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α-Paramodulation Method for a Lattice-Valued Logic $L_nF(X)$ with Equality

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Abstract In this paper, α-paramodulation and α-GH paramodulation methods are proposed for handling logical formulas with equality in a lattice-valued logic $L_nF(X)$, which has unique ability for representing and reasoning uncertain information from a logical point of view. As an extension of the work of [10,11], a new form of α-equality axioms set is proposed. The equivalence between α-equality axioms set and E_α-interpretation in $L_nF(X)$ with an appropriate level is also established, which may provide a key foundation for equality reasoning in lattice-valued logic. Based on its equivalence, E_α-unsatisfiability equivalent transformation is given. Furthermore, α-paramodulation and its restricted method, i.e., α-GH paramodulation are given. The soundness and completeness of the proposed methods are also obtained.

Keywords Lattice-valued logic · Equality · α-Equality axioms · α-Paramodulation · α-GH paramodulation

1 Introduction

The general aim of decision making in big data is to reduce large-scale problems to a scale that humans can comprehend and act upon [21]. The credibility of the data is also an important issue to be guaranteed. Some methods or branches are proposed

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to solve this problem such as deductive methods by mathematics or formal logics, empirical methods by statistical analysis, and computational methods by large scale simulations or data driven methods. Among them, automated reasoning can provide a strict and theoretical foundation for validating its correctness from a formal way.

Resolution in classical logic [24], due to its simplicity and completeness for unsatisfiability validation, is a main inference rule used in many famous automated theorem provers (ATPs) such as Prover9 [17], E [25], Vampire [22], etc. In these ATPs, saturation algorithm and its extended forms are their main frameworks for implementation resolution methods. Due to no restriction in literals and clauses selection, many redundant clauses generates. To solve this problem, many restricted resolution methods [7] are proposed, for instance, lock resolution, hyper-resolution, semantic resolution, extension rule[19], etc. Different from restrictions on literals and clauses, contradiction separation based automated deduction [28] is an extension of binary resolution, where dynamic and multiple (two or more) clauses and literals involving in every deductive step, and its implementation CSE-E[6] also has a good performance among others[5,26].

Equality is very common and well known to be useful in many subjects such as mathematics, logic, computer science, etc. Strictly, equality is a congruence relation between two quantities, or more generally two mathematical expressions, asserting that these quantities have the same value, or that the expressions represent the same mathematical object. Unfortunately, the E-unsatisfiability [4,7], which is the unsatisfiability of logical formula S with equality, cannot be judged if we only use the resolution like methods including contradiction separation based methods. There exist two solutions for this problem. The first way is to add the equality axioms set to S, and a new logical formula S_1 is obtained. Then the E-unsatisfiability of S is equivalent to the unsatisfiability of S_1, and hence it can be judged by the resolution and its extended methods. However, the increasing size of S_1 will cause searching space explosion if S includes many function or predicate symbols. The alternative way is called paramodulation [1–3,8,18,23], which is a new inference rule in which the equality symbol satisfies the congruence relation by means of reasoning. Compared with the former method, the paramodulation method can decrease the complexity of logical formula.

As we know, the mental activities of humans are often involved in uncertain information processing, and it is difficult to represent and reason this kind of phenomena of real world in classical logic [16]. To deal with uncertainty especially for incomparability in the intelligent information processing from a symbolism point of view, lattice implication algebra (LIA) [27] and lattice-valued logic [31] based on LIA are proposed by extending the classical logic in many ways such as the truth-valued field, the implication connective and language. Uncertainty reasoning and automated reasoning [29,30] in lattice-valued logic based on LIA is given, and applied in many areas [20,31] such as rule bases, decision making, natural language processing, etc. Concretely, for the automated reasoning aspect, the α-resolution principle is developed in lattice-valued propositional logic $LP(X)$ [15,30] and lattice-valued first-order logic $LF(X)$ [29] as well as their soundness and weak completeness. Its approximate reasoning scheme was also investigated and reported in [9,12–14,31–34].

The equality in lattice-valued logic based on LIA is also an important and special predicate symbol. If we treat the equation as an ordinary one, and only use the α-resolution methods to judge the α-unsatisfiability of S, then the completeness of α-resolution does not hold. Similar to classical logic, for judging the α-unsatisfiability of logical formula S with equality in lattice-valued logic, two main alternatives exist. One is adding the α-equality axioms to the original clauses set S, and get a new clauses set S_1. Then the E_α-unsatisfiability of S is equivalent to the α-unsatisfiability of S_1, which can be judged by the α-resolution principle. However, this method may increase the complexity of S by adding the equality axioms set. The clauses set may become too large if S includes many different predicate symbols or functional symbols. The other is dealing with the logical formula S directly. Of course, it is incomplete if only α-resolution principle is used. We should extend the α-resolution method and develop some complete automated reasoning methods for handling the logical formula with equality in lattice-valued logic.

By combining α-resolution and paramodulation, α-paramodulation was proposed to handle equality logical formulae directly in [10,11]. Two types of α-equality axioms sets were respectively given to guarantee the equivalence of α-equality axioms set K_α and E_α-interpretation for $LF(X)$. However, many conditions should be added to keep its equivalence, and these conditions are too rigor for logical formulae and resolution level α. In this sense, we propose a new form of K_α for $L_nF(X)$ in this paper as an extension of the work [10,11], which can keep the equivalence of α-equality axioms set and α-congruence relation naturally with an appropriate level. Based on this equivalence, we proposed α-paramodulation and α-GH paramodulation methods. The soundness and completeness of the proposed methods are also given.

The remained part of this paper is organized as follows. After a brief overview about lattice-valued logic based on LIA and α-Gv semantic resolution in lattice-valued logic in Section 2, the α_E-unsatisfiability for a lattice-valued logic $L_nF(X)$ is given including equivalence of α-equality axioms set and α-congruence relation, and α_E-unsatisfiability transformation in Section 3. The concepts of α-paramodulation and α-GH paramodulation are given. Their soundness and completeness are obtained in Section 4. Section 5 concludes this paper.

2 Preliminaries

In this section, we only recall some elementary definitions and properties needed in the following discussions, more detailed notations and results about lattice-valued logic based on LIA and α-resolution principle can be seen in [27,29–31].

2.1 α-Resolution principle in lattice-valued logic based on LIA

Definition 1 [27,31] Let $(L,\vee,\wedge,O,I)$ be a bounded lattice with an order-reversing involution "$'$", I and O the greatest and the smallest element of L, respectively, and $\rightarrow: L\times L \longrightarrow L$ a mapping. $\mathscr{L}=(L,\vee,\wedge,',\rightarrow,O,I)$ is called a lattice implication algebra (LIA) if the following conditions hold for any $x,y,z\in L$:

(I_1) $x \to (y \to z) = y \to (x \to z)$,
(I_2) $x \to x = I$,
(I_3) $x \to y = y' \to x'$,
(I_4) $x \to y = y \to x = I$ implies $x = y$,
(I_5) $(x \to y) \to y = (y \to x) \to x$,
(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

In order to deal with quantifiers, in what follows, we suppose that L is a complete lattice.

Definition 2 [27,31] (Łukasiewicz implication algebra on a finite chain L_n) Let L_n be a finite chain, $L_n = \{a_i | 1 \leq i \leq n\}$ and $a_1 < a_2 < \ldots < a_n$, define for any $a_j, a_k \in L_n$,

$$a_j \vee a_k = a_{\max\{j,k\}}, a_j \wedge a_k = a_{\min\{j,k\}}, (a_j)' = a_{n-j+1}, a_j \to a_k = a_{\min\{n-j+k,n\}}.$$

Then $\mathscr{L}_n = (L_n, \vee, \wedge, ', \to, a_1, a_n)$ is an LIA.

