Mapgerms of $\mathcal{A}_e$-codimension One

by

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Summary

We investigate the structure of multigerms of complex analytic and real smooth maps. The results proved are then used to classify multigerms of maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ with the property that each component has corank at most one. We then show that all the maps we have classified are quasihomogeneous, have image Milnor number one and have good real forms.

Contents

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Declaration

The work in this thesis is, as far as I am aware, original except where the contrary is stated.

§0 Introduction

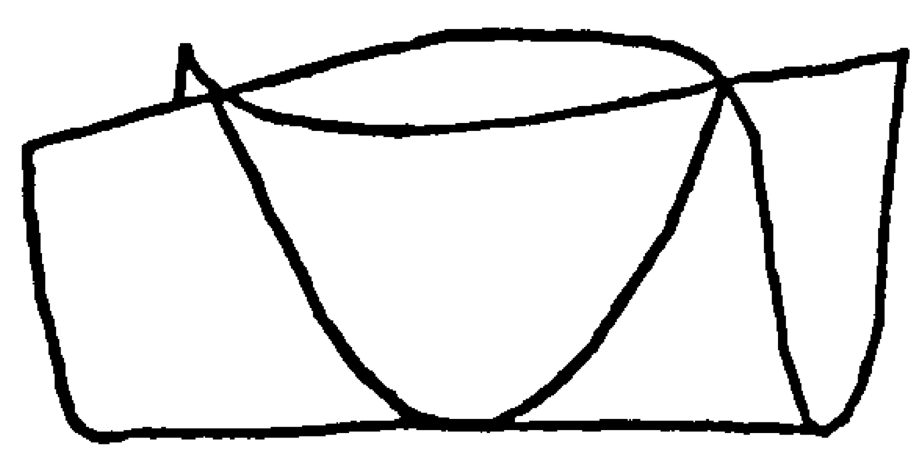
Let S be a finite subset of $\mathbf{C}^n$. Let $[\mathbf{C}^n, S \to \mathbf{C}^p, \{0\}]$ denote the set of pairs (U, f) where U is a neighbourhood of S and $f: U \to \mathbf{C}^p$ is an analytic map such that $f(S) \subseteq \{0\}$. We define an equivalence relation on $[\mathbf{C}^n, S \to \mathbf{C}^p, \{0\}]$ by making (U, f) equivalent to (V, g) if and only if f and g agree on some neighbourhood of S. Equivalence classes of this relation are called multigerms of mappings from $\mathbf{C}^n, S$ to $\mathbf{C}^p, \{0\}$.

Two multigerms $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g: \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are said to be $\mathcal{A}$-equivalent if there are bianalytic maps $\phi: \mathbf{C}^n, S \to \mathbf{C}^n, T$ and $\psi: \mathbf{C}^p, \{0\} \to \mathbf{C}^p, \{0\}$ such that $f \circ \phi = \psi \circ g$. This thesis is concerned with properties of multigerms which hold up to $\mathcal{A}$-equivalence.

An analytic multigerm $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ is stable if every map sufficiently close to f (i.e., for any family $\{f_\lambda\}$ with $f = f_0$, f_λ for λ sufficiently small) is equivalent to f, up to a change of coordinates in the source and target (i.e., there are bianalytic maps in the source and target of f that when composed with f give f_λ. Stable multigerms have been classified by Mather.

A multigerm f is said to have $\mathcal{A}_e$-codimension one if there is one degree of freedom in the maps sufficiently close to f, up to a change of coordinates in the source and target (a multigerm is stable if and only if it has $\mathcal{A}_e$-codimension zero). In order to have some examples we give pictures of the images of the real parts of the five different $\mathcal{A}_e$-codimension one multigerms $\mathbf{C}^2 \to \mathbf{C}^3$. You should be able to see that each picture has one degree of freedom.

1



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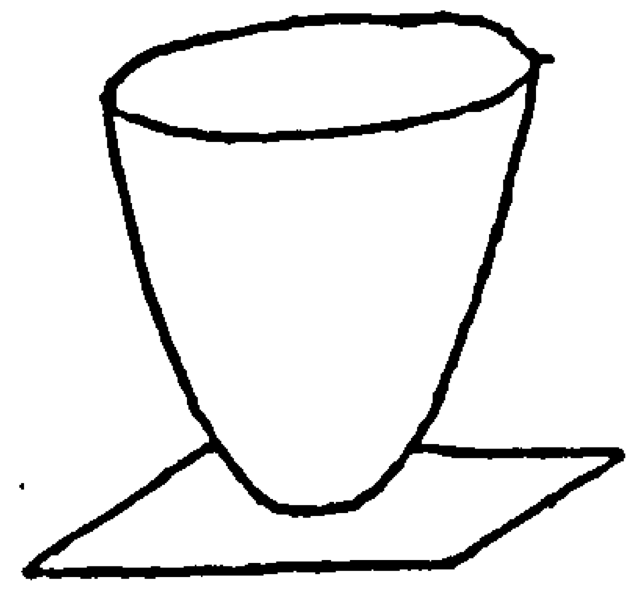

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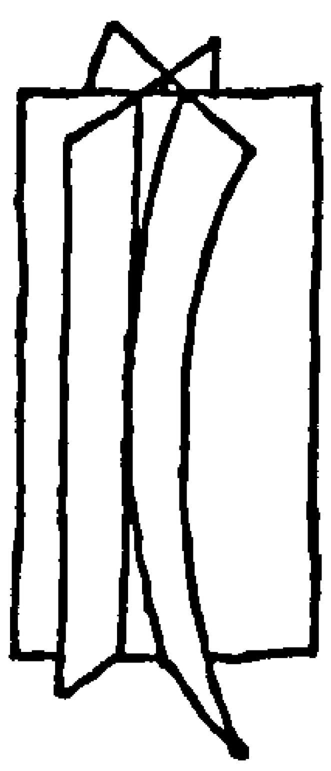

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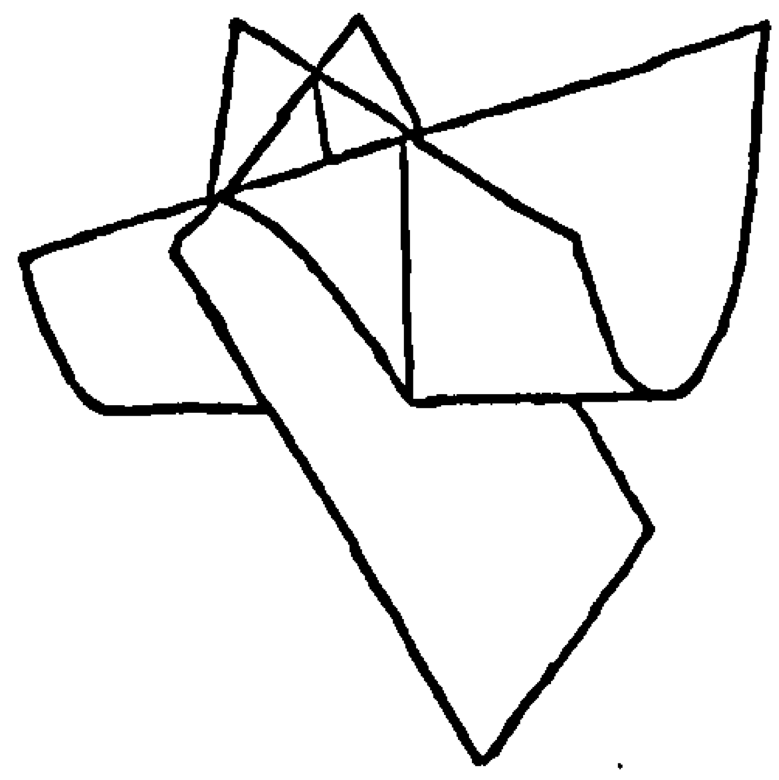

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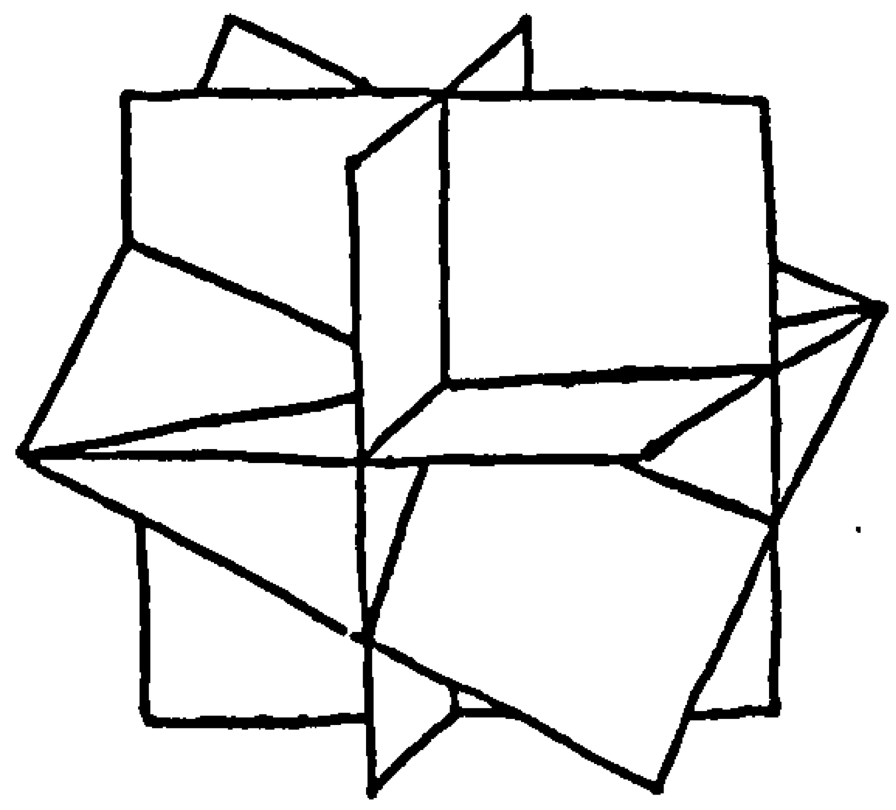

For example, picture 4 is a cross cap ($(x, y) \mapsto (x, y^2, xy)$) and a plane through the singular point, the deformation parameter moves the plane away.

This thesis is about understanding the diverse phenomena that occur in these pictures.

Theorem

If $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ has $\mathcal{A}_e$-codimension one, let

$$F: \mathbf{C} \times \mathbf{C}^n, \{0\} \times S \to \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\}$$
$$(\lambda, x) \mapsto (\lambda, f_\lambda(x))$$

be a miniversal unfolding of $f = f_0$ and let

$$\mathrm{A}_F f: \mathbf{C} \times \mathbf{C}^n, \{0\} \times S \to \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\}$$
$$(\lambda, x) \mapsto (\lambda, f_{\lambda^2}(x)).$$

Then $\mathrm{A}_F f$ has $\mathcal{A}_e$-codimension one, and up to a change of coordinates in source and target depends only on f; furthermore it is even independent of a change of coordinates in the source and target of f.

We call $A_F f$ the augmentation of f and call multigerms which are not augmentations (up to a change of coordinates) primitive. The codimension one germs shown in pictures 4 and 5 are primitive. Those shown in pictures 1 2 and 3 are the augmentations of the three germs whose real pictures are shown here.

6

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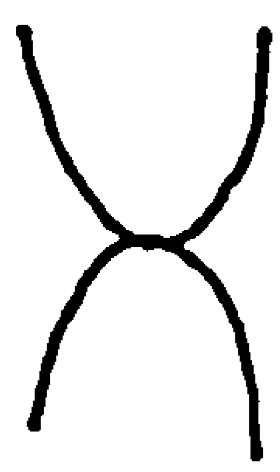

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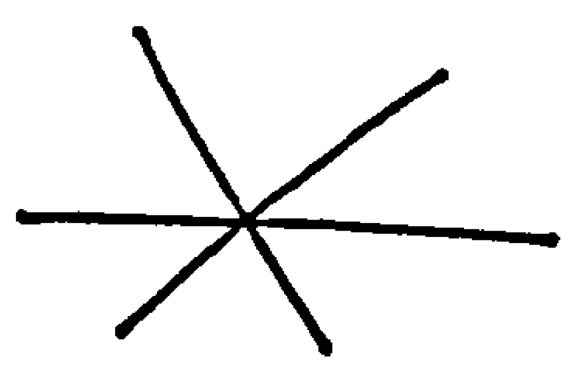


Furthermore the first of these pictures is itself the augmentation of the map from two copies of $\mathbf{C}^0$ to $\mathbf{C}$ sending both points to $0 \in \mathbf{C}$ (we shall refer to this map as the double point). We now concentrate on primitive multigerms and for simplicity assume $n \leq p - 1$ (for this introduction only).

Theorem

If $h: \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$ is a codimension one multigerm made up of $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g: \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ then there is a partition $p = a+b+1$ and a change of coordinates in the source and target of h to reduce f to one of the following forms:

i) $f: \mathbf{C}^a = \mathbf{C}^a \times \{0\} \times \{0\} \subseteq \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$

ii) $F: \mathbf{C}^a \times \mathbf{C}^{n-a-1} \times \mathbf{C} \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}, \qquad (\mu, x, \lambda) \mapsto (\mu, \hat{f}_\lambda(x), \lambda)$ where $\hat{f}: \mathbf{C}^{n-a-1} \to \mathbf{C}^b$ has codimension one and $\mathrm{id}_\mathbf{C} \times \hat{f}_\lambda$ is a miniversal unfolding of $\hat{f}$.

and g to an analogous form with a and b swapped etcetera.

Under reasonable conditions, a converse to this theorem exists.

Applying this theorem to picture 4 we could have, for example, g of type i) and f of type ii) with $\hat{f}$ the map depicted in picture 7. For picture 5 we could have g of type i) and f of type ii) with $\hat{f}$ depicted in picture 6. For picture 8 we could have g of type i) and f of type ii) with $\hat{f}$ equal to the double point. Finally the double point itself can be put into the form of this theorem with both f and g having type i).

Using these results we reduce problems about general codimension one multigerms to questions about primitive monogerms. In this thesis certain codimension one monogerms are classified and this leads to a classification of the corresponding codimension one multigerms as well as some topological information about them.

§1 Overview

In chapter two we introduce a procedure (which we call *augmentation*) which can be applied to any $\mathcal{A}_e$-codimension one analytic mapgerm from $\mathbb{C}^n$ to $\mathbb{C}^p$ (for n and p in $\mathbb{N}_0$) to produce an $\mathcal{A}_e$-codimension one mapgerm from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{p+1}$. This procedure is a generalisation of one in [4].

Chapter three is devoted to giving a characterisation of which $\mathcal{A}_e$-codimension one germs are augmentations of others. We call those codimension one germs that are not augmentations *primitive*. A motif of this thesis is that to prove all codimension one germs possess a property it is sufficient to prove i) that all primitive germs have the property and ii) if a germ has the property then so also does its augmentation.

Chapter four can be thought of as the central chapter of this thesis. Theorem 4.28. is the main result of this thesis and is used repeatedly to gain information about the structure of multigerms. It describes, under certain circumstances, a codimension one multigerm in terms of simpler codimension one multigerms. This is formally analogous to section one of [11] where Mather describes a stable (i.e. a codimension zero) multigerm in terms of stable monogerms.

Chapter five remarks on two generalisations of the results so far. The most important of these generalisations is that, modulo some minor modifications, the results also apply to the real-smooth category.

In order to apply the results of chapter four in a concrete situation it is necessary to know something about codimension one monogerms. Therefore in chapter six we classify the codimension one monogerms of corank one from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ (for n in $\mathbb{N}_0$). In particular this includes all codimension one monogerms from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ where n is less than six. The main results of this chapter (namely Theorems 6.9. and 6.10.) should be compared to [13] where Mond classifies all monogerms of maps from $\mathbb{C}^2$ to $\mathbb{C}^3$ of codimension at most 6; see also [17] which further extends this classification. The proofs of Theorems 6.1. and 6.1. are rather technical and I suggest you skip them on a first reading; it is possible that these proofs can be partially simplified by the use of Mather's lemma (3.1 of [11]).

Chapter seven uses the results of chapters four and six to give a classification of multigerms of maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ each of whose components has corank at most one. Also we give a complete list of the corresponding real smooth multigerms (but without uniqueness) and make a conjecture about what the classification statement should be in this case. Both the real and complex germs are quasihomogeneous.

Finally in chapter eight we use the results of chapter four again, this time to prove that the complex multigerms we have classified all have Milnor number one (this adds evidence to a conjecture of Mond that quasihomogeneous map germs have Milnor number equal to their codimension). We also show that these complex multigerms all have real forms with good perturbations (a good perturbation is one whose n^{th} homology group has the same rank as the n^{th} homology group of the complexification of the multigerm), compare [9].

§2 Augmentations

Let $\mathcal{O}_{\mathbf{C}^n}$ be the ring of germs of analytic functions on $\mathbf{C}^n, \{0\}$ and let $\mathbf{m}_n$ be its maximal ideal. Let $\theta(n)$ be the $\mathcal{O}_{\mathbf{C}^n}$-module of germs of analytic vector fields on $\mathbf{C}^n, \{0\}$. If S is a finite subset of $\mathbf{C}^n$ and $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ is a multigerm, let $\theta(f)$ be the set of germs of analytic vector fields along f. Let $tf: \theta(n)^s \to \theta(f)$, $\xi \mapsto Tf \circ \xi$ and $wf: \theta(p) \to \theta(f)$, $\eta \mapsto \eta \circ f$. Two multigerms $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g: \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are said to be $\mathcal{A}$-equivalent if there are bianalytic maps $\phi: \mathbf{C}^n, S \to \mathbf{C}^n, T$ and $\psi: \mathbf{C}^p, \{0\} \to \mathbf{C}^p, \{0\}$ such that $\psi \circ f = g \circ \phi$.

$T\mathcal{A}f$ is defined to be $tf(\mathbf{m}_n\theta(n)^s) + wf(\mathbf{m}_p\theta(p))$ and the $\mathcal{A}$-codimension of f is $\dim_{\mathbf{C}}[\mathbf{m}_n\theta(f)/T\mathcal{A}f]$. $T\mathcal{A}_e f$ is defined to be $tf(\theta(n)^s) + wf(\theta(p))$ and the $\mathcal{A}_e$-codimension of f is $\dim_{\mathbf{C}}[\theta(f)/T\mathcal{A}_e f]$.

An unfolding F of f is said to be versal if every unfolding of f can be induced from F up to a change of coordinates in the source and target of F parallel to the source and target of f respectively.

Throughout this section let S be a finite subset of $\mathbf{C}^n$, let $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ be a multi-germ of $\mathcal{A}_e$-codimension one and let

$$\begin{aligned} F = \mathrm{id}_{\mathbf{C}} \times f_\lambda : \mathbf{C} \times \mathbf{C}^n, \{0\} \times S &\to \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \\ (\lambda, x) &\mapsto (\lambda, f_\lambda(x)) \end{aligned}$$

(where $f_0 = f$) be an $\mathcal{A}_e$-versal unfolding of f.

Let $k(n,p)$ denote the set of $\mathcal{A}$-equivalence classes of multigerms of $\mathcal{A}_e$-codimension k maps from $\mathbf{C}^n, S$ to $\mathbf{C}^p, \{0\}$. We define a function $\mathrm{A}: 1(n,p) \to 1(n+1, p+1)$. Define $\mathrm{A}_F f: \mathbf{C} \times \mathbf{C}^n, \{0\} \times S \to \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\}$ by $(\lambda, x) \mapsto (\lambda, f_{\lambda^2}(x))$.

Lemma 2.1

If $\alpha: \mathbf{C}, \{0\} \to \mathbf{C}, \{0\}$ is bi-analytic then there is a bi-analytic $\beta: \mathbf{C}, \{0\} \to \mathbf{C}, \{0\}$ such that this diagram commutes.

$$\begin{array}{ccc} \mathbf{C}, \{0\} & \xrightarrow{x \mapsto x^2} & \mathbf{C}, \{0\} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\ \mathbf{C}, \{0\} & \xrightarrow{x \mapsto x^2} & \mathbf{C}, \{0\} \end{array}$$

Proof

Let $\sum_{i=0}^{\infty} a_i x^i$ be the power series expansion of α about 0. Since $\alpha(0) = 0$, $a_0 = 0$ and since α is bi-analytic, $a_1 \neq 0$. Now $\gamma := \sum_{i=1}^{\infty} a_i x^{i-1}$ defines an analytic function on a neighbourhood of 0 and $x \cdot \gamma = \alpha$. Since $a_1 \neq 0$, $\gamma(0) \neq 0$ so in some neighbourhood of $\gamma(0)$ there is an analytic square root function δ. Now the function β defined by $\beta(x) := x \cdot (\delta \circ \gamma(x^2))$ satisfies our requirements. ⋈

Proposition 2.2

The $\mathcal{A}$-equivalence class of $\mathrm{A}_F f$ is independent of the unfolding F of f chosen.

Proof

We prove a stronger result: that the equivalence class of $\mathrm{A}_F f$ *as an unfolding of* f is independent of F. Let $F' = f'_\lambda \times \mathrm{id}_{\mathbf{C}}$ (where $f'_0 = f$) be another $\mathcal{A}$-versal unfolding of f. Clearly F and F' are both mini-versal and so they are isomorphic as unfoldings i.e., there is a bi-analytic map $\alpha: \mathbf{C}, \{0\} \to \mathbf{C}, \{0\}$ and there are germs of one-parameter families of bi-analytic maps $\phi_\lambda: \mathbf{C}^n \to \mathbf{C}^n$ and $\psi_\lambda: \mathbf{C}^p \to \mathbf{C}^p$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} \arrow[r, "F"] \arrow[d, "\alpha \times \phi_\lambda"] & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \arrow[d, "\alpha \times \psi_\lambda"] \\
\mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} \arrow[r, "F'"] & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\}
\end{tikzcd}$$

Now, if we choose β as in Lemma 2.1 then all but the front face of this next diagram commute, so the front face does also.

$$\begin{tikzcd}
\mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} \arrow[rr, "\mathrm{A}_F f"] \arrow[dr, "(x \mapsto x^2) \times \mathrm{id}_{\mathbf{C}^n}"] \arrow[dd, "\beta \times \phi_\lambda"] & & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \arrow[dr, "(x \mapsto x^2) \times \mathrm{id}_{\mathbf{C}^p}"] \arrow[dd, "\beta \times \psi_\lambda"] & \\
& \mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} \arrow[rr, "F"] \arrow[dd, "\alpha \times \phi_\lambda"] & & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \arrow[dd, "\alpha \times \psi_\lambda"] \\
\mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} \arrow[rr, "\mathrm{A}_{F'} f"] \arrow[dr, "(x \mapsto x^2) \times \mathrm{id}_{\mathbf{C}^n}"] & & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \arrow[dr, "(x \mapsto x^2) \times \mathrm{id}_{\mathbf{C}^p}"] & \\
& \mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} \arrow[rr, "F'"] & & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\}
\end{tikzcd}$$

$\bowtie$

We shall write $\mathrm{A}f$ for the $\mathcal{A}$-equivalence class of $\mathrm{A}_F f$

Proposition 2.3

If f is $\mathcal{A}$-equivalent to f' then $\mathrm{A}f = \mathrm{A}f'$.

Proof

If f is $\mathcal{A}$-equivalent to f' then there are germs of bi-analytic maps $\phi\colon \mathbf{C}^n, S \to \mathbf{C}^n, S$ and $\psi\colon \mathbf{C}^p, \{0\} \to \mathbf{C}^p, \{0\}$ such that $\psi \circ f = f' \circ \phi$. Now $F' := (\psi \times \mathrm{id}_\mathbf{C}) \circ F \circ (\phi^{-1} \times \mathrm{id}_\mathbf{C}) = (\psi \circ f_\lambda \circ \phi^{-1}) \times \mathrm{id}_\mathbf{C}$ is a versal unfolding of f' and $(\psi^{-1} \times \mathrm{id}_\mathbf{C}) \circ \mathrm{A}_{F'} f' \circ (\phi \times \mathrm{id}_\mathbf{C}) = (\psi^{-1} \times \mathrm{id}_\mathbf{C}) \circ ([\psi \circ f_{\lambda^2} \circ \phi^{-1}] \times \mathrm{id}_\mathbf{C}) \circ (\phi \times \mathrm{id}_\mathbf{C}) = f_{\lambda^2} \times \mathrm{id}_\mathbf{C} = \mathrm{A}_F f$. Thus $\mathrm{A}_{F'} f' \sim_\mathcal{A} \mathrm{A}_F f$ and so $\mathrm{A}f' = \mathrm{A}f$.

$\bowtie$

We call $\mathrm{A}f$ the augmentation of f and say that a multi-germ is an augmentation if and only if it is the augmentation of some multi-germ f.

The instability locus of a map f is the set of points y in the target such that $f\colon \mathbf{C}^n, f^{-1}(y) \to \mathbf{C}^p, \{0\}$ is unstable.

Lemma 2.4

$\mathrm{A}f$ is finitely $\mathcal{A}$-determined.

Proof

The multi-germ f has $\mathcal{A}_e$-codimension one and hence is finitely determined. This implies that the instability locus of f is contained in $\{0\}$. Now the instability locus of $\mathrm{A}_F f = \mathrm{id}_\mathbf{C} \times f_{\lambda^2}$ is contained in $\mathbf{C} \times \{0\}$ because the vertical arrows in this diagram are bi-analytic away from $\lambda = 0$ and the bottom map is stable.

$$\begin{array}{ccc}
\mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} & \stackrel{\mathrm{id}_\mathbf{C} \times f_{\lambda^2}}{\longrightarrow} & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \\
\Big\downarrow{\scriptstyle (\lambda \mapsto \lambda^2) \times \mathrm{id}_{\mathbf{C}^n}} & & \Big\downarrow{\scriptstyle (\lambda \mapsto \lambda^2) \times \mathrm{id}_{\mathbf{C}^p}} \\
\mathbf{C} \times \mathbf{C}^n, \{0\} \times \{0\} & \stackrel{\mathrm{id}_\mathbf{C} \times f_{\lambda}}{\longrightarrow} & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\}
\end{array}$$

At points $(\lambda, x) \in \mathbf{C} \times \mathbf{C}^p$ where $\lambda = 0$ but $x \neq 0$, f is stable so $\mathrm{A}_F f$ is stable there as well. Therefore the instability locus of $\mathrm{A}_F f$ is contained in $\{0\} \times \{0\} \subseteq \mathbf{C} \times \mathbf{C}^p$ so $\mathrm{A}_F f$ is finitely determined.

$\bowtie$

Theorem 2.5

The $\mathcal{A}_e$-codimension of $\mathrm{A}f$ is one.

Proof

Let the co-ordinates of the source $\mathbf{C} \times \mathbf{C}^n$ of $\mathrm{A}_F f$ be $\lambda, x_1, \ldots, x_n$ and those of the target $\mathbf{C} \times \mathbf{C}^p$ be $\Lambda, X_1, \ldots, X_p$. We will show $T\mathcal{A}_e \mathrm{A}_F f = M$ where

$$M := \left[\mathcal{O}_{\mathbf{C} \times \mathbf{C}^n} \frac{\partial}{\partial \Lambda} + \lambda \theta(\mathrm{A}_F f)\right] \oplus T\mathcal{A}_e f.$$

The result will follow because

$$\theta(\mathrm{A}_F f) = \left[\mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}\frac{\partial}{\partial\Lambda} + \lambda\theta(\mathrm{A}_F f)\right] \oplus \theta(f).$$

Notice that for $i \in \{1,\dots n\}$, $t\mathrm{A}_F f(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}(\mathrm{id}_{\mathbf{C}} \times f_{\lambda^2}) = \frac{\partial}{\partial x_i}(f_{\lambda^2})$ and $tf(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}f$. Since λ divides $f_{\lambda^2} - f$, it must also divide $\frac{\partial}{\partial x_i}f_{\lambda^2} - \frac{\partial}{\partial x_i}f$ and hence

$$t\mathrm{A}_F f\left(\frac{\partial}{\partial x_i}\right) - tf\left(\frac{\partial}{\partial x_i}\right) \in \lambda\theta(\mathrm{A}_F f). \tag{1}$$

Also

$$t\mathrm{A}_F f\left(\frac{\partial}{\partial\lambda}\right) = \frac{\partial}{\partial\lambda}(\mathrm{id}_{\mathbf{C}} \times f_{\lambda^2}) = \frac{\partial}{\partial\Lambda} + 2\lambda\frac{\partial f_t}{\partial t}. \tag{2}$$

Finally $\mathrm{A}_F f|_{\lambda=0} = f$ so for $a \in \mathcal{O}_{\mathbf{C}\times\mathbf{C}^p}$, the difference $a \circ \mathrm{A}_F f - a \circ f$ is divisible by λ, hence for $\eta \in \theta_{\mathbf{C}\times\mathbf{C}^p}$,

$$w\mathrm{A}_F f(\eta) - wf(\eta) \in \lambda\theta(\mathrm{A}_F f). \tag{3}$$

$\subseteq$: It is sufficient to show that $t\mathrm{A}_F f(\theta_{\mathbf{C}\times\mathbf{C}^n})$ and $w\mathrm{A}_F f(\theta_{\mathbf{C}\times\mathbf{C}^p})$ are both contained in M.

Let $\xi = a\frac{\partial}{\partial\lambda} + \sum_{i=1}^n a_i\frac{\partial}{\partial x_i}$ be an arbitrary element of $\theta_{\mathbf{C}\times\mathbf{C}^n}$ (where $a, a_i \in \mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}$), then, for i in $\{1,\dots,n\}$, there is a b_i in $\mathcal{O}_{\mathbf{C}^n}$ such that $a_i - b_i \in \lambda\mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}$ (namely $b_i := a_i|_{\lambda=0}$). Now by (2),

$$\begin{aligned} t\mathrm{A}_F f\left(a\frac{\partial}{\partial\lambda}\right) &= a\cdot t\mathrm{A}_F f\left(\frac{\partial}{\partial\lambda}\right) \\ &= a\frac{\partial}{\partial\Lambda} + \lambda\cdot 2a\frac{\partial f_t}{\partial t} \in \mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}\frac{\partial}{\partial\Lambda} + \lambda\theta(\mathrm{A}_F f) \subseteq M. \end{aligned}$$

Also, for i in $\{1,\dots,n\}$, (1) implies

$$t\mathrm{A}_F f\left(b_i\frac{\partial}{\partial x_i}\right) - tf\left(b_i\frac{\partial}{\partial x_i}\right) = b_i\left[t\mathrm{A}_F f\left(\frac{\partial}{\partial x_i}\right) - tf\left(\frac{\partial}{\partial x_i}\right)\right] \in \lambda\theta(\mathrm{A}_F f)$$

so

$$t\mathrm{A}_F f\left(b_i\frac{\partial}{\partial x_i}\right) \in \lambda\theta(\mathrm{A}_F f) + T\mathcal{A}_e f \subseteq M.$$

Finally for $i \in \{1,\dots,n\}$ again,

$$t\mathrm{A}_F f\left((a_i - b_i)\frac{\partial}{\partial x_i}\right) \in \lambda\theta(\mathrm{A}_F f) \subseteq M.$$

Now we can see

$$\begin{aligned}
tA_Ff(\xi) &= tA_Ff\left(a\frac{\partial}{\partial\lambda}+\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+\sum_{i=1}^{n}(a_i-b_i)\frac{\partial}{\partial x_i}\right)\\
&= tA_Ff\left(a\frac{\partial}{\partial\lambda}\right)+\sum_{i=1}^{n}tA_Ff\left(b_i\frac{\partial}{\partial x_i}\right)\\
&\quad+\sum_{i=1}^{n}tA_Ff\left((a_i-b_i)\frac{\partial}{\partial x_i}\right)\in M.
\end{aligned}$$

Since ξ was arbitrary, $tA_Ff(\theta_{\mathbf{C}\times\mathbf{C}^n})$ is contained in M.

Let η be an arbitrary element of $\theta_{\mathbf{C}\times\mathbf{C}^p}$, then by (3), $wA_Ff(\eta)\in\lambda\theta(A_Ff)+T\mathcal{A}_ef\subseteq M$. Since η was arbitrary, $wA_Ff(\theta_{\mathbf{C}\times\mathbf{C}^p})$ is contained in M.

$\supseteq$: It suffices to show that each of $\mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}\frac{\partial}{\partial\Lambda}$, $\lambda\theta A_Ff$, $tf(\theta_{\mathbf{C}^n})$ and $wf(\theta_{\mathbf{C}^p})$ is contained in $T\mathcal{A}_eA_Ff$.

By Lemma 2.4, $T\mathcal{A}_eAf$ contains $\mathbf{m}_n^l\theta(A_Ff)$ for some natural number l. Then in particular, $\lambda^l\theta(A_Ff)$ is contained in $T\mathcal{A}_eAf$. Let m be the least natural number such that $\lambda^m\theta(A_Ff)\subseteq T\mathcal{A}_eA_Ff$.

Suppose $m\geq 2$ and let $\nu\in\lambda^{m-1}\theta(A_Ff)$ be homogeneous in λ of degree $m-1$ in λ. Then $\nu=\lambda^{m-1}\nu'$ for some $\nu'\in\mathcal{O}_{\mathbf{C}^n}\frac{\partial}{\partial\Lambda}\oplus\theta(f)$. Say $\nu'=e\frac{\partial}{\partial\Lambda}+\nu''$ for $e\in\mathcal{O}_{\mathbf{C}^n}$ and $\nu''\in\theta(f)$. Since $F=\mathrm{id}_{\mathbf{C}}\times f_t$ is a versal unfolding of f, we see that $\mathbf{C}\left\langle\frac{\partial f_t}{\partial t}\right\rangle+T\mathcal{A}_ef=\theta(f)$ so there is a $\mu\in\mathbf{C}$, a $\xi'\in\theta_{\mathbf{C}^n}$ and an $\eta'\in\theta_{\mathbf{C}^p}$ such that

$$\nu''=\mu\frac{\partial f_t}{\partial t}+tf(\xi')+wf(\eta')$$

and therefore

$$\begin{aligned}
\nu=\lambda^{m-1}\nu' &= \lambda^{m-1}e\frac{\partial}{\partial\Lambda}+\lambda^{m-1}\nu''\\
&= \lambda^{m-1}e\frac{\partial}{\partial\Lambda}+\lambda^{m-1}\mu\frac{\partial f_t}{\partial t}+\lambda^{m-1}tf(\xi')+\lambda^{m-1}wf(\eta').
\end{aligned}$$

In order to prove that ν is in $t\mathcal{A}_eA_Ff$ it is sufficient to show that each of these summands is. By (2),

$$\begin{aligned}
&\lambda^{m-1}e\frac{\partial}{\partial\Lambda}-tA_Ff\left(\lambda^{m-1}e\frac{\partial}{\partial\lambda}\right)\\
&=\lambda^{m-1}e\left[\frac{\partial}{\partial\Lambda}-tA_Ff\left(\frac{\partial}{\partial\lambda}\right)\right]\in\lambda^m\theta(A_Ff)\subseteq T\mathcal{A}_eAf
\end{aligned}$$

so $\lambda^{m-1}e\frac{\partial}{\partial\Lambda}\in T\mathcal{A}_eA_Ff$. By (2) again and because $\Lambda\circ A_Ff=\lambda$,

$$\begin{aligned}
\lambda^{m-1}\mu\frac{\partial f_t}{\partial t}-tA_Ff\left(\frac{1}{2}\lambda^{m-2}\mu\frac{\partial}{\partial\lambda}\right) &= \frac{1}{2}\lambda^{m-2}\mu\left[2\lambda\frac{\partial f_t}{\partial t}-tA_Ff\left(\frac{\partial}{\partial\lambda}\right)\right]\\
&= -\frac{1}{2}\lambda^{m-2}\mu\frac{\partial}{\partial\Lambda}\\
&= wA_Ff\left(\frac{1}{2}\Lambda^{m-2}\frac{\partial}{\partial\Lambda}\right)
\end{aligned}$$

so $\lambda^{m-1}\mu\frac{\partial f_t}{\partial t} \in T\mathcal{A}_e\mathrm{A}_F f$. By (1), $\lambda^{m-1}tf(\xi') - t\mathrm{A}_F f(\lambda^{m-1}\xi') = \lambda^{m-1}[tf(\xi') - t\mathrm{A}_F f(\xi')] \in \lambda^m\theta(\mathrm{A}_F f) \subseteq T\mathcal{A}_e\mathrm{A}_F f$ so $\lambda^{m-1}tf(\xi') \in T\mathcal{A}_e\mathrm{A}_F f$. Since $\Lambda \circ \mathrm{A}_F f = \lambda$ and by (3), $\lambda^{m-1}wf(\eta') - w\mathrm{A}_F f(\Lambda^{m-1}\eta') = \lambda^{m-1}[wf(\eta') - w\mathrm{A}_F f(\eta')] \in \lambda^m\theta(\mathrm{A}_F f) \subseteq T\mathcal{A}_e\mathrm{A}_F f$ so $\lambda^{m-1}wf(\eta') \in T\mathcal{A}_e\mathrm{A}_F f$.

We deduce that $\nu \in T\mathcal{A}_e\mathrm{A}_F f$. Since any element of $\lambda^{m-1}\theta(\mathrm{A}_F f)$ can be written as the sum of an element in $\lambda^{m-1}\theta(\mathrm{A}_F f)$ homogeneous in λ of degree $m-1$ in λ and an element of $\lambda^m\theta(\mathrm{A}_F f)$, it follows that $\lambda^{m-1}\theta(\mathrm{A}_F f) \subseteq T\mathcal{A}_e\mathrm{A}_F f$ contradicting the definition of m. Our hypothesis that $m \geq 2$ must be wrong so $m \leq 1$ and $\lambda\theta(\mathrm{A}_F f) \subseteq T\mathcal{A}_e\mathrm{A}_F f$.

Let $\alpha = s\frac{\partial}{\partial\Lambda}$ where $s \in \mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}$, be an arbitrary element of $\mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}\frac{\partial}{\partial\Lambda}$, then by (2),

$$\alpha - t\mathrm{A}_F f\left(s\frac{\partial}{\partial\lambda}\right) = s\left[\frac{\partial}{\partial\Lambda} - t\mathrm{A}_F f\left(\frac{\partial}{\partial\lambda}\right)\right] = 2s\lambda\frac{\partial f_t}{\partial t} \in \lambda\theta(\mathrm{A}_F f) \subseteq T\mathcal{A}_e\mathrm{A}_F f$$

so $\alpha \in T\mathcal{A}_e\mathrm{A}_F f$ and therefore $\mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}\frac{\partial}{\partial\Lambda} \subseteq T\mathcal{A}_e\mathrm{A}_F f$.

Let ξ'' be an arbitrary element of $\theta_{\mathbf{C}^n}$, then by (1); $tf(\xi'') - t\mathrm{A}_F f(\xi'') \in \lambda\theta(\mathrm{A}_F f) \subseteq T\mathcal{A}_e\mathrm{A}_F f$ so $tf(\xi'') \in T\mathcal{A}_e\mathrm{A}_F f$ and therefore $tf(\theta_{\mathbf{C}^n}) \subseteq T\mathcal{A}_e\mathrm{A}_F f$.

Let η'' be an arbitrary element of $\theta_{\mathbf{C}^p}$, then by (3); $wf(\eta'') - w\mathrm{A}_F f(\eta'') \in \lambda\theta(\mathrm{A}_F f) \subseteq t\mathcal{A}_e\mathrm{A}_F f$ so $wf(\eta'') \in T\mathcal{A}_e\mathrm{A}_F f$ and therefore $wf(\theta_{\mathbf{C}^p}) \subseteq T\mathcal{A}_e\mathrm{A}_F f$.

$\bowtie$

We now prove a partial converse to Theorem 2.5, the next chapter is devoted to another.

