older $c_{2,H}$	pensioner $c_{3,H}$	younger $c_{1,L}$	older $c_{2,L}$	pensioner $c_{3,L}$
1.00	1.00	0.500	0.500	1.000	0.500	0.500	1.000
1.00	0.75	0.538	0.538	0.942	0.538	0.538	0.942
0.75	1.00	0.527	0.527	1.053	0.683	0.512	0.769
0.75	0.75	0.559	0.559	0.978	0.725	0.544	0.713

In summary, we have seen that the neoclassical model of life-cycle saving is much more cumbersome than our ad-hoc learning model. Moreover, it cannot be applied to a large domain of the parameter space, namely when the workers are too shortsighted or the matching rate is moderate. Finally, its welfare properties are far from being attractive: middle-age consumption is too low with respect to old-age consumption.

References

Auerbach, A. J. and Kotlikoff, L. J. (1987): *Dynamic Fiscal Policy*, Cambridge, Cambridge University Press.

Campbell, Y. J. (2006): “Household Finance”, *The Journal of Finance, 56*, 1553–1604.

Chetty, R. (2015): "Behavioral Economics and Public Policy: A Pragmatic Approach", *American Economic Review, Papers and Proceedings, 105*, 1–33.

Barr, N. and Diamond, P. (2008): *Reforming Pensions: Principles and Policy Choices,* Oxford, Oxford University Press.

Duflo, E. and Saez, E. (2003): "The Role of Information and Social Interaction in Retirement Decisions", *Quarterly Journal of Economics, 111*, 815–842.

Duflo, E.; Gale, W.; Liebman, J; Orszag, P. and Saez, E. (2007): "Saving Incentives for Low- and Moderate Income Families in the United States: Why is Saver's Credit not More Effective?", *Journal of European Economic Association 5*, 647–661.

Engen, E. M., Gale, W. G. and Scholz, J. (1996): "The Illusory Effects of Savings Incentives on Saving", *Journal of Economic Perspectives 10:4*, 111–138.

Erdős, P. and Rényi, A. (1959): "Random Graphs I", *Publicationes Mathematicae 6*, 290–297.

Fehr, H., Habermann, C. and Kindermann, F. (2008): "Tax-Favored Retirement Accounts: Are they Efficient in Increasing Savings and Growth?, *FinanzArchiv, Public Finance Analysis 64*, 171–198.

Feldstein, M. S. (1985): "The Optimal Level of Social Benefits", *Quarterly Journal of Economics 100*, 302–320.

Findley, T. S. and Caliendo, F. N. (2008): "Short Horizons, Time Inconsistency and Optimal Social Security", *International Taxation and Public Finance 16*, 487–513.

Gigerenzen, G. and Selten, R. (2002): Bounded Rationality: The Toolkit, Boston, MA, MIT Press.

Hubbard, R. G. and Skinner, J. S. (1996): "Assessing the Effectiveness of Saving Incentives", *Journal of Economic Perspectives 10:4*, 73–90.

Laibson, D. (1998): "Life-Cycle Consumption and Hyperbolic Discount Functions", *European Economic Review 42*, 861–871.

Lusardi, A. and Mitchell, O. (2014): "The Economic Importance of Financial Literacy: Theory and Evidence", *Journal of Economic Literature* 52, 5–44.

Méder, Zs.; Simonovits, A. and Vincze, J. (2012): "Tax Morale and Tax Evasion: Social Preferences and Bounded Rationality", *Economic Analysis and Policy, 42,* 171–188.

Modigliani, F. and Brumberg, R. (1954): "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data', Kurihara, K. K., ed. (1954): *Post-Keynesian Economics*, New Brunswick, Rutgers University Press, 388–436.

Molnár, Gy. and Simonovits, A. (1998): "Expectations, (In)stability and (In)viability in Realistic Overlapping Cohorts Models", *Journal of Economic Dynamics and Control 23*, 303–332.

OECD (2005): *Tax-Favored Retirement Saving*, OECD Economic Studies, 39, Paris.

Poterba, J. M.; Venti, S. F. and Wise, D. A. (1996): "How Retirement Savings Program Increase Savings", *Journal of Economic Perspectives 10:4*, 91–112.

Samuelson, P. A. (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money", *Journal of Political Economy 66* 467–482.

Simonovits, A. (2011): "When are Voluntary Pensions Indifferent", *Economics Letters 111,* 153–155.

Tesfatsion, L. (2006): "Agent-Based Computational Economics: A Constructive Approach to Economic Theory," Tesfatsion, L. and Judd, K. L, eds. *Handbook of Computational Economics, Vol. II.* 831–880.

Varga, G. and Vincze, J. (2015): "Ants and Crickets: Arbitrary Saving Rates in an Agent-Based Model with Infinitely Lived Agents", IE-CERS-HAS Working Paper 4.