Definition 3 [30,31] Let X be a set of propositional variables, T = $L \cup \{', \to\}$ be a type with ar($'$) = 1, ar($\to$) = 2 and ar(a) = 0 for every $a \in L$. The propositional algebra of the lattice-valued propositional calculus on the set X of propositional variables is the free T algebra on X and is denoted by LP(X).

Remark 1 Specially, when $\mathscr{L} = \mathscr{L}_n, LP(X)$ is denoted as $L_nP(X)$.

Definition 4 [30,31] Let $F \in LP(X)$, $\alpha \in L$. If there exists a valuation γ_0 of LP(X) such that $\gamma_0(F) \geq \alpha$, F is satisfiable by a truth-value level α, in short, α-satisfiable. If $\gamma(F) \geq \alpha$ for every valuation γ of LP(X), F is valid by the truth-value level α, in short, α-valid. If $\gamma(F) \leq \alpha$ for every valuation γ of LP(X), F is always false by the truth-value level α, in short, α-false.

Definition 5 [30,31] $F \in LP(X)$ is called an extremely simple form, in short ESF, if $F^* \in$ LP(X) obtained by deleting any constant or literal or implication term appearing in F is not equivalent to F.

Definition 6 [30,31] $F \in LP(X)$ is called an indecomposable extremely simple form, in short IESF, if

(1) F is an ESF containing connectives $\to$ and $'$ at most.
(2) For any $G \in LP(X)$, if $G \in \overline{F}$ in $\overline{LP(X)}$, then G is an ESF containing connectives $\to$ and $'$ at most, where $\overline{LP(X)} = (LP(X)/_=, \vee, \wedge, ', \to)$ is an LIA, $LP(X)/_= = \{\overline{p} \mid p \in LP(X)\}$, $\overline{p} = \{q \mid q \in LP(X), q = p\}$, for any $\overline{p}, \overline{q} \in LP(X)/_=, \overline{p} \vee \overline{q} = \overline{p \vee q}, \overline{p} \wedge \overline{q} = \overline{p \wedge q}, (\overline{p})' = \overline{p'}, \overline{p} \to \overline{q} = \overline{p \to q}$.

Definition 7 [30,31] All the constants, literals and IESFs in $LP(X)$ are called generalized literals.

In LP(X), a disjunction of finite generalized literals is called a generalized clause, and a conjunction of finite generalized clauses is called a generalized conjunctive normal form.

Definition 8 [30,31] Let $\alpha \in L$, G_1 and G_2 be two generalized clauses in $LP(X)$ of the forms $G_1 = g_1 \vee \ldots \vee g_i \vee \ldots \vee g_m$ and $G_2 = h_1 \vee \ldots \vee h_j \vee \ldots \vee h_n$, respectively. If for any valuation I such that $I(g_i \wedge h_j) \leq \alpha$, then

$$G = g_1 \vee \ldots \vee g_{i-1} \vee g_{i+1} \vee \ldots \vee g_m \vee h_1 \vee \ldots \vee h_{j-1} \vee h_{j+1} \vee \ldots \vee h_n$$

is called an α-resolvent of G_1 and G_2, denoted by $G = R_\alpha(G_1, G_2)$, g_i and h_j form an α-resolution pair, denoted by (g_i, h_j)-α. The generation of an α-resolvent from two clauses, called α-resolution, is the sole rule of the α-resolution principle inference.

Definition 9 [30,31] Suppose a generalized conjunctive normal form $S = G_1 \wedge G_2 \wedge \ldots \wedge G_n$ in $LP(X)$, $\alpha \in L$. $w = \{D_1, D_2, \ldots, D_m\}$ is called an α-resolution deduction from S to the generalized clause D_m, if

(1) $D_i \in \{G_1, G_2, \ldots, G_n\}$ or
(2) There exist $j, k < i$, such that $D_i = R_\alpha(D_j, D_k)$.

If there exists an α-resolution deduction from S to the empty clause (denoted by α-$\square$), then w is called an α-refutation of S.

The truth-value domain of lattice-valued first-order logic $LF(X)$ is an LIA. This logic system can be used to deal with propositions with quantifiers [31]. Specially, if the valuation field of $LF(X)$ $\mathscr{L}$ is $\mathscr{L}_n$, then $LF(X)$ is denoted as $L_nF(X)$.

Definition 10 [29] Suppose V and F are the set of variable symbols and that of functional symbols in $LF(X)$, respectively, the set of terms of $LF(X)$ is defined as the minimal set $\mathscr{J}$ satisfying the following conditions:

(1) $V \subseteq \mathscr{J}$.
(2) For any $n \in N \cup \{0\}$, if $f^{(n)} \in F$, then $f^{(n)}(t_0, t_1, \ldots, t_n) \in \mathscr{J}$ for any $t_0, t_1, \ldots, t_n \in \mathscr{J}$.

Definition 11 [29] Suppose P is the predicate symbol set in $LF(X)$. The set of atoms of $LF(X)$ is defined as the smallest set $\mathscr{A}_t$ satisfying the following conditions: For any $n \in N \cup \{0\}$, if $P^{(n)} \in P$, then $P^{(n)}(t_0, t_1, \ldots, t_n) \in \mathscr{A}_t$ for any $t_0, t_1, \ldots, t_n \in \mathscr{J}$.

Definition 12 [29] The set of logical formulae of $LF(X)$ is defined as the smallest set $\mathscr{F}$ satisfying the following conditions:

(1) $\mathscr{A}_t \subseteq \mathscr{F}$.
(2) If $p, q \in \mathscr{F}$, then $p \to q \in \mathscr{F}$.
(3) If $p \in \mathscr{F}$, x is a free variable in p, then $(\forall x)p$, $(\exists x)p \in \mathscr{F}$.

Definition 13 [29] A logical formula G in $LF(X)$ is a g-literal, if

(1) G is a literal, or
(2) G is constructed only by some literals and some implication connectives with the condition that G can not be represented by connectives $\vee$ or $\wedge$ and G can not be decomposed into a simpler form (G is called an indecomposable form).

In $LF(X)$, a disjunction of finite g-literals is called a g-clause, and a conjunction of finite g-clauses is called a g-conjunctive normal form.

2.2 α-Gv semantic resolution for lattice-valued logic based on LIA

Definition 14 [32] Let S be a set of g-clauses in $LF(X)$, $\alpha \in L$, G an order of g-literals of S, I an interpretation in $LF(X)$, then $(E_1, E_2, \ldots, E_q, N)$ is called an α-Gv semantic clash if it satisfies the following conditions.

(1) $I(E_i) \leq \alpha$ $(1 \leq i \leq q)$.
(2) Let $R_1 = N$, for any $i = 1, 2, \ldots q$, there exist g-clauses R_i and E_i, such that $R_{i+1} = R_\alpha(R_i, E_i)$.
(3) The resolved g-literal in E_i has the maximal order in E_i with respect to G.
(4) $I(R_{q+1}) \leq \alpha$.

Then R_{q+1} is called the α-Gv semantic resolvent of $(E_1, E_2, \ldots, E_q, N)$, i.e., $R_{q+1} = R_{\alpha-Gv}(E_1, E_2, \ldots, E_q, N)$.

Definition 15 [32] Suppose S is a set of g-clauses $S = G_1 \wedge G_2 \wedge \cdots \wedge G_n$ in $LF(X)$, $\alpha \in L$. $w = \{D_1, D_2, \ldots, D_m\}$ is called an α-Gv semantic resolution deduction of S from D_1 to D_m, if

(1) $D_i \in \{G_1, G_2, \ldots, G_n\}$, or
(2) there exist $k_1, k_1, \ldots, k_n < i$, such that $D_i = R_{\alpha-Gv}(D_{k_1}, D_{k_2}, \ldots, D_{k_n})$.