Proposition 2.6

Suppose that $G = \mathrm{id}_{\mathbf{C}} \times g_\lambda$ is a one dimensional unfolding of a multi-germ $g = g_0$ (where $\lambda \in \mathbf{C}$), and suppose that $H = id_{\mathbf{C}} \times g_{\lambda^2}$ has $\mathcal{A}_e$-codimension one. Then g has $\mathcal{A}_e$-codimension one and G is a versal unfolding of g. Thus H is the augmentation of g.

Proof

The first part of the proof of Theorem 2.5 is valid in this situation as well and it shows that

$$T\mathcal{A}_e H \subseteq \left[\mathcal{O}_{\mathbf{C}\times\mathbf{C}^n}\frac{\partial}{\partial\Lambda} + \lambda\left(T\mathcal{A}_e g + \mathcal{O}_{\mathbf{C}^n}\frac{\partial g_t}{\partial t}\right) + \lambda^2\theta(H)\right] \oplus T\mathcal{A}_e g.$$

It follows that g has $\mathcal{A}_e$-codimension zero or one. g can't have $\mathcal{A}_e$-codimension zero because then it would be stable and any unfolding of a stable map is stable. Now, using the same containment we see that $T\mathcal{A}_e g + \mathcal{O}_{\mathbf{C}^n}\frac{\partial g_t}{\partial t} = \theta(g)$. It follows that G is a versal unfolding of g.

$\bowtie$

§3 A Characterisation of Augmentations

Notice that $0(n,p)$ is the set of $\mathcal{A}$-equivalence classes of stable multi-germs. Define $\mathrm{U}: 1(n,p) \to 0(n+1,p+1)$ to be the function induced by taking mini-versal unfoldings of $\mathcal{A}_e$-codimension one multi-germs. Since mini-versal unfoldings are unique up to isomorphism, U is well defined.

Define $\mathrm{P}: 0(n,p) \to 0(n+1,p+1)$ to be the map induced from that taking a multi-germ $f: \mathbb{C}^n, S \to \mathbb{C}^p, \{0\}$ to

$$\mathrm{P}f = \mathrm{id}_{\mathbb{C}} \times f: \mathbb{C} \times \mathbb{C}^n, \{0\} \times S \to \mathbb{C} \times \mathbb{C}^p, \{0\} \times \{0\}$$
$$(\lambda, x) \mapsto (\lambda, f(x)).$$

If $f \sim_{\mathcal{A}} f'$ then there exist germs of bi-analytic maps $\phi: \mathbb{C}^n, S \to \mathbb{C}^n, S$ (preserving S) and $\psi: \mathbb{C}^p, \{0\} \to \mathbb{C}^p, \{0\}$ such that $\psi \circ f = f' \circ \phi$. Then $\mathrm{id}_{\mathbb{C}} \times \phi: \mathbb{C} \times \mathbb{C}^n, \{0\} \times S \to \mathbb{C} \times \mathbb{C}^n, \{0\} \times S$ and $\mathrm{id}_{\mathbb{C}} \times \psi: \mathbb{C} \times \mathbb{C}^p, \{0\} \times \{0\} \to \mathbb{C} \times \mathbb{C}^p, \{0\} \times \{0\}$ are germs of bi-analytic maps and $(\mathrm{id}_{\mathbb{C}} \times \psi) \circ \mathrm{P}f = \mathrm{P}f' \circ (\mathrm{id}_{\mathbb{C}} \times \phi)$ so $\mathrm{P}f \sim_{\mathcal{A}} \mathrm{P}f'$. Since f is stable, $\mathrm{P}f$ is a versal unfolding of f and therefore is stable itself. Thus P is well defined. We call $\mathrm{P}f$ the prism on f and say that a multi-germ is a prism if and only if it is the prism of some other multi-germ f.

Lemma 3.1

$T\mathcal{A}_e\mathrm{P}f = \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n}\frac{\partial}{\partial \Lambda} \oplus \mathcal{O}_{\mathbb{C}} T\mathcal{A}_e f$

Proof

Notice $t\mathrm{P}f(\frac{\partial}{\partial \lambda}) = \frac{\partial}{\partial \lambda}(\mathrm{P}f) = \frac{\partial}{\partial \Lambda}$, $t\mathrm{P}f(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}(\mathrm{P}f) = \frac{\partial f}{\partial x_i} = tf(\frac{\partial}{\partial x_i})$, $\Lambda \circ \mathrm{P}f = \lambda$ and $X_i \circ \mathrm{P}f = X_i \circ f$. It is now an easy check.

⋈

Corollary 3.2

If f is stable then $\mathrm{P}f$ is too. However, if f is not stable then $\mathrm{P}f$ is not even finitely determined.

Proof

The first part follows from Lemma 3.1 and the observation that

$$\theta(\mathrm{P}f) = \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n}\frac{\partial}{\partial \Lambda} \oplus \mathcal{O}_{\mathbb{C}}\theta(f).$$

For the second part, if $\eta \in \theta(f)\backslash T\mathcal{A}_e f$ then by Lemma 3.1 again the infinite set $\{\lambda^i \eta \mid i \in \mathbb{N}\}$ is linearly independent over $T\mathcal{A}_e\mathrm{P}f$.

⋈

The multi-germ f can be reconstructed, up to $\mathcal{A}$-equivalence, from $\mathrm{P}f$ as the top arrow in the pullback of this diagram

$$\begin{array}{ccc} & & \mathbb{C}^p, \{0\} \\ & & \big\downarrow i \\ \mathbb{C} \times \mathbb{C}^n, \{0\} \times S & \xrightarrow{\mathrm{P}f} & \mathbb{C} \times \mathbb{C}^p, \{0\} \times \{0\} \end{array}$$

where i is a generic immersion (and hence is transverse to the trivial direction of the prism). It follows that P is injective. Notice that P, U and A all preserve the number of components of a multi-germ and their co-ranks.

Theorem 3.3

$\mathrm{U} \circ \mathrm{A} = \mathrm{P} \circ \mathrm{U}$, *that is, the following diagram commutes.*

$$\begin{array}{ccc} 1(n,p) & \xrightarrow{\mathrm{A}} & 1(n+1,p+1) \\ \big\downarrow \mathrm{U} & & \big\downarrow \mathrm{U} \\ 0(n+1,p+1) & \xrightarrow{\mathrm{P}} & 0(n+2,p+2) \end{array}$$

Proof

Let $f \in 1(n,p)$ then $\mathrm{A}_{\mathrm{U}f} f = \mathrm{id}_{\mathbb{C}} \times f_{\lambda^2} : (\lambda, x) \mapsto (\lambda, f_{\lambda^2}(x))$. It follows from the proof of Theorem 2.5 that a miniversal unfolding of $\mathrm{A}_{\mathrm{U}f} f$ is given by $\mathrm{id}_{\mathbb{C}} \times \mathrm{id}_{\mathbb{C}} \times f_{(\lambda^2+\mu)}$. This is an unfolding of $\mathrm{U}f = \mathrm{id}_{\mathbb{C}} \times f_\mu$ and since $\mathrm{U}f$ is stable (because it is itself a versal unfolding), is $\mathcal{A}$-equivalent to a prism on it.

⋈

It follows from this theorem that if a multi-germ is an augmentation then its mini-versal unfolding is a prism. The next theorem says that this property characterises augmentations. Notice that if a multi-germ is an augmentation then we can recover the mini-versal unfolding of the multi-germ of which it is an augmentation by Theorem 3.3 and the comment after Corollary 3.2.

Lemma 3.4

Suppose that $f: \mathbb{C}^n, \{0\} \to \mathbb{C}^p, \{0\}$ *is a germ of a map. Then if the* $\mathcal{K}_e$-*codimension of* f *is zero then* f *is a submersion and if the* $\mathcal{K}_e$-*codimension of* f *is one then* $n \geq p-1$ *and up to* $\mathcal{A}$-*equivalence* f *is*

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{p-1} \\ \sum_{i=p}^{n} x_i^2 \end{pmatrix}.$$

Proof

First suppose that the $\mathcal{K}_e$-codimension of f is zero, then $\theta(f) = tf(\theta(n)) + f^*\mathbf{m}_p\theta(f)$. Since $f^*\mathbf{m}_p \subseteq \mathbf{m}_n$, $Tf(T_0\mathrm{C}^n) = T_0\mathrm{C}^p$. It follows that f is a submersion.

Now suppose that the $\mathcal{K}_e$-codimension of f is one, then since $f^*\mathbf{m}_p \subseteq \mathbf{m}_n$, the dimension of $T_0\mathrm{C}^p/Tf(T_0\mathrm{C}^n)$ as a C vector space is zero or one. This dimension cannot be zero because if it were then as before f would be a submersion and hence would have $\mathcal{K}_e$-codimension zero. Therefore the dimension of $T_0\mathrm{C}^p/Tf(T_0\mathrm{C}^n)$ is one. Now we see that $n \geq p-1$ and that by an application of the inverse function theorem we can reduce f to the following form.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{p-1} \\ g \end{pmatrix}$$

where $g \in \mathbf{m}_n^2$. If $p \leq i \leq n$ then $x_i\frac{\partial}{\partial X_p} \in tf\left(\theta(n)\right) + f^*\mathbf{m}_p\theta(f)$. Now $x_i \notin f^*\mathbf{m}_p$ so there is some term of the form $\lambda x_i x_j$ in g ($\lambda \neq 0, p \leq j \leq n$). We may take $j = i$ here because if there is no λx_i^2 term then the change of coordinates $x_j \mapsto x_j + x_i$ creates one, after that the change $x_i \mapsto x_i + \mu x_k$ removes the $x_i x_k$ term from g ($1 \leq k \leq n, i \neq k$). We can remove terms of the form $x_i x_j$ ($1 \leq i, j \leq p-1$) from g by a change of coordinates in the target. Now our germ has the same two-jet as the germ whose formula is given in the statement of the lemma. The result now follows because this germ is two-determined by corollary 3.4 of [3].

$\bowtie$

The next result uses Damon's theory of $\mathcal{K}_V$-equivalence and pullbacks of maps which is explained in [2] and also in [16].

Theorem 3.5

Suppose that g is an $\mathcal{A}_e$-codimension one multi-germ and that the mini-versal unfolding Ug *of g is a prism. Then g is an augmentation.*

Proof

There is a multi-germ h and a natural number l such that Ug=P$^l h$ and h is not a prism. By Corollary 3.2, h is stable and the remark following Corollary 3.2 implies that l and h are uniquely determined. We have the following commutative diagram in which i and i' are the natural inclusions, ϕ and ψ are bi-analytic, π and π' are the natural projections and S' is a subset of C^{n+1-l} of the same cardinality as S:

$$
\begin{array}{ccc}
\mathbf{C}^n, S & \xrightarrow{g} & \mathbf{C}^p, \{0\} \\
\downarrow i & & \downarrow i' \\
\mathbf{C} \times \mathbf{C}^n, \{0\} \times S & \xrightarrow{\mathrm{id}_{\mathbf{C}} \times g_\lambda} & \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \\
\downarrow \phi & & \downarrow \psi \\
\mathbf{C}^l \times \mathbf{C}^{n+1-l}, \{0\} \times S' & \xrightarrow{id_{\mathbf{C}^l} \times h} & \mathbf{C}^l \times \mathbf{C}^{p+1-l}, \{0\} \times \{0\} \\
\downarrow \pi & & \downarrow \pi' \\
\mathbf{C}^{n+1-l}, S' & \xrightarrow{h} & \mathbf{C}^{p+1-l}, \{0\}
\end{array}
\qquad (A)
$$

Each of the three squares of the diagram is Cartesian (i.e. a pullback) so the outside rectangle is Cartesian as well. By theorem 3.2 of [16], the $\mathcal{A}_e$-codimension of g (which is one) is equal to the $\mathcal{K}_{D(h)}$-codimension of $\pi' \circ \psi \circ i'$ where $D(h)$ is the discriminant of h. Since h is stable it is Thom transversal so any vector field parallel to $D(h)$ lifts by 6.14 of [7]. Therefore, since h is not a prism, every vector field parallel to $D(h)$ vanishes at 0, i.e. Derlog $D(h) \subseteq \mathbf{m}_{p+1-l}\theta_{\mathbf{C}^{p+1-l}}$. Now

$$
\begin{aligned}
T\mathcal{K}D(h)(\pi' \circ \psi \circ i') &= t(\pi' \circ \psi \circ i')(\theta_{\mathbf{C}^p}) + (\pi' \circ \psi \circ i')^* D(h) \\
&\subseteq t(\pi' \circ \psi \circ i')(\theta^p_{\mathbf{C}}) + (\pi' \circ \psi \circ i')^*(\mathbf{m}_{p+1-l}\theta_{\mathbf{C}^{p+1-l}}) \\
&= T\mathcal{K}_e(\pi' \circ \psi \circ i').
\end{aligned}
$$

Since the $\mathcal{K}_{D(h)}$-codimension of $\pi' \circ \psi \circ i'$ is one, the $\mathcal{K}_e$-codimension of $\pi' \circ \psi \circ i'$ is zero or one. We can apply Lemma 3.4 to deduce that $\pi' \circ \psi \circ i'$ is either a submersion or is $\mathcal{A}$-equivalent to

$$
\begin{aligned}
&\gamma: \mathbf{C}^n, 0 \to \mathbf{C}^p, 0 \\
&(x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_{p-1}, \sum_{i=p}^{n} x_i^2 \right).
\end{aligned}
$$

But $\pi' \circ \psi \circ i'$ cannot be a submersion, because if it were then g would be $\mathcal{A}$-equivalent to $\mathrm{P}^l h$ (for some $l \geq 0$) and hence stable. We are left with the second alternative. Let ϕ' and ψ' be germs of diffeomorphisms such that this diagram commutes.

$$
\begin{array}{ccc}
\mathbf{C}^p, \{0\} & \xrightarrow{\pi' \circ \psi \circ i'} & \mathbf{C}^{p+1-l}, \{0\} \\
\downarrow \phi' & & \downarrow \psi' \\
\mathbf{C}^p, \{0\} & \xrightarrow{\gamma} & \mathbf{C}^{p+1-l}, \{0\}
\end{array}
$$

Let the co-ordinates in the source of γ be $y_1, \dots, y_{p-l}, z_1, \dots, z_l$ and those of the target, $Y_1, \dots Y_{p-l}, Z$. Since $\mathrm{id}_{\mathbf{C}} \times g_\lambda$ is transverse to i' (that is to say each component of $\mathrm{id}_{\mathbf{C}} \times g_\lambda$ is transverse to i'), h is transverse to $\pi' \circ \psi \circ i'$. $d(Z \circ \psi' \circ \pi' \circ \psi \circ i')|_0 = 0$ so $d(Z \circ \psi' \circ h)|_{S'} \neq 0$. It follows that for λ near 0, $(Z \circ \psi' \circ h)^{-1}(\lambda) \cong \mathbf{C}^{n-l}$, $(Z \circ \psi')^{-1}(\lambda) \cong \mathbf{C}^{p-l}$ and, if we define $h_\lambda :=$ $h|_{(Z\circ\psi'\circ h)^{-1}(\lambda)}: (Z \circ \psi' \circ h)^{-1}(\lambda) \to (Z \circ \psi')(\lambda)$ then $h = \mathrm{id}_{\mathbf{C}} \times h_\lambda$ is a germ of an unfolding of h_0. Using again the fact that the outside rectangle of diagram (A) is a pullback, we deduce that g is $\mathcal{A}$-equivalent to

$$\mathrm{id}_{\mathbf{C}^l} \times h_{\sum_{i=1}^{l} \lambda_i^2} : \mathbf{C}^l \times \mathbf{C}^{n+1-l}, \{0\} \times S' \to \mathbf{C}^l \times \mathbf{C}^{p+1-l}, \{0\} \times \{0\}.$$

Since $\mathrm{id}_{\mathbf{C}} \times g_\lambda$ is a prism, $l \geq 1$, therefore by Proposition 2.6, g is an augmentation.

⋈

We can see from the proof of Theorem 3.5 that if the unfolding of an $\mathcal{A}_e$-codimension one germ f is in the image of P^l then f is in the image of A^l. This can also be seen directly from the statement of Theorem 3.5 by diagram chasing up this ladder.

$$\begin{tikzcd}
1(n,p) \arrow[r, "\mathrm{U}"] \arrow[d, "\mathrm{A}"] & 0(n+1,p+1) \arrow[d, "\mathrm{P}"] \\
1(n+1,p+1) \arrow[r, "\mathrm{U}"] \arrow[d, "\mathrm{A}"] & 0(n+2,p+2) \arrow[d, "\mathrm{P}"] \\
1(n+2,p+2) \arrow[r, "\mathrm{U}"] \arrow[d, "\mathrm{A}"] & 0(n+3,p+3) \arrow[d, "\mathrm{P}"] \\
\vdots \arrow[d, "\mathrm{A}"] & \vdots \arrow[d, "\mathrm{P}"] \\
1(n+l,p+l) \arrow[r, "\mathrm{U}"] & 0(n+l+1,p+l+1).
\end{tikzcd}$$

If $\mathrm{A}: 1(n,p) \to 1(n+1,p+1)$ and $\mathrm{U}: 1(n+1,p+1) \to 0(n+2,p+2)$ are both injective then $\mathrm{U}: 1(n,p) \to 0(n+1,p+1)$ is also injective. If $\mathrm{U}: 1(n,p) \to 0(n+1,p+1)$ is injective then $\mathrm{A}: 1(n,p) \to 1(n+1,p+1)$ is injective. $\mathrm{A}: 1(n,p) \to 1(n+1,p+1)$ is injective if and only if the commutative square of Theorem 3.3 is Cartesian.

§4 Multigerms of $\mathcal{A}_e$-codimension at most one

In this section we shall analyse multigerms of maps from $\mathbf{C}^n$ to $\mathbf{C}^p$ of $\mathcal{A}_e$-codimension at most one. We reduce the problem of their classification to questions about monogerms. First we consider the case of stable multigerms which was dealt with by Mather in [11]. We give a transcript of part of that paper here.

For a stable multigerm $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ with components $f^{(i)}$, set

$$\tau(f^{(i)}) := \mathrm{ev}_0\left[(wf^{(i)})^{-1}\{f^{(i)*}\mathbf{m}_p\theta(f^{(i)}) + tf^{(i)}(\theta(n))\}\right]$$

where $\mathrm{ev}_0\colon \theta(p) \to T_0\mathbf{C}^p, \eta \mapsto \eta|_0$.

Define *regular intersection* by saying that a finite set $E_1, \ldots, E_s$ of vector subspaces of a finite dimensional vector space F have regular intersection (with respect to F) if and only if

$$\mathrm{codim}(E_1 \cap \ldots \cap E_s) = \mathrm{codim}E_1 + \cdots + \mathrm{codim}E_s$$

(where 'codim' denotes the codimension in F).

Lemma 4.1

$E_1, \ldots, E_s$ have regular intersection if and only if the natural mapping

$$F \to (F/E_1) \oplus \cdots \oplus (F/E_s)$$

is surjective.

Proof

The kernel of this mapping is $E_1 \cap \ldots \cap E_s$ so the lemma follows from comparison of dimensions.

$\bowtie$

Proposition 4.2

f is stable if and only if each $f^{(i)}$ is stable and $\tau(f^{(1)}), \ldots, \tau(f^{(s)})$ have regular intersection with respect to $T_0\mathbf{C}^p$.

Proof

$wf\colon \theta(p) \to \theta(f)$ induces a mapping

$$\bar{w}f\colon T_0\mathbf{C}^p = \frac{\theta(p)}{\mathbf{m}_p\theta(p)} \to \frac{\theta(f)}{f^*\mathbf{m}_p\theta(f) + tf(\theta(n))}$$
$$= \bigoplus_{i=1}^{s} \frac{\theta(f^{(i)})}{f^{(i)*}\mathbf{m}_p\theta(f^{(i)}) + tf^{(i)}(\theta(n))}.$$

f is stable if and only if $\theta(f) = f^*\mathbf{m}_y\theta(f) + tf(\theta(n)) + \mathbf{C}\langle\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_p}\rangle$ which is true if and only if $\bar{w}f$ is surjective. Similarly $f^{(i)}$ is stable if and only if $\bar{w}f^{(i)}$ is surjective. Furthermore

$$(a) \qquad \bar{w}f(\eta) = \bar{w}f^{(1)}(\eta) \oplus \ldots \oplus \bar{w}f^{(s)}(\eta)$$

for any $\eta \in T_0\mathbf{C}^p$. Hence if $\bar{w}f$ is surjective then each $wf^{(i)}$ is surjective, which shows that if f is stable then each $f^{(i)}$ is stable.

Conversely assume that each $f^{(i)}$ is stable. Then $\bar{w}f^{(i)}$ induces an isomorphism

$$e_i\colon \frac{T_0\mathbf{C}^p}{\tau(f^{(i)})} \cong \frac{\theta(f^{(i)})}{f^{(i)*}\mathbf{m}_y\theta(f^{(i)}) + tf^{(i)}(\theta(n))}$$

(since $\bar{w}f^{(i)}$ is onto and $\tau(f^{(i)})$ is the kernel of $\bar{w}f^{(i)}$ by definition). Then f is stable if and only if

$$(e_1^{-1} \oplus \cdots \oplus e_s^{-1}) \circ \bar{w}f\colon T_0\mathbf{C}^p \to \frac{T_0\mathbf{C}^p}{\tau(f^{(1)})} \oplus \cdots \oplus \frac{T_0\mathbf{C}^p}{\tau(f^{(s)})}$$

is surjective, since $e_1^{-1} \oplus \cdots \oplus e_s^{-1}$ is an isomorphism. By (a), this is the "natural mapping" referred to in Lemma 4.1; hence by this lemma, f is stable if and only if $\tau(f^{(1)}), \ldots, \tau(f^{(s)})$ have regular intersection.

$\bowtie$

This is the end of the transcript. Thus Mather has shown how to reduce the classification of stable multigerms to questions about monogerms. We now stop referring directly to Mather. It will be necessary to investigate the operation τ more throughly but first we extend the definition of τ to multigerms. For a multigerm f with components $f^{(1)}, \ldots, f^{(s)}$, define

$$\tau(f) := \mathrm{ev}_0\big[(wf)^{-1}\{f^*\mathbf{m}_p\theta(f) + tf(\theta(n)^s)\}\big].$$

When f has just one component this reduces to the old definition of τ for a monogerm. We also define

$$\tau'(f) := \mathrm{ev}_0\big[(wf)^{-1}\{tf(\theta(n)^s)\}\big].$$

Lemma 4.3

Suppose that $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g\colon \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are multigerms and that the multigerm $h\colon \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$ has as components the components of f together with those of g. Then h is stable if and only if (i) both f and g are stable and (ii) $\tau(f)$ and $\tau(g)$ have regular intersection with respect to $T_0\mathbf{C}^p$. Furthermore, in this case $\tau(h) = \tau(f) \cap \tau(g)$.

Proof

The first part is analogous to Proposition 4.2 and the proof is the same. In these circumstances we have the following commutative diagram.

$$\begin{array}{ccc}
\dfrac{\theta(h)}{h^*\mathbf{m}_p\theta(h)+th\big(\theta(n)^{s+t}\big)} & = & \dfrac{\theta(f)}{f^*\mathbf{m}_p\theta(f)+tf\big(\theta(n)^s\big)} \oplus \dfrac{\theta(g)}{g^*\mathbf{m}_p\theta(g)+tg\big(\theta(n)^t\big)} \\
\Big\downarrow \mathrm{ev}_0 & & \Big\downarrow \mathrm{ev}_0 \\
\dfrac{T_0\mathbf{C}^p}{\tau(h)} & \longrightarrow & \dfrac{T_0\mathbf{C}^p}{\tau(f)} \oplus \dfrac{T_0\mathbf{C}^p}{\tau(g)}
\end{array}$$

The vertical maps are isomorphisms by the definition of τ so the bottom map must also be an isomorphism, and thus $\tau(h) = \tau(f) \cap \tau(g)$.

$\bowtie$

Corollary 4.4

If f is a stable multigerm with components $f^{(1)}, \ldots, f^{(s)}$ then

$$\tau(f) = \tau(f^{(1)}) \cap \ldots \cap \tau(f^{(s)}).$$

Proof

By induction on s using Lemma 4.3.

$\bowtie$

We shall now investigate the geometrical significance of τ.

Lemma 4.5

If $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ is a stable multigerm then $\tau(f) = \tau'(f)$.

Proof

Clearly $\tau'(f) \subseteq \tau(f)$. Conversely if $a \in \tau(f)$ then there is a germ of a vector field α on $\mathbf{C}^p, \{0\}$ such that $wf(\alpha) \in f^*\mathbf{m}_p\theta(f) + tf(\theta(n)^s)$ and $\mathrm{ev}_0(\alpha) = a$. Since f is stable,

$$f^*\mathbf{m}_p\theta(f) + tf(\theta(n)^s) = wf(\mathbf{m}_p\theta(p)) + tf(\theta(n)^s)$$

so we can write $wf(\alpha) = wf(\beta) + tf(\gamma)$ for some $\beta \in \mathbf{m}_p\theta(p)$ and $\gamma \in \theta(n)^s$. But now $wf(\alpha - \beta) = tf(\gamma)$ and so

$$a = \mathrm{ev}_0(\alpha - \beta) \in \tau'(f).$$

$\bowtie$

Lemma 4.6

If $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ is a multigerm which is not a prism then $\tau'(f) = 0$.

Proof

Suppose that $\tau'(f) \neq 0$, then there are germs of vector fields $\xi \in \theta(n)^s$ and $\eta \in \theta(p)$ such that $tf(\xi) = wf(\eta)$ and $\mathrm{ev}_0(\eta) \neq 0$. Since $\mathrm{ev}_0(\eta) \neq 0$, $\mathrm{ev}_0[tf^{(i)}(\xi)] = \mathrm{ev}_0[wf^{(i)}(\eta)] \neq 0$ and so $ev_0(\xi)$ is non zero in each component (here we are pretending, for ease of notation, that each component of f is a germ at 0). We integrate ξ and η to get germs of one parameter families of bi-analytic maps ϕ_t of $\coprod^s \mathbf{C}^n$ and ψ_t of $\mathbf{C}^p$ respectively (where $t \in \mathbf{C}$ and both ϕ_0 and ψ_0 are identity functions).

As usual we take representatives of the germs. For $x \in \mathbf{C}^n$ in a neighbourhood of S and $t \in \mathbf{C}$ in a neighbourhood of 0,

$$\begin{aligned}\frac{d}{dt}\big((f \circ \phi_t)(x)\big) = Tf\left(\frac{d\phi_t(x)}{dt}\right) &= (Tf \circ \xi)\big(\phi_t(x)\big) = tf(\xi)\big(\phi_t(x)\big) \\ &= wf(\eta)\big(\phi_t(x)\big) = \eta\big((f \circ \phi_t)(x)\big).\end{aligned}$$

This can be thought of as a differential equation for the function $(f \circ \phi_t)(x)$ of t and with the initial condition $(f \circ \phi_0)(x) = f(x)$ it has the unique solution $(f \circ \phi_t)(x) = (\psi_t \circ f)(x)$. Therefore as germs, $f \circ \phi_t = \psi_t \circ f$.

Let the coordinate functions in the target be $X_1, \ldots, X_p$. Since $\mathrm{ev}_0(\eta) \neq 0$, at least one of these functions has a non-zero derivative with respect to $\mathrm{ev}_0(\eta)$. We may suppose that X_1 is such a function. For $s^{(i)} \in S$, let $u_i = \mathrm{ev}_{s^{(i)}}[\xi]$, then

$$Tf(u_i) = Tf(\mathrm{ev}_{s^{(i)}}[\xi]) = \mathrm{ev}_{s^{(i)}}[tf(\xi)] = \mathrm{ev}_{s^{(i)}}[wf(\eta)] = \mathrm{ev}_0(\eta)$$

so

$$T(X_1 \circ f)(u_i) = TX_1\big(Tf(u_i)\big) = TX_1(\mathrm{ev}_0[\eta]) \neq 0.$$

Therefore f is transverse to the $\{X_1 = 0\}$ hyperplane and we can find a system of coordinates $x_1, \ldots, x_n$ of the source such that $x_1 = X_1 \circ f$. Notice that $f(\{x_1 = 0\}) \subseteq \{X_1 = 0\}$ and define $\hat{f}\colon \{x_1 = 0\} \to \{X_1 = 0\}$ by $\hat{f} = f|_{\{x_1=0\}}$.

Define maps

$$\begin{aligned} \gamma\colon \{x_1 = 0\} \times \mathbf{C}, S &\to \mathbf{C}^n, S \qquad & \delta\colon \{X_1 = 0\} \times \mathbf{C}, \{0\} &\to \mathbf{C}^p, \{0\} \\ (x,t) &\mapsto \phi_t(x) & (X,t) &\mapsto \psi_t(X). \end{aligned}$$

We can see that γ and δ are both bi-analytic by calculating their Jacobians. For $x \in \{x_1 = 0\}$ and $t \in \mathbf{C}$,

$$\delta^{-1} \circ f \circ \gamma(x,t) = \delta^{-1} \circ f \circ \phi_t(x) = \delta^{-1} \circ \psi_t \circ f(x) = \big(f(x), t\big) = \big(\hat{f}(x), t\big)$$

so $\delta^{-1} \circ f \circ \gamma$ is a prism (on $\hat{f}$).

⋈

Lemma 4.7

If $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ is a multigerm and

$$\begin{aligned} \mathrm{id}_{\mathbf{C}} \times f\colon \mathbf{C} \times \mathbf{C}^n, \{0\} \times S &\to \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \\ (t,x) &\mapsto \big(t, f(x)\big) \end{aligned}$$

then $\tau'(\mathrm{id}_{\mathbf{C}} \times f) \cap (\{0\} \times \mathbf{C}^p) = \tau'(f)$.

Proof

If $v \in \tau(f)$ then there are germs of vector fields $\eta \in \theta(p)$ and $\xi \in \theta(n)^s$ such that $wf(\eta) = tf(\xi)$ and $\mathrm{ev}_0(\eta) = v$. But since $\theta(n) \subseteq \theta(1+n)$, $\theta(p) \subseteq \theta(1+p)$ and $\theta(f) \subseteq \theta(\mathrm{id}_{\mathbf{C}} \times f)$, we have $w(\mathrm{id}_{\mathbf{C}} \times f)(\eta) = t(\mathrm{id}_{\mathbf{C}} \times f)(\xi)$ and thus $v \in \tau'(\mathrm{id}_{\mathbf{C}} \times f)$. Clearly $v \in \{0\} \times \mathbf{C}^p$ so since v was arbitrary, $\tau'(\mathrm{id}_C \times f) \cap (\{0\} \times \mathbf{C}^p) \supseteq \tau'(f)$.

Conversely if $v \in \tau'(\mathrm{id}_C \times f) \cap (\{0\} \times \mathbf{C}^p)$ then there are germs of vector fields $\eta \in \theta(1+p)$ and $\xi \in \theta(1+n)$ such that $w(\mathrm{id}_{\mathbf{C}} \times f)(\eta) = t(\mathrm{id}_{\mathbf{C}} \times f)(\xi)$. If we set η' to be the projection of η onto $\{0\} \times \mathbf{C}^p$ and ξ' to be the projection of ξ onto $\{0\} \times \mathbf{C}^p$ then since both $w(\mathrm{id}_{\mathbf{C}} \times f)$ and $t(\mathrm{id}_{\mathbf{C}} \times f)$ commute with this projection, $w(\mathrm{id}_{\mathbf{C}} \times f)(\eta') = t(id_C \times f)(\xi')$ and this vector field has no component in the direction of the first coordinate. Now setting $\eta'' = \eta'|_{\{0\} \times \mathbf{C}^p}$ and $\xi'' = \xi'|_{\{0\} \times \mathbf{C}^n}$, $wf(\eta'') = tf(\xi'')$ and thus since $v \in \{0\} \times \mathbf{C}^p$,

$$v = \mathrm{ev}_0(\eta) = \mathrm{ev}_0(\eta'') \in \tau'(f).$$

Since v was arbitrary, $\tau'(\mathrm{id}_{\mathbf{C}} \times f) \cap (\{0\} \times \mathbf{C}^p) \subseteq \tau'(f)$.

⋈

Corollary 4.8

If $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ *is a multigerm and*

$$\begin{aligned} \mathrm{id}_{\mathbf{C}} \times f\colon \mathbf{C} \times \mathbf{C}^n, \{0\} \times S &\to \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\} \\ (t, x) &\mapsto \big(t, f(x)\big) \end{aligned}$$

then $\tau'(\mathrm{id}_{\mathbf{C}} \times f) = \mathbf{C}\frac{\partial}{\partial\Lambda} \oplus \tau'(f)$ *where* Λ *is the first coordinate of the target.*

Proof

By Lemma 4.7 it is sufficient to prove that $\frac{\partial}{\partial\Lambda} \in \tau'(\mathrm{id}_{\mathbf{C}} \times f)$. This follows from the fact that as a vector field, $\frac{\partial}{\partial\Lambda} \in \big(w(\mathrm{id}_{\mathbf{C}} \times f)\big)^{-1}\big\{t(\mathrm{id}_{\mathbf{C}} \times f)\big(\theta(n)^s\big)\big\}$ which is true because $w(\mathrm{id}_{\mathbf{C}} \times f)(\frac{\partial}{\partial\Lambda}) = \frac{\partial}{\partial\Lambda} = t(\mathrm{id}_{\mathbf{C}} \times f)(\frac{\partial}{\partial\lambda})$ where λ is the first coordinate of the source.

$\bowtie$

Proposition 4.9

If $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ *is a multigerm and* $m = \dim_{\mathbf{C}} \tau'(f)$ *then there is a multigerm* $g\colon \mathbf{C}^{n-m} \to \mathbf{C}^{p-m}, \{0\}$ *which is not a prism, such that* $\mathrm{P}^m g = f$. *In other words there are bi-analytic maps* ϕ *and* ψ *as shown in this diagram.*

$$\begin{array}{ccc} \mathbf{C}^m \times \mathbf{C}^{n-m}, \{0\} \times T & \xrightarrow{\mathrm{id}_{\mathbf{C}^m} \times g} & \mathbf{C}^m \times \mathbf{C}^{p-m}, \{0\} \times \{0\} \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\psi} \\ \mathbf{C}^n, S & \xrightarrow{f} & \mathbf{C}^p, \{0\} \end{array}$$

Furthermore $T_0\big(\psi(\mathbf{C}^m \times \{0\})\big) = \tau'(f)$.

Proof

The first part follows from Lemma 4.6 and Corollary 4.8. For the second: the definition of τ' is coordinate free, so considering ϕ and ψ as changes of coordinates, we see that it is sufficient to prove that $\tau'(\mathrm{id}_{\mathbf{C}^m} \times g) = \mathbf{C}^m \times \{0\}$. We do this by induction on m. The case $m = 0$ is Lemma 4.6 and the induction step follows from Corollary 4.8.

$\bowtie$

Proposition 4.10

If $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ *and* $g\colon \mathbf{C}^{n'}, S' \to \mathbf{C}^{p'}, \{0\}$ *are multigerms neither of which are prisms and if* $\mathrm{P}^k f = \mathrm{P}^{k'} g$ *then* $|S| = |S'|$, $n = n'$, $p = p'$, $k = k'$ *and* $f \sim_{\mathcal{A}} g$. *Furthermore, if the* $\mathcal{A}$*-equivalence is given by bi-analytic maps as in the following diagram then* $\psi(\{0\} \times \mathbf{C}^k) = \{0\} \times \mathbf{C}^{k'}$.

$$\begin{array}{ccc}
\mathbf{C}^n \times \mathbf{C}^k, S \times \{0\} & \stackrel{f\times \mathrm{id}_{\mathbf{C}^k}}{\longrightarrow} & \mathbf{C}^p \times \mathbf{C}^k, \{0\} \times \{0\} \\
\downarrow \phi & & \downarrow \psi \\
\mathbf{C}^{n'} \times \mathbf{C}^{k'}, S' \times \{0\} & \stackrel{g\times \mathrm{id}_{\mathbf{C}^{k'}}}{\longrightarrow} & \mathbf{C}^{p'} \times \mathbf{C}^{k'}, \{0\} \times \{0\}
\end{array}$$

Proof

By the comment before Theorem 3.3, we can recover k and f from $\mathrm{P}^k f$. The first part of the result follows from this.

For a representative $\overline{f \times \mathrm{id}_{\mathbf{C}^k}}$ of $f \times \mathrm{id}_{\mathbf{C}^k}$, and for $x \in \mathbf{C}^p \times \mathbf{C}^k$ write $\tau_x(\overline{f \times \mathrm{id}_{\mathbf{C}^k}})$ for τ of the germ of $\overline{f \times \mathrm{id}_{\mathbf{C}^k}}$ at $(\overline{f \times \mathrm{id}_{\mathbf{C}^k}})^{-1}(x)$. By Proposition 4.9, for a sufficiently small representative $\overline{g \times \mathrm{id}_{\mathbf{C}^k}}$ of $g \times \mathrm{id}_{\mathbf{C}^k}$, and for $b \in \mathbf{C}^k$ sufficiently near 0,

$$\tau_{\psi(0,b)}(g \times \mathrm{id}_{\mathbf{C}^k}) = T_{(0,b)}\psi(\{0\} \times \mathbf{C}^k)$$

but also, by the same result, $\tau_{\psi(0,b)}(g \times \mathrm{id}_{\mathbf{C}^k}) \supseteq \{0\} \times \mathbf{C}^k$ so by comparing dimensions

$$T_{(0,b)}\psi(\{0\} \times \mathbf{C}^k) = \{0\} \times \mathbf{C}^k.$$

Let $\gamma : \mathbf{C} \to \mathbf{C}^k$, $t \mapsto tb$. Then

$$\psi(b) = \psi(b) - \psi(0) = (\psi \circ \gamma)(1) - (\psi \circ \gamma)(0) = \int_0^1 \frac{d}{dt}[(\psi \circ \gamma)(t)]dt$$
$$= \int_0^1 T\psi \circ \frac{d\gamma(t)}{dt} dt = \int_0^1 T_{\gamma(t)}\psi(b)dt.$$

Since $tb \in \{0\} \times \mathbf{C}^k$, $T_{\gamma(t)}\psi(b) \in \{0\} \times \mathbf{C}^k$ and therefore $\psi(b) \in \{0\} \times \mathbf{C}^k$. Since b was arbitrary, as germs, $\psi(\{0\} \times \mathbf{C}^k) \subseteq \{0\} \times \mathbf{C}^k$.

The converse inclusion can be seen by applying the same reasoning to this diagram.

$$\begin{array}{ccc}
\mathbf{C}^n \times \mathbf{C}^k, S' \times \{0\} & \stackrel{g\times \mathrm{id}_{\mathbf{C}^k}}{\longrightarrow} & \mathbf{C}^p \times \mathbf{C}^k, \{0\} \times \{0\} \\
\downarrow \phi^{-1} & & \downarrow \psi^{-1} \\
\mathbf{C}^n \times \mathbf{C}^k, S \times \{0\} & \stackrel{f\times \mathrm{id}_{\mathbf{C}^k}}{\longrightarrow} & \mathbf{C}^p \times \mathbf{C}^k, \{0\} \times \{0\}
\end{array}$$

$\bowtie$

Proposition 4.10 says that if f is a multigerm then there is a well defined (i.e. coordinate independent) maximal sub-manifold along which f is trivial (i.e. a prism). We call this sub-manifold the analytic stratum of f. By Proposition 4.9, $\tau'(f)$ is the tangent space at 0 to the analytic stratum of f.

Lemma 4.11

If $g: \mathbb{C}^n, S \to \mathbb{C}^p, \{0\}$ is an $\mathcal{A}_e$-codimension one multigerm which is an augmentation and if $G: \mathbb{C} \times \mathbb{C}^n, \{0\} \times S \to \mathbb{C} \times \mathbb{C}^p, \{0\} \times \{0\}$ is a miniversal unfolding of g then the intersection of $\tau(G)$ with $\{0\} \times \mathbb{C}^p \subseteq \mathbb{C} \times \mathbb{C}^p$ is nontrivial.

Proof

Since τ is defined in a coordinate free way, it suffices to take g and G as in the proof of Theorem 3.3, i.e. we take

$$g = \mathrm{id}_C \times f_{\lambda^2}: (\lambda, x) \mapsto (\lambda, f_{\lambda^2}(x))$$

and

$$G = \mathrm{id}_C \times \mathrm{id}_C \times f_{\lambda^2+\mu}: (\mu, \lambda, x) \mapsto (\mu, \lambda, f_{\lambda^2+\mu}(x)).$$

Define the germ Ψ of a bi-analytic map of the target of G by $(\mu, \lambda, X) \mapsto (\lambda^2 + \mu, \lambda, X)$ and the germ Φ of a bi-analytic map of the source of G by $(\mu, \lambda, x) \mapsto (\lambda^2 + \mu, \lambda, x)$ (these maps are bi-analytic by the inverse function theorem).

Now $\Psi \circ G \circ \Phi^{-1}: (\lambda^2 + \mu, \lambda, x) \mapsto (\lambda^2 + \mu, \lambda, f_{\lambda^2+\mu}(x))$ which is a prism in the second coordinate, thus by Proposition 4.9, $\frac{\partial}{\partial \lambda} \in \tau(\Psi \circ G \circ \Phi^{-1})$. Therefore, using again the fact that the definition of τ is coordinate free, and considering Φ and Ψ as changes of coordinates, $\frac{\partial}{\partial \lambda} = T_0(\Psi^{-1})(\frac{\partial}{\partial \lambda}) \in \tau(G)$. This is the non-trivial element in the intersection that we need.

$\bowtie$

We return to the question of $\mathcal{A}_e$-codimension one multigerms. As far as the $\mathcal{A}$-equivalence theory is concerned, components that are submersions only affect a multigerm in a simple way and they can almost be ignored. More precisely we have:

Lemma 4.12

If $f: \mathbb{C}^n, S \to \mathbb{C}^p, \{0\}$ is a multigerm then f has the same $\mathcal{A}_e$-codimension as the multigerm f' which has all the components of f and some extra submersive ones. If $g: \mathbb{C}^n, T \to \mathbb{C}^p, \{0\}$ is also a multigerm then f is $\mathcal{A}$-equivalent to g if and only if both i) they have the same number of submersive components and ii) the multigerms $\hat{f}$ and $\hat{g}$, got from f and g by removing their submersive components, are $\mathcal{A}$-equivalent.

Proof

Notice that

$$\theta(f) = \bigoplus_{s^{(i)} \in S} \theta(f^{(i)})$$

and that

$$T\mathcal{A}_e f = \bigoplus_{s^{(i)} \in S} tf^{(i)}(\theta(n)) + \{(wf^{(1)}(\eta), \ldots, wf^{(s)}(\eta)) \mid \eta \in \theta(p)\}.$$

For a submersion, $tf^{(i)}(\theta(n)) = \theta(f^{(i)})$, it follows that the inclusion $\theta(f) \subseteq \theta(f')$ induces an isomorphism

$$\frac{\theta(f)}{T\mathcal{A}_e f} \cong \frac{\theta(f')}{T\mathcal{A}_e f'}$$

and the first part of the lemma follows.

For the second part: it is clear that $f \sim_{\mathcal{A}} g \Rightarrow \hat{f} \sim_{\mathcal{A}} \hat{g}$. Conversely suppose that $\psi \circ \hat{f} = \hat{g} \circ \phi$ and that f and g have the same number of submersive components so we can pair them up. Then f is $\mathcal{A}$-equivalent to g via a change of coordinates which is ψ in the target, and ϕ in the source of the non-submersive components, because any two germs of submersions mapping $\mathbf{C}^n$ to $\mathbf{C}^p, \{0\}$ are right equivalent by the implicit function theorem.

$\bowtie$

This lemma shows that a multigerm of $\mathcal{A}_e$-codimension one must have at least one non-submersive component and reduces the classification of those with precisely one to the classification of monogerms. We therefore now turn our attention to multigerms with more than one non-submersive component.

Lemma 4.13

Suppose that $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g: \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are multigerms each of which has at least one component which is not a submersion. Suppose also that the multigerm $h: \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$, whose components are those of f together with those of g, has $\mathcal{A}_e$-codimension one. Then if ψ_t is a germ of a one-parameter family of bi-analytic maps of $\mathbf{C}^p$ at 0 such that $\psi_0 = \mathrm{id}_{\mathbf{C}^p}$ and

$$\mathrm{ev}_0\left[\left.\frac{d\psi_t}{dt}\right|_{t=0}\right] \notin \tau'(f) + \tau'(g)$$

then $H: \mathbf{C} \times \mathbf{C}^n, \{0\} \times (S \cup T) \to \mathbf{C} \times \mathbf{C}^p, \{0\} \times \{0\}$ is a versal unfolding of h, where H is made up of $F = \mathrm{id}_{\mathbf{C}} \times f$ and $G = \mathrm{id}_{\mathbf{C}} \times (\psi_\lambda \circ g)$.

Proof

If we take $H(= \mathrm{id}_{\mathbf{C}} \times h_\lambda)$ as described in the statement of the lemma then $\frac{dh_\lambda}{d\lambda}|_{\lambda=0}$ is a vector field along h which is zero along the components of f and $wg^{(j)}(\frac{d\psi_t}{dt}|_{t=0})$ along the components $g^{(j)}$ of g. If this vector field were in $T\mathcal{A}_e h$ then it could be expressed as $th(\xi) + wh(\eta)$ for some germs of vector fields $\xi \in \theta(n)^{s+t}$ and $\eta \in \theta(p)$. It would follow that $wf(\eta) = -tf(\xi) = tf(-\xi)$ and that $tg(-\xi) = wg(\eta - \frac{d\psi_t}{dt}|_{t=0})$ so $\mathrm{ev}_0[\eta] \in \tau'(f)$ and $\mathrm{ev}_0[\eta - \frac{d\psi_t}{dt}|_{t=0}] \in \tau'(g)$ and therefore that $\mathrm{ev}_0[\frac{d\psi_t}{dt}|_{t=0}] \in \tau'(f) + \tau'(g)$ which contradicts our hypothesis.

It follows that $\frac{dh_\lambda}{d\lambda}|_{\lambda=0} \notin T\mathcal{A}_e h$ and thus, since h has $\mathcal{A}_e$-codimension one, that H is an $\mathcal{A}_e$-versal unfolding of h.

$\bowtie$

Corollary 4.14

Suppose that $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ *and* $g: \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ *are multigerms each of which has at least one component which is not a submersion. Suppose also that the multigerm* $h: \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$*, whose components are those of* f *together with those of* g*, has* $\mathcal{A}_e$*-codimension one. Then both* f *and* g *are stable.*

Proof

Suppose that one of f and g is not stable. Since the situation is symmetrical with respect to swapping f and g, we may suppose that it is f which is unstable. The $\mathcal{A}_e$-codimension of f is not greater than that of h so f has $\mathcal{A}_e$-codimension one. By Corollary 3.2, f is not a prism and therefore, by Lemma 4.6, $\tau'(f) = 0$. Since g does not consist entirely of submersions, $\tau'(g) \neq T_0\mathbf{C}^p$ by Proposition 4.9. Choose $v \in T_0\mathbf{C}^p \backslash \tau'(g)$ and extend v to a vector field ν on $\mathbf{C}^p$. We may integrate ν to give a germ of a one parameter family ψ_t of bi-analytic maps of $\mathbf{C}^p$ at 0 satisfying the conditions of Lemma 4.13.

Applying Lemma 4.13 we see that H, as described there, is a versal unfolding of h. But then $\mathrm{id}_{\mathbf{C}} \times f$ is an $\mathcal{A}_e$-versal unfolding of f and so f is stable—a contradiction. We are forced to conclude that both f and g are stable.

$\bowtie$

Corollary 4.15

If h *is a multigerm with at least two non-submersive components and* h *has* $\mathcal{A}_e$*-codimension one then every component of* h *is stable.*

Proof

Split the components of h into f and g in such a way that f and g each have at least one non-submersive component. Corollary 4.14 proves that both f and g are stable and then Proposition 4.2 shows that every component of h must be stable.

$\bowtie$

Define *almost regular intersection* by saying that a finite set $E_1, \ldots, E_s$ of vector subspaces of a finite dimensional vector space F have almost regular intersection (with respect to F) if and only if

$$\mathrm{codim}(E_1 \cap \ldots \cap E_s) = \mathrm{codim} E_1 + \cdots + \mathrm{codim} E_s - 1$$

(where 'codim' denotes the codimension in F).

Lemma 4.16

$E_1, \ldots, E_s$ *have almost regular intersection if and only if the cokernel of the natural mapping*

$$F \to (F/E_1) \oplus \cdots \oplus (F/E_s)$$

has dimension one.

Proof

The kernel of this mapping is $E_1 \cap \ldots \cap E_s$ so we have an exact sequence

$$0 \to E_1 \cap \ldots \cap E_s \to (F/E_1) \oplus \cdots \oplus (F/E_s) \to C \to 0$$

where C is the cokernel. The result follows because the alternating sum of the dimensions of a finite exact sequence is zero.

$\bowtie$

Proposition 4.17

Suppose that $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g\colon \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are multigerms each of which has at least one component which is not a submersion. Suppose also that the multigerm $h\colon \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$, whose components are those of f together with those of g, has $\mathcal{A}_e$-codimension one. Then $\tau(f)$ and $\tau(g)$ have almost regular intersection with respect to $T_0\mathbf{C}^p$.

Proof

By Corollary 4.14, both f and g are stable so by Lemma 4.5 we may replace τ with τ'. Let H be a versal unfolding of h consisting of a versal unfolding F of f and a versal unfolding G of g. Since f is stable, F is, up to a change of coordinates, a prism on f and hence by Corollary 4.8, the inclusion $T_0\mathbf{C}^p \subseteq T_0(\mathbf{C} \times \mathbf{C}^p)$ induces an isomorphism

$$\frac{T_0\mathbf{C}^p}{\tau(f)} \cong \frac{T_0(\mathbf{C} \times \mathbf{C}^p)}{\tau(F)}.$$

A similar result holds for g and G. Therefore we have a commutative diagram

$$\begin{array}{ccc}
T_0\mathbf{C}^p & \longrightarrow & \frac{T_0\mathbf{C}^p}{\tau(f)} \oplus \frac{T_0\mathbf{C}^p}{\tau(g)} \\
\downarrow & & \downarrow \\
T_0(\mathbf{C} \times \mathbf{C}^p) & \longrightarrow & \frac{T_0(\mathbf{C}\times\mathbf{C}^p)}{\tau(F)} \oplus \frac{T_0(\mathbf{C}\times\mathbf{C}^p)}{\tau(G)}
\end{array}$$

in which the right hand map is bijective. The bottom map is surjective by Lemma 4.3 so the top map is either surjective or it has a one dimensional cokernel. Since h is not stable, by Lemma 4.3 we may discount the first alternative. Therefore by Lemma 4.16, $\tau(f)$ and $\tau(g)$ have almost regular intersection with respect to $T_0\mathbf{C}^p$.

$\bowtie$

Corollary 4.18

If h is a multigerm with components $h^{(1)}, \ldots, h^{(r)}$ at least two of which are non-submersive and if h has $\mathcal{A}_e$-codimension one then $\tau(h^{(1)}), \ldots, \tau(h^{(r)})$ have almost regular intersection with respect to to $T_0\mathbf{C}^p$.

Proof

Split the components of h into f and g in such a way that each of f and g has at least one non-submersive component. Name the components of f, $f^{(1)}, \ldots, f^{(s)}$ and those of g, $g^{(1)}, \ldots, g^{(t)}$. By Corollary 4.14 f is stable, so by Proposition 4.2 there is a natural bijection

$$\alpha: \frac{T_0\mathbf{C}^p}{\tau(f)} \to \bigoplus_{i=1}^{s} \frac{T_0\mathbf{C}^p}{\tau(f^{(i)})}.$$

Similarly there is another natural bijection

$$\beta: \frac{T_0\mathbf{C}^p}{\tau(g)} \to \bigoplus_{j=1}^{t} \frac{T_0\mathbf{C}^p}{\tau(g^{(j)})}.$$

Consider the composition

$$T_0\mathbf{C}^p \to \frac{T_0\mathbf{C}^p}{\tau(f)} \oplus \frac{T_0\mathbf{C}^p}{\tau(g)} \xrightarrow{\alpha\oplus\beta} \bigoplus_{i=1}^{r} \frac{T_0\mathbf{C}^p}{\tau(h^{(i)})}.$$

The first map has a one dimensional cokernel by Proposition 4.17 and Lemma 4.16 so the composition has a one dimensional cokernel also, and the result follows by using Lemma 4.16 again.

⋈

Corollary 4.19

Suppose that $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g: \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are multigerms each of which has at least one component which is not a submersion. Suppose also that the multigerm $h: \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$, whose components are those of f together with those of g, has $\mathcal{A}_e$-codimension one. Then the codimension of $\tau(f) + \tau(g)$ in $T_0\mathbf{C}^p$ is one.

Proof

By Proposition 4.17 and the definition of almost regular intersection,

$$p - \dim(\tau(f) \cap \tau(g)) = (p - \dim \tau(f)) + (p - \dim \tau(g)) - 1$$

so

$$p - \dim(\tau(f) + \tau(g)) = p - [\dim \tau(f) + \dim \tau(g) - \dim(\tau(f) \cap \tau(g))] = 1.$$

⋈

It is natural to ask whether we can spot when our codimension one multigerm is primitive.

Proposition 4.20

Suppose that $f: \mathbb{C}^n, S \to \mathbb{C}^p, \{0\}$ and $g: \mathbb{C}^n, T \to \mathbb{C}^p, \{0\}$ are multigerms each of which has at least one component which is not a submersion. Suppose also that the multigerm $h: \mathbb{C}^n, S \cup T \to \mathbb{C}^p, \{0\}$, whose components are those of f together with those of g, has $\mathcal{A}_e$-codimension one. Then h is a primitive map augmented k times, where $k = \dim_{\mathbb{C}}(\tau(f) \cap \tau(g))$.

Proof

By Corollary 4.19 we can choose $v \in T_0\mathbb{C}^p \backslash (\tau(f) + \tau(g))$. Choose a germ of a one parameter family ψ_t of bi-analytic maps $\mathbb{C}^p \to \mathbb{C}^p$ at 0 such that $\mathrm{ev}_0(\frac{d\psi_t}{dt}|_0) = v$. Then choose a versal unfolding H of h as in Lemma 4.13. If Λ is the first coordinate in the target $\mathbb{C} \times \mathbb{C}^p$ of H then by Corollary 4.8, $\tau(F) = \tau(f) \oplus \mathbb{C}\frac{\partial}{\partial\Lambda}$ and $\tau(G) = \tau(g) \oplus \mathbb{C}(\frac{\partial}{\partial\Lambda} + v)$. Now by Lemma 4.3, $\tau(H) = \tau(F) \cap \tau(G)$. Clearly $\tau(f) \cap \tau(g) \subseteq \tau(H)$, conversely if $u \in \tau(H)$ then there exist $a, b \in \mathbb{C}$, $\alpha \in \tau(f)$ and $\beta \in \tau(g)$ such that $u = \alpha + a\frac{\partial}{\partial\Lambda} = \beta + b(\frac{\partial}{\partial\Lambda} + v)$. Taking the $\frac{\partial}{\partial\Lambda}$ component shows that $a = b$, but then $\alpha = \beta + bv$ so $bv \in \tau(f) + \tau(g)$ and thus $b = 0$. Therefore $u = \alpha = \beta \in \tau(f) \cap \tau(g)$. We have proved that $\tau(H) = \tau(f) \cap \tau(g)$, the result follows from the comment after Theorem 3.5.

$\bowtie$

We will find it useful to study primitive codimension one multigerms because results about these often generalise to the not-necessarily-primitive case. In the primitive case Corollary 4.19 and Proposition 4.20 show that we have a decomposition

$$T_0\mathbb{C}^p = \tau(f) \oplus \tau(g) \oplus \mathbb{C}v.$$

We choose parameterizations $\alpha: \mathbb{C}^a, \{0\} \to \mathbb{C}^p, \{0\}$ and $\beta: \mathbb{C}^b, \{0\} \to \mathbb{C}^p, \{0\}$ of the analytic strata of f and g respectively, where $a = \dim_{\mathbb{C}} \tau(f)$ and $b = \dim_{\mathbb{C}} \tau(g)$. Now by the inverse function theorem, the map

$$\begin{aligned} \psi: \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}, \{0\} &\to \mathbb{C}^p, \{0\} \\ (x, y, \lambda) &\mapsto \alpha(x) + \beta(y) + \lambda v \end{aligned}$$

is bi-analytic. If we compose h with ψ^{-1} (i.e. $\psi^{-1} \circ h$) and consider it as a change of coordinates in the target, then the analytic stratum of f becomes $\mathbb{C}^a \times \{0\} \times \{0\}$, that of g becomes $\{0\} \times \mathbb{C}^b \times \{0\}$ and v becomes $(0, 0, 1) \in \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$. We shall suppose for the remainder of this chapter that this change of coordinates has been made.

We shall say that a multigerm f is transverse to a vector subspace V of $T_0\mathbb{C}^p$ if and only if every component of f is transverse to V. Our analysis falls into two cases characterised by whether g is or is not transverse to $\tau(f)$. First we will study the case where g is not transverse to $\tau(f)$.

Lemma 4.21

A stable map germ of rank zero is either a morse singularity or at least one of the domain or the codomain has dimension zero.

Proof

Suppose that $a\colon \mathbf{C}^n, \{0\} \to \mathbf{C}^p, \{0\}$ is such a germ and let the coordinates of the source be $x_1, \ldots, x_n$ and those of the target $X_1, \ldots, X_p$. Since a is stable, $ta(\theta(n)) + wa(\theta(p)) = \theta(a)$ and therefore the same equation holds modulo $\mathbf{m}_n^2\theta(a)$. Since a has rank zero, $ta(\mathbf{m}_n\theta(n)) \subseteq \mathbf{m}_n^2\theta(p)$, $ta(\theta(n)) \subseteq \mathbf{m}_n\theta(a)$ and $wa(m_n\theta(p)) \subseteq \mathbf{m}_n^2\theta(a)$. Therefore the map

$$\overline{ta}\colon \frac{\theta(a)}{\mathbf{m}_n\theta(n)} \to \frac{\mathbf{m}_n\theta(a)}{\mathbf{m}_n^2\theta(a)}$$

is surjective (where $\overline{ta}$ is induced by ta). $\theta(n)/\mathbf{m}_n\theta(n)$ has $\mathbf{C}$-basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ and therefore has dimension n as a $\mathbf{C}$-vector space. $\mathbf{m}_n\theta(a)/\mathbf{m}_n^2\theta(a)$ has a $\mathbf{C}$-basis consisting of the elements of the form $x_i\frac{\partial}{\partial X_j}$ and therefore has dimension $n \cdot p$ as a $\mathbf{C}$-vector space. Since the map $\overline{ta}$ is linear we deduce that either n is zero or that p is either zero or one.

If p is one then we can consider a as a function germ; it is singular because it has rank zero. Since $\overline{ta}$ is surjective, we can apply Nakayama's lemma to deduce that $ta(\theta(n)) = \mathbf{m}_n\theta(a)$ and thus that a is stable as a function germ as well. The result follows because the only stable function singularities are morse singularities.

⋈

Proposition 4.22

Suppose that $f\colon \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g\colon \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are multigerms each of which has at least one component which is not a submersion. Suppose also that the multigerm $h\colon \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$, whose components are those of f together with those of g, is primitive of $\mathcal{A}_e$-codimension one and that g is not transverse to $\tau(f)$. Then g has precisely one non-submersive component and that is either a prism on a morse singularity or it is an immersion.

Proof

At least one of the components of g ($g^{(1)}$ say) is not transverse to $\tau(f)$ so $\tau(g) \subseteq \tau(g^{(1)}) \subseteq \mathrm{im}(Tg^{(1)}) \subset \tau(g) + \mathbf{C}v$ because $T_0\mathbf{C}^p = \tau(f) \oplus \tau(g) \oplus \mathbf{C}v$. Therefore $\tau(g) = \tau(g^{(1)}) = \mathrm{im}(Tg^{(1)})$. It now follows from Proposition 4.9 that $g^{(1)}$ is a prism on a map of rank zero and we may apply Lemma 4.21. We deduce that $g^{(1)}$ is either a prism on a morse singularity or an immersion (it cannot be a submersion because $v \notin \tau(g^{(1)})$).

By Corollary 4.4, for each component $g^{(i)}$ of g, $\tau(g^{(i)}) \supseteq \tau(g) = \tau(g^{(1)})$ but since g is stable, by Proposition 4.2, the map

$$T_0\mathbf{C}^p \to \frac{T_0\mathbf{C}^p}{\tau(g^{(1)})} \oplus \cdots \oplus \frac{T_0\mathbf{C}^p}{\tau(g^{(s)})}$$

is surjective. It follows that for $i \neq 1$, $\tau(g^{(i)}) = T_0\mathbf{C}^p$, i.e. every component of g except $g^{(1)}$ is a submersion.

⋈

If g is as in this last proposition and $g^{(1)}$ is a prism on a morse singularity then the codimension of the analytic stratum of g is one and so $\tau(f) = 0$. Therefore we have a decomposition of the target as $\mathbf{C}^{p-1} \times \mathbf{C}$ where $\mathbf{C}^{p-1} \times \{0\}$ is the analytic stratum of g. There is some neighbourhood U of 0 in $\mathbf{C}^{p-1}$ such that for all $u \in U$, the pullback of $g^{(1)}$ along the inclusion of the subset $\{u\} \times \mathbf{C}$ is a morse singularity and so by changing the coordinates in the source as in 2.2 of [12] we can reduce this pullback to the form $\sum_{i=1}^{m} x_i^2$. In fact we can say more than this. The changes of coordinates in the source (if they are chosen as they are in the reference) depend analytically on u and so together they give a change of coordinates in the source of $g^{(1)}$ which reduces it to the form

$$\mathrm{id}_{\mathbf{C}^{p-1}} \times \sum_{i=1}^{m} x_i^2 : \mathbf{C}^{p-1} \times \mathbf{C}^m \to \mathbf{C}^{p-1} \times \mathbf{C}$$
$$(\lambda, (x_1, \ldots, x_m)) \mapsto \left(\lambda, \sum_{i=1}^{m} x_i^2\right).$$

Alternatively if g is as in the last proposition and $g^{(1)}$ is an immersion then the image of $g^{(1)}$ is the analytic stratum of g and we can change coordinates in the source of $g^{(1)}$ to make it the inclusion of $\mathbf{C}^b$ in $\mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$.

Now we shall treat the case where g is transverse to $\tau(f)$. Recall that we have decomposed the target $\mathbf{C}^p$ into $\mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ where $\mathbf{C}^a$ is the analytic stratum of f and $\mathbf{C}^b$ is the analytic stratum of g. By the implicit function theorem we can decompose the source $\mathbf{C}^n$ of g into $\mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C}$ in such a way that g preserves the last $b+1$ coordinates. Now for $\lambda \in \mathbf{C}^b \times \mathbf{C}$, define $g_\lambda : \mathbf{C}^{n-b-1} \to \mathbf{C}^a$ by $g_\lambda = \pi \circ g \circ i_\lambda$ where $i_\lambda : \mathbf{C}^{n-b-1} \to \mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C}$, $x \mapsto (x, \lambda)$ and $\pi : \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C} \to \mathbf{C}^a$ is the natural projection. Then $g = g_\lambda \times \mathrm{id}_{\mathbf{C}^b \times \mathbf{C}}$ is an unfolding of g_0.

Proposition 4.23

In the circumstances described above, the map

$$\bar{g} : \mathbf{C}^{n-b-1} \times \mathbf{C} \to \mathbf{C}^a \times \mathbf{C}$$
$$(x, \lambda) \mapsto (g_{(0,\lambda)}(x), \lambda) \qquad \text{where } (0, \lambda) \in \mathbf{C}^b \times \mathbf{C}$$

is an $\mathcal{A}_e$-versal unfolding of g_0.

Proof

It suffices to show that if $G : \mathbf{C}^{n-b-1} \times \mathbf{C}^d \to \mathbf{C}^a \times \mathbf{C}^d$ is an unfolding of g_0, then G can be induced from $\bar{g}$. Notice that g_λ is the pullback of g by the inclusion γ_λ of $\mathbf{C}^a$ in $\mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ (as $\mathbf{C}^a \times \{\lambda\}$). Also $\bar{g}$ is the pullback of g by the inclusion $\bar{\gamma}$ of $\mathbf{C}^a \times \mathbf{C}$ in $\mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ (as $\mathbf{C}^a \times \{0\} \times \mathbf{C}$). By Proposition 1.2 of [16], there is a map $\Gamma: \mathbf{C}^a \times \mathbf{C}^d \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ such that $\Gamma|_{\mathbf{C}^a\times\{0\}} = \gamma_0$ and such that the pullback of g by Γ is G. Define

$$\bar{\omega}: \mathbf{C}^p \times \mathbf{C} \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$$
$$(X,\mu) \mapsto X + (0,0,\mu).$$

Then $\bar{\omega}|_{\mathbf{C}^p\times\{0\}} = \mathrm{id}_{\mathbf{C}^p}$ and $\bar{\omega}|_{\mathbf{C}^a\times\mathbf{C}} = \bar{\gamma}$ (where we consider $\mathbf{C}^a$ as a subset of $\mathbf{C}^p = \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$). We can find a map Ω that is to Γ as $\bar{\omega}$ is to $\bar{\gamma}$. More precisely, there is a map $\Omega: \mathbf{C}^p \times \mathbf{C}^d \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ such that $\Omega|\mathbf{C}^p \times \{0\} = \mathrm{id}_{\mathbf{C}^p}$ and $\Omega|_{\mathbf{C}^a\times\mathbf{C}^d} = \Gamma$. We now consider two unfoldings of h. Define $H_{\bar{\omega}}: \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}^p \times \mathbf{C}$ to be the unfolding of h made up of $G_{\bar{\omega}} = g \times \mathrm{id}_{\mathbf{C}}$ and

$$F_{\bar{\omega}}: \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}^p \times \mathbf{C}$$
$$(x,\mu) \mapsto (\bar{\omega}[f(x),\mu],\mu).$$

Also define $H_\Omega: \mathbf{C}^n \times \mathbf{C}^d \to \mathbf{C}^p \times \mathbf{C}^d$ to be the unfolding of h made up of $G|_\Omega = g \times \mathrm{id}_{\mathbf{C}^d}$ and

$$F_\Omega: \mathbf{C}^n \times \mathbf{C}^d \to \mathbf{C}^p \times \mathbf{C}^d$$
$$(x,\nu) \mapsto (\Omega[f(x),\nu],\nu).$$

By Lemma 4.13, $H_{\bar{\omega}}$ is an $\mathcal{A}$-versal unfolding of h and therefore there is a map $\alpha: \mathbf{C}^d \to \mathbf{C}$ such that the unfolding H_Ω of h is induced from $H_{\bar{\omega}}$ via the map α of their parameter spaces (up to changes of coordinates in the sources and targets of the maps parallel to h). Now notice that g_0 is the pullback of g along the analytic stratum of f, that the unfolding $\bar{g}$ of g_0 is the pullback of the unfolding $G_{\bar{\omega}}$ of g along the analytic stratum of the unfolding $F_{\bar{\omega}}$ of f and that similarly the unfolding G of g_0 is the pullback of the unfolding G_Ω of g along the analytic stratum of the unfolding F_Ω of f. However since analytic strata and pullbacks are respected by $\mathcal{A}$-equivalences, for any multigerm $\hat{h}$ made up of the multigerms $\hat{f}$ and $\hat{g}$, the $\mathcal{A}$-equivalence class of the pullback of $\hat{g}$ along the analytic stratum of $\hat{f}$ is determined by the $\mathcal{A}$-equivalence class of $\hat{h}$. It follows that G can be induced from $\bar{g}$ by the map α (up to a change of coordinates in the source and target of the maps parallel to g_0).

$\bowtie$

Corollary 4.24

In the circumstances that are described before Proposition 4.23, g_0 has $\mathcal{A}_e$-codimension one.

Proof

By Proposition 4.23, g_0 has a one-dimensional $\mathcal{A}_e$-versal unfolding so g_0 either has $\mathcal{A}_e$-codimension one or it is stable. However if g_0 were stable then the unfolding g of g_0 would be trivial, and thus $\tau(g)$ would be transverse to $\tau(f)$. We know that this is not the case.

$\bowtie$

This result suggests another way (besides augmentation) that we may be able to build codimension one multigerms from simpler ones.

Corollary 4.25

In the circumstances described before Proposition 4.23, g is an $\mathcal{A}_e$-versal unfolding of g_0.

Proof

g contains the $\mathcal{A}_e$-versal unfolding $\bar{g}$ of Proposition 4.23.

$\bowtie$

Lemma 4.26

In the circumstances described before Proposition 4.23, g_0 is primitive.

Proof

Suppose not, then by Proposition 4.23 and Theorem 3.3, $\tau(\bar{g})$ intersects the codomain of g_0 non-trivially, and then by Corollary 4.8, $\tau(g)$ intersects the codomain of g_0 non-trivially. But the codomain of g_0 is $\tau(f)$ and so by Proposition 4.20, h is an augmentation. However we have assumed that h is primitive—a contradiction.

$\bowtie$

We would now like to reduce g to some normal form analogously to what we did after Proposition 4.22, but it turns out that we cannot do this for an arbitrary g_0. Suppose that $\hat{f}: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ is a quasihomogeneous multigerm of $\mathcal{A}_e$-codimension one. Then we can find a quasihomogeneous versal unfolding $\hat{F} = \mathrm{id}_{\mathbf{C}} \times \hat{f}_\lambda$ of $\hat{f}$. Let the weights in the target of $\hat{F}$ be $w, w_1, \ldots, w_p$ and let those in the source of the i^{th} component of $\hat{F}$ be $w, d_1^{(i)}, \ldots, d_n^{(i)}$ (for all i, $1 \le i \le s$). For $\mu \in \mathbf{C}$ let $\psi_\mu: \mathbf{C}^p \to \mathbf{C}^p$ be the linear map whose matrix is

$$\begin{pmatrix} \mu^{w_1} & 0 & 0 & \cdots & 0 \\ 0 & \mu^{w_2} & 0 & \cdots & 0 \\ 0 & 0 & \mu^{w_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu^{w_p} \end{pmatrix},$$

let $\Psi_\mu: \mathbf{C}^{1+p} \to \mathbf{C}^{1+p}, (\lambda, x) \mapsto (\mu^w, \lambda, \psi_\mu(x))$ and let $\phi_\mu^{(i)}$ and $\Phi_\mu^{(i)}$ be the analogues of these maps in the source of the i^{th} component of $\hat{f}$ and $\hat{F}$ respectively. Then for each component $\hat{F}^{(i)}$ of $\hat{F}$ and for all $\mu \in \mathbf{C}$, $\hat{F}^{(i)} \circ \Phi_\mu^{(i)} = \Psi_\mu \circ \hat{F}^{(i)}$. So for each component $\hat{f}^{(i)}$ of $\hat{f}$ and all $\lambda, \mu \in \mathbf{C}$, $\hat{f}^{(i)}_{\lambda\mu^w} \circ \phi_\mu^{(i)} = \psi_\mu \circ \hat{f}^{(i)}_\lambda$. Let ψ_μ be the change of coordinates in the source made up of the various $\phi_\mu^{(i)}$, then

$$\hat{f}_{\lambda\mu^w} \circ \phi_\mu = \psi_\mu \circ \hat{f}_\lambda.$$

We define the description *nice* by saying that a multigerm $\hat{f}$ of $\mathcal{A}_e$-codimension one is nice if and only if it is quasihomogeneous and has a quasihomogeneous versal unfolding $(\lambda, x) \mapsto (\lambda, \hat{f}_\lambda(x))$, where the degree of the first coordinate (λ) is non-zero.

Proposition 4.27

Let $\bar{f}: \mathbb{C}^n \to \mathbb{C}^p$ and $\tilde{f}: \mathbb{C}^n \to \mathbb{C}^p$ be nice codimension one multigerms which are $\mathcal{A}$-equivalent to each other. Let $\bar{F} = \mathrm{id}_{\mathbb{C}\times\mathbb{C}^{d-1}} \times \bar{f}_\lambda: \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p$ be a versal unfolding of $\bar{f} = \bar{f}_0$ with analytic stratum $\{0\} \times \mathbb{C}^{d-1} \times \{0\} \subseteq \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p$. Finally let $\tilde{F} = \mathrm{id}_{\mathbb{C}} \times \tilde{f}_\lambda: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^p$ be a miniversal unfolding of $\tilde{f} = \tilde{f}_0$. Then there is a family α_λ of bi-analytic maps $\mathbb{C}^n \to \mathbb{C}^n$ where $\lambda \in \mathbb{C} \times \mathbb{C}^{d-1}$ and a family β_λ of bi-analytic maps $\mathbb{C}^p \to \mathbb{C}^p$ such that if we define $\alpha: \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n$ by $(\mu, \nu, x) \mapsto (\mu, \alpha_{(\mu,\nu)}(x))$ and $\beta: \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p \to \mathbb{C} \times \mathbb{C}^p$ by $(\mu, \nu, X) \mapsto (\mu, \beta_{(\mu,\nu)}(X))$ then the following diagram commutes.

$$\begin{tikzcd} \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n \arrow[r, "\bar{F}"] \arrow[d, "\alpha"] & \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p \arrow[d, "\beta"] \\ \mathbb{C} \times \mathbb{C}^n \arrow[r, "\tilde{F}"] & \mathbb{C} \times \mathbb{C}^p \end{tikzcd}$$

Proof

We introduce a multigerm $\hat{f}: \mathbb{C}^n \to \mathbb{C}^p$ which is $\mathcal{A}$-equivalent to $\bar{f}$ but with coordinates chosen to exhibit the quasihomogeneity. Also we let $\hat{F}: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^p$ be a quasihomogeneous miniversal unfolding of $\hat{f}$ (as described before this proposition). Let $\bar{\phi}: \mathbb{C}^n \to \mathbb{C}^n$ and $\bar{\psi}: \mathbb{C}^p \to \mathbb{C}^p$ be bi-analytic maps such that $\bar{\psi} \circ \bar{f} = \hat{f} \circ \bar{\phi}$. Then in the following (rather obviously) commutative diagram, the bottom arrow is a versal unfolding of $\hat{f}$ with analytic stratum $\{0\} \times \mathbb{C}^{d-1} \times \{0\} \subseteq \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p$.

$$\begin{tikzcd} \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n \arrow[r, "\bar{F}"] \arrow[d, "\mathrm{id}_{C\times\mathbb{C}^{d-1}}\bar{\phi}"] & \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p \arrow[d, "\mathrm{id}_{C\times\mathbb{C}^{d-1}}\bar{\psi}"] \\ \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n \arrow[r, "(\mathrm{id}_{\mathbb{C}\times\mathbb{C}^{d-1}}\times\bar{\psi})\circ\bar{F}\circ(\mathrm{id}_{\mathbb{C}\times\mathbb{C}^{d-1}}\times\bar{\phi})^{-1}"] & \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p \end{tikzcd}$$

Since $\hat{F}$ is a miniversal unfolding of $\hat{f}$, there is a map $\gamma: \mathbb{C} \times \mathbb{C}^{d-1} \to \mathbb{C}$ and there are families of bi-analytic maps $\bar{\phi}_\lambda$ of $\mathbb{C}^n$ and $\bar{\psi}_\lambda$ of $\mathbb{C}^p$, where λ varies through $\mathbb{C} \times \mathbb{C}^{d-1}$ such that if we define $\Lambda: \mathbb{C} \times \mathbb{C}^{d-1} \to \mathbb{C} \times \mathbb{C}^{d-1}$ by $(\mu, \nu) \mapsto (\gamma(\mu, \nu), \nu)$ then we have the following commutative diagram.

$$
\begin{tikzcd}
\mathbf{C} \times \mathbf{C}^{d-1} \times \mathbf{C}^n \arrow[rr, "(\mathrm{id}_{\mathbf{C}\times\mathbf{C}^{d-1}} \times \bar{\psi}) \circ \bar{F} \circ (\mathrm{id}_{\mathbf{C}\times\mathbf{C}^{d-1}} \times \bar{\phi})^{-1}"] \arrow[d, "\Gamma \times \bar{\phi}_\lambda"] & & \mathbf{C} \times \mathbf{C}^{d-1} \times \mathbf{C}^p \arrow[d, "\Gamma \times \bar{\psi}_\lambda"] \\
\mathbf{C} \times \mathbf{C}^{d-1} \times \mathbf{C}^n \arrow[rr, "\hat{F} \times \mathrm{id}_{\mathbf{C}^{d-1}}"] & & \mathbf{C} \times \mathbf{C}^{d-1} \times \mathbf{C}^p
\end{tikzcd}
$$

Since the top arrow of this diagram is a versal unfolding of $\hat{f}$ and $\hat{F}$ is a *mini*versal unfolding of $\hat{f}$, γ is a submersion. Also we see that $\gamma^{-1}(0) = \{0\} \times \mathbf{C}^{d-1}$ so by the inverse function theorem, Γ is bi-analytic. Since Γ commutes with projection onto $\mathbf{C}^{d-1}$, Γ^{-1} does also, so there is a map $\gamma' \colon \mathbf{C} \times \mathbf{C}^{d-1} \to \mathbf{C}$ such that $\Gamma^{-1}(\mu, \nu) = (\gamma'(\mu, \nu), \nu)$. Since for all μ in $\mathbf{C}^{d-1}$, $\gamma(0, \mu) = 0$, we see that for all μ in $\mathbf{C}^{d-1}$, $\gamma'(0, \mu) = 0$. Therefore if the function $\mu \colon \mathbf{C} \times \mathbf{C}^{d-1} \to \mathbf{C}$ is projection onto the first coordinate, μ divides γ'—say $\gamma' = \mu\gamma''$. Since Γ^{-1} is bi-analytic, γ' is a submersion, so γ'' is non-zero in a neighbourhood of $0 \in \mathbf{C} \times \mathbf{C}^{d-1}$. Now for $(\mu, \nu, x) \in \mathbf{C} \times \mathbf{C}^{d-1} \times \mathbf{C}^n$ we have

$$
\begin{aligned}
[(\gamma' \times \psi_{\sqrt[w]{\gamma''}}) \circ (\hat{F} \times \mathrm{id}_{\mathbf{C}^{d-1}})](\mu, \nu, x) &= (\gamma' \times \psi_{\sqrt[w]{\gamma''}})(\mu, \nu, \hat{f}_\mu(x)) \\
&= (\gamma'(\mu, \nu), [\psi_{\sqrt[w]{\gamma''(\mu,\nu)}} \circ \hat{f}](x)) \\
&= (\gamma'(\mu, \nu), [\hat{f}_{\gamma'(\mu,\nu)} \circ \phi_{\sqrt[w]{\gamma''(\mu,\nu)}}](x)) \\
&= \hat{F}(\gamma'(\mu, \nu), \phi_{\sqrt[w]{\gamma''(\mu,\nu)}}(x)) \\
&= [\hat{F} \circ (\gamma' \times \phi_{\sqrt[w]{\gamma''}})](\mu, \nu, x)
\end{aligned}
$$

where we have used notation developed just before the statement of this proposition and $\sqrt[w]{}$ is chosen analytically in a neighbourhood of $\gamma''(0,0)$. We have a commutative diagram.

$$
\begin{tikzcd}
\mathbf{C} \times \mathbf{C}^{d-1} \times \mathbf{C}^n \arrow[r, "\hat{F} \times \mathrm{id}_{\mathbf{C}^{d-1}}"] \arrow[d, "\gamma' \times \phi_{\sqrt[w]{\gamma''}}"] & \mathbf{C} \times \mathbf{C}^{d-1} \times \mathbf{C}^p \arrow[d, "\gamma' \times \psi_{\sqrt[w]{\gamma''}}"] \\
\mathbf{C} \times \mathbf{C}^n \arrow[r, "\hat{F}"] & \mathbf{C} \times \mathbf{C}^p
\end{tikzcd}
$$

We define

$$
\bar{\alpha} := (\gamma' \times \phi_{\sqrt[w]{\gamma''}}) \circ (\Gamma \times \bar{\phi}_\lambda) \circ (\mathrm{id}_{\mathbf{C}\times\mathbf{C}^{d-1}} \times \bar{\phi})
$$

and

$$
\bar{\beta} := (\gamma' \times \psi_{\sqrt[w]{\gamma''}}) \circ (\Gamma \times \bar{\psi}_\lambda) \circ (\mathrm{id}_{\mathbf{C}\times\mathbf{C}^{d-1}} \times \bar{\psi}).
$$

Then we have the following commutative diagram (it is commutative because it is the outside of the diagram got by stacking the previous three diagrams on top of each other).

$$\begin{tikzcd}
\mathbf{C}\times\mathbf{C}^{d-1}\times\mathbf{C}^n \arrow[r, "\bar{F}"] \arrow[d, "\bar{\alpha}"] & \mathbf{C}\times\mathbf{C}^{d-1}\times\mathbf{C}^p \arrow[d, "\bar{\beta}"] \\
\mathbf{C}\times\mathbf{C}^n \arrow[r, "\hat{F}"] & \mathbf{C}\times\mathbf{C}^p
\end{tikzcd}$$

As a special case of what we have proved so far, if we take $d = 1$ and $\bar{F} = \tilde{F}$ we get a commutative diagram

$$\begin{tikzcd}
\mathbf{C}\times\mathbf{C}^n \arrow[r, "\tilde{F}"] \arrow[d, "\tilde{\alpha}"] & \mathbf{C}\times\mathbf{C}^p \arrow[d, "\tilde{\beta}"] \\
\mathbf{C}\times\mathbf{C}^n \arrow[r, "\hat{F}"] & \mathbf{C}\times\mathbf{C}^p
\end{tikzcd}$$

where $\tilde{\alpha}(\lambda, x) = \big(\lambda, \tilde{\alpha}_\lambda(s)\big)$ for bi-analytic $\tilde{\alpha}_\lambda$. We deduce that $\tilde{\alpha}$ is bi-analytic (similarly for $\tilde{\beta}$).