Theorem 1 [32] *Let S be a set of g-clauses in $LF(X)$, $\alpha \in L$, $\{D_1, D_2, \ldots, D_m\}$ an α-Gv resolution deduction from S to a g-clause D_m. If $D_m \leq \alpha$, then $S \leq \alpha$.*

Theorem 2 [32] *Let S be a set of g-clauses in $LF(X)$, $\alpha \in L$, I an interpretation in $LF(X)$. Then there exists an α-Gv semantic resolution deduction from S to α-$\Box$ if S is α-unsatisfiable and satisfies the following conditions.*

(1) *For any g-literals $g_1, g_2, \ldots, g_n$ in S, if $g_1 \wedge g_2 \wedge \cdots \wedge g_n \leq \alpha$, then there exist g_i and g_j $(1 \leq i, j \leq n)$, such that $g_i \wedge g_j \leq \alpha$,*
(2) *If for any interpretation I, $I(g_i \wedge g_j) \leq \alpha$, then $I(g_i) \leq \alpha$ and $I(g_j) \leq \alpha$ do not hold simultaneously,*
(3) *If the g-literal g has the minimal order in S, then $I(g) \leq \alpha$,*

3 α_E-Unsatisfiability for a lattice-valued logic $L_nF(X)$

3.1 Equality relation in $L_nF(X)$

Definition 16 Let S be a set of g-clauses in $L_nF(X)$, $\alpha \in L_n$, W the set of all the interpretations of S, $Q \subseteq W (Q \neq \emptyset)$. Then S is α_Q-unsatisfiable if and only if $S \leq \alpha$ with the interpretation Q.

Example 1 Let $g_1 = (\forall x)(P_1(x) \rightarrow a_2)$ be a g-literal in $L_9F(X)$, $\alpha = a_5$, where x is a variable symbol, a_2 is a constant symbol, P_1 is a predicate symbol. For the predicate P_1, we take a special assignment for partial interpretation W of g_1, that is, for any interpretation field D, $W = \{I_W$: assign P_1 to a_7, that is, for any $x \in D$, $I_W(P_1(x)) = a_7\}$, assign constant symbol a_2 to constant $a_2 \in L_9$. Therefore, with the interpretation I_W, we have $I_w(g_1)$= $a_7 \rightarrow a_2 = a_4$. Therefore, g_1 is α_w-unsatisfiable in $L_9F(X)$. However, g_1 is α-satisfiable in $L_9F(X)$.

Remark 2 The interpretation mentioned in this paper is the Herbrand interpretation of S in $L_nF(X)$ [31].

Equality is an important relation in mathematic logic. Especially, in classical logic, the equality predicate symbol satisfies the properties of congruence relation of equality, that is, reflexity, symmetry, transitivity and monotonicity. In [7], a special partial interpretation, E-interpretation, is given, which satisfies its congruence relation. Now we extend the concept of E-interpretation in [23] to E_α-interpretation for $L_nF(X)$.

For convenience, we denote the equation $s = t$ as $E(s,t)$, where E is the equality predicate symbol in $L_nF(X)$.

Definition 17 Let S be a set of g-clauses in $L_nF(X)$. Then the interpretation I is an E_α-interpretation if it satisfies the following conditions.

(1) $I(E(x_1,x_1)) \geq \alpha$.
(2) If $I(E(x_1,x_2)) \geq \alpha$, then $I(E(x_2,x_1)) \geq \alpha$.
(3) If $I(E(x_1,x_2)) \geq \alpha$ and $I(E(x_2,x_3)) \geq \alpha$, then $I(E(x_1,x_3)) \geq \alpha$.
(4) If $I(E(x_j,x_0)) \geq \alpha$ and $I(P(x_1,x_2,\ldots,x_j,\ldots,x_n)) \geq \alpha$, then $I(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$.
(5) If $I(E(x_j,x_0)) \geq \alpha$, then $I(E(f(x_1,x_2,\ldots,x_j,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$.

Where $x_1,x_2,\ldots,x_0,\ldots,x_n$ are variable symbols in $L_nF(X)$, P is an n-ary predicate symbol in S, f is an n-ary function symbol in S.

Remark 3 It is shown from (1), (2), (3), (4), (5) in Definition 17 that the equality predicate E in $L_nF(X)$ should satisfy properties of α-reflexity, α-symmetry, α-transitivity and α-monotonicity for function and predicate symbols, respectively.

Generally, if a clauses set S in $L_nF(X)$ includes equality symbol E, then E should satisfy appropriate logical formulas. Hence an α-equality axioms set K_α is given, that is, if I is an E_α-interpretation, then for every formula $g \in K_\alpha$, $I(g) \geq \alpha$. On the other hand, if every formula in K_α is α-valid with the interpretation I, then I is an E_α-interpretation. In [10,11], two types of α-equality axioms set are given to guarantee the equivalence of α-equality axioms set K_α and E_α-interpretation for $LF(X)$. However, many conditions should be added to keep its equivalence. For example, in [10], if there exists a valuation I_0 such that $I_0(E(x,y)) = a_7$, $I_0(E(y,x)) = a_2$, then $I_0(E(x,y)) \to I_0(E(y,x)) = a_7 \to a_2 = a_4 < a_5$. Therefore, we should take $\alpha = I \in L_n$. This condition is too rigor, because if $\alpha = I$ is a resolution level, then all g-clauses can be resolved. To solve this problem, in this section we propose a new form of α-equality axioms set for $L_nF(X)$, which can keep its equivalence naturally with an appropriate resolution level.

Definition 18 Let S be a set of g-clauses in $L_nF(X)$, $\alpha \in L_n$. Then K_α is an α-equality axioms set of S if the following logical formulas are α-valid clauses.

e_1. $E(x_1,x_1)$,
e_2. $(\alpha \to E(x_1,x_2))' \vee E(x_2,x_1)$,

e_3. $(\alpha \to E(x_1,x_2))' \vee (\alpha \to E(x_2,x_3))' \vee E(x_1,x_3)$,
e_4. $(\alpha \to E(x_j,x_0))' \vee (\alpha \to P(x_1,x_2,\ldots,x_j,\ldots,x_n))' \vee P(x_1,x_2,\ldots,x_0,\ldots,x_n)$,
e_5. $(\alpha \to E(x_j,x_0))' \vee E(f(x_1,x_2,\ldots,x_j,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))$,

where $x_1,x_2,\ldots,x_0,\ldots,x_n$ are variable symbols in $L_nF(X)$, P is an n-ary predicate symbol in S, f is an n-ary function symbol in S.

Remark 4 Specially, if $\alpha = I$, then all formulas in K_α are valid.

To keep the equivalence of α-equality axioms set and E_α-interpretation, we should consider a special set of resolution level α as shown in Definition 19.

Definition 19 Let $(L,\vee,\wedge,',\to,O,I)$ be an LIA. α is called an appropriate level if satisfies: for any $a \in L$, if $a \leq \alpha$, then $(\alpha \to a)' \geq \alpha$.

Proposition 1 *Let $(L_n,\vee,\wedge,',\to,O,I)$ be an LIA. Then $\alpha \in L_n$ is an appropriate level if and only if $\alpha \in \{a \in L_n | a \leq a_{[n/2]}\}$.*

Proof The sufficiency can be easily validated, we only prove the necessity.

Since $\alpha, a \in L_n$, let $\alpha = a_m$, $a = a_i$, then $\alpha \to a = a_m \to a_i = a_{\min\{m,n-m+i\}}$, hence $\alpha \to a = a_{n-\min\{m,n-m+i\}}$. If $\alpha \in L_n$ is an appropriate level, then $n - \min\{m, n-m+i\} \geq m$, that is, $n-m \geq \min\{m,n-m+i\}$ for any $i \leq m$. In this sense, two cases exist as follows.

(1) *If $m \leq [n/2]$, then $n-m \geq [n/2]$, hence $\min\{m,n-m+i\} = m$, that is, $n-m \geq \min\{m,n-m+i\}$ for any $i \leq m$.*
(2) *If $m \geq [n/2]$, then $n-m \leq [n/2]$, hence $\min\{m,n-m+i\} = m$ for some i. In this case, $n-m < \min\{m,n-m+i\}$.*

Therefore, $\alpha \in L_n$ is an appropriate level if and only if $\alpha \in \{a \in L_n | a \leq a_{[n/2]}\}$.