Now if we define $\alpha := \tilde{\alpha}^{-1} \circ \bar{\alpha}$, $\alpha_{(\mu,\nu)} := \tilde{\alpha}_\mu^{-1} \circ \bar{\alpha}_{(\mu,\nu)}$ and make a similar definition for β then we have the required commutative diagram as we can see by putting the last-but-one commutative diagram on top of the reflection of the last one in the x-axis.

$\bowtie$

Theorem 4.28

Suppose that $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ and $g: \mathbf{C}^n, T \to \mathbf{C}^p, \{0\}$ are multigerms each of which has at least one component which is not a submersion. Suppose also that the multigerm $h: \mathbf{C}^n, S \cup T \to \mathbf{C}^p, \{0\}$, whose components are those of f together with those of g, has $\mathcal{A}_e$-codimension one and is primitive. Further suppose that a) either g is not transverse to $\tau(f)$ or, if it is, that the pullback of g by the analytic stratum of f (which we know has $\mathcal{A}_e$-codimension one) is nice and b) either f is not transverse to $\tau(g)$ of the pullback of f by the analytic stratum of g is nice. Then there is a change of coordinates in the source and target of h which reduces it to the following form:

$f: \mathbf{C}^n, S \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}, \{0\}$ and $g: \mathbf{C}^n, T \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ are stable multigerms, the analytic stratum of f is $\mathbf{C}^a$ and the analytic stratum of g is $\mathbf{C}^b$. Also one of the following three statements holds.

i) f has only one non-submersive component $f^{(1)}$ and that is of the form

$$f^{(1)}: \mathbf{C}^a, \{0\} \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}, \{0\} \times \{0\} \times \{0\}$$
$$x \mapsto (x, 0, 0)$$

ii) $b = 0$ *and* f *has only one non-submersive component* $f^{(1)}$. *Furthermore that is of the form*

$$f^{(1)}: \mathbf{C}^a \times \mathbf{C}^{n-a}, \{0\} \times \{0\} \to \mathbf{C}^a \times \mathbf{C}, \{0\} \times \{0\}$$
$$(\lambda, (x_1, \ldots, x_{n-a})) \mapsto \left(\lambda, \sum_{i=1}^{n-a} x_i^2\right)$$

iii) *There is a nice, primitive multigerm* $\tilde{f}$ *(which we can choose to be any representative of the* $\mathcal{A}$*-equivalence class of the pullback of* f *by the analytic stratum of* g*) and a versal unfolding* $\tilde{F} = \mathrm{id}_{\mathbf{C}} \times \tilde{f}_\mu$ *of* $\tilde{f} = \tilde{f}_0$ *(we can choose* $\tilde{F}$ *to be any miniversal unfolding of* $\tilde{f}$*) such that*

$$f: \mathbf{C}^a \times \mathbf{C}^{n-a-1} \times \mathbf{C}, \{0\} \times S \times \{0\} \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}, \{0\} \times \{0\} \times \{0\}$$
$$(\lambda, x, \mu) \mapsto (\lambda, \tilde{f}_\mu(x), \mu).$$

Finally one of the three analogous statements holds in which f *and* g *are exchanged,* a *and* b *are exchanged and so on.*

Proof

Recall that we have decomposed the target $\mathbf{C}^p$ into $\mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ where $\mathbf{C}^a$ is the analytic stratum of f and $\mathbf{C}^b$ is the analytic stratum of g.

Suppose first that g is not transverse to $\tau(f)$; then by Proposition 4.22, every component of g except one is a submersion. By the implicit function theorem we can change the coordinates in the source of a submersion to make it a projection onto the first p coordinates (where p, of course, is the dimension of the target). The remaining component of g can be reduced to a standard form by a change of coordinates in the source as described after Proposition 4.22. Notice that these coordinate changes do not affect f.

Now suppose that g is transverse to $\tau(f)$ and that $\tilde{G} = \mathrm{id}_{\mathbf{C}} \times \tilde{g}_\mu$ is a miniversal unfolding of a normal form $\tilde{g} = \tilde{g}_0$ of the pullback of g by the analytic stratum of f. By Proposition 4.27 there is a change of coordinates in the source and target of g such that the change of coordinates in the target respects projection onto $\mathbf{C}^b \times \mathbf{C}$ (i.e. it is parallel to the analytic stratum of f) and such that after this change, $g: \mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C} \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$, $(x, \lambda, \mu) \mapsto (\tilde{g}_\mu(x), \lambda, \mu)$. Since the change of coordinates is parallel to the analytic stratum of f, this stratum is not moved. It is clear that the analytic stratum of g is not moved either.

We now normalise f in the same manner. In the case that f is not transverse to $\tau(g)$, this process does not affect g, but in the case that f is transverse to $\tau(g)$, g is altered. However, the alteration is due to a change of coordinates in the target parallel to the analytic stratum of g and can therefore be undone by a change of coordinates in the source of g. More explicitly:

$$g: \mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C} \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$$
$$(x, \lambda, \mu) \mapsto (\tilde{g}_\mu(x), \lambda, \mu)$$

and the change of coordinates in the target is given by

$$\begin{aligned}\psi\colon \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C} &\to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}\\ (\nu, \lambda, \mu) &\mapsto \big(\nu, \psi_{\nu,\mu}(\lambda), \mu\big)\end{aligned}$$

for a bi-analytic family $\psi_{\nu,\mu}\colon \mathbf{C}^b \to \mathbf{C}^b$. Now if for $\nu \in \mathbf{C}^{n-b-1}$ and $\mu \in \mathbf{C}$ we define $\phi\colon \mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C} \to \mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C}$ by $(\nu, \lambda, \mu) \mapsto \big(\nu, \psi_{\tilde{g}_\mu(\nu),\mu}(\lambda), \mu\big)$ then we have the following commutative diagram.

$$\begin{tikzcd} \mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C} \arrow[r, "g"] \arrow[d, "\phi"] & \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C} \arrow[d, "\psi"] \\ \mathbf{C}^{n-b-1} \times \mathbf{C}^b \times \mathbf{C} \arrow[r, "g"] & \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C} \end{tikzcd}$$

Finally by Lemma 4.26, if they exist, $\tilde{f}$ and $\tilde{g}$ are primitive.

⋈

This result shows that if we have an $\mathcal{A}_e$-codimension one multigerm, then under certain circumstances it can be reduced to a standard form (modulo some choices). We now prove a converse.

Theorem 4.29

Suppose that h is a multigerm in the form to which h was reduced in Theorem 4.28. Then h has $\mathcal{A}_e$-codimension one, is primitive and is nice.

Proof

Using Proposition 4.27 as in the proof of Theorem 4.28 we see that in case iii) we can choose $\tilde{f}$ and $\tilde{F}$ to be quasihomogeneous relative to the standard coordinates of their sources and targets.

In order to show that h has $\mathcal{A}_e$-codimension one it is sufficient to prove that $\theta(h) = T\mathcal{A}_e h + \mathbf{C}\cdot\nu$ where ν is the vector field along h which is $wf(\frac{\partial}{\partial\mu})$ along f and 0 along g (where μ is the last coordinate function of $\mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$). We may think of $\theta(h)$ as $\theta(f) \oplus \theta(g)$ and since f and g are both stable, $\theta(f) = tf\big(\theta(n)\big) + wf\big(\theta(a+b+1)\big)$ and $\theta(g) = tg\big(\theta(n)\big) + wg\big(\theta(a+b+1)\big)$. It is therefore sufficient to prove that for all $\eta', \eta'' \in \theta(a+b+1)$, the pair $\big(wf(\eta'), wg(\eta'')\big) \in T\mathcal{A}_e h + \mathbf{C}\cdot\nu$. This would follow if we could show that for all $\eta \in \theta(a+b+1)$, $wf(\eta) \in T\mathcal{A}_e h + \mathbf{C}\cdot\nu$ because if we set $\eta = \eta' - \eta''$ then

$$\big(wf(\eta'), wg(\eta'')\big) = \big(wf(\eta''), wg(\eta'')\big) + wf(\eta) = wh(\eta'') + wf(\eta).$$

Let the coordinates of $\mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ be $X_1, \ldots, X_a, Y_1, \ldots, Y_b, \mu$; then write

$$\eta = \sum_{i=1}^{a} a_i \frac{\partial}{\partial X_i} + \sum_{j=1}^{b} b_j \frac{\partial}{\partial Y_j} + c\frac{\partial}{\partial \mu}.$$

Now $wf(\sum_{i=1}^{a} a_i \frac{\partial}{\partial X_i})$ is in the trivial direction of f and is thus in the $\mathcal{A}_e$-tangent space of h.

$$wf\left(\sum_{j=1}^{b} b_j \frac{\partial}{\partial Y_j}\right) = wg\left(\sum_{j=1}^{b} -b_j \frac{\partial}{\partial Y_j}\right) + wh\left(\sum_{j=1}^{b} b_j \frac{\partial}{\partial Y_j}\right)$$

and this vector field is in the $\mathcal{A}_e$-tangent space of h because each of the two summands is. We now only have to prove that $wf(c\frac{\partial}{\partial \mu}) \in T\mathcal{A}_e h + \mathbf{C}\nu$ and to do this it suffices to show that $wf(\mathbf{m}\frac{\partial}{\partial \mu}) \subseteq T\mathcal{A}_e h$ (where $\mathbf{m}$ is the maximal ideal of $\mathcal{O}_{a+b+1}$).

Let c be an arbitrary element of $\mathbf{m}$. Since $\mathbf{m}$ is generated by the coordinate functions we can write

$$c = \sum_{i=1}^{a} \alpha_i X_i + \sum_{j=1}^{b} \beta_j Y_j + m\mu$$

where $\alpha_i, \beta_j, m \in \mathcal{O}_{a+b+1}$. It now suffices to show that for i between 1 and a, $wf(\alpha_i X_i \frac{\partial}{\partial \mu}) \in T\mathcal{A}_e h$, that for j between 1 and b, $wf(\beta_j Y_j \frac{\partial}{\partial \mu}) \in T\mathcal{A}_e h$ and that $wf(m\mu\frac{\partial}{\partial \mu}) \in T\mathcal{A}_e h$.

We deal first with $wf(\beta_j Y_j \frac{\partial}{\partial \mu})$ and consider the three cases (as enumerated in the statement of Theorem 4.28) separately. In case i), this term is zero on the only non-submersive component (because Y_j vanishes on the image of this component) and therefore is in $T\mathcal{A}_e h$. In case ii), $b = 0$ and so this term does not exist. We are left with case iii).

$\tilde{f}: \mathbf{C}^{n-a-1} \to \mathbf{C}^b$ is induced from the stable map $\tilde{F}: \mathbf{C}^{n-a-1} \times \mathbf{C} \to \mathbf{C}^b \times \mathbf{C}$ by the inclusion $\gamma: \mathbf{C}^b \to \mathbf{C}^b \times \mathbf{C}$. Therefore the $\mathcal{K}_{D(\tilde{F})}$-codimension of γ is equal to the $\mathcal{A}_e$-codimension of $\tilde{F}$ which is one. That is,

$$\dim_{\mathbf{C}}\left[\frac{\gamma^*\theta(b+1)}{t\gamma\big(\theta(b)\big) + \gamma^*\mathrm{Derlog}D(\tilde{F})}\right] = 1.$$

We can identify $\gamma^*\theta(b+1)$ with $(\mathcal{O}_{\mathbf{C}^b})^{1+b}$ and $t\gamma\big(\theta(b)\big)$ with $(\mathcal{O}_{\mathbf{C}^b})^b = \{0\} \times (\mathcal{O}_{\mathbf{C}^b})^b \subseteq \mathcal{O}_{\mathbf{C}^b} \times (\mathcal{O}_{\mathbf{C}^b})^b$. $\gamma^*\mathrm{Derlog}D(\tilde{F}) \subseteq \mathbf{m}_b\frac{\partial}{\partial \mu} + t\gamma\big(\theta(b)\big)$ where μ is the first coordinate of $\mathbf{C} \times \mathbf{C}^b$ (because if not,some element of $\mathrm{ev}_0[\mathrm{Derlog}D(\tilde{F})]$ would be transverse to the image of γ and so $\tilde{F}$ would be a prism on $\tilde{f}$ which is impossible). Now we see that

$$\gamma^*\mathrm{Derlog}D(\tilde{F}) + t\gamma\big(\theta(b)\big) = \mathbf{m}_b\frac{\partial}{\partial \mu} + t\gamma\big(\theta(b)\big).$$

In particular $Y_j\frac{\partial}{\partial \mu} \in \gamma^*\mathrm{Derlog}D(\tilde{F}) + t\gamma\big(\theta(b)\big)$ (as a vector field along γ), say $Y_j\frac{\partial}{\partial \mu} = \alpha + \beta$ where $\alpha \in \gamma^*\mathrm{Derlog}D(\tilde{F})$ and $\beta \in t\gamma\big(\theta(b)\big) = (\mathcal{O}_{\mathbf{C}^b})^b$.

We now return to vector fields along h. On $\mathbf{C}^a \times \mathbf{C}^{n-a-1} \times \{0\}$, $wf(\beta_j Y_j \frac{\partial}{\partial\mu}) = wf(\beta_j\alpha) + wf(\beta_j\beta)$. $\beta_j\alpha$ lifts along f (see the proof of Theorem 3.5) so the first of the summands is in $\operatorname{im}(tf) \subseteq T\mathcal{A}_e h$. $wf(\beta_j\beta)$ has no component in the $\frac{\partial}{\partial\mu}$ direction and so by what we have already proved is in $T\mathcal{A}_e h$. Since $wf(\beta_j Y_j \frac{\partial}{\partial\mu})$ differs from a germ of a vector field in $T\mathcal{A}_e h$ by a vector field which vanishes on $\mathbf{C}^a \times \mathbf{C}^{n-a-1} \times \{0\}$ (and hence is a multiple of μ), we reduce the problem of showing that $wf(\beta_j Y_j \frac{\partial}{\partial\mu})$ is in $T\mathcal{A}_e h$ to that of showing that $wf(m'\mu\frac{\partial}{\partial\mu})$ is in $T\mathcal{A}_e h$ for some $m' \in \mathcal{O}_{a+b+1}$ and this is the same as showing that $wf(m\mu\frac{\partial}{\partial\mu}) \in T\mathcal{A}_e h$ which we will do later in this proof.

Now we move onto the term $wf(\alpha_i X_i \frac{\partial}{\partial\mu})$. It is sufficient to show that $wg(-\alpha_i X_i \frac{\partial}{\partial\mu})$ (which is equal to $wf(\alpha_i X_i \frac{\partial}{\partial\mu}) - wh(\alpha_i X_i \frac{\partial}{\partial\mu})$) is in $T\mathcal{A}_e h$. This is completely analogous to showing that $wf(\beta_j Y_j \frac{\partial}{\partial\mu}) \in T\mathcal{A}_e h$ and therefore we can use the method above.

Lastly we have to show that $wf(m\mu\frac{\partial}{\partial\mu}) \in T\mathcal{A}_e h$. In case i) this is because this vector field vanishes on the only non-submersive component. In case ii) this vector field is $tf^{(1)}\left((m \circ f^{(1)}) \cdot \sum_{i=1}^{n-a} x_i \frac{\partial}{\partial x_i}\right)$.

Again we are left with case iii). Since $\tilde{F}$ is quasihomogeneous, as before we have, $\forall \lambda \in \mathbf{C}$, $F \circ \Phi_\lambda = \Psi_\lambda \circ \hat{F}$. Differentiating with respect to λ we have $t\tilde{F}(\xi) = w\tilde{F}(\eta)$ where

$$\eta = \left.\frac{\partial \Psi_\lambda}{\partial \lambda}\right|_{\lambda=0} = w\mu\frac{\partial}{\partial\mu} + \sum_{i=1}^{b} w_i X_i \frac{\partial}{\partial X_i}$$

and ξ is the corresponding multigerm in the source. Now returning to vector fields along f and h, $wf(m\mu\frac{\partial}{\partial\mu}) - tf(\frac{m \circ f}{w}\xi)$ has no component in the $\frac{\partial}{\partial\mu}$ direction and so, as before, is in $T\mathcal{A}_e h$. It follows that $wf(m\mu\frac{\partial}{\partial\mu})$ is in $T\mathcal{A}_e h$ also.

Since $\tilde{f}$ is primitive, $\tau(\tilde{F}) = 0$ and so by Proposition 4.9, $\tau(f) = \mathbf{C}^a \times \{0\} \times \{0\}$. Similarly $\tau(g) = \{0\} \times \mathbf{C}^b \times \{0\}$. Now by Proposition 4.20, h is primitive.

If in case iii) we let $\mathbf{C}^a \subseteq \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ inherit its weights from $\tilde{g}$, $\mathbf{C}^b$ inherit its weights from $\tilde{f}$ and $\mathbf{C} = \{0\} \times \{0\} \times \mathbf{C} \subseteq \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ inherit its weight from either $\tilde{F}$ or $\tilde{G}$ (we assume that the weight of the μ coordinate in f and g is the same which we can arrange by taking their lowest common multiple and multiplying the weights up as necessary) then both f and g are quasihomogeneous with respect to these weights and hence h is also. Cases i) and ii) are easier. Now if we set $\psi_t \colon \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C} \to \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$, $(x, y, z) \mapsto (x, y, z+t)$ then using Lemma 4.13 we can construct a miniversal unfolding H of h which is quasihomogeneous and whose unfolding parameter has non-zero weight. Hence h is nice.

$\bowtie$

§5 Generalisations

We notice two generalisations of the results so far. The first we will not use, but it is nevertheless interesting. We have not used the hypothesis that the dimensions of the sources of the various components are the same, therefore our results apply even when they are not (strictly speaking Lemma 4.12 is an exception but it is obvious what modifications have to be made).

The second generalisation requires more work, but is for our purposes more useful. If we replace $\mathbb{C}$, wherever it occurs in the previous chapters, by $\mathbb{R}$ and replace analytic maps by smooth ones (in other words, if we change from the complex analytic category to the real smooth one) then our results and proofs still hold, modulo some fairly minor alterations which we outline now.

Firstly we find that we have to define two augmentations $\mathrm{A}_F^+ f:(\lambda, x) \mapsto (\lambda, f_{\lambda^2}(x))$ and $\mathrm{A}_F^- f:(\lambda, x) \mapsto (\lambda, f_{-\lambda^2}(x))$ instead of just one. The proof of Lemma 2.4 requires some modification. In fact we will provide an alternative, less elementary proof of Theorem 2.5 that will generalise to the smooth case. Lemma 2.4 is an easy corollary of Theorem 2.5.

Theorem 5.1

If $f: \mathbb{C}^n, S \to \mathbb{C}^p, \{0\}$ is a multigerm of $\mathcal{A}_e$-codimension one then the $\mathcal{A}_e$-codimension of $\mathrm{A}f$ is also one.

Proof

Let

$$F: \mathbb{C} \times \mathbb{C}^n, \{0\} \times S \to \mathbb{C} \times \mathbb{C}^p, \{0\} \times \{0\}$$
$$(\lambda, x) \mapsto (\lambda, f_\lambda(x))$$

be a versal unfolding of $f = f_0$. Then we may take $\mathrm{A}_F f$ for our representative of $\mathrm{A}f$. f is induced from F by the map

$$\gamma: \mathbb{C}^p \to \mathbb{C} \times \mathbb{C}^p$$
$$X \mapsto (0, X)$$

and $\mathrm{A}_F f$ is induced from F by the map

$$\Gamma: \mathbb{C} \times \mathbb{C}^p \to \mathbb{C} \times \mathbb{C}^p$$
$$(\lambda, X) \mapsto (\lambda^2, X).$$

So using again the results of [2] we have that the $\mathcal{K}_{D(F)}$-codimension of γ is one and we have to prove that the $\mathcal{K}_{D(F)}$-codimension of Γ is one. The surjection $\Gamma^*\theta(1+p) \to \gamma^*\theta(1+p)$ given by restriction induces a surjection

$$\frac{\Gamma^*\theta(1+p)}{t\Gamma(\theta(1+p)) + \Gamma^*\mathrm{Derlog}(F)} \to \frac{\gamma^*\theta(1+p)}{t\gamma(\theta(p)) + \gamma^*\mathrm{Derlog}(F)}.$$

It is sufficient for us to show that this function is also injective. This is so because if $\nu \in \Gamma^*\theta(1+p)$ and $\nu|_{\{0\}\times\mathbb{C}^p} \in t\gamma(\theta(p))$ then $\nu \in t\Gamma(\theta(1+p))$ and also because if $\mu \in \Gamma^*\theta(1+p)$ and $\mu|_{\{0\}\times\mathbb{C}^p} \in \gamma^*\mathrm{Derlog}(F)$ then $\mu \in \Gamma^*\mathrm{Derlog}(F) + t\Gamma(\theta(1+p))$. $\bowtie$

In the smooth case Lemma 3.4 is the same as the analytic case except that if the $\mathcal{K}_e$-codimension of f is one then $n \geq p-1$ and up to $\mathcal{A}$-equivalence f is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{p-1} \\ \sum_{i=p}^{n} \pm x_i^2 \end{pmatrix}.$$

In the proof of Proposition 4.27, if w is even then we cannot necessarily define ψ properly. Consequently instead of defining α and β by $(\mu, \nu, x) \mapsto (\mu, \alpha_{(\mu,\nu)}(x))$ and by $(\mu, \nu, X) \mapsto (\mu, \beta_{(\mu,\nu)}(X))$ respectively we may have to define them by $(\mu, \nu, x) \mapsto (-\mu, \alpha_{(\mu,\nu)}(x))$ and $(\mu, \nu, X) \mapsto (-\mu, \beta_{(\mu,\nu)}(X))$ respectively in order to get the diagram to commute.

Theorem 4.28 is the same except that in case ii), $f^{(1)}$ is of the form

$$f^{(1)}: \mathbb{C}^a \times \mathbb{C}^{n-a}, \{0\} \times \{0\} \to \mathbb{C}^a \times \mathbb{C}, \{0\} \times \{0\}$$
$$(\lambda, (x_1, \ldots, x_{n-a})) \mapsto \left(\lambda, \sum_{i=1}^{n-a} \pm x_i^2\right)$$

and in case iii) we may need to take f to be of the form $(\lambda, x, \mu) \mapsto (\lambda, f_{-\mu}(x), \mu)$ instead of $(\lambda, x, \mu) \mapsto (\lambda, f_\mu(x), \mu)$.

§6 The Classification of Certain Monogerms

In this section we classify germs of maps from $\mathbb{C}^n, \{0\}$ to $\mathbb{C}^{n+1}, \{0\}$ of co-rank one and with $\mathcal{A}_e$-codimension at most one. We will do this is by calculating the partition into orbits of the pre-image in successively higher jet-spaces of some orbit in a lower jet space and then, using two results of Gafney, translating the information gained about jets into information about germs.

The first two theorems of this section have very technical proofs and, in order to make these proofs easier to understand, we have adopted some conventions which we will now explain. The theorems are both of the following form: If k is a natural number then for a jet $f \in J^k(n, n+1)$ of a certain type, if $g \in J^{k+1}(n,p)$ is a jet lying over f then g is $\mathcal{A}$-equivalent to one of a finite list of $(k+1)$-jets. Both theorems are proved by taking an arbitrary map-germ with k-jet f and finding changes of co-ordinates in both source and target that make the $(k+1)$-jet of the map germ equal to one of the entries of the list. These changes of co-ordinates are found as a sequence of changes of co-ordinates each one of which changes (and usually simplifies) the form of the germ.

Since all changes of co-ordinates preserve the origin, the $(k+1)$-jet of a map-germ after a change of co-ordinates depends only on the $(k+1)$-jet of the germ before and is independent of terms of degree $k+2$ and higher. Therefore, at any stage of the calculation, terms of degree $k+2$ and higher are irrelevant, and we ignore them throughout. We consider each change of co-ordinates to operate on the coefficients of the monomials of the jet, that is to say, although each change of co-ordinates changes these coefficients, we do not rename them but consider the old name to have a new value.

We present each co-ordinate change as follows:

- First a number, to help with referencing.
- Second the change of co-ordinates itself, using the '$\mapsto$' symbol preceded by a co-ordinate function and followed by some expression in the co-ordinate functions to mean that change of co-ordinates which takes a point to the point whose co-ordinates are given in terms of those of the old point by the expression on the right. We do not bother with writing down co-ordinates that do not change.
- Third a more or less explicit description of its effects on the coefficients (except where the coefficient remains unchanged or remains undetermined).

If i is a natural number then $O(i)$ will stand for some unknown function of order i or higher.

It is a corollary of the inverse function theorem that a map germ $f\colon \mathbb{C}^n, \{0\} \to \mathbb{C}^{n+1}, \{0\}$ has corank one if and only if it is $\mathcal{A}$-equivalent to a germ $f'\colon \mathbb{C}^n, \{0\} \to \mathbb{C}^{n+1}, \{0\}$ with 1-jet: $J^1 f' = (x_1, \ldots, x_{n-1}, 0, 0)$. We are investigating maps of this sort so it is logical to start by enumerating the orbits lying in the pre-image of this 1-jet in 2-jet space. In fact, for reasons that will become clear later, the next theorem enumerates orbits lying in a pre-image of any of a class of jets and this 1-jet is a particular instance of this class.

Theorem 6.1

Let $n, k \in \mathbb{N}_0$, $n \geq 2k+1$. Let $\mathbb{C}^n$ have co-ordinates $(x_1, \ldots, x_{n-1}, y)$ and suppose that $f\colon \mathbb{C}^n, \{0\} \to \mathbb{C}^{n+1}, \{0\}$ has $(k+1)$-jet

$$J^{k+1}f = \left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i, \sum_{i=1}^{k} x_{2i}y^i\right),$$

then $J^{k+2}f$ is $\mathcal{A}$-equivalent to one of the following five $(k+2)$-jets:

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i}y^i + x_{2k+1}y^{k+1}\right), \tag{A}$$

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k+1} x_{2i-1}y^i, \sum_{i=1}^{k+1} x_{2i}y^i\right), \tag{B}$$

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i}y^i\right), \tag{C}$$

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k+1} x_{2i-1}y^i, \sum_{i=1}^{k} x_{2i}y^i\right) \tag{D}$$

or

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i, \sum_{i=1}^{k} x_{2i}y^i\right). \tag{E}$$

Proof

Let the co-ordinates of the co-domain $\mathbb{C}^{n+1}$ be $X_1, \ldots, X_{n-1}, Y, Z$ and suppose that f has the given $(k+1)$-jet.

1. A change of co-ordinates of the form $x_i \mapsto x_i + O(k+2)$, for $i \in \{1, \ldots, n-1\}$ can be found to put $J^{k+2}f$ into the following form:

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i + \sum_{j=0}^{k+2} \theta_{k+2-j}(x_1, \ldots, x_{n-1})y^j,\right.$$
$$\left.\sum_{i=1}^{k} x_{2i}y^i + \sum_{j=0}^{k+2} \eta_{k+2-j}(x_1, \ldots, x_{n-1})y^j\right)$$

where, for $j \in \{0, \ldots, k+2\}$, θ_{k+2-j} and η_{k+2-j} are homogeneous of degree $k+2-j$. The functions $\theta_1(x_1, \ldots, x_{n-1})$ and $\eta_1(x_1, \ldots, x_{n-1})$ are homogeneous of degree one so there are $\lambda_l, \mu_l \in \mathbb{C}$ for $1 \leq l \leq n-1$ such that

$$\theta_1(x_1, \ldots, x_{n-1}) = \sum_{l=1}^{n-1} \lambda_l x_l$$

and

$$\eta_1(x_1,\ldots,x_{n-1})=\sum_{l=1}^{n-1}\mu_l x_l.$$

2. $Y\mapsto Y-\theta_{k+2}(X_1,\ldots,X_{n-1})$ sets $\theta_{k+2}=0$.
3. $Z\mapsto Z-\eta_{k+2}(X_1,\ldots,X_{n-1})$ sets $\eta_{k+2}=0$.

If θ_0 and η_0 are not both 0 (if they are then skip 4 to 10) then either θ_0 is non-zero or it can be made non-zero by

4. $Y\mapsto Z$, $Z\mapsto Y$.
5. For all i such that $1\le i\le k$, $x_{2i-1}\mapsto\theta_0 x_{2i-1}$, $X_{2i-1}\mapsto X_{2i-1}/\theta_0$ and $Y\mapsto Y/\theta_0$ gives $\theta_0=1$.
6. For all i such that $1\le i\le k$, $x_{2i}\mapsto x_{2i}+\eta_0 x_{2i-1}$, $X_{2i}\mapsto X_{2i}-\eta_0 X_{2i-1}$ and $Z\mapsto Z-\eta_0 Y$ gives $\eta_0=0$.
7. $y\mapsto y-\frac{1}{k+2}\sum_{l=1}^{n-1}\lambda_l x_l$ gives $\lambda_l=0$ for $1\le l\le n-1$ but introduces extra terms to Y and Z each of which is of degree ≥ 2 in the x_l's and of degree $q\le k$ in y.
8. A change of the form $Y\mapsto Y-\delta(X_1,\ldots,X_{n-1})$ will remove those extra terms in Y for which $q=0$.
9. A change of the form $Z\mapsto Z-\epsilon(X_1,\ldots,X_{n-1})$ will remove those extra terms in Z for which $q=0$.

For $1\le i\le 2k$, define the polynomials $\chi_i(x_1,\ldots,x_{n-1})$ to be solutions of the two equations $Y=\sum_{i=1}^{k}\chi_{2i-1}(x_1,\ldots,x_{n-1})y^i+O(k+2)$ and $Z=\sum_{i=1}^{k}\chi_{2i}(x_1,\ldots,x_{n-1})y^i+O(k+2)$ (which are unique up to $O(k-i+2)$) and, for $2k+1\le i\le n-1$, define the function $\chi_i(x_1,\ldots,x_{n-1})$ by $\chi_i:=x_i$. Now define the map

$$\begin{aligned}\Xi\colon \mathbf{C}^{n-1}&\to\mathbf{C}^{n-1}\\(x_1,\ldots,x_{n-1})&\mapsto\big(\chi_1(x_1,\ldots,x_{n-1}),\ldots,\chi_{n-1}(x_1,\ldots,x_{n-1})\big).\end{aligned}$$

The Jacobian of Ξ is the identity at 0 so, by the inverse function theorem, Ξ has an inverse in a neighbourhood of 0.

10. The change $(x_1,\ldots,x_{n-1})\mapsto\Xi^{-1}(x_1,\ldots,x_{n-1})$ and $(X_1,\ldots,X_{n-1})\mapsto\Xi(X_1,\ldots,X_{n-1})$ removes the remaining extra terms in Y and Z.

Whether $(\theta_0,\eta_0)=(0,0)$ or not, repeat steps 11 to 16 for all a such that $1\le a\le k$ and $2a>k+1$ in order of decreasing a. Their purpose is to set $\mu_{2a}=0$.

11. In order of increasing i for $1\le i\le a-1$ and $i\ne k-a+1$, the change $x_{2k-2a+2+2i}\mapsto x_{2k-2a+2+2i}+\mu_{2a}x_{2i}$, $X_{2k-2a+2+2i}\mapsto X_{2k-2a+2+2i}-\mu_{2a}X_{2i}$ adds the extra terms $\mu_{2a}\sum_{\substack{i=1\\i\ne k-a+1}}^{a-1}x_{2i}y^{i+k-a+1}$ to Z.

12. $x_{2k-2a+2}\mapsto x_{2k-2a+2}-\mu_{2a}\sum_{\substack{i=1\\i\ne k-a+1}}^{k}x_{2i}y^i$ removes the extra terms from Z, also it removes $\mu_{2a}x_{2a}y^{k+1}$ from Z (i.e. it sets $\mu_{2a}=0$), finally it adds the extra terms $-\mu_{2a}\sum_{\substack{i=1\\i\ne k-a+1}}^{k}x_{2i}y^i$ to $X_{2k-2a+2}$.

13. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \mu_{2a} Z$ removes the extra terms from $X_{2k-2a+2}$ but it adds $\mu_{2a} x_{2k-2a+2} y^{k-a+1} + O(k+2)$ to $X_{2k-2a+2}$.

14. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - O(k+2)$ removes the $O(k+2)$ terms from $X_{2k-2a+2}$.

15. The co-ordinate change $x_{2k-2a+2} \mapsto x_{2k-2a+2}/(1+\mu_{2a} y^{k-a+1})$ removes the term $\mu_{2a} x_{2k-2a+2} y^{k-a+1}$ from $X_{2k-2a+2}$ but adds the extra terms $x_{2k-2a+2} y^{k-a+1} \sum_{b=1}^{\infty} (-\mu_{2a} y^{k-a+1})^b$ to Z.

Those extra terms in Z whose degree c in y is not greater than k can be removed by

16. $x_{2c} \mapsto x_{2c} - t x_{2k-2a+2}$, $X_{2c} \mapsto X_{2c} + t X_{2k-2a+2}$ (where $t \in \mathrm{C}$ depends on c). Those extra terms in Z whose degree c in y is greater than $k+1$ are $O(k+3)$ and can be ignored. Finally, that extra term in Z whose degree c in y is $k+1$ is of the form $t x_{2k-2a+2} y^{k+1}$ so this term changes $\mu_{2k-2a+2}$. Since $2a > k+1$, $2k-2a+2 < 2a$ so, as we are working in order of decreasing a, $\mu_{2k-2a+2}$ is, as yet, undetermined.

Again whether $(\theta_0, \eta_0) = (0,0)$ or not, repeat steps 17 to 21 for all a such that $1 \leq a \leq k$ and $2a \leq k+1$ in order of decreasing a. Their purpose is to set $\mu_{2a} = 0$.

17. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \mu_{2a} \sum_{i=a}^{k} x_{2i} y^i$ subtracts $\mu_{2a} x_{2a} y^a$ from Z (i.e. it sets $\mu_{2a} = 0$) and subtracts $\mu_{2a} \sum_{i=a}^{k} x_{2i} y^i$ from $X_{2k-2a+2}$.

18. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \mu_{2a} Z$ adds $\mu_{2a} \sum_{i=a}^{k} x_{2i} y^i$ back on to $X_{2k-2a+2}$ but also adds $\mu_{2a} \sum_{i=1}^{a-1} x_{2i} y^i + O(k+2)$ to $X_{2k-2a+2}$.

19. A co-ordinate change of the form $x_{2k-2a+2} \mapsto x_{2k-2a+2} - O(k+2)$ removes the $O(k+2)$ terms from $X_{2k-2a+2}$.

20. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \mu_{2a} \sum_{i=1}^{a-1} x_{2i} y^i$ removes the terms $\sum_{i=1}^{a-1} x_{2i} y^i$ from $X_{2k-2a+2}$ but adds the extra terms $-\mu_{2a} \sum_{i=1}^{a-1} x_{2i} y^{k-a+1+i}$ to Z.

21. For all i such that $1 \leq i \leq a-1$, the co-ordinate changes $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \mu_{2a} x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \mu_{2a} X_{2i}$ remove these extra terms from Z.

Again whether $(\theta_0, \eta_0) = (0,0)$ or not, repeat steps 22 to 25 for all a such that $1 \leq a \leq k$. Their purpose is to set $\mu_{2a-1} = 0$.

22. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \mu_{2a-1} \sum_{i=1}^{k} x_{2i-1} y^i$ removes $\mu_{2a-1} x_{2a-1} y^{k+1}$ from Z (i.e. it sets $\mu_{2a-1} = 0$), subtracts $\mu_{2a-1} \sum_{i=1}^{a-1} x_{2i-1} y^{2k-2a+2i+2}$ from Z and subtracts $\mu_{2a-1} \sum_{i=1}^{k} x_{2i-1} y^i$ from $X_{2k-2a+2}$.

23. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \mu_{2a-1} Y$ replaces the terms removed from $X_{2k-2a+2}$ but adds $O(k+2)$.

This last can be removed by

24. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - O(k+2)$.

25. For all i such that $1 \leq i \leq a-1$, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \mu_{2a-1} x_{2a-1}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \mu_{2a-1} X_{2i-1}$ adds the subtracted terms $\mu_{2a-1} \sum_{i=1}^{a-1} x_{2i-1} y^{2k-2a+2i+2}$ back on to Z.

If θ_0 and η_0 are both zero then

26. $Y \mapsto Z$ and $Z \mapsto Y$ followed by stages 11 to 25 again sets $\lambda_l = 0$ for all l such that $1 \le l \le 2k$.

Whether $(\theta_0, \eta_0) = (0,0)$ or not, define $\Xi\colon \mathbf{C}^{n-1}, \{0\} \to \mathbf{C}^{n-1}, \{0\}$ to be the map whose co-ordinate functions are given by

$$\begin{aligned}
\Xi_{2i-1}(x_1, \ldots, x_{n-1}) &= x_{2i-1} + \theta_{k+2-i}(x_1, \ldots, x_{n-1}) \qquad 1 \le i \le k \\
\Xi_{2i}(x_1, \ldots, x_{n-1}) &= x_{2i} + \eta_{k+2-i}(x_1, \ldots, x_{n-1}) \qquad 1 \le i \le k \\
\Xi_j(x_1, \ldots, x_{n-1}) &= x_j \qquad 2k+1 \le j \le n-1.
\end{aligned}$$

Then the Jacobian of Ξ at 0 is the identity so, by the inverse function theorem, Ξ has an inverse in a neighbourhood of 0.

27. The change of co-ordinates given by $(x_1, \ldots, x_{n-1}) \mapsto \Xi^{-1}(x_1, \ldots, x_{n-1})$ and $(X_1, \ldots, X_{n-1}) \mapsto \Xi(X_1, \ldots, X_{n-1})$ sets $\theta_{k-j+2} = 0$ and $\eta_{k-j+2} = 0$ for all j such that $1 \le j \le k$.

If θ_0 and η_0 are both zero (if not then skip 28 to 31) then either: for all l such that $2k+1 \le l \le n-1$, $\lambda_l = \mu_l = 0$ (in which case the jet is of type (E)) or, possibly after

28. $Y \mapsto Z$, $Z \mapsto Y$, we may assume that some $\lambda_l \ne 0$.

29. The change of co-ordinates $(x_{2k+1}, \ldots, x_{n-1}) \mapsto M(x_{2k+1}, \ldots, x_{n-1})$, $(X_{2k+1}, \ldots, X_{n-1}) \mapsto M^{-1}(X_{2k+1}, \ldots, X_{n-1})$ for some $M \in GL_{n-2k-2}(\mathbf{C})$ gives $\lambda_{2k+1} = 1$ and $\lambda_l = 0$ for all l such that $2k+2 \le l \le n-1$.

30. $x_{2i} \mapsto x_{2i} + \mu_{2k+1} x_{2i-1}$, $X_{2i} \mapsto X_{2i} - \mu_{2k+1} X_{2i-1}$, $Z \mapsto Z - \mu_{2k+1} Y$ sets $\mu_{2k+1} = 0$.

Either for all l such that $2k+2 \le l \le n-1$, $\mu_l = 0$ (in which case the jet is of type (D)) or

31. The change of co-ordinates $(x_{2k+2}, \ldots, x_{n-1}) \mapsto N(x_{2k+2}, \ldots, x_{n-1})$, $(X_{2k+2}, \ldots, X_{n-1}) \mapsto N^{-1}(X_{2k+2} \ldots, X_{n-1})$ for some $N \in GL_{n-2k-3}$ gives $\mu_{2k+2} = 1$ and $\mu_l = 0$ for all l such that $2k+3 \le l \le n-1$. In this case the jet is of type (B).

The case for which $(\theta_0, \eta_0) \ne (0,0)$ remains. Either: for all l such that $2k \le l \le n-1$, $\mu_l = 0$ (in which case the jet is of type (C)) or

32. The change of co-ordinates $(x_{2k+1}, \ldots, x_{n-1}) \mapsto L(x_{2k+1}, \ldots, x_{n-1})$, $(X_{2k+1}, \ldots, X_{n-1}) \mapsto L^{-1}(X_{2k+1}, \ldots, X_{n-1})$ for some map $L \in GL_{n-2k-2}$ gives $\mu_{2k+1} = 1$ and $\mu_l = 0$ for all l such that $2k+2 \le l \le n-1$. This jet is of type (A).

$\bowtie$

Theorem 6.1 holds if $\mathbf{C}$ is replaced throughout by $\mathbf{R}$. The proof is the same except that, of course, you must replace $\mathbf{C}$ by $\mathbf{R}$ there as well. We now consider each of the germs (A) to (E) and decide which germs of $\mathcal{A}_e$-codimension at most one they each contain. We will use the following result of Gafney's while investigating class (A).

Proposition 6.2 [corollary 3.4 of [3]]

Let $f: \mathbf{C}^n, \{0\} \to \mathbf{C}^p, \{0\}$ be an analytic map germ and suppose that

$$tf(\theta(n)) + wf(\theta(p)) \supseteq \mathbf{m}_n^j \theta(f)$$
$$\text{and} \qquad tf(\theta(n)) + f^* \mathbf{m}_p \theta(f) \supseteq \mathbf{m}_n^i \theta(f)$$

where i and j are both at least one. Then f is $i+j$-determined for $\mathcal{A}$.

$\bowtie$

Let

$f: \mathbf{C}^n, \{0\} \to \mathbf{C}^{n+1}, \{0\}$

$$(x_1, \ldots, x_{n-1}, y) \mapsto \left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1} y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i} y^i + x_{2k+1} y^{k+1} \right)$$

for some k such that $2k+1 \leq n$. The germ f is a $\mathcal{K}_e$-miniversal unfolding of $g: \mathbf{C}, \{0\} \to \mathbf{C}^2, \{0\}$; $(y) \mapsto (y^{k+2}, 0)$ and so it is stable. It follows that $tf(\theta(n)) + wf(\theta(p)) = \theta(f)$. $f^*(\mathbf{m}_{n+1}) = (x_1, \ldots, x_{n-1}, y^{k+2})$ so $tf(\theta(n)) + f^* \mathbf{m}_{n+1} \theta(f) \supseteq \mathbf{m}_n^{k+2} \theta(f)$. Now Proposition 6.2 implies that f is $k+2$-determined. In other words, any germ with $(k+2)$-jet of type (A) is $\mathcal{A}$-equivalent to the stable germ f.

To investigate type (B) we should classify the orbits in $(k+3)$-jet space of the $(k+2)$-jet (B). But this is precisely what Theorem 6.1 is about. In other words, Theorem 6.1 says something recursively about jet (B).

Jet (C) is, as far as we are concerned, the most complicated of the five jets so we defer its investigation until after that of jets (D) and (E).

It is a straightforward check that the $\mathcal{A}_e$-tangent space of jet (D) is contained in the linear subspace displayed in figure 1. Where we follow the notation of [13]. The codimension of this linear subspace is two and so any germ with a jet of type (D) has $\mathcal{A}_e$-codimension at least two.

We may also check that the $\mathcal{A}_e$-tangent space of jet (E) is contained in the linear subspace displayed in figure 2. The codimension of this linear subspace is two and so any germ with a jet of type (E) has $\mathcal{A}_e$-codimension at least two.

Theorem 6.3

Let $n, k \in \mathbf{N}_0$, $n \geq 2k+1$. Let $\mathbf{C}^n$ have co-ordinates $(x_1, \ldots, x_{n-1}, y)$ and suppose that $f: \mathbf{C}^n, \{0\} \to \mathbf{C}^{n+1}, \{0\}$ has $(k+2)$-jet

$$J^{k+2} f = \left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1} y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i} y^i \right),$$

then there is an r (where $0 \leq r \leq n-2k-1$) such that $J^{k+3} f$ is $\mathcal{A}$-equivalent to one of the following two $(k+3)$-jets:

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1} y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i} y^i + y^{k+1} \sum_{l=2k+1}^{2k+r} x_l^2 + y^{k+3} \right) \qquad (U)$$

$$
\begin{array}{rc}
\begin{matrix}
\\ \\ \\ 2k+1 \\ \\ \\ \\ \\ \\ n-2k-1 \\ \\ \\ \\ \\
\end{matrix}
&
\begin{bmatrix}
\mathcal{O}_{\mathbf{C}^n}-\{y^{k+1}\} \\
\mathcal{O}_{\mathbf{C}^n}-\{y^{k+1}\} \\
\mathcal{O}_{\mathbf{C}^n}-\{y^{k}\} \\
\mathcal{O}_{\mathbf{C}^n}-\{y^{k}\} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n}-\{y^{2}\} \\
\mathcal{O}_{\mathbf{C}^n}-\{y^{2}\} \\
\mathcal{O}_{\mathbf{C}^n}-\{y\} \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n}-\{y^{k+2}\} \\
\mathcal{O}_{\mathbf{C}^n}-\{y^{k+2}\}
\end{bmatrix}
\end{array}
$$

$$
+\mathbf{C}\left\{
\begin{bmatrix} y^{k+1}\\0\\0\\0\\\vdots\\0\\0\\0\\0\\0\\\vdots\\0\\y^{k+2}\\0 \end{bmatrix},
\begin{bmatrix} 0\\y^{k+1}\\0\\0\\\vdots\\0\\0\\0\\0\\0\\\vdots\\0\\0\\y^{k+2} \end{bmatrix},
\begin{bmatrix} 0\\0\\y^{k}\\0\\\vdots\\0\\0\\0\\0\\0\\\vdots\\0\\y^{k+2}\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\y^{k}\\\vdots\\0\\0\\0\\0\\0\\\vdots\\0\\0\\y^{k+2} \end{bmatrix}
,\ldots,
\begin{bmatrix} 0\\0\\0\\0\\\vdots\\y^{2}\\0\\0\\0\\0\\\vdots\\0\\y^{k+2}\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\\vdots\\0\\y^{2}\\0\\0\\0\\\vdots\\0\\0\\y^{k+2} \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\\vdots\\0\\0\\y\\0\\0\\\vdots\\0\\0\\y^{k+2} \end{bmatrix}
\right\}
$$

figure 1

$$
\begin{matrix}
2k \left\updownarrow \vphantom{\begin{matrix} a \\ a \\ a \\ a \\ a \\ a \\ a \end{matrix}} \right. \\
n-2k-1 \left\updownarrow \vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}} \right. \\
\vphantom{\begin{matrix} a \\ a \end{matrix}}
\end{matrix}
\begin{bmatrix}
\mathcal{O}_{\mathbf{C}^n} - \{y^{k+1}\} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k+1}\} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k}\} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k}\} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{2}\} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{2}\} \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k+2}\} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k+2}\}
\end{bmatrix}
$$

$$
+\mathbb{C}\left\{
\begin{bmatrix} y^{k+1} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y^{k+2} \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ y^{k+1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+2} \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ y^{k} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y^{k+2} \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ y^{k} \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+2} \end{bmatrix},
\ldots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ y^{2} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y^{k+2} \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y^{2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+2} \end{bmatrix}
\right\}
$$

figure 2

or

$$\left(x_1,\ldots,x_{n-1},\sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2},\sum_{i=1}^{k} x_{2i}y^i + y^{k+1}\sum_{l=2k+1}^{2k+r} x_l^2\right) \qquad (V)$$

Proof

Let the co-ordinates of the co-domain $\mathbf{C}^{n+1}$ be $X_1,\ldots,X_{n-1},Y,Z$ and suppose that f has the given $(k+2)$-jet.

1. A change of co-ordinates of the form $x_i \mapsto x_i+O(k+3)$ for $i \in \{1,\ldots,n-1\}$ can be found to put $J^{k+3}f$ into the following form:

$$\left(x_1,\ldots,x_{n-1},\sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2} + \sum_{j=0}^{k+3}\theta_{k+3-j}(x_1,\ldots,x_{n-1})y^j,\right.$$
$$\left.\sum_{i=1}^{k} x_{2i}y^i + \sum_{j=0}^{k+3}\eta_{k+3-j}(x_1,\ldots,x_{n-1})y^j\right)$$

where, for $j \in \{0,\ldots,k+3\}$, θ_{k+3-j} and η_{k+3-j} are homogeneous of degree $k+3-j$.

2. $Y \mapsto Y - \theta_{k+3}(X_1,\ldots,X_{n-1})$ sets $\theta_{k+3}=0$.

3. $Z \mapsto Z - \eta_{k+3}(X_1,\ldots,X_{n-1})$ sets $\eta_{k+3}=0$.

4. $y \mapsto y - \frac{1}{2}\theta_0 y^2$ sets $\theta_0 = -\frac{1}{2}k\theta_0$ (the left hand θ_0 refers to its new value) and adds the extra terms: $\sum_{i=1}^{k}\sum_{j=1}^{i}\binom{i}{j}(-\frac{1}{2}\theta_0)^j x_{2i-1}y^{i+j}$ to Y and $\sum_{i=1}^{k}\sum_{j=1}^{i}\binom{i}{j}(-\frac{1}{2}\theta_0)^j x_{2i}y^{i+j}$ to Z. In particular, the coefficient of $x_{2k-1}y^{k+1}$ in Y is also $-\frac{1}{2}\theta_0$.

5. A change of the form $x_{2i-1} \mapsto x_{2i-1} - \alpha_i(x_1,x_3,\ldots,x_{2i-3})$, $X_{2i-1} \mapsto X_{2i-1} + \alpha_i(X_1,X_3,\ldots,X_{2i-3})$ for all i such that $1 \le i \le k$ and linear α_i removes the extra terms from Y whose degree in y is not greater than k.

6. A change of the form $x_{2i} \mapsto x_{2i} - \beta_i(x_2,x_4,\ldots,x_{2i-2})$, $X_{2i} \mapsto X_{2i} + \beta_i(X_2,X_4,\ldots,X_{2i-2})$ for all i such that $1 \le i \le k$ and linear β_i removes the extra terms from Z whose degree in y is not greater than k.

Repeat steps 7 to 14 for all a such that $1 \le a \le k$ and $2a > k+1$, in order of decreasing a. Their purpose is to remove the $x_{2a-1}y^{k+1}$ term from Y. Suppose that the coefficient of this term is μ_{2a-1}.

7. For all i such that $1 \le i \le a-1$ and $i \ne k-1+1$ in order of increasing i, $x_{2k-2a+1+2i} \mapsto x_{2k-2a+1+2i} + \mu_{2a-1}x_{2i-1}$, $X_{2k-2a+1+2i} \mapsto X_{2k-2a+1+2i} - \mu_{2a-1}X_{2i-1}$ adds $\mu_{2a-1}\sum_{\substack{i=1\\ i\ne k-a+1}}^{a-1} x_{2i-1}y^{k-a+1+i}$ to Y.

8. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - \mu_{2a-1}\sum_{\substack{i=1\\ i\ne k-a+1}}^{k} x_{2i-1}y^i$ removes these terms from Y, removes $\mu_{2a-1}x_{2a-1}y^{k+1}$ from Y (i.e. it sets $\mu_{2a-1}=0$) and subtracts $\mu_{2a-1}\sum_{\substack{i=1\\ i\ne k-a+1}}^{k} x_{2i-1}y^i$ from $X_{2k-2a+1}$.

9. $X_{2k-2a+1} \mapsto X_{2k-2a+1}+\mu_{2a-1}Y$ adds these terms back onto $X_{2k-2a+1}$ but also adds $\mu_{2a-1}x_{2k-2a+1}y^{k-a+1}+\gamma(x_1,x_3,\ldots,x_{2a-3})y^{k+1}+\mu_{2a-1}y^{k+2}+O(k+3)$ where γ is linear.

10. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - \gamma(x_1,x_3,\ldots,x_{2a-3})y^{k+1}$ removes the second of these terms.

11. $x_{2k-2a+1} \mapsto x_{2k-2a+1}/(1+\mu_{2a-1}y^{k-a+1})$ removes the first but adds $x_{2k-2a+1}y^{k-a+1}\sum_{b=1}^{\infty}(-\mu_{2a-1}y^{k-a+1})^b$ to Y.

12. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - O(k+3)$ removes the fourth.

13. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - \mu_{2a-1}y^{k+2}$ removes the third but disturbs Y. However the changes to Y are $O(k+4)$ unless $a=k$ in which case, since $\mu_{2k-1} = -\frac{1}{2}k\theta_0$, this adds $\frac{1}{2}k\theta_0 y^{k+2}$ (i.e. it removes the y^{k+2} term from Y).

Of those terms that step 11 added to Y, those whose degree c in y is not greater than k can be removed by

14. $x_{2c-1} \mapsto x_{2c-1} + tx_{2k-2a+1}$, $X_{2c-1} \mapsto X_{2c-1} - tX_{2k-2a+1}$ where $t \in \mathbf{C}$ depends on c. Since $2a > k+1$, $2a-1 > 2k-2a+1$ and so $\mu_{2k-2a+1}$ is at this stage undetermined. It follows that those terms whose degree c in y is $k+1$ make no significant changes. Finally θ_1 is also as yet undetermined so those terms whose degree c in y is not less than $k+2$ also make no significant changes.

Repeat steps 15 to 21 for all a such that $1 \leq a \leq k$ and $2a \leq k+1$, in order of decreasing a. Their purpose is to remove the x_{2a-1} term from Y. Suppose that the coefficient of this term is μ_{2a-1}.

15. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - \mu_{2a-1}\sum_{i=a}^{k} x_{2i-1}y^i$ removes $\mu_{2a-1}x_{2a-1}y^{k+1}$ from Y (i.e. sets $\mu_{2a-1}=0$) and subtracts $\mu_{2a-1}\sum_{i=a}^{k} x_{2i-1}y^i$ from $X_{2k-2a+1}$.

16. $X_{2k-2a+1} \mapsto X_{2k-2a+1} + \mu_{2a-1}Y$ adds these terms back onto $X_{2k-2a+1}$ but also adds $\mu_{2a-1}\sum_{i=1}^{a-1} x_{2i-1}y^i + \mu_{2a-1}y^{k+2} + \gamma(x_1,x_3,\ldots,x_{2a-3})y^{k+1} + O(k+3)$ where γ is linear.

17. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - \gamma(x_1,x_3,\ldots,x_{2a-3})y^{k+1}$ removes the third of these terms.

18. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - \mu_{2a-1}\sum_{i=1}^{a-1} x_{2i-1}y^i$ removes the first but also adds $-\mu_{2a-1}\sum_{i=1}^{a-1} x_{2i-1}y^{k-a+1+i}$ to Y.

19. For all i such that $1 \leq i \leq a-1$, $x_{2k-2a+1+2i} \mapsto x_{2k-2a+1+2i} + \mu_{2a-1}x_{2i-1}$, $X_{2k-2a+1+2i} \mapsto X_{2k-2a+1+2i} - \mu_{2a-1}x_{2i-1}$ removes this term from Y.

20. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - O(k+3)$ removes the fourth term introduced by step 16.

21. $x_{2k-2a+1} \mapsto x_{2k-2a+1} - \mu_{2a-1}y^{k+2}$ removes the second term introduced by 16 but adds $-\mu_{2a-1}y^{2k-a+3}$ to Y. This term is $O(k+4)$ unless $a=k$ in which case, since $\mu_{2k-1} = -\frac{1}{2}k\theta_0$, we have added $\frac{1}{2}k\theta_0$ to Y (i.e. we have removed the y^{k+3} term from Y).

Repeat steps 22 to 28 for all a such that $1 \leq a \leq k$ and $2a > k+1$, in order of decreasing a. Their purpose is to remove the $x_{2a}y^{k+1}$ term from Z. Suppose that the coefficient of this term is μ_{2a}.

22. For all i such that $1 \leq i \leq k-a+1$ and $i \neq k-a+1$ in order of increasing i, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \mu_{2a}x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \mu_{2a}X_{2i}$ adds $\mu_{2a}\sum^{k}_{\substack{i=1\\ i\neq k-a+1}} x_{2i}y^{i+k-a+1}$ to Z.

23. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \mu_{2a}\sum^{k}_{\substack{i=1\\ i\neq k-a+1}} x_{2i}y^{i}$ removes this term from Z, removes $\mu_{2a}x_{2a}y^{k+1}$ from Z (i.e. sets $\mu_{2a}=0$) and subtracts $\mu_{2a}\sum^{k}_{\substack{i=1\\ i\neq k-a+1}} x_{2i}y^{i}$ from $X_{2k-2a+2}$.

24. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \mu_{2a}Z$ adds these terms back onto $X_{2k-2a+2}$ but also adds $\mu_{2a}x_{2k-2a+2}y^{k-a+1} + \gamma(x_2, x_4, \ldots, x_{2a-2})y^{k+1} + O(k+3)$ where γ is linear.

25. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \gamma(x_2, x_4, \ldots, x_{2a-2})y^{k+1}$ removes the second of these terms.

26. $x_{2k-2a+2} \mapsto x_{2k-2a+2}/(1+\mu_{2a}y^{k-a+1})$ removes the first but adds $x_{2k-2a+2}y^{k-a+1}\sum_{b=1}^{\infty}(-\mu_{2a}y^{k-a+1})^{b}$ to Z.

27. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - O(k+3)$ removes the fourth.

Of those terms that 26 added to Z, those whose degree c in y is not greater than k can be removed by

28. $x_{2c} \mapsto x_{2c} + tx_{2k-2a+2}$, $X_{2c} \mapsto X_{2c} - tX_{2k-2a+2}$ where $t \in \mathbb{C}$ depends on c. Since $2a > k+1$, $2a > 2k-2a+2$ and so $\mu_{2k-2a+2}$ is at this stage undetermined. It follows that those extra terms whose degree c in y is $k+1$ make no significant changes. Finally η_1 is also as yet undetermined so that those terms whose degree c in y is not less that $k+2$ also make no significant changes.

Repeat steps 29 to 34 for all a such that $1 \leq a \leq k$ and $2a \leq k+1$ in order of decreasing a. Their purpose is to remove the $x_{2a}y^{k+1}$ term from Z. Suppose that the coefficient of this term is μ_{2a}.

29. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \mu_{2a}\sum_{i=a}^{k} x_{2i}y^{i}$ removes $\mu_{2a}x_{2a}y^{k+1}$ from Z (i.e. it sets $\mu_{2a}=0$) and subtracts $\mu_{2a}\sum_{i=a}^{k} x_{2i}y^{i}$ from $X_{2k-2a+2}$.

30. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \mu_{2a}Z$ adds these terms back onto $X_{2k-2a+2}$ but also adds $\mu_{2a}\sum_{i=1}^{a-1} x_{2i}y^{i} + \gamma(x_2, x_4, \ldots, x_2a-2)y^{k+1} + O(k+3)$ where γ is linear.

31. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \gamma(x_2, x_4, \ldots, x_{2a-2})y^{k+1}$ removes the second of these terms.

32. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \mu_{2a}\sum_{i=1}^{a-1} x_{2i}y^{i}$ removes the first but also adds $-\mu_{2a}\sum_{i=1}^{a-1} x_{2i}y^{k-a+1+i}$ to Z.

33. For all i such that $1 \leq i \leq a-1$, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \mu_{2a}x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \mu_{2a}X_{2i}$ removes this term from Z.

34. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - O(k+3)$ removes the third term introduced by 30.

35. $Z \mapsto Z - \eta_1(X_1, \dots, X_{n-1})Y$ sets $\eta_1(x_1, \dots, X_{n-1}) = 0$ but also adds $-\eta_1(x_1, \dots, x_{n-1}) \sum_{i=1}^{k} x_{2i-1} y^i$ to Z.

36. $Y \mapsto Y\big(1 - \theta_1(X_1, \dots, X_{n-1})\big)$ sets $\theta_1(x_1, \dots, x_{n-1}) = 0$ but adds $-\theta_1(x_1, \dots, x_{n-1}) \sum_{i=1}^{k} x_{2i-1} y^i$ to Y.

Define $\Xi\colon \mathbf{C}^{n-1}, \{0\} \to \mathbf{C}^{n-1}, \{0\}$ to be the map whose co-ordinate functions are given by

$$
\begin{aligned}
\Xi_{2i-1}(x_1, \dots, x_{n-1}) &= x_{2i-1}\big(1 - \theta_1(x_1, \dots, x_{n-1})\big) \qquad & 1 \le i \le k \\
\Xi_{2i}(x_1, \dots, x_{n-1}) &= x_{2i} - x_{2i-1}\theta_1(x_1, \dots, x_{n-1}) \qquad & 1 \le i \le k \\
\Xi_j(x_1, \dots, x_{n-1}) &= x_j \qquad 2k+1 \le j \le n-1. &
\end{aligned}
$$

Then the Jacobian of Ξ at 0 is the identity, so by the inverse function theorem, Ξ has an inverse in a neighbourhood of 0.

37. $(x_1, \dots, x_{n-1}) \mapsto \Xi^{-1}(x_1, \dots, x_{n-1})$ followed by $(X_1, \dots, X_{n-1}) \mapsto \Xi(X_1, \dots, X_{n-1})$ removes the terms introduced by 35 and 36.

38. $y \mapsto y - \frac{1}{k+2}\theta_2(x_1, \dots, x_{n-1})$ sets $\theta_2(x_1, \dots, x_{n-1}) = 0$ but introduces extra terms to Y and Z all of which have degree not less than 3 in the x_l's and degree not greater than k in y.

39. A change of the form $Y \mapsto Y - \alpha(X_1, \dots, X_{n-1})$ removes the extra terms of degree 0 in y from Y.

40. A change of the form $Z \mapsto Z - \beta(X_1, \dots, X_{n-1})$ removes the extra terms of degree 0 in y from Z

For $1 \le i \le k$, define the polynomials $\xi_i(x_1, \dots, x_{n-1})$ to be solutions of the two equations $Y = \sum_{i=1}^{k} \xi_{2i-1}(x_1, \dots, x_{n-1}) y^i + y^{k+2} + O(k+3)$ and $Z = \sum_{i=1}^{k} \xi_{2i}(x_1, \dots, x_{n-1}) y^i + O(k+3)$ (which are unique up to $O(k-i+3)$) and for $2k+1 \le i \le n-1$ define the function $\xi_i(x_1, \dots, x_{n-1})$ by $\xi_i = x_i$. Now define the map

$$
\begin{aligned}
&\Xi\colon \mathbf{C}^{n-1} \to \mathbf{C}^{n-1} \\
&(x_1, \dots, x_{n-1}) \mapsto \big(\xi_1(x_1, \dots, x_{n-1}), \dots, \xi_{n-1})\big).
\end{aligned}
$$

The Jacobian of Ξ is the identity at 0 so by the inverse function theorem, Ξ has an inverse in a neighbourhood of 0.

41. $(x_1, \dots, x_{n-1}) \mapsto \Xi^{-1}(x_1, \dots, x_{n-1})$ followed by $(X_1, \dots, X_{n-1}) \mapsto \Xi(X_1, \dots, X_{n-1})$ removes the remaining extra terms in Y and Z.

Repeat steps 42 to 45 for all a such that $1 \le a \le k$ and all l such that $1 \le l \le n-1$ and $l \ne 2a'$ for $1 \le a' \le k$. Their purpose is to remove the $x_{2a-1} x_l y^{k+1}$ term from Z. Suppose that the coefficient of this term is λ.

42. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \lambda x_l \sum_{i=a}^{k} x_{2i-1} y^i$ removes the $x_{2a-1} x_l y^{k+1}$ term from Z but adds $-\lambda \sum_{i=a}^{k} x_{2i-1} y^i$ to $X_{2k-2a+2}$.

43. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \lambda X_l Y$ removes this term but also adds the terms $\lambda x_l \sum_{i=1}^{a-1} x_{2i-1} y^i + \lambda x_l y^{k+2}$ to $X_{2k-2a+2}$.

44. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \lambda x_l \sum_{i=1}^{a-1} x_{2i-1}y^i - \lambda x_l y^{k+2}$ takes this term back off but adds $-\lambda x_l \sum_{i=1}^{a-1} x_{2i-1}y^{k-a+1+i} - \lambda x_l y^{2k+3-a}$ to Z. $-\lambda x_l y^{2k+3-a}$ is $O(k+4)$.

For all i such that $1 \leq i \leq a-1$,

45. $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \lambda x_l x_{2i-1}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \lambda_l X_l X_{2i-1}$ removes this.

Repeat steps 46 to 50 for a such that $1 \leq a \leq k$ and $2a > k+1$ in order of increasing a and all l such that $1 \leq l \leq n-1$ and $l \neq 2a'$ for any a' such that $1 \leq a' \leq k$. Their purpose is to remove the $x_{2a}x_l y^{k+1}$ term from Z. Suppose that the coefficient of this term is λ.

46. For all i such that $1 \leq i \leq a-1$ and $i \neq k-a+1$ in order of increasing i, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \lambda x_l x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \lambda X_l X_{2i}$ adds $\lambda x_l \sum_{\substack{i=1 \\ i \neq k-a+1}}^{a-1} x_{2i}y^{k-a+1+i}$ to Z.

47. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \lambda x_l \sum_{\substack{i=1 \\ i \neq k-a+1}}^{k} x_{2i}y^i$ removes this term from Z, removes the $x_{2a}x_l y^{k+1}$ term from Z but adds $-\lambda x_l \sum_{\substack{i=1 \\ i \neq k-a+1}}^{k} x_{2i}y^i$ to $X_{2k-2a+2}$.

48. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \lambda X_l Z$ removes this term but adds an extra $\lambda x_l x_{2k-2a+2}y^{k-a+1}$ to $X_{2k-2a+2}$.

49. $x_{2k-2a+2} \mapsto x_{2k-2a+2}/(1+\lambda x_l y^{k-a+1})$ removes this term from $X_{2k-2a+2}$ but adds $x_{2k-2a+2}y^{k-a+1} \sum_{b=1}^{\infty}(-\lambda x_l y^{k-a+1})^b$ to Z.

Those extra terms in Z whose degree c in y is not greater than k can be removed by

50. $x_{2c} \mapsto x_{2c} - t x_{2k-2a+2}$, $X_{2c} \mapsto X_{2c} + t X_{2k-2a+2}$ (where t depends on c). Those terms whose degree x in y is greater than $k+1$ are $O(k+4)$ and the only possible term whose degree c in y in $k+1$ and which is not $O(k+4)$ is $-\lambda x_{2k-2a+2}x_l y^{k+1}$. Since $2a > k+1$, $2k-2a+2 < 2a$ so, as we are working in order of decreasing a, the $x_{2k-2a+2}x_l y^{k+1}$ term has not been removed yet.

Repeat steps 51 to 53 for all a such that $1 \leq a \leq k$ and $2a \geq k+1$ and all l such that $1 \leq l \leq n-1$ and $l \neq 2a'$ for any a' such that $1 \leq a' \leq k$. Their purpose is to remove the $x_{2a}x_l y^{k+1}$ term from Z. Suppose that the coefficient of this term is λ.

51. For all i such that $1 \leq i \leq a-1$, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \lambda x_l x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \lambda X_l X_{2i}$ adds $\lambda x_l \sum_{i=1}^{a-1} x_{2i}y^{k-a+1+i}$ to Z.

52. $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} - \lambda x_l \sum_{i=1}^{k} x_{2i}y^i$ removes these terms from Z, removes $\lambda x_l x_{2a}y^{k+1}$ from Z and adds the terms $-\lambda x_l \sum_{i=1}^{k} x_{2i}y^i$ to $X_{2k-2a+2}$.

53. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \lambda X_l Z$ removes these terms.

Repeat steps 54 to 56 for all a and a' such that $1 \leq a \leq a' \leq k$ and $a + a' \leq k$. Their purpose is to remove the $x_{2a}x_{2a'}y^{k+1}$ term from Z. Suppose that the coefficient of this term is λ.

54. For all i such that $1 \leq i \leq a-1$, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \lambda x_{2a'}x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - X_{2a'}X_{2i}$ adds $\lambda x_{2a'} \sum_{i=1}^{a-1} x_{2i}y^{k-a+1+i}$ to Z because $a-1 < a' < k-a+1$.

55. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \lambda x_{2a'} \sum_{i=1}^{k} x_{2i}y^{i}$ removes this term, removes the $x_{2a'}x_{2a}y^{k+1}$ term from Z but adds $-\lambda x_{2a'} \sum_{i=1}^{k} x_{2i}y^{i}$ to $X_{2k-2a+2}$.

56. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \lambda X_{2a'}Z$ takes this term away again.

Repeat steps 57 to 64 for all a and a' such that $1 \leq a \leq a' \leq k$, $a+a' \geq k+2$ and $2a \leq k+1$. Their purpose is to remove the $x_{2a}x_{2a'}y^{k+1}$ term from Z. Suppose that the coefficient of this term is λ.

57. For all i such that $1 \leq i \leq a-1$ and $i \neq a+a'-k-1$, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \lambda x_{2a'}x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \lambda X_{2a'}X_{2i}$ adds the extra term $\lambda x_{2a'} \sum_{i \neq a+a'-k-1}^{i=1} x_{2i}y^{k-a+1+i}$ to Z.

58. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \lambda x_{2a'} \sum_{\substack{i=1 \\ i \neq a+a'-k-1}}^{k} x_{2i}y^{i}$ removes this, removes the $x_{2a}x_{2a'}y^{k+1}$ term from Z but adds $\lambda x_{2a'} \sum_{\substack{i=1 \\ i \neq a+a'-k-1}}^{k} x_{2i}y^{i}$ to $X_{2k-2a+2}$.

59. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \lambda X_{2a'}Z$ takes this away again but adds the term $\lambda x_{2a'}x_{2a+2a'-2k-2}y^{a+a'-k-1}$ to $X_{2k-2a+2}$.

There are now two cases depending on whether $2a+a' = 2k+2$ or not. For the case when $2a+a' \neq 2k+2$,

60. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \lambda x_{2a'}x_{2a+2a'-2k-2}y^{a+a'-k-1}$ takes this term away again but adds $\lambda x_{2a'}x_{2a+2a'-2k-2}y^{a'}$ to Z.

61. $x_{2a'} \mapsto x_{2a'}/(1-\lambda x_{2a+2a'-2k-2})$, $X_{2a'} \mapsto X_{2a'}(1-\lambda X_{2a+2a'-2k-2})$ removes this term.

In the case when $2a+a' = 2k+2$,

62. $x_{2k-2a+2} \mapsto x_{2k-2a+2}/(1+\lambda x_{2a'}y^{k-a+1})$ removes the term introduced by 59 but itself adds $x_{2k-2a+2}y^{k-a+1}\sum_{b=1}^{\infty}(-\lambda x_{2a'}y^{k-a+1})^{b}$ to Z.

Of these new terms, those whose degree c in y is not greater than k and not equal to a' can be removed by

63. $x_{2c} \mapsto x_{2c} - tx_{2k-2a+2}$, $X_{2c} \mapsto X_{2c} + tX_{2k-2a+2}$.

The term whose degree c in y is a' is $-\lambda x_{2k-2a+2}x_{2a'}y^{a'}$. This can be removed by

64. $x_{2a'} \mapsto x_{2a'}/(1-\lambda x_{2k-2a+2})$, $X_{2a'} \mapsto X_{2a'}(1-\lambda X_{2k-2a+2})$. Terms introduced by 62 whose degree c in y is not less than $k+2$ are $O(k+4)$ and since $2(k-a+1) = a' < k+1$, terms whose degree c in y is $k+1$ are $O(k+4)$ as well.

Repeat steps 65 to 70 for all a and a' such that $1 \leq a \leq a' \leq k$ and $2a \geq k+2$. Their purpose is to remove the $x_{2a}x_{2a'}y^{k+1}$ term from Z. Suppose that the coefficient of this term is λ.

65. For all i such that $1 \leq i \leq a-1$, $i \neq k-a+1$ and $i \neq a+a'-k-1$ in order of increasing i, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \lambda x_{2a'}x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \lambda X_{2a'}X_{2i}$ adds $\lambda x_{2a'} \sum_{\substack{i=1 \\ i \neq k-a+1,\, a+a'-k-1}}^{k} x_{2i}y^{k-a+1+i}$ to Z.

66. $x_{2k-2a+2} \mapsto x_{2k-2a+2} - \lambda x_{2a'} \sum_{\substack{i=1 \\ i \neq k-a+1, a+a'-k-1}}^{k} x_{2i} y^i$ removes this term from Z but adds $-\lambda x_{2a'} \sum_{\substack{i=1 \\ i \neq k-a+1, a+a'-k-1}}^{k} x_{2i} y^i$ to $X_{2k-2a+2}$.

67. $X_{2k-2a+2} \mapsto X_{2k-2a+2} + \lambda X_{2a'} Z$ removes this term but adds the terms $\lambda x_{2a'} x_{2k-2a+2} y^{k-a+1}$ and $\lambda x_{2a'} x_{2a+2a'-2k-2} y^{a+a'-k-1}$ to $X_{2k-2a+2}$ except if $2a + a' = 2k+2$ in which case these two terms are the same and only one of them is added.

There are now two cases depending on whether $2a + a' = 2k + 2$ or not. For the case when $2a + a' \neq 2k + 2$ (for the other, skip to 70),

68. $x_{2k-2a+2} \mapsto (x_{2k-2a+2} - \lambda x_{2a'} x_{2a+2a'-2k-2} y^{a+a'-k-1})/(1 + \lambda x_{2a'} y^{k-a+1})$ takes away the two surplus terms that step 67 introduced to $X_{2k-2a+2}$ but does itself introduce the extra terms $x_{2k-2a+2} y^{k-a+1} \sum_{b=1}^{\infty} (-\lambda x_{2a'} y^{k-a+1})^b$ and $-\lambda x_{2a'} x_{2a+2a'-2k-2} y^{a'} \sum_{b=0}^{\infty} (-\lambda x_{2a'} y^{k-a+1})^b$ to Z. Of these new terms, those whose degree c in y is greater than k are $O(k+4)$ because $2a \geq k+2$ implies $2k - 2a + 2 \leq k$.

For all i such that $1 \leq i \leq k$, define the polynomials $\chi_i(x_2, x_4, \ldots, x_{2k})$ to be solutions of the equation $Z = \sum_{i=1}^{k} \chi_i(x_2, x_4, \ldots, x_{2k}) y^i + O(k+3)$ (which are unique up to $O(k - i + 3)$. Now define the map

$$
\begin{aligned}
&\Xi \colon \mathbf{C}^k \to \mathbf{C}^k \\
&(z_1, \ldots, z_k) \mapsto \big(\chi_1(z_1, \ldots, z_k), \ldots, \chi_k(z_1, \ldots, z_k)\big).
\end{aligned}
$$

The Jacobian of Ξ is the identity at 0 so, by the inverse function theorem, Ξ has an inverse in a neighbourhood of 0.

69. The co-ordinate change $(x_2, x_4, \ldots, x_{2k}) \mapsto \Xi^{-1}(x_2, x_4, \ldots, x_{2k})$ followed by $(X_2, X_4, \ldots, X_{2k}) \mapsto \Xi(X_2, X_4, \ldots, X_{2k})$ removes the remaining extra terms from Z.

For the case when $2a + a' = 2k + 2$,

70. $x_{2k-2a+2} \mapsto x_{2k-2a+2}/(1 + \lambda x_{2a'} y^{k-a+1})$ removes the term added to $X_{2k-2a+2}$ by step 67, but adds $x_{2k-2a+2} y^{k-a+1} \sum_{b=1}^{\infty} (-\lambda x_{2a'} y^{k-a+1})^b$ to Z. Of these terms, those whose degree c in y is greater than k are $O(k+4)$ because $2a \geq k+2$ implies $2k - 2a + 2 \leq k$. Step 69 removes the extra terms from Z again.

Repeat steps 71 to 73 for all a and a' such that $1 \leq a \leq a' \leq k$ and $a + a' = k+1$. Their purpose is to remove the $x_{2a} x_{2a'} y^{k+1}$ term from Z. Suppose that the coefficient of this term is λ.

71. For all i such that $1 \leq i \leq a-1$, $x_{2k-2a+2+2i} \mapsto x_{2k-2a+2+2i} + \lambda x_{2a'} x_{2i}$, $X_{2k-2a+2+2i} \mapsto X_{2k-2a+2+2i} - \lambda X_{2a'} X_{2i}$ adds $\lambda x_{2a'} \sum_{i=1}^{a-1} x_{2i} y^{k-a+1+i}$ to Z.

72. $x_{2a'} \mapsto x_{2a'}/(1 + \lambda \sum_{i=1}^{k} x_{2i} y^i)$ removes these terms from Z, removes the $x_{2a} x_{2a'} y^{k+1}$ term from Z but changes $X_{2a'}$ to $x_{2a'}/(1 + \lambda \sum_{i=1}^{k} x_{2i} y^i)$.

73. $X_{2a'} \mapsto X_{2a'}(1 + Z)$ changes $X_{2a'}$ back to $x_{2a'}$.

Define $\Xi: \mathbf{C}^{n-1}, \{0\} \to \mathbf{C}^{n-1}, \{0\}$ to be the map whose co-ordinate functions are given by

$$\Xi_{2i-1}(x_1, \ldots, x_{n-1}) = x_{2i-1} + \theta_{k+3-a}(x_1, \ldots, x_{n-1}) \qquad 1 \leq i \leq k$$
$$\Xi_{2i}(x_1, \ldots, x_{n-1}) = x_{2i} + \eta_{k+3-a}(x_1, \ldots, x_{n-1}) \qquad 1 \leq i \leq k$$
$$\Xi_j(x_1, \ldots, x_{n-1}) = x_j \qquad 2k+1 \leq j \leq n-1$$

Then the Jacobian of Ξ at 0 is the identity so, by the inverse function theorem, Ξ has an inverse in a neighbourhood of 0.

74. The change of co-ordinates given by $(x_1, \ldots, x_{n-1}) \mapsto \Xi^{-1}(x_1, \ldots, x_{n-1})$ and $(X_1, \ldots, X_{n-1}) \mapsto \Xi(X_1, \ldots, X_{n-1})$ sets $\theta_{k-j+3} = 0$ and $\eta_{k-j+3} = 0$ for all j such that $1 \leq j \leq k$.