Remark 5 The appropriate levels set $\{a \in L_n | a \leq a_{[n/2]}\}$ is reasonable because we can choose a small truth value α in L_n and it satisfies the sense of the definition of α-resolution.

In the following, we take α as an appropriate level to keep the equivalence of α-equality axioms and E_α-interpretation.

Theorem 3 *Let S be a set of g-clauses in $L_nF(X)$, α an appropriate level, K_α an α-equality axioms set of S. Then I_E is an E_α-interpretation if and only if $I_E(K_\alpha) \geq \alpha$.*

Proof (Sufficiency) e_1) It holds obviously.
e_2) For any E_α-interpretation I_E, two cases exist.

(*i*) *If $I_E(E(x,y)) \geq \alpha$, then $I_E(E(y,x)) \geq \alpha$ since I_E is an E_α-interpretation. Hence $I_E((\alpha \to E(x,y))' \vee E(y,x)) = I_E((\alpha \to E(x,y))') \vee I_E(E(y,x)) \geq I_E(E(y,x)) \geq \alpha$.*
(*ii*) *If $I_E(E(x,y)) \leq \alpha$, then $I_E((\alpha \to E(x,y))' \vee E(y,x)) = I_E((\alpha \to E(x,y))') \vee I_E(E(y,x)) \geq I_E((\alpha \to E(x,y))')$. Since α is an appropriate level, we have $I_E((\alpha \to E(x,y))' \vee E(y,x)) \geq \alpha$.*

Therefore, for any E_α-interpretation I_E, $I_E((\alpha \to E(x,y))' \vee E(y,x)) \geq \alpha$.
e_3) For any E_α-interpretation I_E, two cases exist.

(*i*) *If* $I_E(E(x,y)) \geq \alpha$ *and* $I_E(E(y,z)) \geq \alpha$, *then* $I_E(E(x,z)) \geq \alpha$, *and thus* $I_E((\alpha \to E(x,y))' \vee (\alpha \to E(y,z))' \vee E(x,z)) \geq I_E(E(x,z)) \geq \alpha$.

(*ii*) *If* $I_E(E(x,y)) \leq \alpha$ *or* $I_E(E(y,z)) \leq \alpha$, *without loss of generality, let* $I_E(E(x,y)) \leq \alpha$. *Since* α *is an appropriate level, we have* $I_E((\alpha \to E(x,y))') \leq \alpha$. *Thus we have* $I_E((\alpha \to E(x,y))' \vee (\alpha \to E(y,z))' \vee E(x,z)) = I_E((\alpha \to E(x,y))') \vee I_E((\alpha \to E(y,z))') \vee I_E(E(x,z)) \geq I_E((\alpha \to E(x,y))') \geq \alpha$.

Therefore, for any E_α*-interpretation* I_E, $I_E((\alpha \to E(x,y))' \vee (\alpha \to E(y,z))' \vee E(x,z)) \geq \alpha$.
e_4) *For any* E_α*-interpretation* I_E, *two cases exist.*

(*i*) *If* $I_E(E(x_j,x_0)) \geq \alpha$ *and* $I_E(P(x_1,x_2,\ldots,x_j,\ldots,x_n)) \geq \alpha$, *then* $I_E(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$, *and thus* $I_E((\alpha \to E(x_j,x_0))' \vee (\alpha \to P(x_1,x_2,\ldots,x_j,\ldots,x_n))' \vee P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq I_E(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$.

(*ii*) *If* $I_E(E(x_j,x_0)) \leq \alpha$ *or* $I_E(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \leq \alpha$, *without loss of generality, let* $I_E(E(x_j,x_0)) \leq \alpha$. *Since* α *is an appropriate level, we have* $I_E((\alpha \to E(x_j,x_0))') \geq \alpha$. *Then we have* $I_E((\alpha \to E(x_j,x_0))' \vee (\alpha \to P(x_1,x_2,\ldots,x_0,\ldots,x_n))' \vee P(x_1,x_2,\ldots,x_0,\ldots,x_n)) = I_E(\alpha \to E(x_j,x_0)) \vee I_E((\alpha \to P(x_1,x_2,\ldots,x_0,\ldots,x_n))') \vee I_E(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq I_E((\alpha \to E(x,y))') \geq \alpha$.

Therefore, for any E_α*-interpretation* I_E, $I_E((\alpha \to E(x_j,x_0))' \vee (\alpha \to P(x_1,x_2,\ldots,x_0,\ldots,x_n))' \vee P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$.
e_5) *For any* E_α*-interpretation* I_E, *two cases exist.*

(*i*) *If* $I_E(E(x_j,x_0)) \geq \alpha$, *then* $I_E(E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq \alpha$. *Since* I_E *is an* E_α*-interpretation, we have* $I_E((\alpha \to E(x_j,x_0)') \vee E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq I_E(E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq \alpha$.

(*ii*) *If* $I_E(E(x_j,x_0)) \leq \alpha$, *then since* α *is an appropriate level, we have* $I_E((\alpha \to E(x_j,x_0)') \geq \alpha$. *Therefore,* $I_E((\alpha \to E(x_j,x_0))') \vee E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq I_E((\alpha \to E(x_j,x_0))') \geq \alpha$.

Therefore, for any E_α*-interpretation* I_E, $I_E((\alpha \to E(x_j,x_0))' \vee E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq \alpha$.
According to the proof of e_1), e_2), e_3), e_4) *and* e_5), *for any* E_α*-interpretation* I_E, *we have* $I_E(K_\alpha) \geq \alpha$.
(Necessity) (1) *If* $I(K_\alpha) \geq \alpha$, *then* $I(E(x,x)) \geq \alpha$.

(2) *If* $I(K_\alpha) \geq \alpha$, *then* $I((\alpha \to E(x,y))' \vee E(y,x)) \geq \alpha$, *that is,* $I((\alpha \to E(x,y))') \vee I(E(y,x)) \geq \alpha$. *If* $I(E(x,y)) \geq \alpha$, *then* $I((\alpha \to E(x,y))') = O$, *hence* $I((\alpha \to E(x,y))') \vee I(E(y,x)) = I(E(y,x)) \geq \alpha$, *that is,* $I(E(y,x)) \geq \alpha$.

(3) *If* $I(K_\alpha) \geq \alpha$, *then* $I((\alpha \to E(x,y))' \vee (\alpha \to E(y,z))' \vee E(x,z)) \geq \alpha$, *that is,* $I((\alpha \to E(x,y))') \vee I((\alpha \to E(y,z))') \vee I(E(x,z)) \geq \alpha$. *If* $I(E(x,y)) \geq \alpha$ *and* $I(E(y,z)) \geq \alpha$, *then* $I((\alpha \to E(x,y))') = O$ *and* $I((\alpha \to E(y,z))') = O$. *Hence* $I((\alpha \to E(x,y))') \vee I((\alpha \to E(y,z))') \vee I(E(x,z)) = I(E(x,z)) \geq \alpha$, *that is,* $I(E(x,z)) \geq \alpha$.

(4) *If* $I(K_\alpha) \geq \alpha$, *then* $I((\alpha \to E(x_j,x_0))' \vee (\alpha \to P(x_1,x_2,\ldots,x_0,\ldots,x_n))' \vee P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$, *that is,* $I((\alpha \to E(x_j,x_0))') \vee I((\alpha \to P(x_1,x_2,\ldots,x_0,\ldots,x_n))') \vee$

$P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$. If $I(E(x_j,x_0)) \geq \alpha$ and $I(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$, then $I((\alpha \rightarrow E(x_j,x_0))') = O$ and $I((\alpha \rightarrow P(x_1,x_2,\ldots,x_0,\ldots,x_n))') = O$. Hence $I((\alpha \rightarrow E(x_j,x_0))') \vee I((\alpha \rightarrow P(x_1,x_2,\ldots,x_0,\ldots,x_n))') \vee I(P(x_1,x_2,\ldots, x_0,\ldots,x_n)) = I(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$, that is, $I(P(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$.

(5) *If $I(K_\alpha) \geq \alpha$, then $I((\alpha \rightarrow E(x_j,x_0))') \vee E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n)) \geq \alpha$, that is, $I((\alpha \rightarrow E(x_j,x_0))') \vee I(E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq \alpha$. If $I(E(x_j,x_0)) \geq \alpha$, then $I((\alpha \rightarrow E(x_j,x_0))') = O$, hence $I((\alpha \rightarrow E(x_j,x_0))') \vee I(E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) = I(E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq \alpha$, that is, $I(E(f(x_1,x_2,\ldots,x_0,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))) \geq \alpha$.*

According to the proof of $(1)-(5)$, if $I(K_\alpha) \geq \alpha$, then I is an E_α-interpretation.

3.2 α_E-Unsatisfiability for $L_nF(X)$

Definition 20 Let S be a set of g-clauses in $L_nF(X)$. S is α_E-unsatisfiable if for any E_α-interpretation I_E such that $I_E(S) \leq \alpha$. S is α_E-satisfiable if there exists an E_α-interpretation I_E such that $I_E(S) \geq \alpha$. S is α_E-true if for any E_α-interpretation I_E such that $I_E(S) \geq \alpha$.

Theorem 4 *Let S be a set of g-clauses in $L_nF(X)$, K_α an α-equality axiom set of S, $\alpha \in L_n$. Then S is α_E-unsatisfiable if and only if for any interpretation I, we have $I(S \wedge K_\alpha) \leq \alpha$.*

Proof (Necessity) Since S is α_E-unsatisfiable in $L_nF(X)$, then for any interpretation I_E, we have $I_E(S) \leq \alpha$. If there exists an interpretation I_0, such that $I_0(S \wedge K_\alpha) \geq \alpha$, then $I_0(S) \geq \alpha$ and $I_0(K_\alpha) \geq \alpha$. Since $I_0(K_\alpha) \geq \alpha$, we have I_0 is an E_α-interpretation. However, $I_0(S) \geq \alpha$, which is contradictory to the fact that S is α_E-unsatisfiable. Therefore, for any interpretation I, we have $I(S \wedge K_\alpha) \leq \alpha$.

(Sufficiency) If for any interpretation I, we have $I(S \wedge K_\alpha) \leq \alpha$. If for the interpretation I, such that $I(K_\alpha) > \alpha$, then I is an E_α-interpretation by the definition of E_α-interpretation, and we denote it by I_E. Since $\alpha \in L_n$ is a dual numerator, hence if $I_E(K_\alpha) \geq \alpha$, then $I_E(S) \leq \alpha$, that is, for any interpretation I_E, $I_E(S) \leq \alpha$. Therefore, S is α_E-unsatisfiable.

Theorem 5 *Let S be a set of g-clauses in $L_nF(X)$. Then S is α_E-unsatisfiable if and only if there exists a set of finite ground instance S_1 of S in $L_nP(X)$, such that S_1 is α_E-unsatisfiable.*

Proof (Necessity) Since S is α_E-unsatisfiable in $L_nF(X)$, we have $S \wedge K_\alpha \leq \alpha$ by Theorem 4. By Herbrand Theorem [29] in $L_nF(X)$, there exists a set of finite ground instances $S_1 \wedge K^0_\alpha$ in $L_nP(X)$, such that $S_1 \wedge K^0_\alpha \leq \alpha$. By Theorem 4, S_1 is α_E-unsatisfiable.

(Sufficiency) If there exists a set of finite ground instance S_1 in $L_nP(X)$, such that S_1 is α_E-unsatisfiable, that is, S_1 is α_E-unsatisfiable with the interpretation I_E, i.e., $I_E(S_1) \leq \alpha$. On the other hand, for any interpretation I, we have $I(S) \leq I(S_1)$, hence $I_E(S) \leq I_E(S_1) \leq \alpha$. Therefore, S is α_E-unsatisfiable.

Remark 6 By Theorem 4 and 5, validating the α_E-unsatisfiability of S can be equivalently converted to discussing the α-unsatisfiability of $S \wedge K_\alpha$, but this transformation process may increase the complexity of validating the α-unsatisfiability because more clauses are added to S. If S includes too many predicate or functional symbols, then $S \wedge K_\alpha$ is relatively complex.

4 α-Paramodulation and α-GH paramodulation for $L_nF(X)$

In this section, we consider an inference rule to avoid adding all logical formulas in α-equality axioms set to S, that is, the α-paramodulation in $L_nF(X)$, and therefore α_E-unsatisfiability of S can be validated by combining α-resolution and α-paramodulation. Furthermore, α-GH paramodulation is also proposed to improve the efficiency of α-paramodulation, its soundness and completeness are also shown.

4.1 α-Paramodulation for $L_nF(X)$

Definition 21 Let G_1 and G_2 be g-clauses without the same variables in $L_nF(X)$, $G_1 = g_1[t] \vee G_1^0$, $G_2 = E(s_1, s_2) \vee G_2^0$, where $g_1[t]$ is the g-literal including term t, G_1^0 and G_2^0 are g-clauses. If t and s_1 have an mgu σ, then

$$PR_\alpha(G_1, G_2) = g_1^\sigma[s_2^\sigma] \vee G_1^{0\sigma} \vee G_2^{0\sigma}$$

is called an α-paramodulator of G_1 and G_2, where $g_1^\sigma[s_2^\sigma]$ represents that t^σ in g_1^σ is substituted by s_2^σ.

Example 2 Let $L_9 = \{a_i | 1 \leq i \leq 9\}$ be a Łukasiewicz implication algebra, $G_1 = (P(y) \to P(x))' \vee E(f(a_4), x)$ and $G_2 = E(a_6, a_4)$ g-clauses in $L_9F(X)$, $\alpha = a_5$, where x, y are variable symbols, a_4, a_6 are constant symbols, f is a function symbol, and P is a predicate symbol. Then there exist an α-paramodulator $PR_\alpha(G_1, G_2) = (P(y) \to P(a_6))' \vee E(f(a_6), a_6)$.

Definition 22 Let G_1 and G_2 be g-clauses in $L_nF(X), \alpha \in L_n$. G_1 E_α-implies G_2 if and only if $G_1 \to G_2$ is α_E-true, and denoted by $G_1 \Rightarrow_{\alpha_E} G_2$.

Theorem 6 *Let G_1 and G_2 be g-clauses in $L_nF(X), \alpha \in L_n$, then $G_1 \wedge G_2 \Rightarrow_{\alpha_E} PR_\alpha(G_1, G_2)$.*

Proof For any E_α-interpretation I_E in $L_nF(X)$, if $I_E(G_1 \wedge G_2) \geq \alpha$, then $I_E(G_1) \wedge I_E(G_2) \geq \alpha$, hence $I_E(G_1) \geq \alpha$ and $I_E(G_2) \geq \alpha$. Hence two cases exist.