If η_0 is non-zero then

75. $Z \mapsto Z/\eta_0$, $x_{2i} \mapsto \eta_0 x_{2i}$, $X_{2i} \mapsto X_{2i}/\eta_0$ for all i such that $1 \leq i \leq k$ sets $\eta_0 = 1$.

η_2 is a quadratic form in the variables $x_{2k+1}, \ldots, x_{n-1}$ so by a well known theorem about quadratic forms,

76. Some linear change in the co-ordinates $x_{2k+1}, \ldots, x_{n-1}$ reduces η_2 to $\sum_{l=2k+1}^{2k+1+r} x_l^2$ where $0 \leq r \leq n-2k-2$.

Now if $\eta_0 = 0$ then $J^{k+3}f$ is of type (U) and if $\eta_0 \neq 0$ then $J^{k+3}f$ is of type (V).

$\bowtie$

Theorem 6.3holds if $\mathbf{R}$ is replaced by $\mathbf{C}$ except that in the jets (U) and (V) the term $\sum_{l=2k+1}^{2k+r} x_l^2$ must be replaced by $\sum_{l=2k+1}^{2k+r} \pm x_l^2$. The proof is the same.

The $\mathcal{A}_e$-tangent space of jet (V) is contained in the linear subspace displayed in figure 3 which has codimension two. Therefore any germ with jet (V) has $\mathcal{A}_e$-codimension at least two.

If $r \leq n-2k-2$ then the $\mathcal{A}_e$-tangent space of the jet (U) is contained in the linear subspace displayed in figure 4 which has $\mathcal{A}_e$-codimension two. Therefore any germ with this jet has $\mathcal{A}_e$-codimension at least two. The only remaining candidate for a jet that is the jet of a germ of $\mathcal{A}_e$-codimension at most one is jet (U) when $r = n-2k-1$. In order to calculate the $\mathcal{A}_e$-tangent space (and hence the $\mathcal{A}_e$-codimension) of a germ with this jet we use the following result of Gafney.

Proposition 6.4 [Gafney 3.2 of [13]]
Let $f: \mathbf{C}^n, \{0\} \to \mathbf{C}^p\{0\}$ be an analytic map germ such that

$$tf(\theta(n)) + f^*\mathbf{m}_p\theta(f) \supseteq \mathbf{m}_n^i\theta(f)$$

and let C be an $\mathcal{O}_{\mathbf{C}^p}$-module via f^* such that $C \supseteq \mathbf{m}_n^j\theta(f)$ for $j \geq 1$. Then

$$C = tf(\theta(n)) + wf(\theta(p))$$

if and only if

$$C = tf(\theta(n)) + wf(\theta(p)) + f^*\mathbf{m}_p \cdot C + \mathbf{m}_n^{k+i}\theta(f).$$

$\bowtie$

$$
\begin{array}{r}
2k \\ \\ \\ \\ \\ \\ \\ \\ n-2k-1 \\ \\ \\ \\ \\
\end{array}
\begin{bmatrix}
\mathcal{O}_{\mathbb{C}^n} \\
\mathcal{O}_{\mathbb{C}^n} - \{y^k, x_{n-1}y^k\} \\
\mathcal{O}_{\mathbb{C}^n} \\
\mathcal{O}_{\mathbb{C}^n} - \{y^{k-1}, x_{n-1}y^{k-1}\} \\
\vdots \\
\mathcal{O}_{\mathbb{C}^n} \\
\mathcal{O}_{\mathbb{C}^n} - \{y, x_{n-1}y\} \\
\\
\mathcal{O}_{\mathbb{C}^n} \\
\mathcal{O}_{\mathbb{C}^n} \\
\vdots \\
\mathcal{O}_{\mathbb{C}^n} \\
\\
\mathcal{O}_{\mathbb{C}^n} \\
\mathcal{O}_{\mathbb{C}^n} - \{y^{k+1}, x_{n-1}y^{k+1}\}
\end{bmatrix}
$$

$$
+\mathbb{C}\left\{
\begin{bmatrix} 0 \\ y^k \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ y^{k+1} \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ y^{k-1} \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ y^{k+1} \end{bmatrix},
\ldots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ y^{k+1} \end{bmatrix},
\begin{bmatrix} 0 \\ x_{n-1}y^k \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ x_{n-1}y^{k+1} \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ x_{n-1}y^{k-1} \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ x_{n-1}y^{k+1} \end{bmatrix},
\ldots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ x_{n-1}y \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ x_{n-1}y^{k+1} \end{bmatrix}
\right\}
$$

figure 3

$$
\begin{array}{l}
\begin{array}{c}
2k\left\{\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right. \\
n-2k-1\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\
\\
\\
\end{array}
\begin{bmatrix}
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^k, y^{k+2}, x_1 y, x_3, y^2, \ldots, x_{2i-1} y^i, \ldots, x_{2k-1} y^k\} \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k-1}, y^{k+1}\} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k-i+1}, y^{k-i+3}\} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y, y^3\} \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k+1}, y^{k+3}, x_1 y^2, x_3 y^3, \ldots, x_{2i-1} y^i, \ldots, x_{2k-1} y^{k+1}\}
\end{bmatrix}
\\
+\mathbf{C}\left\{
\begin{bmatrix} 0 \\ y^k \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+1} \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ y^{k-1} \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+1} \end{bmatrix},
\ldots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y^{k-i+1} \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+1} \end{bmatrix},
\ldots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ y \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+1} \end{bmatrix},
\right.
\end{array}
$$

figure 4 (continued overleaf)

$$
\begin{bmatrix} 0 \\ y^{k+2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ y^{k+3} \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ y^{k+1} \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ y^{k+3} \end{bmatrix}, \dots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y^{k-i+3} \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ y^{k+3} \end{bmatrix}, \dots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ y^{3} \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ y^{k+3} \end{bmatrix},
$$

$$
\left.
\begin{bmatrix} 0 \\ y^{k+2} + \sum_{i=1}^{k} x_{2i-1} y^{i} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ x_1 y \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ x_1 y^2 \end{bmatrix},
\begin{bmatrix} 0 \\ x_3 y^2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ x_3 y^3 \end{bmatrix}, \dots,
\begin{bmatrix} 0 \\ x_{2i-1} y^{i} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ x_{2i-1} y^{i+1} \end{bmatrix}, \dots,
\begin{bmatrix} 0 \\ x_{2k-1} y^{k} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \\ x_{2k-1} y^{k+1} \end{bmatrix}
\right\}
$$

figure 4 (continued from previous page)

Lemma 6.5

If $f\colon \mathbf{C}^n, \{0\} \to \mathbf{C}^p, \{0\}$, *sending* $(x_1, \ldots, x_{n-1}, y)$ *to*

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i}y^i + y^{k+3} + y^{k+1} \sum_{j=2k+1}^{n-1} x_j^2\right)$$

and t *is a natural number then*

$$\mathcal{O}_{\mathbf{C}^n}\frac{\partial}{\partial Y} + \mathcal{O}_{\mathbf{C}^n}\frac{\partial}{\partial Z} \subseteq T\mathcal{A}_e f + \mathbf{C}\left\langle y^{k+1}\frac{\partial}{\partial Z}\right\rangle + \mathbf{m}_n^t\theta(f).$$

Proof

It is sufficient to check for all monomials $a \in \mathbf{C}[x_1, \ldots, x_{n-1}]$ and natural numbers l, that $ay^l\frac{\partial}{\partial Y}$ and $ay^l\frac{\partial}{\partial Z}$ are in M. Suppose that this is not so and let a be the greatest monomial such that, for some l, either $ay^l\frac{\partial}{\partial Y} \notin M$ or $ay^l\frac{\partial}{\partial Z} \notin M$ (where 'greatest' refers to the order that we are about to describe).

If a and b are monomials in $\mathbf{C}[x_1, \ldots, x_{n-1}]$ then we shall say that a is greater than b in the following circumstances; firstly if a has a higher degree than b, secondly if a has the same degree as b but has a higher degree in x_1 than b, thirdly if a has the same degree as b, the same degree in x_1 as b but a higher degree in x_2 than b, and so on.

Let l be maximal with respect to the condition that not both $ay^l\frac{\partial}{\partial Y} \in M$ and $ay^l\frac{\partial}{\partial Z} \in M$. Firstly we will consider the case when $ay^l\frac{\partial}{\partial Y} \notin M$.

If $1 \leq l \leq k$ then

$$ay^l - tf\left(a(x_1, \ldots, x_{n-1})\frac{\partial}{\partial x_{2l-1}}\right) + wf\left(a(X_1, \ldots, X_{n-1})\frac{\partial}{\partial X_{2l-1}}\right) \in M.$$

It follows that $ay^l\frac{\partial}{\partial Y}$ is in M—a contradiction.

If $l \geq k+1$ then

$$\begin{aligned}
ay^l\frac{\partial}{\partial Y} &- tf\left(a(x_1, \ldots, x_{n-1})y^{l-k-1}\frac{\partial}{\partial y}\right) \\
&= \sum_{i=1}^{k}\frac{i}{k+2}x_{2i-1}a(x_1, \ldots, x_{n-1})y^{l-k+1-2}\frac{\partial}{\partial Y} \\
&+ \frac{k+3}{k+2}a(x_1, \ldots, x_{n-1})y^{l+1}\frac{\partial}{\partial Z} \\
&+ \sum_{i=1}^{k}\frac{i}{k+2}x_{2i}a(x_1, \ldots, x_{n-1})y^{l-k+i-2}\frac{\partial}{\partial Z} \\
&+ \frac{k+1}{k+2}\sum_{j=k+1}^{n-1}x_j^2a(x_1, \ldots, x_{n-1})y^k\frac{\partial}{\partial Z}.
\end{aligned}$$

All the monomials in this expression either have a higher degree in the x_j's (for $1 \leq j \leq n-1$) than ay^l or the same degree in each of the x_j's and a higher degree in y. Either way, by our initial assumptions on a and l, each of these monomials is in M. Therefore $ay^l\frac{\partial}{\partial Y} \in M$—a contradiction.

Now we will consider the case when $ay^l\frac{\partial}{\partial Z} \notin M$. Write $l = d(k+2)+r$ where $1 \leq r \leq k+1$.

If $r = 0$ then all the monomials in

$$ay^l\frac{\partial}{\partial Z} - wf\left(a(X_1,\ldots,X_{n-1})Y^d\frac{\partial}{\partial Z}\right)$$

have a higher degree in x_j's than ay^l and so by our initial assumptions on a, they are in M. It follows that $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

If $1 \leq r \leq k$ then all the monomials in

$$ay^l\frac{\partial}{\partial Z} - tf\left(a(x_1,\ldots,x_{n-1})\left(y^{k+2}+\sum_{i=1}^{k}x_{2i-1}y^i\right)^d\frac{\partial}{\partial x_{2r}}\right)$$
$$+wf\left(a(X_1,\ldots,X_{n-1})Y^d\frac{\partial}{\partial X_{2r}}\right)$$

have a higher degree in x_j's than ay^l and so by our initial assumptions on a, they are in M. It follows that $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

If $r = k+1$ and $d \geq 1$ then all the monomials in

$$ay^l\frac{\partial}{\partial Z}$$
$$-tf\left(a(x_1,\ldots,x_{n-1})\left(y^{k+2}+\sum_{i=1}^{k}x_{2i-1}y^i\right)^{d-1}\left(y^{k+3}+\sum_{i=1}^{k}x_{2i}y^i\right)\frac{\partial}{\partial x_{2k}}\right)$$
$$+wf\left(a(X_1,\ldots,X_{n-1})Y^{d-1}Z\frac{\partial}{\partial X_{2k}}\right)$$

have a higher degree in x_j's than ay^l and so by our initial assumptions on a, they are in M. It follows that $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

If $r = k+1$ and $d = 1$ then $l = k+1$. Since $\mathbb{C}\langle y^{k+1}\frac{\partial}{\partial Z}\rangle \subseteq M$,$a \notin \mathbb{C}$. Therefore x_j divides a for some j such that $1 \leq j \leq n-1$. Write $a(x_1,\ldots,x_{n-1}) = x_jb(x_1,\ldots,x_{n-1})$.

If $2k+1 \leq j \leq n-1$ then

$$ay^l\frac{\partial}{\partial Z} = tf\left(\frac{1}{2}b(x_1,\ldots,x_{n-1})\frac{\partial}{\partial x_j}\right) - wf\left(\frac{1}{2}b(X_1,\ldots,X_{n-1})\frac{\partial}{\partial X_j}\right).$$

It follows that $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

If $j = 2i$ for $1 \leq i \leq k$ then

$$ay^l \frac{\partial}{\partial Z} - tf\left(\frac{k+3}{k-i+3} b(x_1, \ldots, x_{n-1}) \left(y^{k+3} + \sum_{l=1}^{k} x_{2l} y^l\right) \frac{\partial}{\partial x_{2k-2i+2}}\right)$$
$$+ wf\left(\frac{k+3}{k-i+3} b(X_1, \ldots, X_{n-1}) Z \frac{\partial}{\partial X_{2k-2i+2}}\right)$$
$$= -\frac{k+3}{k-i+3} by^{2k-i+4} \frac{\partial}{\partial Z} - \sum_{\substack{h=1 \\ h \neq i}}^{k} bx_{2h} y^{k-i+h+1} \frac{\partial}{\partial Z}$$
$$- \frac{i}{k-i+3} bx_{2i} y^{k+1} \frac{\partial}{\partial Z}.$$

The terms of the form $\frac{k+3}{k-i+3} bx_{2h} y^{k-i+h+1} \frac{\partial}{\partial Z}$ for $h \neq i$ are in M by what we have already proved.

$$\frac{k+3}{k-i+3} by^{2k-i+4} + \frac{i}{k-i+3} x_{2i} y^{k+1} \frac{\partial}{\partial Z}$$
$$- tf\left(\frac{1}{k-i+3} b(x_1, \ldots, x_{n-1}) y^{k-i+2} \frac{\partial}{\partial y}\right)$$

$$= -\frac{k+2}{k-i+3} by^{2k-i+3} \frac{\partial}{\partial Y} - \sum_{h=1}^{k} \frac{h}{k-i+3} x_{2h-1} y^{k-i+h+1} \frac{\partial}{\partial Y}$$
$$- \sum_{\substack{h=1 \\ h \neq i}}^{k} \frac{h}{k-i+3} bx_{2h} y^{k-i+h+1} \frac{\partial}{\partial Z}.$$

We have already proved that the terms $\frac{h}{k-i+3} bx_{2h} y^{k-i+h+1} \frac{\partial}{\partial Z}$ and the terms $\frac{h}{k-i+3} bx_{2h} y^{k-i+h+1} \frac{\partial}{\partial Y}$ for $h \neq i$ are in M.

$$\frac{k+2}{k-i+3} by^{2k-i+3} \frac{\partial}{\partial Y} + \frac{i}{k-i+3} bx_{2i-1} y^{k+1} \frac{\partial}{\partial Y}$$
$$-tf\left(\frac{i}{k-i+3} b(x_1, \ldots, x_{n-1}) \left(y^{k+2} + \sum_{h=1}^{k} x_{2h-1} y^h\right) \frac{\partial}{\partial x_{2k-2i+1}}\right)$$
$$+wf\left(\frac{i}{k-i+3} b(X_1, \ldots, X_{n-1}) Y \frac{\partial}{\partial X_{2k-2i+1}}\right)$$
$$= \frac{k-i+2}{k-i+3} by^{2k-i+3} \frac{\partial}{\partial Y} - \sum_{\substack{h=1 \\ h \neq i}}^{k} \frac{i}{k-i+3} bx_{2h-1} y^{k-i+h+1} \frac{\partial}{\partial Y}.$$

We have already proved that the terms $\frac{i}{k-i+3}bx_{2h-1}y^{k-i+h+1}\frac{\partial}{\partial Y}$ are in M.

If $i = k$ then

$$\frac{k-i+2}{k-i+3}by^{2k-i+3}\frac{\partial}{\partial Y} - wf\left(\frac{k-i+2}{k-i+3}bZ\frac{\partial}{\partial Y}\right) = \sum_{h=1}^{k}\frac{k-i+2}{k-i+3}bx_{2h}y^{h}$$

and we have already shown that each of these terms is in M. Hence $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

If $i \neq k$ then

$$\begin{aligned}
&\frac{k-i+2}{k-i+3}by^{2k-i+3}\frac{\partial}{\partial Y}\\
&-tf\left(\frac{k-i+2}{k-i+3}b(x_1,\ldots,x_{n-1})\left(y^{k+3}+\sum_{h=1}^{k}x_{2h}y^h\right)\frac{\partial}{\partial x_{2k-2i-1}}\right)\\
&+wf\left(\frac{k-i+2}{k-i+3}b(X_1,\ldots,X_{n-1})Z\frac{\partial}{\partial X_{2k-2i+2}}\right)\\
&=\sum_{h=1}^{k}\frac{k-i+2}{k-i+3}bx_{2h}y^{k-i+h}\frac{\partial}{\partial Y}.
\end{aligned}$$

We have already shown that each of the terms $\frac{k-i+2}{k-i+3}bx_{2h}y^{k-i+h}$ for $h \neq i+1$ is in M but the remaining term $\frac{k-i+2}{k-i+3}bx_{2i+2}y^{k+1}$ is in M as well because $x_{2i+2}b$ is greater than $x_{2i}b$ in our order. It follows again that $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

Finally if $j = 2i-1$ for $1 \leq i \leq k$ then

$$\begin{aligned}
ay^l\frac{\partial}{\partial Z} &- tf\left(b(x_1,\ldots,x_{n-1})\left(y^{k+2}+\sum_{h=1}^{k}x_{2h-1}y^h\right)\frac{\partial}{\partial x_{2k-2i+2}}\right)\\
&+wf\left(b(X_1,\ldots,X_{n-1})Y\frac{\partial}{\partial X_{2k-2i+2}}\right)\\
&= -by^{2k-i+3}\frac{\partial}{\partial Z} - \sum_{\substack{h=1\\h\neq i}}^{k} bx_{2h-1}y^{k-i+h+1}\frac{\partial}{\partial Z}
\end{aligned}$$

We have already proved that each of the terms $bx_{2h-1}y^{k-i+h+1}\frac{\partial}{\partial Z}$ for $h \neq i$ is in M.

If $i = k$ then

$$by^{2k-i+3}\frac{\partial}{\partial Z} = wf\left(b(X_1,\dots,X_{n-1})Z\frac{\partial}{\partial Z}\right)$$
$$-tf\left(\sum_{h=1}^{k} b(x_1,\dots,x_{n-1})x_{2h}\frac{\partial}{\partial x_{2h}}\right)$$
$$+wf\left(\sum_{h=1}^{k} b(X_1,\dots,X_{n-1})X_{2h}\frac{\partial}{\partial X_{2h}}\right) \in M$$

hence $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

If $i \neq k$ then

$$by^{2k-i+3}\frac{\partial}{\partial Z} - tf\left(b(x_1,\dots,x_{n-1})\left(y^{k+3}+\sum_{h=1}^{k}x_{2h}y^h\right)\frac{\partial}{\partial x_{2k-2i}}\right)$$
$$+wf\left(b(X_1,\dots,X_{n-1})Z\frac{\partial}{\partial X_{2k-2i}}\right)$$
$$=\sum_{h=1}^{k} bx_{2h}y^{k-i+h}.$$

We have already shown that each of the terms $bx_{2h}y^{k-i+h}$ for $1 \leq h \leq k$ is in M. It follows that $ay^l\frac{\partial}{\partial Z} \in M$—a contradiction.

We are forced to conclude that our initial supposition was incorrect, in other words that the lemma is true.

$\bowtie$

Proposition 6.6

If $f: \mathbf{C}^n, \{0\} \to \mathbf{C}^{n+1}, \{0\}$, and sends $(x_1, \dots, x_{n-1}, y)$ to

$$\left(x_1,\dots,x_{n-1},\sum_{i=1}^{k}x_{2i-1}y^i+y^{k+2},\sum_{i=1}^{k}x_{2i}y^i+y^{k+3}+y^{k+1}\sum_{j=2k+1}^{n-1}x_j^2\right)$$

then $T\mathcal{A}_e f$ is the linear space M displayed in figure 5.

Proof

It is a straightforward check that $T\mathcal{K}_e f + \mathbf{C}\langle\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_{n-1}},\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\rangle = M$ so M is an $\mathcal{O}_{\mathbf{C}^{n+1}}$-module. The spaces $f^*\mathbf{m}_{n+1}$ and M both contain $\mathbf{m}_n^{k+2}$ so by Proposition 6.4 with $C = M$ and $i = j = k+2$ we need only check that $T\mathcal{A}_e f + \mathbf{m}_n^{2k+4}\theta(f) = M$. It is not hard to see that $T\mathcal{A}_e f + \mathbf{m}_n^{2k+4}\theta(f) \subseteq M$ so we only need to check the other inclusion. Since M has codimension one in $\theta(f)$, it is sufficient to check that

$$T\mathcal{A}_e f + \mathbf{C}\left\langle y^{k+1}\frac{\partial}{\partial Z}\right\rangle + \mathbf{m}_n^{2k+4} \supseteq \theta(f).$$

$$
\begin{array}{r}
\\ \\ \\ 2k \updownarrow \\ \\ \\ \\ \\ n-2k-1 \updownarrow \\ \\ \\ \\ \\
\end{array}
\begin{bmatrix}
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^k\} \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k-1}\} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y\} \\
\mathcal{O}_{\mathbf{C}^n} \\
\vdots \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} \\
\mathcal{O}_{\mathbf{C}^n} - \{y^{k+1}\}
\end{bmatrix}
+ \mathbf{C}\left\{
\begin{bmatrix} 0 \\ y^k \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+1} \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ y^{k-1} \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+1} \end{bmatrix},
\ldots,
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y \\ 0 \\ \vdots \\ 0 \\ 0 \\ y^{k+1} \end{bmatrix}
\right\}
$$

figure 5

By Lemma 6.5 with $t = 2k+4$, $\mathcal{O}_{\mathbf{C}^n}\frac{\partial}{\partial Y}$ and $\mathcal{O}_{\mathbf{C}^n}\frac{\partial}{\partial Z}$ are contained in $T\mathcal{A}_e f + \mathbf{C}\langle y^{k+1}\frac{\partial}{\partial Z}\rangle + \mathbf{m}_n^{2k+4}\theta(f)$. It is now sufficient to show that for $a \in \mathcal{O}_{\mathbf{C}^n}$ and j such that $1 \leq j \leq n-1$,

$$a\frac{\partial}{\partial X_i} \in T\mathcal{A}_e f + \mathbf{C}\left\langle y^{k+1}\frac{\partial}{\partial Z}\right\rangle + \mathbf{m}_n^{2k+4}\theta(f).$$

This is so because

$$
\begin{aligned}
a\frac{\partial}{\partial X_i} - tf\left(a\frac{\partial}{\partial X_i}\right) &\in \mathcal{O}_{\mathbf{C}^n}\frac{\partial}{\partial Y} + \mathcal{O}_{\mathbf{C}^n}\frac{\partial}{\partial Z} \\
&\subseteq T\mathcal{A}_e f + \mathbf{C}\left\langle y^{k+1}\frac{\partial}{\partial Z}\right\rangle + \mathbf{m}_n^{2n+4}\theta(f).
\end{aligned}
$$

$\bowtie$

In order to complete our classification, we will need to refer to the following lemma of Mather (and its complex analytic version).

Lemma 6.7 [3.1 of [11]]
Let $\alpha: G \times U \to U$ be a C^∞ action of a Lie group on a C^∞ manifold U, and let V be a connected C^∞ sub-manifold of U. Then necessary and sufficient conditions for V to be contained in a single orbit of α are that:
a) $T(Gv)_v \supseteq TV_v$, if $v \in V$.
b) $\dim T(Gv)_v$ is independent of the choice of $v \in V$.

$\bowtie$

Theorem 6.8

If n and k are natural numbers and $2k+1 \le n$ then the germ of the map $f\colon \mathbf{C}^n\{0\} \to \mathbf{C}^{n+1}\{0\}$ which takes $(x_1, \ldots, x_{n-1}, y)$ to

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1} y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i} y^i + y^{k+3} + y^{k+1} \sum_{j=2k+1}^{n-1} x_j^2 \right)$$

has $\mathcal{A}_e$-codimension one and is $(k+3)$-determined for $\mathcal{A}$.

Proof

It follows from Proposition 6.6 that the $\mathcal{A}_e$-codimension of f is one. It remains to show that f is $(k+3)$-determined.

It is easy to check that a map f' with the same $(k+3)$-jet as f has $\mathcal{K}_e f' + \mathbf{C}\langle \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_{n-1}}, \frac{\partial}{\partial Y} \frac{\partial}{\partial Z} \rangle$ equal to the space M shown in figure 5. It follows from this that for such an f', $T\mathcal{A}_e f' \subseteq M$. We will show that if the f' is sufficiently close, in some sense, to f then we have equality. In order to do this it is sufficient to check that $M \subseteq T\mathcal{A}_e f' + \mathrm{m}_n^{2k+4}$ by Proposition 6.4. By Proposition 6.6 we have equality when $f = f'$.

Name all possible products of the form $b\frac{\partial}{\partial X}$ where b is a monomial of degree strictly between $k+3$ and $2k+4$ in $\mathcal{O}_{C^n}$ and W is a coordinate in the target: $a_1, \ldots, a_t$. Since $T\mathcal{A}_e f + \mathrm{m}_n^{2k+4}\theta(f) = M$ and $M/\mathrm{m}_n^{2k+4}\theta(f)$ is finitely generated as a $\mathbf{C}$-vector space, there are a finite number of vectors of the form $tf(\xi_1), \ldots, tf(\xi_u)$ and $wf(\eta_1) \ldots, wf(\eta_v)$ which form a basis for $M/\mathrm{m}_n^{2k+4}\theta(f)$ over $\mathbf{C}$. Since they form a basis, the number of these vectors is equal to the $\mathbf{C}$-dimension of $M/\mathrm{m}_n^{2k+4}\theta(f)$ and any linearly independent set of vectors of this size will form a basis. The condition that such a set of vectors be linearly independent is equivalent to the non-vanishing of the determinant of the matrix whose columns are these vectors (or rather their components) written with respect to some fixed basis of M. Since the determinant is a continuous function of the components of a square matrix and since the components of our matrix depend continuously on the function f, there is an $\epsilon > 0$ such that if $\lambda_1, \ldots, \lambda_t$ are complex numbers each of whose magnitudes is less than ϵ then $f' := f + \sum_{i=1}^{t} \lambda_i a_i$ has the vectors $tf'(\xi_1), \ldots, tf'(\xi_u)$ and $wf'(\eta_1) \ldots, wf'(\eta_v)$ linearly independent. It follows that they generate the whole of M/m_n^{2k+4} and hence that $T\mathcal{A}_e f' + \mathrm{m}_n^{2k+4}\theta(f) = M$. Since $M \supseteq \mathrm{m}_n^{2k+4}\theta(f)$, we can apply Lemma 6.7 with $G = \mathcal{A}^{2k+4}$ (the group $\mathcal{A}$ considered modulo terms of order at least $2k+5$), $U = J^{2k+4}(n, n+1)$ and

$$V = \left\{ j^{2k+4}(f + \sum_{i=1}^{t} \lambda_i a_i) : |\lambda_i| < \epsilon \quad \forall i \right\}$$

to deduce that all such f' are $\mathcal{A}$-equivalent to f modulo m_n^{2k+4}. We can then apply Proposition 6.2 with $i = j = k+2$ to deduce that they are actually $\mathcal{A}$-equivalent to f.

Notice that f is quasihomogeneous with respect to the following weights.

<table>
<tr><td></td><td colspan="3">Coordinate</td><td>Weight</td></tr>
<tr><td rowspan="4">Source</td><td rowspan="3">x_j</td><td>$j = 2i - 1$</td><td rowspan="2">for $1 \leq i \leq k$</td><td>$k - i + 2$</td></tr>
<tr><td>$j = 2i$</td><td>$k - i + 3$</td></tr>
<tr><td colspan="2">$2k + 1 \leq j \leq n - 1$</td><td>1</td></tr>
<tr><td colspan="3">y</td><td>1</td></tr>
<tr><td rowspan="5">Target</td><td rowspan="3">X_j</td><td>$j = 2i - 1$</td><td rowspan="2">for $1 \leq i \leq k$</td><td>$k - i + 2$</td></tr>
<tr><td>$j = 2i$</td><td>$k - i + 3$</td></tr>
<tr><td colspan="2">$2k + 1 \leq j \leq n - 1$</td><td>1</td></tr>
<tr><td colspan="3">Y</td><td>$k + 2$</td></tr>
<tr><td colspan="3">Z</td><td>$k + 3$</td></tr>
</table>

Let f' be the germ of a map with the same $(k+3)$-jet as f. There are complex numbers $\lambda_1, \ldots, \lambda_t$ such that $f' = f + \sum_{i=1}^{t} \lambda_i a_i$ modulo terms of order $2k + 4$ and higher. Since all the weights in the source are positive and the degree of each a_i is at least $k + 4$, the weight of each a_i is at least $k + 4$ which is greater than the weight of any of the coordinates in the target. Let the difference between the weight of a_i and the weight of the corresponding coordinate in the target be q_i, so $q_i > 0$.

Let the weights of the coordinates in the source be $w_1, \ldots, w_n$ and of those in the target $d_1, \ldots, d_{n+1}$. For $\mu \in \mathbf{C}$ let $\phi_\mu : \mathbf{C}^n, \{0\} \to \mathbf{C}^n, \{0\}$ be the linear map from the source to itself whose matrix is

$$\begin{pmatrix} \mu^{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \mu^{d_2} & 0 & \cdots & 0 \\ 0 & 0 & \mu^{d_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu^{d_n} \end{pmatrix}$$

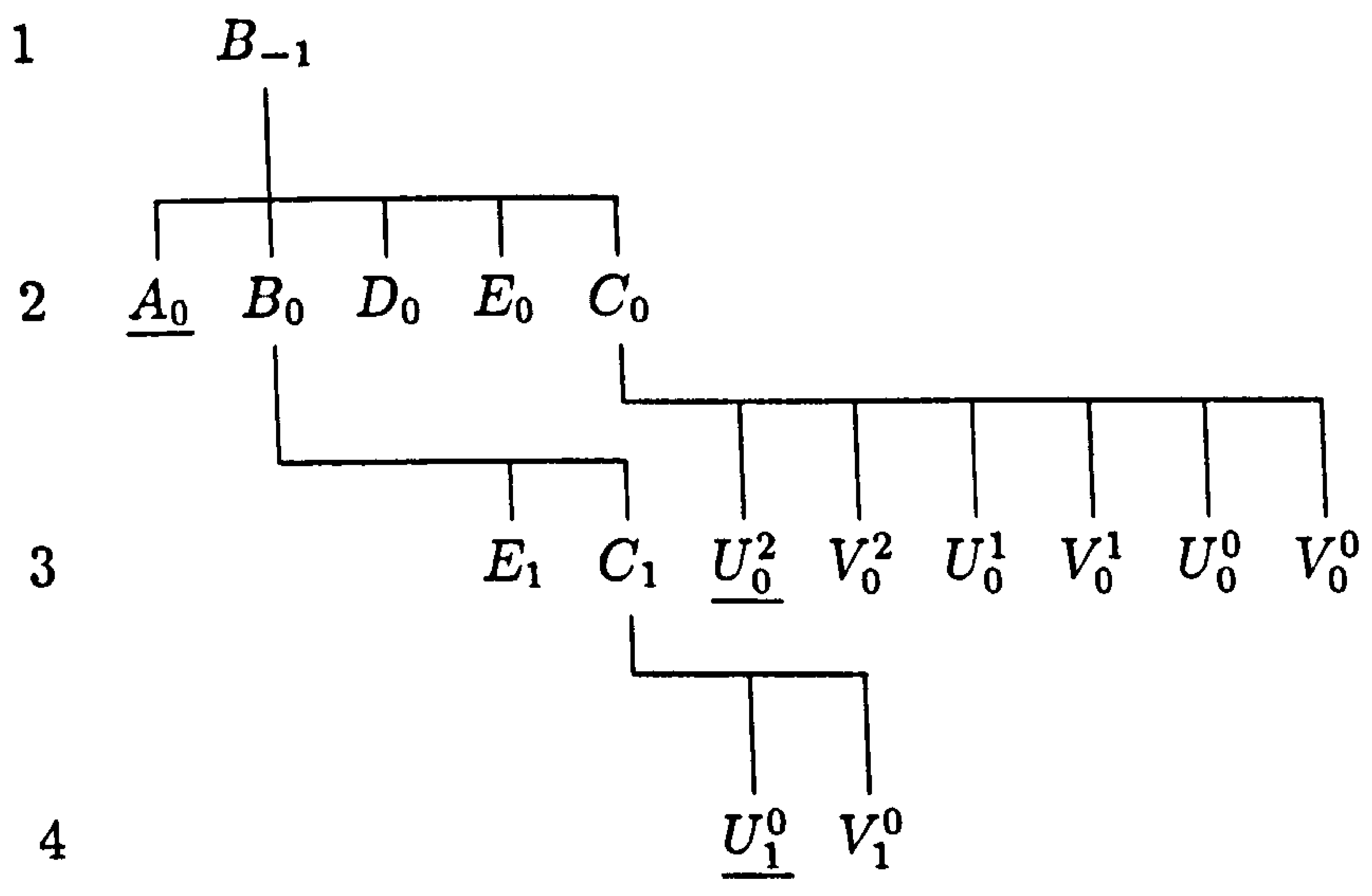
and let ψ_μ be the analogous map in the target. That f is quasihomogeneous with respect to these weights is equivalent to the statement: $\forall \mu \in \mathbf{C}, f \circ \phi_\mu = \psi_\mu \circ f$. By the definition of the q_i, $\forall \mu \in \mathbf{C}$

$$\begin{aligned} \left(f + \sum_{i=1}^{t} \lambda_i a_i \right) \circ \phi_\mu &= f \circ \phi_\mu + \sum_{i=1}^{t} \lambda_i (a_i \circ \phi_\mu) \\ &= \psi_\mu \circ f + \sum_{i=1}^{t} \lambda_i (\mu^{q_i} \psi_\mu \circ a_i) \\ &= \psi_\mu \circ \left(f + \sum_{i=1}^{t} (\mu^{q_i} \lambda_i) a_i \right). \end{aligned}$$

It follows that $f + \sum_{i=1}^{t} \lambda_i a_i$ is $\mathcal{A}$-equivalent to $f + \sum_{i=1}^{t}(\mu^{q_i}\lambda_i)a_i$. If we let μ be sufficiently small then the magnitude of each $\mu^{q_i}\lambda_i$ is less than ϵ and so by what we have already said (in the third paragraph of this proof), $f + \sum_{i=1}^{t} \mu^{q_i}\lambda_i$ is $\mathcal{A}$-equivalent to f. Now we see that f' (i.e., $f + \sum_{i=1}^{t} \lambda_i a_i$) is $\mathcal{A}$-equivalent to f. Since f' was an arbitrary map with the same $(k+3)$-jet as f, it follows that f is $(k+3)$-determined.

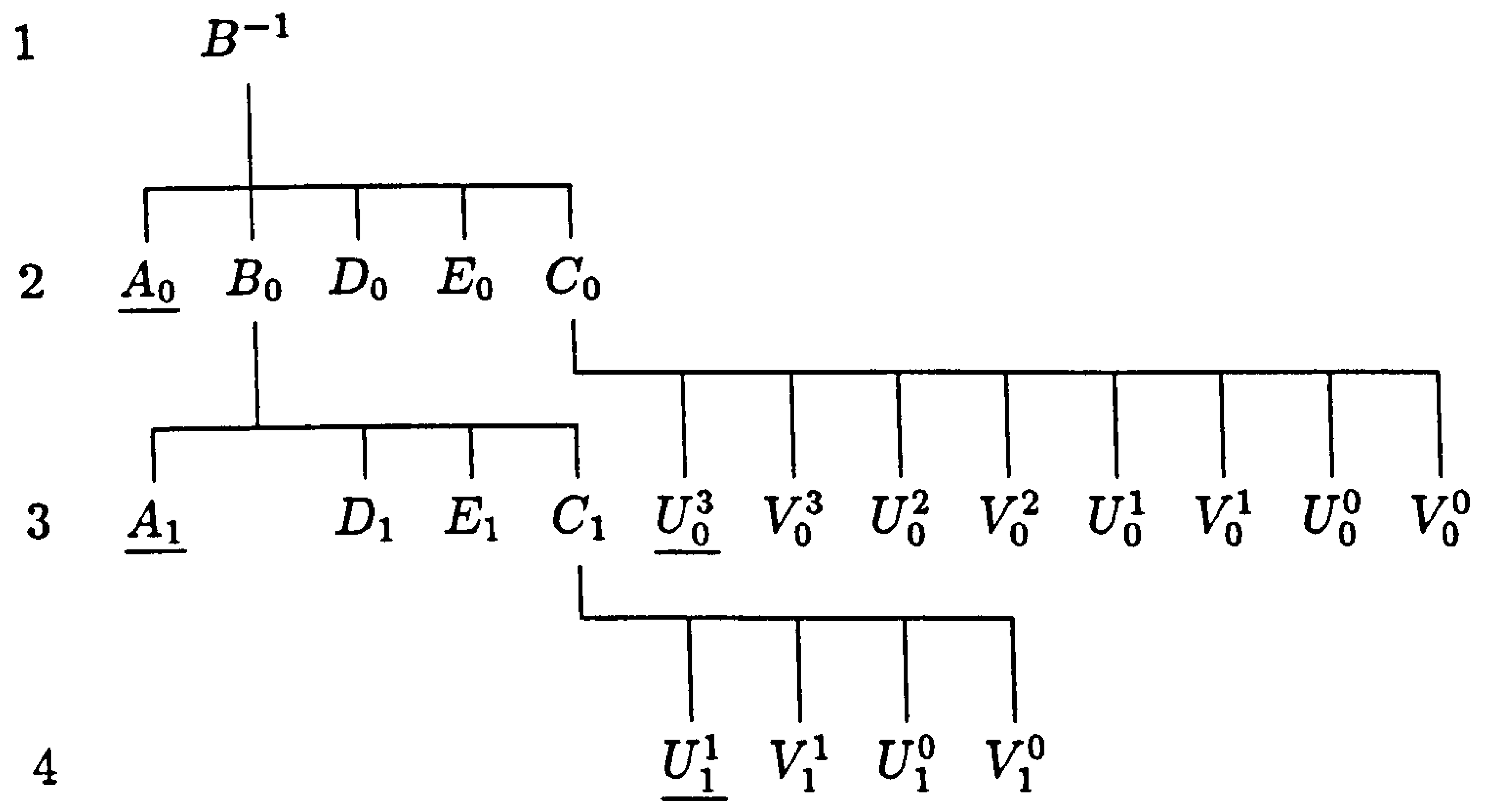
$\bowtie$

Summing up the results of this chapter so far, we have a complete classification of $\mathcal{A}$-orbits of jets in $J(n, n+1)$ with $\mathcal{A}$-codimension zero or one and corank one, and also a complete classification of the orbits of jets in the preimage of these jets in the jet space of degree one higher. Rather than present this classification formally in a theorem, which would be cumbersome, we present diagrams for the cases $n = 4$ and $n = 5$ from which the structure of the general case should be clear.



The diagram above shows the orbits that we have described for the case $n = 4$. The letters refer to the notation for certain jets introduced in Theorem 6.1 and Theorem 6.3, their subscripts refer to the values of k in the same theorems and the superscripts of the U and V jets refer to the value of r in Theorem 6.3. The numbers on the left refer to the degree of the jet space which contains the orbits and the connecting lines show when one jet is contained in the preimage of another. If a jet is underlined, it is sufficient (that means that if for example it is a k-jet then it is k-determined). The determinacy degrees shown in this way are exact.