(1) *If $I_E(G_1^{0\sigma}) \geq \alpha$ or $I_E(G_2^{0\sigma}) \geq \alpha$, then $I_E(PR_\alpha(G_1, G_2)) = I_E(g_1^\sigma[s_2^\sigma] \vee G_1^{0\sigma} \vee G_2^{0\sigma}) \geq I_E(G_i^{0\sigma}) \geq \alpha$, where $i = 1, 2$.*
(2) *If $I_E(G_1^{0\sigma}) \leq \alpha$, and $I_E(G_2^{0\sigma}) \leq \alpha$. Since $I_E(g_1^\sigma[t^\sigma] \vee G_1^{0\sigma}) = I_E(g_1^\sigma[t^\sigma]) \vee I_E(G_1^{0\sigma}) \geq \alpha$, we have $I_E(g_1^\sigma[t^\sigma]) \geq \alpha$. Similarly, $I_E(E(s_1^\sigma, s_2^\sigma)) \geq \alpha$. Since I_E is an E_α-interpretation and t^σ is equal to s_1^σ, we have $I_E(g_1^\sigma[s_2^\sigma]) \geq \alpha$. Hence $I_E(PR_\alpha(G_1, G_2)) \geq \alpha$.*

Therefore, for any E_α-interpretation I_E, if $I_E(G_1 \wedge G_2) \geq \alpha$, then $I_E(PR_\alpha(G_1, G_2)) \geq \alpha$, that is, $G_1 \wedge G_2 \Rightarrow_{\alpha_E} PR_\alpha(G_1, G_2)$.

Definition 23 Suppose S is a set of g-clauses $S = G_1 \wedge G_2 \wedge \cdots \wedge G_n$ in $L_nF(X)$, $\alpha \in L_n$. $w = \{D_1, D_2, \ldots, D_m\}$ is called an α-paramodulation deduction of S from D_1 to D_m, if

(1) $D_i \in \{G_1, G_2, \ldots, G_n\}$, or
(2) there exist $j, k < i$, such that $D_i = R_\alpha(D_j, D_k)$, or
(3) there exist $j, k < i$, such that $D_i = PR_\alpha(D_j, D_k)$.

Theorem 7 *Suppose S is a set of g-clauses $S = G_1 \wedge G_2 \wedge \cdots \wedge G_n$ in $L_nF(X)$, $\alpha \in L_n$. $w = \{D_1, D_2, \ldots, D_m\}$ is an α-paramodulation deduction of S from D_1 to D_m. If D_m is α_E-unsatisfiable, then S is α_E-unsatisfiable.*

Proof *According to the soundness of α-resolution and Theorem 6 in $L_nF(X)$, Theorem 7 follows immediately.*

4.2 α-GH paramodulation for $L_nF(X)$

Definition 24 Let S be a set of g-clauses in $L_nF(X), \alpha \in L_n$. S is called an α-Gv complete clauses set if it satisfies conditions of completeness of α-Gv semantic resolution.

In what follows, the g-clauses sets mentioned are all α-Gv complete clauses sets if without any special statement.

Definition 25 (α-GH resolution) In an α-Gv semantic resolution, if the interpretation I satisfies $I(g) \geq \alpha$ in case g has the form of $g = F'$, where F is a g-clause, then the α-Gv semantic resolution is an α-GH resolution.

Remark 7 Since the conditions of α-Gv complete clauses set only restrict the interpretations for $I(g) \leq \alpha$, not for $I(g) \geq \alpha$, then the conditions in α-GH resolution are not conflict with those in α-Gv semantic resolution. Furthermore, from Definition 25, α-GH resolution is a special case of α-Gv semantic resolution where the involved g-clauses should be their negative forms.

Definition 26 (α-GH resolution deduction) Suppose S is a set of g-clauses $S = G_1 \wedge G_2 \wedge \cdots \wedge G_n$ in $L_nF(X)$, $\alpha \in L_n$. $w = \{D_1, D_2, \ldots, D_m\}$ is called an α-GH resolution deduction of S from D_1 to D_m, if

(1) $D_i \in \{G_1, G_2, \ldots, G_n\}$, or
(2) there exist $j_1, j_2, \ldots, j_k < i$, such that $D_i = R_{\alpha-GH}(D_{j_1}, D_{j_1}, \ldots, D_{j_k})$.

Theorem 8 *(Completeness of α-GH resolution) Let S be a set of g-clauses in $L_nF(X)$, $\alpha \in L_n$. If S is α-unsatisfiable, then there exists an α-GH resolution deduction from S to α-$\Box$.*

Proof *It immediately follows by Theorem 2.*

Definition 27 (α-GH paramodulation) Suppose G is an order of g-literals in G_1 and G_2 in $L_nF(X)$, $\alpha \in L_n$. $PR_{\alpha-GH}(G_1,G_2)$ is called an α-GH paramodulator of G_1 and G_2 if it satisfies the following conditions.

(1) G_1 and G_2 do not include the g-literals with the form F', where F is a g-clause.
(2) The α-GH paramodulated literals in G_1 and G_2 are maximal ones with respect to G.

Definition 28 (α-GH paramodulation deduction) Suppose S is a set of g-clauses $S = G_1 \wedge G_2 \wedge \cdots \wedge G_n$ in $L_nF(X)$, $\alpha \in L_n$. $w = \{D_1, D_2, \ldots, D_m\}$ is called an α-GH paramodulation deduction of S from D_1 to D_m, if

(1) $D_i \in \{G_1, G_2, \ldots, G_n\}$, or
(2) there exist $j_1, j_2, \ldots, j_k < i$, such that $D_i = R_{\alpha-GH}(D_{j_1}, D_{j_2}, \ldots, D_{j_k})$, or
(3) there exist $j_1, j_2, \ldots, j_k < i$, such that $D_i = PR_{\alpha-GH}(D_{j_1}, D_{j_2}, \ldots, D_{j_k})$.

Specially, if w is an α-GH paramodulation deduction from S to α-$\Box$, then w is called an α-GH paramodulation refutation of S.

Theorem 9 *(Soundness) Suppose S is a set of g-clauses $S = G_1 \wedge G_2 \wedge \cdots \wedge G_n$ in $L_nF(X)$, $\alpha \in L_n$, $w = \{D_1, D_2, \ldots, D_m\}$ is an α-GH paramodulation deduction of S from D_1 to D_m. If D_m is α_E-unsatisfiable, then S is α_E-unsatisfiable.*

Proof According to the soundness of α-paramodulationin $L_nF(X)$ discussed in Theorem 7, Theorem 9 follows immediately.

Definition 29 Let S be a set of g-clauses in $L_nF(X)$, $\alpha \in L_n$. F_α is called an α-reflexivity function axioms set if $F_\alpha = \{E(f_i(x_1,x_2,\ldots,x_i), f_i(x_1,x_2,\ldots,x_i)) \mid f_i$ is an i-ary function symbol of $S\}$.

Theorem 10 *(Completeness of α-GH paramodulation deduction) Let S be a set of g-clauses in $L_nF(X)$, $\alpha \in L_n$. If S is α_E-unsatisfiable, and S_1 is the set by adding to $S \cup \{E(x,x)\} \cup F_\alpha$, then there exists an α-GH paramodulation deduction from S_1 to α-$\Box$.*

Proof Since S is α_E-unsatisfiable in $L_nF(X)$, we have $S \cup K_\alpha$ is α-unsatisfiable by Theorem 4, where K_α is the α-equality axioms set of S. By the completeness of α-GH resolution discussed in Theorem 8, there exists an α-GH resolution refutation $w = \{D_1, D_2, \ldots, D_n\}$ of $S \cup K_\alpha$. Therefore, we only need to prove that every resolvent $D_i (i = 1,2,\ldots,n)$ can be also derived by α-GH paramodulation deduction of $S \cup \{E(x,x)\} \cup F_\alpha$. For convenience we denote $S_1 = S \cup \{E(x,x)\} \cup F_\alpha$.

For every D_i, there exists an α-GH clash of $(E_1, E_2, \ldots, E_q, N)$. By the definition of α-GH resolution, we have $v(E_i) \le \alpha (i = 1,2,\ldots,q)$. Hence E_i has not include the literals with the form F'. Therefore, if $N \in S$, then D_i is an α-GH resolvent of S_1. Otherwise, $N \in K_\alpha$, then four cases exist as follows.

(1) *N is the clause of $(\alpha \to E(x_1,x_2))' \vee E(x_2,x_1)$, then there exists $E_1 = E(s_1,s_2) \vee E_1^0$, where E_1^0 is a g-clause, s_1, s_2 are terms in H_S of S, and we have $R_{\alpha-GH}(N,E_1)$ = $E(s_2^\sigma, s_1^\sigma) \vee E_1^{0\sigma}$, where σ is the most general unifier of x_1 and s_1, x_2 and s_2. On the other hand, $PR_{\alpha-GH}(E_1, E(x,x))$ = $E(s_2^\sigma, s_1^\sigma) \vee E_1^{0\sigma}$. Hence, if $N = (\alpha \to E(x_1,x_2))' \vee E(x_2,x_1)$, then $R_{\alpha-GH}(N,E_1)$ =$PR_{\alpha-GH}(E_1,E(x,x))$.*

(2) *Since N is the clause of* $(\alpha \to E(x_1,x_2))' \vee (\alpha \to E(x_2,x_3))' \vee E(x_1,x_3)$*, we know that there exist* $E_1 = E(t_1,t_2) \vee E_1^0$*, and* $E_2 = E(t_3,t_4) \vee E_2^0$*, where* E_1^0 *and* E_2^0 *are g-clauses,* t_1*,* t_2*,* t_3 *and* t_4 *are terms in* H_S *of S, and we have* $R_{\alpha-GH}(N,E_1,E_2)$ $= E(t_1^\sigma,t_4^\sigma) \vee E_1^{0\sigma} \vee E_2^{0\sigma}$*, where* σ *is the most general unifier of* t_2 *and* t_3*. On the other hand, since* σ *is the most general unifier of* t_2 *and* t_3*, we have* $PR_{\alpha-GH}(E_1,E_2)$ $= E(t_1^\sigma,t_4^\sigma) \vee E_1^{0\sigma} \vee E_2^{0\sigma}$*. Hence, if* N=$(\alpha \to E(x_1,x_2))' \vee (\alpha \to E(x_2,x_3))' \vee E(x_1,x_3)$*, then* $R_{\alpha-GH}(N,E_1,E_2) = PR_{\alpha-GH}(E_1,E_2)$*.*

(3) *Since N is the clause of* $(\alpha \to E(x_j,x_0))' \vee (\alpha \to P(x_1,x_2,\ldots,x_j,\ldots,x_n))' \vee P(x_1,x_2,\ldots,x_0,\ldots,x_n)$*, we know that there exist* $E_1 = E(t_j,t_0) \vee E_1^0$ *and* $E_2 = P(s_1,s_2,\ldots,s_j,\ldots,s_n) \vee E_2^0$*, where* E_1^0 *and* E_2^0 *are g-clauses,* t_j*,* t_0*,* s_1*,* s_2*,* $\ldots,s_j,\ldots,s_n$ *are terms in* H_S *of S, and we have* $R_{\alpha-GH}(N,E_1,E_2) = P(s_1^\sigma,s_2^\sigma,\ldots,t_0^\sigma,\ldots,s_n^\sigma) \vee E_1^{0\sigma} \vee E_2^{0\sigma}$*, where* σ *is the most general unifier of* t_j *and* s_j*. On the other hand, since* σ *is the most general unifier of* t_j *and* s_j*, we have* $PR_{\alpha-GH}(E_1,E_2) = P(s_1^\sigma,s_2^\sigma,\ldots,t_0^\sigma,\ldots,s_n^\sigma) \vee E_1^{0\sigma} \vee E_2^{0\sigma}$*. Hence, if* $N = (\alpha \to E(x_j,x_0))' \vee (\alpha \to P(x_1,x_2,\ldots,x_j,\ldots,x_n))' \vee P(x_1,x_2,\ldots,x_0,\ldots,x_n)$*, then* $R_{\alpha-GH}(N,E_1,E_2) = PR_{\alpha-GH}(E_1,E_2)$*.*

(4) *Since N is the clause of* $(\alpha \to E(x_j,x_0))' \vee E(f(x_1,x_2,\ldots,x_j,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))$*, we know that there exists* $E_1 = E(t_j,t_0) \vee E_1^0$*, where* E_1^0 *is a g-clause,* t_j*,* t_0 *are terms in* H_S *of S, and we have* $R_{\alpha-GH}(N,E_1) = E(f(x_1^\sigma,x_2^\sigma,\ldots,t_j^\sigma,\ldots,x_n^\sigma), f(x_1^\sigma,x_2^\sigma,\ldots,t_0^\sigma,\ldots,x_n^\sigma)) \vee E_1^{0\sigma}$*, where* σ *is the most general unifier of* x_j *and* t_j*. On the other hand,* $PR_{\alpha-GH}(E_1,E(f(x_1,x_2,\ldots,x_j,\ldots,x_n), f(x_1,x_2,\ldots,x_j,\ldots,x_n))) = E(f(x_1^\sigma,x_2^\sigma,\ldots,t_j^\sigma,\ldots,x_n^\sigma), f(x_1^\sigma,x_2^\sigma,\ldots,t_0^\sigma,\ldots,x_n^\sigma)) \vee E_1^0$*. Hence, if* N= $(\alpha \to E(x_j,x_0))' \vee E(f(x_1,x_2,\ldots,x_j,\ldots,x_n), f(x_1,x_2,\ldots,x_0,\ldots,x_n))$*, then* $R_{\alpha-GH}(N,E_1) = PR_{\alpha-GH}(E_1,E(f(x_1,x_2,\ldots,x_j,\ldots,x_n), f(x_1,x_2,\ldots,x_j,\ldots,x_n)))$*.*

Therefore, for every α*-GH semantic resolvent* $D_i (i = 1,2,\ldots,n)$*, it can be derived by* α*-GH paramodulation of* S_1*. Furthermore,* $w = \{D_1,D_2,\ldots,D_n\}$ *is an* α*-GH resolution refutation of S, hence we get a corresponding* α*-GH paramodulation deduction from* S_1 *to* α*-□.*

Example 3 Let $L_9 = \{a_i | 1 \le i \le 9\}$ be a Łukasiewicz implication algebra, S be a set of g-clauses in $L_9P(X)$, $\alpha = a_5$. $S = \{(y \to x)' \vee (a_3 \to x) \vee E(a_6,a_4)$, $x \vee E(a_6,a_4)$, $x \to y$, $E(f(a_6),f(a_4))'\}$, where x and y are propositional variables, a_3, a_4 and a_6 are constants, and f is a functional symbol in $L_9F(X)$. Then we get an α-GH paramodulation refutation of S_1 by adding ground term $E(f(a_4),f(a_4))$ to S.

(1) $(y \to x)' \vee (a_3 \to x) \vee E(a_6,a_4)$
(2) $x \vee E(a_6,a_4)$
(3) $x \to y$
(4) $E(f(a_6),f(a_4))'$
(5) $E(f(a_4),f(a_4))$

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(6) $E(a_6,a_4)$ by α-GH resolution of (1), (2) and (3)
(7) $E(f(a_6),f(a_4))$ by α-GH paramodulation of (5) and (6)
(8) α-□ by α-GH resolution of (4) and (7)

Example 4 Let $L_9 = \{a_i | 1 \le i \le 9\}$ be a Łukasiewicz implication algebra, S a set of g-clauses in $L_9F(X)$, $\alpha = a_5$. S = $\{E(f(a),f(b))' \vee P(x)$, $E(f(c),f(d))' \vee (P(x) \to a_2)$,

$(P(y) \to a_2) \vee E(c,d)$, $E(a,b) \vee P(z)\}$, where a, a_2, b, c and d are constant symbols, x, y, z and w are variable symbols, f is a functional symbol and P is a predicate symbol in $L_9F(X)$. Then we get an α-GH paramodulation refutation of S_1 by adding $\{E(x,x)\} \cup \{E(f(x), f(x))\}$ to S.