The diagram below is of the case $n = 5$. The conventions of the the $n = 4$ diagram are used again.



From this classification of $\mathcal{A}$-equivalence classes of jets we can deduce a classification of $\mathcal{A}$-equivalence classes of germs of corank one and $\mathcal{A}_e$-codimension at most one. We have proved the following theorems.

Theorem 6.9

Suppose that n is a natural number. If k is another natural number such that $2k+2 \leq n$ then the germ f mapping $\mathbf{C}^n, \{0\}$ to $\mathbf{C}^{n+1}, \{0\}$ sending $(x_1, \ldots, x_{n-1}, y)$ to

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i}y^i + x_{2k+1}y^{k+1}\right)$$

is stable and $(k+2)$-determined. Conversely any stable mapgerm of corank one between these spaces is $\mathcal{A}$-equivalent to the above germ for some natural number k satisfying $2k+2 \leq n$.

$\bowtie$

Stable germs have already been classified in greater generality (see for example [11]).

Theorem 6.10

Suppose that n is a natural number. If k is another natural number such that $2k+1 \leq n$ then the germ f mapping $\mathbf{C}^n, \{0\}$ to $\mathbf{C}^{n+1}, \{0\}$ sending $(x_1, \ldots, x_{n-1}, y)$ to

$$\left(x_1, \ldots, x_{n-1}, \sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i}y^i + y^{k+1} \sum_{l=2k+1}^{n-1} x_l^2 + y^{k+3}\right)$$

has $\mathcal{A}_e$-codimension one and is $(k+3)$-determined. Conversely any mapgerm of $\mathcal{A}_e$-codimension one and corank one between these spaces is $\mathcal{A}$-equivalent to the above germ for some natural number k satisfying $2k+1 \leq n$.

$\bowtie$

Corollary 6.11

If $f\colon \mathbf{C}^n,\{0\} \to \mathbf{C}^{n+1},\{0\}$ has corank one and $\mathcal{A}_e$-codimension at most one then it is $\mathcal{A}$-equivalent to a quasihomogeneous germ.

Proof

Our germ is $\mathcal{A}$-equivalent to either the germ explicitly described in Theorem 6.9 or to the one in Theorem 6.10. The germ in the second case is quasihomogeneous with the weights shown in the table in the proof of Theorem 6.8. The germ in the second case is quasihomogeneous with the weights shown below.

<table>
<tr><td></td><td colspan="4">Coordinate</td><td>Weight</td></tr>
<tr><td rowspan="4">Source</td><td rowspan="3">x_j</td><td>$j = 2i - 1$</td><td rowspan="2">for $1 \le i \le k$</td><td></td><td>$k - i + 2$</td></tr>
<tr><td>$j = 2i$</td><td></td><td>$k - i + 3$</td></tr>
<tr><td colspan="3">$2k + 1 \le j \le n - 1$</td><td>2</td></tr>
<tr><td colspan="4">y</td><td>1</td></tr>
<tr><td rowspan="5">Target</td><td rowspan="3">X_j</td><td>$j = 2i - 1$</td><td rowspan="2">for $1 \le i \le k$</td><td></td><td>$k - i + 2$</td></tr>
<tr><td>$j = 2i$</td><td></td><td>$k - i + 3$</td></tr>
<tr><td colspan="3">$2k + 1 \le j \le n - 1$</td><td>2</td></tr>
<tr><td colspan="4">Y</td><td>$k + 2$</td></tr>
<tr><td colspan="4">Z</td><td>$k + 3$</td></tr>
</table>

$\bowtie$

§7 The Classification of Certain Multigerms

In this chapter we classify multigerms of maps from $\mathbf{C}^n$ to $\mathbf{C}^{n+1}$ ($n \geq 0$) of $\mathcal{A}_e$-codimension one, each of whose components has corank at most one.

Suppose that n is a non-negative integer and that s,t,r and $k_1 \leq \ldots, \leq k_s$ are non-negative integers such that

$$n+1 = s+t-1+\sum_{i=1}^{s}(k_i+2)+r$$

and such that if $s=0$ then $t \geq 2$. Define the multigerm $h_{s,t,r,k_1,\ldots,k_s}$ (h for short) from $\mathbf{C}^n$ to $\mathbf{C}^{n+1}$ to be the one with components $f^{(1)},\ldots,f^{(s)},g^{(1)},\ldots,g^{(t)}$ as described below.

Let the coordinate functions of $\mathbf{C}^{n+1}$ be

$$\begin{gathered}
\Lambda_1,\Lambda_2,\ldots,\Lambda_{s+t-1},\\
X_1^{(1)},X_2^{(1)},\ldots,X_{2k_1}^{(1)},Y^{(1)},Z^{(1)},\\
X_1^{(2)},X_2^{(2)},\ldots,X_{2k_2}^{(2)},Y^{(2)},Z^{(2)},\\
\vdots\\
X_1^{(s)},X_2^{(s)},\ldots,X_{2k_s}^{(s)},Y^{(s)},Z^{(s)},\\
V_1,V_2,\ldots,V_r.
\end{gathered}$$

Let the coordinate functions of the source $\mathbf{C}^n$ of each component, $f^{(i)}$ or $g^{(j)}$ of h be

$$\begin{gathered}
\lambda_1,\lambda_2,\ldots,\lambda_{s+t-1},\\
x_1^{(1)},x_2^{(1)},\ldots,x_{2k_1}^{(1)},y^{(1)},z^{(1)},\\
x_1^{(2)},x_2^{(2)},\ldots,x_{2k_2}^{(2)},y^{(2)},z^{(2)},\\
\vdots\\
x_1^{(s)},x_2^{(s)},\ldots,x_{2k_s}^{(s)},y^{(s)},z^{(s)},\\
v_1,v_2,\ldots,v_r
\end{gathered}$$

but with the following modifications: for $f^{(i)}$ remove the two functions $y^{(i)}$ and $z^{(i)}$ and substitute the single function $w^{(i)}$, and for $g^{(j)}$ remove λ_{s+j-1} (unless $s+j-1=0$, in which case remove λ_1).

Let $f^{(1)}$ be that germ $\mathbf{C}^n,\{0\} \to \mathbf{C}^{n+1},\{0\}$ which, when composed with a coordinate function in the target, gives the corresponding coordinate function in the source (i.e. capital letters correspond to their small versions) except that

$$Y^{(1)} \circ f^{(1)} = w^{(1)k_1+2}+\sum_{j=1}^{k_1} x_{2j-1}^{(1)} w^{(1)j}$$

and

$$Z^{(1)} \circ f^{(1)} = w^{(1)k_1+3} + \sum_{j=1}^{k_1} x_{2j}^{(1)} w^{(1)j} + U w^{(1)k_1+1}$$

where the meaning of U is explained below.

Let $f^{(i)}$ (for i greater than one) be be that germ $\mathbf{C}^n, \{0\} \to \mathbf{C}^{n+1}, \{0\}$ which, when composed with a coordinate function in the target, gives the corresponding coordinate function in the source except that

$$\Lambda_{i-1} \circ f^{(i)} = \lambda_{i-1} + U,$$

$$Y^{(i)} \circ f^{(i)} = w^{(i)k_i+2} + \sum_{j=1}^{k_i} x_{2j-1}^{(i)} w^{(i)j}$$

and

$$Z^{(i)} \circ f^{(i)} = w^{(i)k_i+3} + \sum_{j=1}^{k_i} x_{2j}^{(i)} w^{(i)j} + \lambda_{i-1} w^{(i)k_i+1}$$

where the meaning of U is explained below.

Let $g^{(j)}$ be that germ $\mathbf{C}^n, \{0\} \to \mathbf{C}^{n+1}, \{0\}$ which when composed with a coordinate function in the target gives the corresponding coordinate function in the source except that

$$\Lambda_{s+j-1} \circ g^{(j)} = U$$

unless $s + j - 1 = 0$, in which case the exception is that

$$\Lambda_1 \circ g^{(j)} = 0.$$

Finally we define U. For the i^{th} component, U stands for λ_i, unless it is the last component, in which case U stands for $\sum_{j=1}^{r} v_j^2$.

The multigerm h is quasihomogeneous with respect to the following weights.

	Coordinate			Weight
Source	λ_i			2
	$x_l^{(i)}$	$l = 2j - 1$	for $1 \leq j \leq k_i$	$k_i + 2 - j$
		$l = 2j$		$k_i + 3 - j$
	$w^{(i)}$			1
	$y^{(i)}$			$k_i + 2$
	$z^{(i)}$			$k_i + 3$
	$v^{(i)}$			1
Target	Λ_i			2
	$X_l^{(i)}$	$l = 2j - 1$	for $1 \leq j \leq k_i$	$k_i + 2 - j$
		$l = 2j$		$k_i + 3 - j$
	$Y^{(i)}$			$k_i + 2$
	$Z^{(i)}$			$k_i + 3$
	$V^{(i)}$			1

Theorem 7.1

Suppose that n is a non-negative integer and that s, t, r and $k_1 \dots, k_s$ are non-negative integers satisfying

$$n+1 = s+t-1+\sum_{i=1}^{s}(k_i+2)+r$$
$$s=0 \Rightarrow t \geq 2$$
$$k_1 \leq k_2 \leq \cdots \leq k_s$$

then the map $h_{s,t,r,k_1,\dots,k_s}$ has $\mathcal{A}_e$-codimension one. Conversely if h is a multigerm from $\mathbf{C}^n$ to $\mathbf{C}^{n+1}$ of $\mathcal{A}_e$-codimension one, each of whose components has corank at most one, then there are unique integers s, t, r and $k_1, \dots, k_s$, satisfying the above three conditions, such that h is $\mathcal{A}$-equivalent to $h_{s,t,r,k_1,\dots,k_s}$.

Proof

First we show that $h_{s,t,r,k_1,\dots,k_s}$ has $\mathcal{A}_e$-codimension one. It is sufficient to prove that $h_{s,t,r,k_1,\dots,k_s}$ is primitive of $\mathcal{A}_e$-codimension one for the case $r=0$ and to prove that $\frac{\partial h'}{\partial V} \in T\mathcal{A}_e h'$ (where h' is got from $h_{s,t,r,k_1,\dots,k_s}$ by replacing the occurrence of $\sum_{j=1}^{r} v_j^2$ by V); because this is the base step for a proof of the theorem by induction on r, and the inductive step is given by Theorem 2.5 (and the identity for $T\mathcal{A}_e \mathrm{A}_F f$ given in the first paragraph of its proof). We therefore now suppose that $r=0$.

We treat first the case where $s=0$ by induction on t. By the second of our three hypotheses, $t \geq 2$. The case $t=2$ follows from Theorem 4.29 where $a=b=0$ and both f and g are of type i). The statement about h' follows from Lemma 4.13. If $t \geq 3$ and $h_{0,t-1,0}$ has $\mathcal{A}_e$-codimension one then we can apply Theorem 4.29 to $h_{0,t,0}$ where g is just the last component. Here g is of type i) and f is of type i), we see that f is nice by inspection.

We now treat the case where $s \neq 0$ by induction on the number of components $(s+t)$. If $s+t=1$ then $s=1$ and $t=0$ and h has $\mathcal{A}_e$-codimension one by Theorem 6.10. To see that h is primitive: notice that if h were an augmentation then it would have to be mentioned in two different ways in Theorem 6.10; this cannot be because all these germs have different $\mathcal{K}$-codimensions. For the inductive step we suppose that $s+t \geq 2$ and use Theorem 4.29 again, where g is just the last component of h. g is of type i) or iii) depending on whether $t>0$ or not. f is of type iii).

Now we show the converse. By Theorem 2.5 again it is sufficient to treat the case where h is primitive and we do this by induction on the number of components. We treat the cases where h has no non-immersive components and some non-immersive components separately.

Suppose first that h has no non-immersive components, that is to say, that all the components of h are immersions. Since an immersion is stable and h is not, h has at least two components. If h has precisely two components then we apply Theorem 4.28 to h letting f be one component and g be the other. We may apply Theorem 4.28 because g is not transverse to $\tau(f)$ by Proposition 4.2. f is either of type i) or ii) and either way $b = 0$. Similarly $a = 0$. Therefore since $n+1 = a+b+1$, $n = 0$ and therefore h is $h_{0,2,0}$. For the inductive step suppose that h has more than two components. Let g be the last component of h and let f be made up of all of the others. We may apply Theorem 4.28 because g is not transverse to the analytic stratum of f by Proposition 4.2 and if f is transverse to the analytic stratum of g then by the inductive hypothesis, the pullback of f by the analytic stratum of g is some $h_{s,t,r,k_1,\ldots,k_s}$ which is nice by inspection. We see that g is of type i) or ii) (and either way $b = 0$); and that f is of type iii) and therefore is transverse to the analytic stratum of g. We showed in the first part of the proof that $\frac{\partial h'}{\partial V} \notin T\mathcal{A}_e h'$ and therefore in Theorem 4.28 we can take f to be the unfolding of $h_{s,t,r,k_1,\ldots,k_s}$ required to make h $\mathcal{A}$-equivalent to $h_{s,t+1,r,k_1,\ldots,k_s}$.

We now treat the case where h does contain a non-immersive component. If h has only one component then by Theorem 6.10 it is $\mathcal{A}$-equivalent to some $h_{0,1,0,k_1}$. Now for the inductive step we suppose that h has at least two components. If any of the components of h are immersions then we can mimic the inductive step of the last paragraph. Therefore we treat the case where none of them are. Let g be the last component of h and let f be made up of all the other components. As before we may assume inductively that if f is transverse to the analytic stratum of g then the pullback of f by this analytic stratum is nice (and similarly with f and g swapped). Therefore we may apply Theorem 4.28. f and g are both of type iii) and by the inductive hypothesis both are of the standard form. We deduce that h is itself of the standard form.

Finally we notice that the $h_{s,t,r,k_1,\ldots,k_s}$ are all distinct up to $\mathcal{A}$-equivalence because the components of different h's are not $\mathcal{A}$-equivalent.

$\bowtie$

Each of the complex analytic multigerms ($h_{s,t,r,k_1,\ldots,k_s}$ or h for short) that we have described has real coefficients, and therefore the formula for each multigerm is also the formula for a real smooth multigerm. In these circumstances we say that the real $\mathcal{A}$-equivalence class of the real multigerm is a *real form* of the complex $\mathcal{A}$-equivalence class of the complex multigerm. The $\mathcal{A}_e$-codimension of a real form of a complex multigerm is the same as that of the complex multigerm itself, so the formulae we gave for the various h's are also formulae for real smooth $\mathcal{A}_e$-codimension one multigerms. Also, for the same reason, every smooth $\mathcal{A}_e$-codimension one multigerm with components of corank at most one, is a real form of one of our complex hs. However each complex h may have more than one real form.

Suppose that $f: \mathbf{R}^n, S \to \mathbf{R}^p, \{0\}$ has $\mathcal{A}_e$-codimension one and that $F: \mathrm{id}_{\mathbf{R}} \times f_\lambda$ is an $\mathcal{A}_e$-versal unfolding of f. If g is got from f by augmenting it k times, then up to $\mathcal{A}$-equivalence, g has the form:

$$((\lambda_1, \ldots, \lambda_r), x) \mapsto ((\lambda_1, \ldots, \lambda_r), f_\sigma(x))$$

where $\sigma = \sum_{i=1}^{r'} \lambda_i^2 - \sum_{r'+1}^{r} \lambda_i^2$ for some r' such that $0 \leq r' \leq r$.

Suppose that n is a non-negative integer and that s,t,r and $k_1 \leq \ldots, \leq k_s$ are non-negative integers such that

$$n + 1 = s + t - 1 + \sum_{i=1}^{s} (k_i + 2) + r$$

and such that if $s = 0$ then $t \geq 2$. Suppose also that $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_s)$ is a sequence of s terms, each of which is 1 or -1 and that $r' \in \mathbb{N} \cup \{0\}$ satisfies $0 \leq r' \leq r$. Define the multigerm $h^{\Xi, r'}_{s,t,r,k_1,\ldots,k_s}$ (h for short) from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ to be the one with components $f^{(1)}, \ldots, f^{(s)}, g^{(1)}, \ldots, g^{(t)}$ as described below.

Let the coordinate functions of $\mathbb{R}^{n+1}$ be

$$\begin{gathered}
\Lambda_1, \Lambda_2, \ldots, \Lambda_{s+t-1}, \\
X_1^{(1)}, X_2^{(1)}, \ldots, X_{2k_1}^{(1)}, Y^{(1)}, Z^{(1)}, \\
X_1^{(2)}, X_2^{(2)}, \ldots, X_{2k_2}^{(2)}, Y^{(2)}, Z^{(2)}, \\
\vdots \\
X_1^{(s)}, X_2^{(s)}, \ldots, X_{2k_s}^{(s)}, Y^{(s)}, Z^{(s)}, \\
V_1, V_2, \ldots, V_r.
\end{gathered}$$

Let the coordinate functions of the source $\mathbb{R}^n$ of each component, $f^{(i)}$ or $g^{(j)}$ of h be

$$\begin{gathered}
\lambda_1, \lambda_2, \ldots, \lambda_{s+t-1}, \\
x_1^{(1)}, x_2^{(1)}, \ldots, x_{2k_1}^{(1)}, y^{(1)}, z^{(1)}, \\
x_1^{(2)}, x_2^{(2)}, \ldots, x_{2k_2}^{(2)}, y^{(2)}, z^{(2)}, \\
\vdots \\
x_1^{(s)}, x_2^{(s)}, \ldots, x_{2k_s}^{(s)}, y^{(s)}, z^{(s)}, \\
v_1, v_2, \ldots, v_r
\end{gathered}$$

but with the following modifications: for $f^{(i)}$ remove the two functions $y^{(i)}$ and $z^{(i)}$ and substitute the single function $w^{(i)}$, and for $g^{(j)}$ remove λ_{s+j-1} (unless $s + j - 1 = 0$, in which case remove λ_1).

Let $f^{(1)}$ be that germ $\mathbf{R}^n, \{0\} \to \mathbf{R}^{n+1}, \{0\}$ which, when composed with a coordinate function in the target, gives the corresponding coordinate function in the source (i.e. capital letters correspond to their small versions) except that

$$Y^{(1)} \circ f^{(1)} = w^{(1)k_1+2} + \sum_{j=1}^{k_1} x^{(1)}_{2j-1} w^{(1)j}$$

and

$$Z^{(1)} \circ f^{(1)} = w^{(1)k_1+3} + \sum_{j=1}^{k_1} x^{(1)}_{2j} w^{(1)j} - U\Xi_1 w^{(1)k_1+1}$$

where the meaning of U is explained below.

Let $f^{(i)}$ (for i greater than one) be be that germ $\mathbf{R}^n, \{0\} \to \mathbf{R}^{n+1}, \{0\}$ which, when composed with a coordinate function in the target, gives the corresponding coordinate function in the source except that

$$\Lambda_{i-1} \circ f^{(i)} = \lambda_{i-1} + U,$$

$$Y^{(i)} \circ f^{(i)} = w^{(i)k_i+2} + \sum_{j=1}^{k_i} x^{(i)}_{2j-1} w^{(i)j}$$

and

$$Z^{(i)} \circ f^{(i)} = w^{(i)k_i+3} + \sum_{j=1}^{k_i} x^{(i)}_{2j} w^{(i)j} + \lambda_{i-1}\Xi_i w^{(i)k_i+1}$$

where the meaning of U is explained below.

Let $g^{(j)}$ be that germ $\mathbf{R}^n, \{0\} \to \mathbf{R}^{n+1}, \{0\}$ which when composed with a coordinate function in the target gives the corresponding coordinate function in the source except that

$$\Lambda_{s+j-1} \circ g^{(j)} = U$$

unless $s + j - 1 = 0$, in which case the exception is that

$$\Lambda_1 \circ g^{(j)} = 0.$$

Finally we define U. For the i^{th} component, U stands for λ_i, unless it is the last component, in which case U stands for $\sum_{j=1}^{r'} v_j^2 - \sum_{j=r'+1}^{r} v_j^2$.

Theorem 7.2

Suppose that n is a non-negative integer and that s, t, r and $k_1 \ldots, k_s$ are non-negative integers satisfying

$$n + 1 = s + t - 1 + \sum_{i=1}^{s}(k_i + 2) + r$$
$$s = 0 \Rightarrow t \geq 2$$
$$k_1 \leq k_2 \leq \cdots \leq k_s.$$

Suppose also that Ξ is a sequence of elements of $\{1,-1\}$ as described above, and that $r' \in \mathbb{N} \cup \{0\}$ is not greater than r, then the map $h^{\Xi,r'}_{s,t,r,k_1,\ldots,k_s}$ has $\mathcal{A}_e$-codimension one. Conversely if h is a multigerm from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ of $\mathcal{A}_e$-codimension one, each of whose components has corank at most one, then there are unique integers s, t, r and $k_1,\ldots,k_s$, satisfying the above three conditions, a sequence Ξ and a number r' as described above (we do not claim uniqueness for Ξ or r') such that h is $\mathcal{A}$-equivalent to $h^{\Xi,r'}_{s,t,r,k_1,\ldots,k_s}$.

Proof

The proof of this theorem is analogous to that of Theorem 7.1 but we use the results of chapter 5 rather than their complex counterparts from earlier chapters.

$\bowtie$

We have not yet got a classification theorem in the real case because there is some redundancy in our normal forms, for example $h^{(+),()}_{0,2,0}$ is $\mathcal{A}$-equivalent to $h^{(-),()}_{0,2,0}$. Therefore we will now explore the question of which of these normal forms are $\mathcal{A}$-equivalent.

Again suppose that $f\colon \mathbb{R}^n, S \to \mathbb{R}^p, \{0\}$ has $\mathcal{A}_e$-codimension one and that $F = \mathrm{id}_{\mathbb{R}} \times f_\lambda$ is an $\mathcal{A}_e$-versal unfolding of f. We shall say that f is *symmetrical* if there is a diffeomorphism $\alpha\colon \mathbb{R}, \{0\} \to \mathbb{R}, \{0\}$ and diffeomorphisms $\Phi\colon \mathbb{R}\times\mathbb{R}^n, \{0\} \times S \to \mathbb{R}\times\mathbb{R}^n, \{0\}\times S$ and $\Psi\colon \mathbb{R}\times\mathbb{R}^p, \{0\}\times\{0\} \to \mathbb{R}\times\mathbb{R}^p, \{0\}\times\{0\}$ such that Φ is of the form $(\lambda, x) \mapsto (\alpha(\lambda), \phi_\lambda(x))$ and Ψ is of the form $(\lambda, X) \mapsto (\alpha(\lambda), \psi_\lambda(X))$ (for some one parameter families ϕ_λ and ψ_λ of diffeomorphisms with $\phi_0 = \mathrm{id}_{\mathbb{R}^n}$ and $\psi_0 = \mathrm{id}_{\mathbb{R}^p}$), such that this diagram commutes

$$\begin{array}{ccc}
\mathbb{R}\times\mathbb{R}^n, \{0\}\times S & \xrightarrow{F} & R\times\mathbb{R}^p, \{0\}\times\{0\} \\
\downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Psi} \\
\mathbb{R}\times\mathbb{R}^n, \{0\}\times S & \xrightarrow{F} & R\times\mathbb{R}^p, \{0\}\times\{0\}
\end{array}$$

and finally such that α is order reversing. Notice that we do not require that the ϕ_λ be the identity on S.

As we remarked above, if g is got from f by augmenting it k times, then up to $\mathcal{A}$-equivalence, g has the form:

$$((\lambda_1,\ldots,\lambda_r), x) \mapsto ((\lambda_1,\ldots,\lambda_r), f_\sigma(x))$$

where $\sigma = \sum_{i=1}^{r'} \lambda_i^2 - \sum_{r'+1}^{r} \lambda_i^2$ for some r' such that $0 \le r' \le r$.

Lemma 7.3

If f is symmetrical then we can assume that $r' \ge r/2$.

Proof

If $r' \geq r/2$ then we are done so we may assume that $r' < r/2$. Let λ denote $(\lambda_1, \ldots, \lambda_r)$. Since σ has a non-degenerate critical point at 0, so does $\alpha \circ \sigma$. Therefore by 2.1 of [12] there is a change of coordinates ξ of R such that $\alpha \circ \sigma \circ \xi$ is $\sum_{i=1}^{r''} \lambda_i^2 - \sum_{r''+1}^{r} \lambda_i^2$ for some r'' such that $0 \leq r' \leq r$ (we shall write σ'' for this expression). Since α is order reversing, the index of $\alpha \circ \sigma$ is the opposite of the index of α so $r'' = r - r' \geq r$. We have a commutative diagram.

$$\begin{array}{ccc}
\mathbf{R}^k \times \mathbf{R}^n, \{0\} \times S & \xrightarrow{(\lambda,x)\mapsto(\lambda,f_\sigma(x))} & \mathbf{R}^k \times \mathbf{R}^p, \{0\} \times \{0\} \\
\Big\downarrow {\scriptstyle (\lambda,x)\mapsto(\lambda,\phi_\sigma(x))} & & \Big\downarrow {\scriptstyle (\lambda,X)\mapsto(\lambda,\psi_\sigma(X))} \\
\mathbf{R}^k \times \mathbf{R}^n, \{0\} \times S & \xrightarrow{(\lambda,x)\mapsto(\lambda,f_{\alpha(\sigma)}(x))} & \mathbf{R}^k \times \mathbf{R}^p, \{0\} \times \{0\} \\
\Big\downarrow {\scriptstyle (\lambda,x)\mapsto(\xi^{-1}(\lambda),x)} & & \Big\downarrow {\scriptstyle (\lambda,X)\mapsto(\xi^{-1}(\lambda),X)} \\
\mathbf{R}^k \times \mathbf{R}^n, \{0\} \times S & \xrightarrow{(\lambda,x)\mapsto(\lambda,f_{\sigma'}(x))} & \mathbf{R}^k \times \mathbf{R}^p, \{0\} \times \{0\}
\end{array}$$

Each vertical arrow is a diffeomorphism by the inverse function theorem. Therefore the outside rectangle shows that $(\lambda, x) \mapsto (\lambda, f_\sigma(x))$ is $\mathcal{A}$-equivalent to $(\lambda, x) \mapsto (\lambda, f_{\sigma'}(x))$.

$\bowtie$

We will now find out which of the primitive germs h that we have classified are symmetrical. All of our germs are either monogerms or can be split into two parts f and g each of which is of type i) or iii) in the real version of Theorem 4.28.

If both f and g are of type i) then the germ h has two components each of which is the inclusion of the point $\{0\}$ in R. This germ is obviously symmetrical.

If f is of type iii) and g is of type i) then we take a miniversal unfolding $\tilde{F}$ of the pullback $\tilde{f}$ of f by the analytic stratum of g. By Theorem 4.28 we may assume that f and g have the following form.

$$f: \mathbf{R}^{n-1} \times \mathbf{R}, S, \{0\} \to \mathbf{R}^n \times \mathbf{R}, \{0\} \times \{0\}$$
$$(x, \mu) \mapsto (\tilde{f}_\mu(x), \mu)$$
$$g: \mathbf{R}^n, \{0\} \to \mathbf{R}^n \times \mathbf{R}, \{0\} \times \{0\}$$
$$(x) \mapsto (x, 0)$$

By Lemma 4.13, a miniversal unfolding H of h is given by the multigerm made up of F and G where

$$F\colon \mathbf{R}\times\mathbf{R}^{n-1}\times\mathbf{R},\{0\}\times S\times\{0\}\to\mathbf{R}\times\mathbf{R}^n\times\mathbf{R},\{0\}\times\{0\}\times\{0\}$$
$$(\lambda,x,\mu)\mapsto(\lambda,\tilde{f}_\mu(x),\mu)$$
$$G\colon \mathbf{R}\times\mathbf{R}^{n},\{0\}\times\{0\}\to\mathbf{R}\times\mathbf{R}^n\times\mathbf{R},\{0\}\times\{0\}\times\{0\}$$
$$(\lambda,x)\mapsto(\lambda,x,\lambda).$$

If $\tilde{f}$ is symmetrical there are maps α, Φ_λ and Ψ_λ as explained above to make the following diagram commutative.

$$\begin{array}{ccc}
\mathbf{R}\times\mathbf{R}^{n-1},\{0\}\times S & \xrightarrow{(\mu,x)\mapsto(\mu,\tilde{f}_\mu(x))} & \mathbf{R}\times\mathbf{R}^{n},\{0\}\times\{0\} \\
\Big\downarrow{\scriptstyle (\mu,x)\mapsto(\alpha(\mu),\phi_\mu(x))} & & \Big\downarrow{\scriptstyle (\mu,X)\mapsto(\alpha(\mu),\psi_\mu(X))} \\
\mathbf{R}\times\mathbf{R}^{n-1},\{0\}\times S & \xrightarrow{(\mu,x)\mapsto(\mu,\tilde{f}_\mu(x))} & \mathbf{R}\times\mathbf{R}^{n},\{0\}\times\{0\}
\end{array}$$

We then have the following two diagrams which show that h is symmetrical.

$$\begin{array}{ccc}
\mathbf{R}\times\mathbf{R}^{n-1}\times\mathbf{R},\{0\}\times S\times\{0\} & \xrightarrow{F} & \mathbf{R}\times\mathbf{R}^{n}\times\mathbf{R},\{0\}\times\{0\}\times\{0\} \\
\Big\downarrow{\scriptstyle (\lambda,x,\mu)\mapsto(\alpha(\lambda),\phi_\mu(x),\alpha(\mu))} & & \Big\downarrow{\scriptstyle (\lambda,X,\mu)\mapsto(\alpha(\lambda),\psi_\mu(X),\alpha(\mu))} \\
\mathbf{R}\times\mathbf{R}^{n-1}\times\mathbf{R},\{0\}\times S\times\{0\} & \xrightarrow{F} & \mathbf{R}\times\mathbf{R}^{n}\times\mathbf{R},\{0\}\times\{0\}\times\{0\}
\end{array}$$

$$\begin{array}{ccc}
\mathbf{R}\times\mathbf{R}^{n},\{0\}\times\{0\} & \xrightarrow{G} & \mathbf{R}\times\mathbf{R}^{n}\times\mathbf{R},\{0\}\times\{0\}\times\{0\} \\
\Big\downarrow{\scriptstyle (\lambda,x)\mapsto(\alpha(\lambda),\psi_\lambda(x))} & & \Big\downarrow{\scriptstyle (\lambda,x,\mu)\mapsto(\alpha(\lambda),\psi_\mu(x),\alpha(\mu))} \\
\mathbf{R}\times\mathbf{R}^{n},\{0\}\times\{0\} & \xrightarrow{G} & \mathbf{R}\times\mathbf{R}^{n}\times\mathbf{R},\{0\}\times\{0\}\times\{0\}
\end{array}$$

These comments show that in the second part of Theorem 7.2 we can make an additional requirement. If $s=0$ then we may suppose that $r'\geq r/2$. Unfortunately we still do not have uniqueness: for example $h^{(1,1),0}_{2,0,0,0,0}$ is symmetrical so $h^{(1,1),0}_{2,0,1,0,0}$ is $\mathcal{A}$-equivalent to $h^{(1,1),1}_{2,0,1,0,0}$. The classification in the real case is clearly quite complicated, but we make the following conjecture:

Suppose that n is a non-negative integer and that s, t, r and $k_1\dots,k_s$ are non-negative integers satisfying

$$n+1=s+t-1+\sum_{i=1}^{s}(k_i+2)+r$$
$$s=0\Rightarrow t\geq 2$$
$$k_1\leq k_2\leq\cdots\leq k_s$$

For $i \in \mathbb{N} \cup \{0\}$, let a_i be the number of times i occurs in the sequence $k_1, k_2, \ldots, k_s$ and let $b_i \in \mathbb{N} \cup \{0\}$ satisfy $b_i \leq a_i$. Now define Ξ to be the sequence made up of b_1 ones followed by $a_1 - b_1$ minus ones followed by b_2 ones followed by $a_2 - b_2$ minus ones etcetera. Finally let r' satisfy $0 \leq r' \leq r$, then the multigerm $h^{\Xi, r'}_{s,t,r,k_1,\ldots,k_s}$ has $\mathcal{A}_e$-codimension one (this part is a special case of Theorem 7.2). Conversely if h is a multigerm from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ of $\mathcal{A}_e$-codimension one, each of whose components has corank at most one then there are unique terms s, t, r, $k_1, \ldots, k_s$, a_i, b_i, Ξ and r' satisfying all the above identities such that i) if for all $i \in \mathbb{N} \cup \{0\}$, $b_i = a_i/2$ then $r' \geq r/2$, ii) if there is an $i \in \mathbb{N} \cup \{0\}$ such that $b_i \neq a_i/2$ then for the least such i, $b_i > a_i/2$ and finally iii) h is $\mathcal{A}$-equivalent to $h^{\Xi, r'}_{s,t,r,k_1,\ldots,k_s}$.

§8 Topology

In this chapter we investigate the topological properties of the germs we have classified. Firstly we consider the image of a disentanglement of a map germ which is defined as follows (see chapter one of [10])

Let $f: \mathbf{C}^n, S \to \mathbf{C}^{n+1}, \{0\}$ be a finitely determined map germ and let $F: U \to V$ be a proper representative of a miniversal unfolding

$$F = f_\lambda \times \mathrm{id}_{\mathbf{C}^k}: \mathbf{C}^n \times \mathbf{C}^k, S \times \{0\} \to \mathbf{C}^{n+1} \times \mathbf{C}^k, \{0\} \times \{0\}$$

of f such that $F^{-1}(0) = S$. Then the image $Y = F(U)$ is a closed analytic sub-variety of V. For $\lambda \in \mathbf{C}^k$, Define $U_\lambda := \{x \in \mathbf{C}^n | (x, \lambda) \in U\}$, $V_\lambda := \{X \in \mathbf{C}^{n+1} | (X, \lambda) \in V\}$ and $Y_\lambda := \{y \in \mathbf{C}^{n+1} | (y, \lambda) \in Y\}$. Also define an equivalence relation on Y by declaring (y, λ) equivalent to (y', λ') if and only if the map germs $f_\lambda: U_\lambda, f_\lambda^{-1}(y) \to V_\lambda, y$ and $f_{\lambda'}: U_{\lambda'}, f_{\lambda'}^{-1}(y') \to V_{\lambda'}, y'$ are $\mathcal{A}$-equivalent. Now define a stratification S of Y by taking as strata the connected components of the equivalence classes. This stratification induces a stratification of Y_λ which is Whitney regular.

A *Milnor radius* for Y_0 is an $\epsilon > 0$ such that for all ϵ' with $0 < \epsilon' \leq \epsilon$, Y_0 is stratified transverse to the sphere $S_\epsilon^{2n+1} \subseteq \mathbf{C}^{n+1}$. By the conic structure lemma (3.2. of [1]), $Y_0 \cap B_\epsilon$ is a cone on its boundary $Y_0 \cap S_\epsilon^{2n+1}$. It follows that there is a $\delta > 0$ such that for $\lambda \in B_\delta(0) \subseteq \mathbf{C}^k$, y_λ is stratified transverse to S_ϵ^{2n+1} (we call such a δ a *perturbation limit* for F with respect to $B_\epsilon(0) \subseteq \mathbf{C}^{n+1}$). If $\lambda \in B_\delta(0) \subseteq \mathbf{C}^k$ and f_λ is stable then we call Y_λ the image of a disentanglement of f; up to homotopy equivalence Y_λ depends only on f.

For $\epsilon_1, \ldots, \epsilon_{n+1} > 0$ define the set $P_{\epsilon_1,\ldots,\epsilon_{n+1}}(0)$ to be the polycylinder $\{(x_1, \ldots, x_{n-1}) \in \mathbf{C}^{n+1} : |x_i| < \epsilon_i \quad \forall i\}$. A *pseudo Milnor radius* for Y_0 is an $\epsilon > 0$ such that for all $\epsilon_1, \ldots, \epsilon_{n+1}$ with $0 < \epsilon_i < \epsilon$ $(\forall i)$, Y_0 is stratified transverse to the boundary of the polycylinder $P_{\epsilon_1,\ldots,\epsilon_{n+1}}(0)$. The results of the previous paragraph apply with such a polycylinder replacing $B_\epsilon(0)$ and the image of a disentanglement defined this way is the same.

Lemma 8.1

If A, A', B and B' are contractible open subspaces of a topological space and $A \cap B$ and $A' \cap B'$ are homotopy equivalent topological spaces such that $A \cap B$ has collared neighbourhoods in both A and B, and $A' \cap B'$ has collared neighbourhoods in both A' and B', then $A \cup B$ and $A' \cup B'$ are homotopy equivalent.

Proof

Suppose that $\phi: A \cap B \to A' \cap B'$ and $\psi: A' \cap B' \to A \cap B$ together with $\alpha: \psi \circ \phi \simeq \mathrm{id}_{A \cap B}$ and $\beta: \phi \circ \psi \simeq \mathrm{id}_{A' \cap B'}$ are the hypothesised homotopy equivalence. Since A' is contractible and $A \cap B$ has a collared neighbourhood in A, we can extend ϕ to a map $\phi_A: A \to A'$, similarly we extend ϕ to $\phi_B: B \to B'$. Also we define $\psi_{A'}: A' \to A$ and $\psi_{B'}: B' \to B$ each extending ψ. Since A is contractible we can extend α to a homotopy $\alpha_A: A \times I \to A$ of $\psi_{A'} \circ \phi_A$ with id_A, similarly we extend α to a homotopy $\alpha_B: B \times I \to B$ of $\psi_{B'} \circ \phi_B$ with id_B. Also we define homotopies $\beta_A: A' \times I \to A'$ of $\phi_A \circ \psi_{A'}$ with $\mathrm{id}_{A'}$ and $\beta_B: B' \times I \to B'$ of $\phi_B \circ \psi_{B'}$ with $\mathrm{id}_{B'}$; each extending β.

Now define $\bar{\phi}: A \cup B \to A' \cup B'$ by gluing together ϕ_A and ϕ_B, define $\bar{\psi}: A' \cup B' \to A \cup B$ by gluing together $\psi_{A'}$ and $\psi_{B'}$, define $\bar{\alpha}: (A \cup B) \times I \to A \cup B$ by gluing α_A and α_B, and finally define $\bar{\beta}: (A' \cup B') \times I \to A' \cup B'$ by gluing $\beta_{A'}$ and $\beta_{B'}$. Then $\bar{\alpha}$ and $\bar{\beta}$ are the necessary homotopies to show that $\bar{\phi}$ and $\bar{\psi}$ are a homotopy equivalence between $A \cup B$ and $A' \cup B'$.

$\bowtie$

Proposition 8.2

The image of a disentanglement of the augmentation g of an $\mathcal{A}_e$-codimension one multigerm $f: \mathbf{C}^n, S \to \mathbf{C}^{n+1}, \{0\}$ is homotopy equivalent to the suspension of the image of a disentanglement of the multigerm f.

Proof

Suppose that $f = \mathrm{id}_{\mathbf{C}} \times f_\lambda: \mathbf{C} \times \mathbf{C}^n, \{0\} \times S \to \mathbf{C} \times \mathbf{C}^{n+1}, \{0\} \times \{0\}$ is a miniversal unfolding of f. Take a proper representative $\overline{F} = \mathrm{id}_{\mathbf{C}} \times \overline{f_\lambda}$ of F such that $\overline{F}^{-1}(0) = S$, then $\overline{f} := \overline{f}_0$ is a representative of f, $\overline{\mathrm{A}_F f} := \mathrm{id}_{\mathbf{C}} \times f_{\lambda^2}$ is a representative of $\mathrm{A}_F f$ and $\overline{\mathrm{UA}_F f} := \mathrm{id}_{\mathbf{C}} \times \mathrm{id}_{\mathbf{C}} \times \overline{f_{\lambda^2+\mu}}$ is a proper representative of a miniversal unfolding $\mathrm{UA}_F f$ of $\mathrm{A}_F f$ such that $\overline{\mathrm{UA}_F f}^{-1}(0) = S$

Let $\epsilon > 0$ be a pseudo Milnor radius for both $\overline{f}$ and $\overline{F}$, also let $\delta > 0$ be a perturbation limit with respect to the unfolding $\overline{F}$ of $\overline{f}$ and $P_{\epsilon \ldots \epsilon}(0) \subseteq \mathbf{C}^{n+1}$. Let $\epsilon' > 0$ be a pseudo Milnor radius for $\overline{\mathrm{A}_F f}$ and let $\delta' > 0$ be a perturbation limit with respect to the unfolding $\overline{\mathrm{UA}_F f}$ of $\overline{\mathrm{A}_F f}$ and $P_{\epsilon'', \epsilon, \ldots, \epsilon}(0) \subseteq \mathbf{C}^{n+2}$ (where ϵ'' is defined to be the lesser of ϵ and $\sqrt{\delta/2}$). Let $\mu_0 \in \mathbf{C}$ satisfy $\mu_0 \neq 0$, $|\mu_0| < \delta'$, $|\mu_0| < \delta/2$ and $|\mu_0| < \epsilon''^2$.