(1) $E(f(a), f(b))' \vee P(x)$
(2) $E(f(c), f(d))' \vee (P(y) \to a_2)$
(3) $(P(z) \to a_2) \vee E(c,d)$
(4) $E(a,b) \vee P(w)$
(5) $E(f(x), f(x))$

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(6) $E(f(a), f(b)) \vee P(x)$ by α-GH paramodulation of (4) and (5)
(7) $P(x)$ by α-GH resolution of (1) and (6)
(8) $(P(x) \to a_2) \vee E(f(c), f(d))$ by α-GH paramodulation of (3) and (5)
(9) $P(x) \to a_2$ by α-GH resolution of (2) and (8)
(10) α-$\Box$ by α-GH resolution of (7) and (9)

5 Conclusion

This paper proposed α-paramodulation and α-GH paramodulation in a lattice-valued logic $L_nF(X)$ based on LIA for dealing with lattice-valued logical formula with equality. Concretely, a new form of α-equality axioms set was presented to keep the equivalence between α-equality axioms set and E_α-interpretation in $L_nF(X)$, and hence the E_α-unsatisfiability can be transformed. Furthermore, α-paramodulation and α-GH paramodulation were given including their concepts, properties, soundness and completeness. This work may provide a theoretical foundation for more efficient resolution and paramodulation algorithms based automated reasoning in lattice-valued logic with equality since the α-equality axioms set was given. Thus many reasoning methods can be contrived based on it such as new inference rules, restricted methods, etc. The further research will be concentrated on other restricted α-paramodulation methods for handling lattice-valued logical formula with equality and their hybrid ones to further improve the efficiency of automated reasoning in lattice-valued logic.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Bachmair, L., Ganzinger, H., Lynch, C., Snyder, W.: Basic paramodulation. Information and computation **121**(2), 172–192 (1995)
2. Bofill, M., Rubio, A.: Paramodulation with non-monotonic orderings and simplification. Journal of automated reasoning **50**(1), 51–98 (2013)
3. Brand, D.: Proving theorems with the modification method. SIAM Journal on Computing **4**(4), 412–430 (1975)
4. Bridge, J.P., Holden, S.B., Paulson, L.C.: Machine learning for first-order theorem proving. Journal of automated reasoning **53**(2), 141–172 (2014)
5. Brown, C.E., Gauthier, T., Kaliszyk, C., Sutcliffe, G., Urban, J.: Grunge: A grand unified atp challenge. In: P. Fontaine (ed.) Automated Deduction – CADE 27, pp. 123–141. Springer International Publishing, Cham (2019)
6. Cao, F., Xu, Y., Chen, S., Zhong, J., Wu, G.: A contradiction separation dynamic deduction algorithm based on optimized proof search. International Journal of Computational Intelligence Systems **12**(2), 1245–1254 (2019)
7. Chang, C.L., Lee, R.C.T.: Symbolic logic and mechanical theorem proving. Academic press (2014)
8. Echenim, M., Peltier, N., Tourret, S.: An approach to abductive reasoning in equational logic. In: F. Rossi (ed.) Twenty-Third International Joint Conference on Artificial Intelligence, pp. 531–537. AAAI Press, Beijing, China (2013)
9. He, X., Liu, J., Xu, Y., Martínez, L., Ruan, D.: On α-satisfiability and its α-lock resolution in a finite lattice-valued propositional logic. Logic Journal of IGPL **20**(3), 579–588 (2012)
10. He, X., Xu, Y., LI, Y.: α-Lock paramodulation for a lattice-valued first order logic L_nF(X). In: Uncertainty Modelling in Knowledge Engineering and Decision Making: Proceedings of the 12th International FLINS Conference, pp. 477–482. World Scientific (2016)
11. He, X., Xu, Y., Liu, J.: α-Lock paramodulation for lattice-valued propositional logic. In: 2015 10th International Conference on Intelligent Systems and Knowledge Engineering (ISKE), pp. 18–20. IEEE (2015)
12. He, X., Xu, Y., Liu, J., Chen, S.: On compatibilities of α-lock resolution method in linguistic truth-valued lattice-valued logic. Soft Computing **16**(4), 699–709 (2012)
13. He, X., Xu, Y., Liu, J., Chen, S.: A unified algorithm for finding k-IESFs in linguistic truth-valued lattice-valued propositional logic. Soft Computing **18**(11), 2135–2147 (2014)
14. He, X., Xu, Y., Liu, J., Ruan, D.: α-Lock resolution method for a lattice-valued first-order logic. Engineering Applications of Artificial Intelligence **24**(7), 1274–1280 (2011)
15. Liu, J., Ruan, D., Xu, Y., Song, Z.: A resolution-like strategy based on a lattice-valued logic. IEEE Transactions on Fuzzy Systems **11**(4), 560–567 (2003)
16. Lu, J.J., Murray, N.V., Rosenthal, E.: A framework for automated reasoning in multiple-valued logics. Journal of Automated Reasoning **21**(1), 39–67 (1998)
17. McCune, W.: Release of prover9. In: Mile High Conference on Quasigroups, Loops and Nonassociative Systems, Denver, Colorado (2005)
18. Nieuwenhuis, R., Rubio, A.: Paramodulation-based theorem proving. Handbook of automated reasoning **1**, 371–443 (2001)
19. Ouyang, D., Jia, F., Liu, S., Zhang, L.: An algorithm based on extension rule for solving $\sharp$sat using complementary degree. Journal of Computer Research and Development **53**(7), 1596–1604 (2016)
20. Pei, Z., Ruan, D., Liu, J., Xu, Y.: Linguistic values based intelligent information processing: theory, methods, and applications, vol. 259. Springer (2010)
21. Provost, F., Fawcett, T.: Data science and its relationship to big data and data-driven decision making. Big data **1**(1), 51–59 (2013)
22. Riazanov, A., Voronkov, A.: Vampire 1.1. In: International Joint Conference on Automated Reasoning, pp. 376–380. Springer (2001)
23. Robinson, G., Wos, L.: Paramodulation and theorem-proving in first-order theories with equality. In: Automation of Reasoning, pp. 298–313. Springer (1983)
24. Robinson, J.A.: A machine-oriented logic based on the resolution principle. Journal of the ACM (JACM) **12**(1), 23–41 (1965)
25. Schulz, S.: E–a brainiac theorem prover. AI Communications **15**(2, 3), 111–126 (2002)
26. Sutcliffe, G.: The CADE-27 automated theorem proving system competition–CASC-27. AI Communications **32**(5-6), 373–389 (2019)
27. Xu, Y.: Lattice implication algebras. Journal of Southwest Jiaotong University **28**(1), 20–27 (1993)

28. Xu, Y., Liu, J., Chen, S., Zhong, X., He, X.: Contradiction separation based dynamic multi-clause synergized automated deduction. Information Sciences **462**, 93–113 (2018)
29. Xu, Y., Ruan, D., Kerre, E., Liu, J.: α-Resolution principle based on lattice-valued first-order lattice-valued logic LF(X). Information Sciences **132**(1-4), 221–239 (2001)
30. Xu, Y., Ruan, D., Kerre, E.E., Liu, J.: α-Resolution principle based on lattice-valued propositional logic LP(X). Information Sciences **130**(1-4), 195–223 (2000)
31. Xu, Y., Ruan, D., Qin, K., Liu, J.: Lattice-Valued Logic: An Alternative Approach to Treat Fuzziness and Incomparability. Springer-Verlag, Heidelberg (2003)
32. Zhang, J., Xu, Y., He, X.: α-Generalized semantic resolution method in linguistic truth-valued propositional logic $L_{V(n\times 2)}$P(X). International Journal of Computational Intelligence Systems **7**(1), 160–171 (2014)
33. Zhong, X., Xu, Y., Liu, J., Chen, S.: General form of α-resolution principle for linguistic truth-valued lattice-valued logic. Soft Computing **16**(10), 1767–1781 (2012)
34. Zou, L., Li, X., Pan, C., Liu, X.: (α, β)-Ordered linear resolution of intuitionistic fuzzy propositional logic. Information Sciences **414**, 329–339 (2017)