For $\mu \in \mathbf{C}$ define $\overline{g_\mu} := \mathrm{id}_{\mathbf{C}} \times \overline{f_{\lambda^2+\mu}}$. Then $\overline{g_0} = \overline{\mathrm{A}_F f}$ and $\overline{G} := \mathrm{id}_{\mathbf{C}} \times \overline{g_\mu} = \overline{\mathrm{UA}_F f}$. For $\mu \in \mathbf{C}$ let $X'_\mu = \overline{g_\mu}^{-1}(P_{\epsilon'', \epsilon, \ldots, \epsilon}(0))$, then since $|\mu_0| < \delta'$, $Y' := \overline{g_{\mu_0}}(X'_{\mu_0})$ is the image of a disentanglement of $g_0 = \mathrm{A}_F f$. For $\lambda \in \mathbf{C}$ let $X_\lambda := \overline{f_\lambda}^{-1}(P_{\epsilon, \ldots, \epsilon}(0))$ and define $Y_\lambda := \overline{f_\lambda}(X_\lambda)$. If we define $\pi: Y' \to \mathbf{C}$ to be projection onto the first coordinate then $\pi(Y') = B_{\epsilon''}(0)$ and the fibre of π over a point $\lambda \in B_{\epsilon''}(0)$ is naturally homeomorphic to $Y_{\lambda^2+\mu_0}$. Since $|\mu_0| < \epsilon''^2$, both square roots of $-\mu_0$ are in $B_{\epsilon''}(0)$; the fibre over these points is Y_0. If $\lambda \in B_{\epsilon''}(0)$ and $\lambda^2 \neq -\mu_0$ then since $\epsilon'' \leq \sqrt{\delta/2}$, $|\lambda^2| < \delta/2$. Now since $|\mu_0| < \delta/2$, $|\lambda^2 + \mu_0| < \delta$ and therefore the fibre $Y_{\lambda^2+\mu_0}$ of π over λ is the image of a stabilization of f and if we set the square roots of $-\mu_0$ to be a and b, then the restriction of π to $\pi^{-1}(B_{\epsilon''}(0) \backslash \{a, b\})$ is a fibre-bundle.

Let A and B be contractible open subsets of $B_{\epsilon''}(0)$ with a contractible (non-empty) intersection such that $a \in A \backslash B$ and $b \in B \backslash A$. Since Y' is a fibre bundle except over $\{a, b\}$, Y' is homotopy equivalent to $\pi^{-1}(A \cup B)$. At a the map $\gamma: \lambda \mapsto \lambda^2 + \mu_0$ is bi-analytic and so if $\delta'' > 0$ is small enough, the map $\bar{\gamma}: \gamma^{-1}(B_{\delta''}(0)) \to B_{\delta''}(0)$, $\lambda \mapsto \lambda^2 + \mu_0$ is bi-analytic and induces a homeomorphism $\Gamma: \pi^{-1}[\gamma^{-1}(B_{\delta''}(0))] \to \overline{F}(\bar{X})$ where $\bar{X} = \overline{F}^{-1}(P_{\delta'',\epsilon,\ldots,\epsilon}(0))$. Since $\pi^{-1}(A\backslash\{a\})$ is a fibre bundle over $A\backslash\{a\}$, $\pi^{-1}(A)$ is homeomorphic to $\pi^{-1}[\gamma^{-1}(B_{\delta''}(0))]$ for a sufficiently small $\delta'' > 0$. Since ϵ is a pseudo Milnor radius for F, if $\delta'' \leq \epsilon$ then $F(\bar{X})$ is a cone and hence contractible. It follows that $\pi^{-1}(A)$ is contractible. Similarly $\pi^{-1}(B)$ is contractible.

Since $A \cap B$ is contractible and $\pi^{-1}(A \cap B)$ is a fibre bundle over $A \cap B$, $\pi^{-1}(A \cap B)$ is homotopy equivalent to the fibre, which is the image of a disentanglement of f. Now we have two contractible spaces, $\pi^{-1}(A)$ and $\pi^{-1}(B)$ whose intersection is homotopy equivalent to the image of a disentanglement of f. We can now deduce the result from Lemma 8.1 because the suspension of the image of a space can be divided into two contractible subspaces whose intersection has collared neighbourhoods and is homotopy equivalent to the original space.

$\bowtie$

We will use the induction method of Theorem 7.1 to get information about the topology of the image of a disentanglement of our multigerm. Therefore we shall consider the topology of the image of disentanglements of the standard forms of Theorem 4.28.

Lemma 8.3

If h is a multigerm in the standard form of Theorem 4.28 for which both f and g have type i) then the image of a disentanglement of h is homotopy equivalent to a 0-sphere.

Proof

The maps f and g are the inclusions of $\mathbb{C}^a$ and $\mathbb{C}^b$ in $\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$, a disentanglement is therefore made up of f and g' where $g': \mathbb{C}^b \to \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$, $x \mapsto (0, x, \lambda)$ for some small $\lambda \neq 0$. The image of this disentanglement is the disjoint union of two affine subspaces and is therefore homotopy equivalent to S^0.

$\bowtie$

Lemma 8.4

If h is a multigerm in the standard form of Theorem 4.28 for which f has type iii) and g has type i) then the image of the disentanglement of h has the homotopy type of the suspension of the image of a disentanglement of $\tilde{f}_0$.

Proof

This is analogous to Proposition 8.2. By Lemma 4.13 there is a disentanglement of h with components f and $g': \mathbb{C}^b \to \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$, $x \mapsto (0, x, \lambda)$ for a small $\lambda \neq 0$. As in Proposition 8.2 we can split the image into two contractible sets: the image of f and the image of g'. The intersection of these sets is the image of $\tilde{f}_\lambda$: a disentanglement of $\tilde{f}_0$.

$\bowtie$

In order to treat the case when both f and g are of type iii) we need to make a topological definition. If X and Y are topological spaces then following, for example, [18] we define $X * Y$, the *join* of X and Y, to be $(X \times Y \times I)/\sim$ where $(x, y, \lambda) \sim (x', y', \lambda')$ if and only if either $\lambda = \lambda' = 0$ and $y = y'$ or $\lambda = \lambda' = 1$ and $x = x'$.

Lemma 8.5

If X_1 is homotopy equivalent to X_2 and Y_1 is homotopy equivalent to Y_2 then $X_1 * Y_1$ is homotopy equivalent to $X_2 * Y_2$.

Proof

Let the homotopy equivalence between X_1 and X_2 be given by the maps $\phi_X: X_1 \to X_2$ and $\psi_X: X_2 \to X_1$ and the homotopies $F_X: \mathrm{id}_{X_2} \to \phi_X \circ \psi_X$ and $G_X: \mathrm{id}_{X_1} \to \psi_X \circ \phi_X$. Also define the corresponding maps with Y in place of X.

Define

$$\Phi = \phi_X \times \phi_Y \times \mathrm{id}_I: X_1 \times Y_1 \times I \to X_2 \times Y_2 \times I$$

and

$$\Psi = \psi_X \times \psi_Y \times \mathrm{id}_I: X_2 \times Y_2 \times I \to X_1 \times Y_1 \times I,$$

these maps induce $\phi: X_1 * Y_1 \to X_2 * Y_2$ and $\psi: X_2 * Y_2 \to X_1 * Y_1$. ϕ and ψ give the required homotopy equivalence because there is a homotopy $F: \mathrm{id}_{X_2 * Y_2} \to \phi \circ \psi$ induced by F_X, F_Y and id_I and similarly a homotopy $G: \mathrm{id}_{X_1 * Y_1} \to \psi \circ \phi$ induced by G_X, G_Y and id_I.

$\bowtie$

Corollary 8.6

If X_1 is homotopy equivalent to X_2 then $S(X_1)$ is homotopy equivalent to $S(X_2)$.

Proof

This follows from Lemma 8.5 because for any space X, $X * S^0$ is homeomorphic to $S(X)$.

$\bowtie$

Lemma 8.7

Suppose that h is a multigerm in the standard form of Theorem 4.28 for which both f and g are of type iii). Let the images of the disentanglements of h, $\tilde{f}_0$ and $\tilde{g}_0$ be X_h, X_f and X_g respectively. Then X_h is homotopy equivalent to the join of X_f and X_g.

Proof

Take representatives $\overline{\tilde{F}}(= \mathrm{id}_{\mathbf{C}} \times \overline{\tilde{f}_\mu})$ of $\tilde{F}$ and $\overline{\tilde{G}}(= \mathrm{id}_{\mathbf{C}} \times \overline{\tilde{g}_\mu})$ of $\tilde{G}$, then by Lemma 4.13 a representative $\overline{H}$ of a miniversal unfolding H of h is made up of

$$\overline{\tilde{F}}: \mathbf{C} \times \mathbf{C}^a \times \mathbf{C}^{n-a-1} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$$

$$(\nu, \lambda, x, \mu) \mapsto (\nu, \lambda, \overline{\tilde{f}_\mu}(x), \mu)$$

and

$$\overline{G}\colon \mathbf{C} \times \mathbf{C}^{n-b-1} \times \mathbf{C}^{b} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}^{a} \times \mathbf{C}^{b} \times \mathbf{C}$$
$$(\nu, x, \lambda, \mu) \mapsto (\nu, \overline{\tilde{g}_\mu}(x), \lambda, \mu + \nu).$$

This unfolding contains the representative $\overline{h}$ of h, where $\overline{h}$ is made up of

$$\overline{f}\colon \mathbf{C}^{a} \times \mathbf{C}^{n-a-1} \times \mathbf{C} \to \mathbf{C}^{a} \times \mathbf{C}^{b} \times \mathbf{C}$$
$$(\lambda, x, \mu) \mapsto (\lambda, \overline{\tilde{f}_\mu}(x), \mu)$$

and

$$\overline{g}\colon \mathbf{C}^{n-b-1} \times \mathbf{C}^{b} \times \mathbf{C} \to \mathbf{C}^{a} \times \mathbf{C}^{b} \times \mathbf{C}$$
$$(x, \lambda, \mu) \mapsto (\overline{\tilde{g}_\mu}(x), \lambda, \mu).$$

Let $\epsilon > 0$ be a pseudo Milnor radius for $\overline{\tilde{f}_0}$, $\overline{\tilde{g}_0}$ and $\overline{h}$, also let $P_f := P_{\epsilon,\ldots,\epsilon}(0) \subseteq \mathbf{C}^b$ and $P_g := P_{\epsilon,\ldots,\epsilon}(0) \subseteq \mathbf{C}^a$. Now let $\delta > 0$ be a perturbation limit for the unfolding $\overline{\tilde{F}}$ of $\overline{\tilde{f}_0}$ with respect to the set P_f and also for the unfolding $\overline{\tilde{G}}$ of $\overline{\tilde{g}_0}$ with respect to the set P_g. Let ϵ' be the lesser of ϵ and $\delta/2$ and also let $P_h := P_g \times P_f \times B_{\epsilon'}(0) \subseteq \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$. Now let $\delta' > 0$ be a perturbation limit for the unfolding $\overline{H}$ of $\overline{h}$ with respect to the set P_h. Let δ'' be the lesser of δ' and $\delta/2$ and suppose that $\nu_0 \in B_{\delta''}(0)\backslash\{0\} \subseteq \mathbf{C}$.

For $\mu \in \mathbf{C}$ define $\overline{\tilde{g}'_\mu} := \overline{\tilde{g}_{\mu-\nu_0}}$ and $\overline{\tilde{G}'} = \mathrm{id}_{\mathbf{C}} \times \overline{\tilde{g}'_\mu}$, then define

$$\overline{g'}\colon \mathbf{C}^{n-b-1} \times \mathbf{C}^{b} \times \mathbf{C} \to \mathbf{C}^{a} \times \mathbf{C}^{b} \times \mathbf{C}$$
$$(x, \lambda, \mu) \mapsto (\overline{\tilde{g}'_\mu}(x), \lambda, \mu)$$

and define $\overline{h'}$ to be the map made up of $\overline{f}$ and $\overline{g'}$. Then the image X_h of a disentanglement of h is given by $\mathrm{im}\overline{h'} \cap P_h$.

$\mathrm{im}\overline{f} \cap P_h$ and $\mathrm{im}\overline{g'} \cap P_h$ are both cones and thus contractible, so we can apply Lemma 8.1 to deduce that X_h has the homotopy type of the suspension of $Y := \mathrm{im}\overline{f} \cap \mathrm{im}\overline{g'} \cap P_h$. Therefore it is sufficient for us to show that Y has the homotopy type of the join of X_f and X_g.

Let $\pi_h\colon \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C} \to \mathbf{C}$ be projection onto the last coordinate, also let $\pi_f\colon \mathbf{C} \times \mathbf{C}^b \to \mathbf{C}$ and $\pi_g\colon \mathbf{C} \times \mathbf{C}^a \to \mathbf{C}$ both be projections onto the first coordinate. Then for $\mu \in B_{\delta''}(0)$,

$$\pi_h^{-1}(\mu) \cap \mathrm{im}\overline{f} \cap P_h = P_g \times \left(\pi_f^{-1}(\mu) \cap \mathrm{im}\overline{\tilde{F}} \cap P_f\right)$$

and

$$\pi_h^{-1}(\mu) \cap \mathrm{im}\overline{g'} \cap P_h = \left(\pi_g^{-1}(\mu) \cap \mathrm{im}\overline{\tilde{G}'} \cap P_g\right) \times P_f$$

so

$$\pi_h^{-1}(\mu) \cap Y = (\mathrm{im}\overline{\tilde{g}'_\mu} \cap P_g) \times (\mathrm{im}\overline{\tilde{f}_\mu} \cap P_f).$$

$\pi_f^{-1}\left(B_{\delta''}(0)\backslash\{0\}\right) \cap \mathrm{im}\overline{\tilde{F}} \cap \left(B_{\delta''}(0) \times P_f\right)$ is a fibre bundle over $B_{\delta''}(0)\backslash\{0\}$ with fibre X_f and $\pi_g^{-1}\left(B_{\delta''}(0)\backslash\{\nu_0\}\right) \cap \mathrm{im}\overline{\tilde{G}'} \cap \left(B_{\delta''}(0) \times P_g\right)$ is a fibre bundle over $B_{\delta''}(0)\backslash\{\nu_0\}$ with fibre X_g so $\pi_h^{-1}\left(B_{\delta''}(0)\backslash\{0, \nu_0\}\right) \cap Y$ is a fibre bundle over $B_{\delta''}(0)\backslash\{0, \nu_0\}$ with fibre $X_g \times X_f$.

Let A and B be contractible open subsets of $B_{\delta''}(0) \subseteq \mathbf{C}$ with a non-empty contractible intersection such that $0 \in A \backslash B$ and $\nu_0 \in B \backslash A$. Then in analogy with the proof of Proposition 8.2, Y is homotopy equivalent to $Y \cap \pi_h^{-1}(A \cup B)$. Since A is contractible there is a homeomorphism $\bar{\alpha}: A \times X_g \to \mathbf{C} \times \mathbf{C}^a$ with image $\mathrm{im}\overline{\tilde{G}'} \cap (A \times Pg)$ of the form $\mathrm{id}_A \times \alpha_\mu$. Similarly there is a homeomorphism $\bar{\beta}: B \times X_f \to \mathbf{C} \times \mathbf{C}^b$ with image $\mathrm{im}\overline{\tilde{F}} \cap (B \times Pf)$ of the form $\mathrm{id}_B \times \beta_\mu$.

We shall construct a homotopy equivalence of $\pi_h^{-1}(A \cap B) \cap Y$ with $X_f \times X_g \times (\frac{1}{3}, \frac{2}{3}) \subseteq X_f * X_g$. Recall $Y \subseteq \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}$ and define

$$\begin{aligned}\phi_\cap: \pi_h'(A \cap B) \cap Y &\to X_f \times X_g \times (\tfrac{1}{3}, \tfrac{2}{3}) \\ (x, y, \mu) &\mapsto \big(\beta_\mu^{-1}(y), \alpha_\mu^{-1}(x), \tfrac{1}{2}\big).\end{aligned}$$

Now let $c \in A \cap B$ and define

$$\begin{aligned}\psi_\cap: X_f \times X_g \times (\tfrac{1}{3}, \tfrac{2}{3}) &\to \pi_h'(A \cap B) \cap Y \\ (x, y, \mu) &\mapsto \big(\alpha_c(y), \beta_c(x), c\big).\end{aligned}$$

Then

$$\begin{aligned}\psi_\cap \circ \phi_\cap: \pi_h^{-1}(A \cap B) \cap Y &\to \pi_h^{-1}(A \cap B) \cap Y \\ (x, y, \mu) &\mapsto \big((\alpha_c \circ \alpha_\mu^{-1})(x), (\beta_c \circ \beta_\mu^{-1})(y), c\big).\end{aligned}$$

Since $A \cap B$ is contractible there is a homotopy $\Gamma: (A \cap B) \times I \to A \cap B$ between $\mathrm{id}_{A \cap B}$ and $\bar{c}$ the constant map with value c. We now define a homotopy $\bar{\Gamma}$ between the identity map of $\pi_h^{-1}(A \cap B) \cap Y$ and $\psi_\cap \circ \phi_\cap$:Define

$$\begin{aligned}\bar{\Gamma}: [\pi_h^{-1}(A \cap B) \cap Y] \times I &\to \pi_h^{-1}(A \cap B) \cap Y \\ \big((x, y, \mu), t\big) &\mapsto \big((\alpha_{\Gamma(\mu,t)} \circ \alpha_m u^{-1})(x), (\beta_{\Gamma(\mu,t)} \circ \beta_m u^{-1})(y), \Gamma(\mu, t)\big).\end{aligned}$$

Notice that

$$\begin{aligned}\phi_\cap \circ \psi_\cap: X_f \times X_g \times (\tfrac{1}{3}, \tfrac{2}{3}) &\to X_f \times X_g \times (\tfrac{1}{3}, \tfrac{2}{3}) \\ (x, y, \mu) &\mapsto (x, y, \tfrac{1}{2}).\end{aligned}$$

Let $\Lambda: (\frac{1}{3}, \frac{2}{3}) \times I \to (\frac{1}{3}, \frac{2}{3})$ be a homotopy between the identity map of $(\frac{1}{3}, \frac{2}{3})$ and the constant map $(\frac{1}{3}, \frac{2}{3}) \to (\frac{1}{3}, \frac{2}{3})$ with value $\frac{1}{2}$. In order to complete our homotopy equivalence we display a homotopy between the identity of $X_f \times X_g \times (\frac{1}{3}, \frac{2}{3})$ and $\phi_\cap \circ \psi_\cap$: Define

$$\begin{aligned}\bar{\Lambda}: X_f \times X_g \times (\tfrac{1}{3}, \tfrac{2}{3}) \times I &\to X_f \times X_g \times (\tfrac{1}{3}, \tfrac{2}{3}) \\ \big((x, y, \mu), t\big) &\mapsto \big(x, y, \Lambda(\mu, t)\big).\end{aligned}$$

We shall now extend the homotopy equivalence constructed in the last paragraph to one between $\pi_h^{-1}(B) \cap Y$ and $\big(X_f \times X_g \times (\frac{1}{3}, 1]\big)/\sim \subseteq X_f * X_g$ (where

$(x,y,\mu) \sim (x',y',\mu')$ if and only if both $\mu = \mu' = 1$ and $x = x'$). Consider the map

$$\begin{aligned} \phi^g_\cap : \mathrm{im}\overline{\tilde{G}} \cap ((A \cap B) \times P_g) &\to X_g \times (\tfrac{1}{3}, \tfrac{2}{3}) \\ (\mu, x) &\mapsto (\alpha_\mu^{-1}(x), \tfrac{1}{2}). \end{aligned}$$

and the topological space $X_g \times (\frac{1}{3}, 1]/\sim'$ where $(x,t) \sim (x',t')$ if and only if $t = t' = 1$. Since $X_g \times (\frac{1}{3}, 1]/\sim'$ is contractible, $\phi^g_\cap$ can be extended to a map $\phi^g_B : \mathrm{im}\overline{\tilde{G}'} \cap (B \times P_g) \to X_g \times (\frac{1}{3}, 1]/\sim'$. Now we can define a map

$$\begin{aligned} \phi_B : \pi_h^{-1}(B) \cap Y &\to (X_f \times X_g \times (\tfrac{1}{3}, 1])/\sim \\ (x,y,\mu) &\mapsto (\beta_\mu^{-1}(y), \phi^g_B(\mu, y)) \end{aligned}$$

because we can identify $(X_f \times X_g \times (\frac{1}{3}, 1])/\sim$ with $X_f \times [X_g \times (\frac{1}{3}, 1]/\sim']$. Notice that ϕ_B extends $\phi_\cap$. In a similar way we can extend the map $\psi_\cap$ to a map $\psi_B : (X_f \times X_g \times (\frac{1}{3}, 1])/\sim \to \pi_h^{-1}(B) \cap Y$. Now in a similar way we can construct the homotopies necessary to show that ϕ_B and ψ_B give the intended homotopy equivalence.

Analogously to the last paragraph, we can extend the homotopy equivalence given by $\phi_\cap$ and $\psi_\cap$ to one given by

$$\phi_A : \pi_h^{-1}(A) \cap Y \to (X_f \times X_g \times [0, \tfrac{2}{3}))/\sim''$$

and

$$\phi_A : (X_f \times X_g \times [0, \tfrac{2}{3}))/\sim'' \to \pi_h^{-1}(A) \cap Y$$

where $(x,y,t) \sim'' (x',y',t')$ if and only if $t = t' = 0$ and $y = y'$. Now we can glue ϕ_A and ϕ_B together to get a map $\phi_\cup : \pi_h^{-1}(A \cup B) \cap Y \to X_f * X_g$ and similarly we can get a map $\psi_\cup : X_f * X_g \to \pi_h^{-1}(A \cup Y)$. These two maps give a homotopy equivalence as can be seen by gluing together the appropriate homotopies as in Proposition 8.2.

Now we have shown that X_h is homotopy equivalent to the suspension of Y, that Y is homotopy equivalent to $\pi_h^{-1}(A \cup B)$ and that $\pi_h^{-1}(A \cup B)$ is homotopy equivalent to $X_f * X_g$. The result now follows by Corollary 8.6

$\bowtie$

The final bit of information we need is the topology of the image of the disentanglement of a primitive codimension one monogerm. We will use the results of [5] to calculate it. In order to do this we need to know what the multiple point schemes of the monogerms are, and to calculate these we use proposition 2.3. of [8].

Let $m \in \mathbb{N}$, then for $l \in \mathbb{N}$ let p_l be the l^{th} symmetric polynomial in $y_1, y_2, \ldots, y_m$, that is

$$p_l := \sum_{1 \le i_1 < i_2 < \cdots < i_l \le m} y_{i_1} y_{i_2} \cdots y_{i_l},$$

also for $q \in \mathbf{N}_0$ let

$$s_q := \sum_{i=1}^{m} y_i^q.$$

We shall use one of Newton's identities (for example see page 140 of [6] Basic Algebra I): If $q \in \mathbf{N}_0$ then

$$s_{m+q} = \sum_{j=1}^{m} (-1)^{j+1} p_j s_{m+q-j}.$$

Proposition 8.8

Suppose that $k \in \mathbf{N}_0$, that $\lambda \in \mathbf{C}$ and that $f_\lambda \colon \mathbf{C}^{2k+1}, \{0\} \to \mathbf{C}^{2k+2}, \{0\}$ sends $(x_1, \ldots, x_{2k}, y)$ to

$$(x_1, \ldots, x_{2k}, \sum_{i=1}^{k} x_{2i-1} y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i} y^i + y^{k+3} + \lambda y^{k+1})$$

then the m^{th} multiple point space of f_λ is as follows: if $m > k+2$ it is empty, if $m < k+2$ it is contractible and if $m = k+2$ it is a k-sphere which vanishes as $k \to 0$.

Proof

The m^{th} multiple point scheme is a sub-scheme of $\mathcal{O}_{2k+m}$ given by the sheaf of ideals generated by the $m-1$ generators $(1 \le i \le m-1)$.

$$h_i^m := \frac{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{i-1} & (\sum_{j=1}^{k} x_{2j-1} y_1^j + y_1^{k+2}) & y_1^{i+1} & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{i-1} & (\sum_{j=1}^{k} x_{2j-1} y_m^j + y_m^{k+2}) & y_m^{i+1} & \cdots & y_m^{m-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{m-1} \end{vmatrix}}$$

together with the $m-1$ generators $(1 \le i \le m-1)$.

$$h_i'^m := \frac{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{i-1} & (\sum_{j=1}^{k} x_{2j} y_1^j + y_1^{k+3} + \lambda y_1^{k+1}) & y_1^{i+1} & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{i-1} & (\sum_{j=1}^{k} x_{2j} y_m^j + y_m^{k+3} + \lambda y_m^{k+3}) & y_m^{i+1} & \cdots & y_m^{m-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{m-1} \end{vmatrix}}.$$

By expanding out the numerators we see that

$$h_i^m = \sum_{j=1}^{k} x_{2j-1} D_j^i + D_{k+2}^i$$

and

$$h_i'^m = \sum_{j=1}^{k} x_{2j} D_j^i + D_{k+3}^i + \lambda D_{k+1}^i$$

where

$$D_r^i := \frac{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{i-1} & y_1^r & y_1^{i+1} & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{i-1} & y_m^r & y_m^{i+1} & \cdots & y_m^{m-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{m-1} \end{vmatrix}}.$$

This term is zero if $0 \leq r \leq m-1$ unless $r = i$ in which case this term is one. If $r \geq m$ then we can rewrite this term as

$$\frac{\begin{vmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_m \\ \vdots & & \vdots \\ y_1^{m-1} & \cdots & y_m^{m-1} \end{vmatrix} \begin{vmatrix} 1 & y_1 & \cdots & y_1^{i-1} & y_1^r & y_1^{i+1} & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{i-1} & y_m^r & y_m^{i+1} & \cdots & y_m^{m-1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_m \\ \vdots & & \vdots \\ y_1^{m-1} & \cdots & y_m^{m-1} \end{vmatrix} \begin{vmatrix} 1 & y_1 & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_m & \cdots & y_m^{m-1} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} s_0 & s_1 & \cdots & s_{i-1} & s_r & s_{i+1} & \cdots & s_{m-1} \\ s_1 & s_2 & \cdots & s_i & s_{r+1} & s_{i+2} & \cdots & s_m \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ s_{m-1} & s_m & \cdots & s_{m+i-2} & s_{m+r-1} & s_{m+i} & \cdots & s_{2m-2} \end{vmatrix}}{\begin{vmatrix} s_0 & s_1 & \cdots & s_{m-1} \\ s_1 & s_2 & \cdots & s_m \\ \vdots & \vdots & & \vdots \\ s_{m-1} & s_m & \cdots & s_{2m-2} \end{vmatrix}}.$$

We expand out the i^{th} column of the numerator using Newton's identity to get: If $q \in \mathbb{N}_0$ then

$$D^i_{m+q} = \sum_{j=1}^{m} (-1)^{j+1} p_j D^i_{m+q-j}.$$

We now have an inductive definition of D^i_r that gives an expression for D^i_r as a polynomial in the p_j's.

We proceed to calculate the multiple point schemes. If $m > k+2$ then h^m_{k+2} has a constant term—$D^{k+2}_{k+2} = 1$ so the m^{th} multiple point space is empty.

If $2 \leq m \leq k+2$ then h^m_1 contains the term $x_1 D^1_1 = x_1$. Therefore if $\pi\colon \mathbb{C}^{n-1+m} \to \mathbb{C}^{n-2+m}$ is the projection that forgets the x_1 coordinate then π restricted to the m^{th} multiple point space is an isomorphism onto its image. Since x_1 does not occur elsewhere in the defining equations for the m^{th} multiple point scheme, the defining equations for this image are just those of the multiple point scheme with h^m_1 omitted. In a similar way h^m_2 allows us to eliminate the coordinate x_3 and so on until h^m_{m-1} allows us to eliminate x_{2m-3}. Similarly the equations h'^m_i allow us to eliminate $x_2, x_4, \ldots, x_{2m-2}$. The scheme we are left with is $\mathbb{C}^{k-m+2}$ with coordinates $x_{2m-1}, x_{2m}, \ldots, x_{2k}, y_1, y_2, \ldots, y_m$: this is clearly contractible.

We are left with the case in which $m = k+2$. As before $h^m_1, h^m_2, \ldots, h^m_k$ allow us to eliminate $x_1, x_3, \ldots, x_{2k-1}$ and $h'^m_1, h'^m_2, \ldots, h'^m_k$ allow us to eliminate $x_2, x_4, \ldots, x_{2k}$. Our remaining scheme is contained in the ambient space $\mathbb{C}^m$ with coordinates $y_1, \ldots, y_m$; and has defining equations

$$h^{k+2}_{k+1} = D^{k+1}_{k+2} = p_1$$

and

$$h'^{k+2}_{k+1} = D^{k+1}_{k+3} + \lambda D^{k+1}_{k+1} = p_1^2 - p_2 + \lambda.$$

This can be thought of as the intersection of the hyperplane $h^{k+2}_{k+1} (= \sum_{i=1}^m y_i) = 0$ with the $m-1$-sphere $2h'^{k+2}_{k+1} - (h^{k+2}_{k+1})^2 (= p_1^2 - 2p_2 + 2\lambda = \sum_{i=1}^m y_i^2 + 2\lambda) = 0$. If $k \neq 0$ this is a k-sphere because $k = m-2$.

$\bowtie$

Corollary 8.9

Suppose that $k \in \mathbb{N}_0$ and that the map $f_\lambda\colon \mathbb{C}^{2k+1}, \{0\} \to \mathbb{C}^{2k+2}, \{0\}$ sends $(x_1, \ldots, x_{2k}, y)$ to

$$(x_1, \ldots, x_{2k}, \sum_{i=1}^{k} x_{2i-1} y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i} y^i + y^{k+3}).$$

If Y is the image of a disentanglement of f then $H^0(Y, \mathbb{Q}) = \mathbb{Q}$, $H^{2k+1}(Y, \mathbb{Q}) = \mathbb{Q}$ and all the other cohomology groups of Y are trivial.

Proof

We will use the spectral sequence of [5]. Therefore we need to calculate the alternating part of the cohomology of the multiple point schemes. Firstly $H^q_{\mathrm{Alt}}(D^1,\mathbb{Q}) = H^q(D^1,\mathbb{Q})$ because the only representation of S_1 is alternating). So

$$H^0_{\mathrm{Alt}}(D^1,\mathbb{Q}) = \mathbb{Q} \quad \text{and} \quad H^q_{\mathrm{Alt}}(D^1,\mathbb{Q}) = 0 \quad \text{for} \quad q \geq 1.$$

However if $2 \leq m \leq k+1$ then

$$H^q_{\mathrm{Alt}}(D^m,\mathbb{Q}) = 0 \quad \text{for} \quad q \geq 0$$

because the representation of S_m given by the only non-vanishing cohomology of D^m (the zeroth cohomology) is trivial (and therefore has no alternating part). Similarly $H^0_{\mathrm{Alt}}(D^{k+2},\mathbb{Q}) = 0$. Finally we compute $H^k_{\mathrm{Alt}}(D^{k+2},\mathbb{Q})$. If $k \geq 1$ then a transposition in S_{k+2} induces an orientation reversing transformation of the hyperplane $p_1 = 0$ so the co-cycle generating $H^k(D^{k+2},\mathbb{Q})$ is alternating and therefore

$$H^k_{\mathrm{Alt}}(D^{k+2},\mathbb{Q}) = \mathbb{Q}.$$

If $k = 0$ then D^2 is a 0-sphere and S_2 acts as the permutation group of its two points, so again $H^k_{\mathrm{Alt}}(D^{k+2},\mathbb{Q}) = \mathbb{Q}$.

In summary, the only non-zero terms of the first page of the spectral sequence are $H^0_{\mathrm{Alt}}(D^1,\mathbb{Q})$ and $H^k_{\mathrm{Alt}}(D^{k+2},\mathbb{Q})$ and these are both $\mathbb{Q}$. These are the $E_1^{0,0}$ and $E_1^{k+1,k}$ terms.

If $k \geq 1$ then none of the differentials of any page of the spectral sequence map from one of these terms to the other. If $k = 0$ then the differential of the first page maps $E_1^{0,0}$ to $E_1^{1,0}$ but a routine calculation shows that it is the zero map. Either way the spectral sequence collapses at the first page and the result follows.

$\bowtie$

Corollary 8.10

Suppose that $k \in \mathbb{N}_0$ and that the map $f_\lambda\colon \mathbb{C}^{2k+1},\{0\} \to \mathbb{C}^{2k+2},\{0\}$ sends $(x_1,\ldots,x_{2k},y)$ to

$$(x_1,\ldots,x_{2k},\sum_{i=1}^{k} x_{2i-1}y^i + y^{k+2},\sum_{i=1}^{k} x_{2i}y^i + y^{k+3}).$$

If Y is the image of a disentanglement of f then Y has the homotopy type of a $2k+1$-sphere.

Proof

By theorem 1.4 of [14], Y has the homotopy type of a wedge of $2k+1$-spheres. Corollary 8.9 shows that there must be precisely one of them.

$\bowtie$

The number of these spheres is called the *Milnor number* of f.

Lemma 8.11
If $k \in \mathbb{N}_0$ then the suspension of S^k is homeomorphic to S^{k+1}.

Proof
Define

$$\begin{aligned} \phi \colon S(S^k) &\to S^{k+1} \\ ((x_0, \ldots, x_k), t) &\mapsto (\sqrt{1-(2t-1)^2}(x_0, \ldots, x_k), 2t-1) \end{aligned}$$

and

$$\psi \colon S^{k+1} \to S(S^k)$$

$$(x_0, \ldots, x_{k+1}) \mapsto \begin{cases} ((x_0, \ldots, x_k)/\sqrt{1-x_{k+2}^2}, (x_{k+1}+1)/2) & \text{if } |x_{k+1}| \neq 1 \\ ((1,0,\ldots,0), (x_{k+1}+1)/2) & \text{if } |x_{k+1}| = 1 \end{cases}$$

Then ϕ and ψ are both continuous and are inverse to each other.

$\bowtie$

Lemma 8.12 see [18]
If $j, k \in \mathbb{N}_0$ then $S^j * S^k$ is homeomorphic to S^{j+k+1}.

$\bowtie$

Corollary 8.13
If $k \in \mathbb{N}_0$ then $S(S^k)$ is homeomorphic to S^{k+1}.

Proof
This follows from Lemma 8.12 because for any space X, $S^0 * X$ is homeomorphic to $S(X)$.

$\bowtie$

Theorem 8.14
If $n \in \mathbb{N}$ and $h \colon \mathbb{C}^n, S \to \mathbb{C}^{n+1}, \{0\}$ is an $\mathcal{A}_e$-codimension one multigerm each of whose components has corank at most one then the image of a disentanglement of f has the homotopy type of an n-sphere (i.e. h has Milnor number one).

Proof
We follow the inductive framework of the proof of Theorem 7.1. Firstly Proposition 8.2, Corollary 8.6 and Corollary 8.13 allow us to reduce to the case where h is primitive.

We treat the case where all the components of h are immersions by induction on the number t of these immersions. If $t = 2$ then h is the double point and the result is Lemma 8.3. If $t > 2$ then let g be the last component of h and let f be made up of all the others. Then as in Theorem 7.1 we may apply Theorem 4.28. Now by Lemma 8.4, Corollary 8.6 and Corollary 8.13 the result follows.

Now we treat the case where the number s of non-immersive components of h is non-zero by induction on $s+t$, the total number of components. If $s+t=1$ then $s=1$ and $t=0$ and the result we require follows from Theorem 6.10 and Corollary 8.10. If $s+t \geq 2$ then we let g be the last component of h and let f be the other components. Again we can apply Theorem 4.28 but this time g is of type iii). The inductive step follows in this case from Lemma 8.4, Corollary 8.6 and Corollary 8.13 or from Lemma 8.7, Lemma 8.5 and Lemma 8.12 depending on whether f is of type i) or iii).

$\bowtie$

Now we shall consider the real case. Let $f: \mathbf{R}^n, S \to \mathbf{R}^{n+1}, \{0\}$ be a codimension one map germ and let

$$F: \mathrm{id}_{\mathbf{R}} \times f_\lambda: \mathbf{R} \times \mathbf{R}^n, \{0\} \times S \to \mathbf{R} \times \mathbf{R}^{n+1}, \{0\} \times \{0\}$$

be a miniversal unfolding of f. There are two (possibly equivalent) choices for the disentanglement of f, one for a positive λ and one for a negative λ. We shall call these $X^+(F,f)$ and $X^-(F,f)$ respectively. Recall that in the real case, f has two augmentations $\mathrm{A}_F^+ f = \mathrm{id}_{\mathbf{C}} \times f_{\lambda^2}$ and $\mathrm{A}_F^- f = \mathrm{id}_{\mathbf{C}} \times f_{-\lambda^2}$ (see chapter 5). We fix miniversal unfoldings $\mathrm{UA}_F^+ f = \mathrm{id}_{\mathbf{R}} \times \mathrm{id}_{\mathbf{R}} \times f_{\lambda^2+\mu}$ and $\mathrm{UA}_F^- f = \mathrm{id}_{\mathbf{R}} \times \mathrm{id}_{\mathbf{R}} \times f_{-\lambda^2+\mu}$ of $\mathrm{A}_F^+ f$ and $\mathrm{A}_F^- f$ respectively.

Proposition 8.15

With the above notation

$$\begin{array}{ll} X^+(\mathrm{UA}_F^+ f, \mathrm{A}_F^+ f) \simeq X^+(F,f) & X^-(\mathrm{UA}_F^+ f, \mathrm{A}_F^+ f) \simeq S(X^-(F,f)) \\ X^+(\mathrm{UA}_F^- f, \mathrm{A}_F^- f) \simeq S(X^+(F,f)) & X^-(\mathrm{UA}_F^- f, \mathrm{A}_F^- f) \simeq X^-(F,f) \end{array}.$$

Proof

By symmetry it is sufficient to show just the first two homotopy equivalences. The second is analogous to Proposition 8.2 but if we try the same proof with the first we find that $-\mu_0$ has no real square roots and thus that, in the notation of the proof of Proposition 8.2, $\pi^{-1}(B_{\epsilon''}(0))$ is a fibre bundle over $B_{\epsilon''}(0)$ with fibre $X^+(F,f)$. But the total space of a bundle over a contractible space is homotopy equivalent to the fibre.

$\bowtie$

Lemma 8.16

If h is a multigerm in the standard form of the real version of Theorem 4.28 (see chapter 5) for which both f and g have type i), then the image of a disentanglement of h is homotopy equivalent to S^0.

Proof

This is analogous to Lemma 8.3.

$\bowtie$

Lemma 8.17

If h is a multigerm in the standard form of the real version of Theorem 4.28 (see chapter 5) for which f has type iii) and g has type i) then the images of the two disentanglements of h are homotopy equivalent to the suspensions of the two disentanglements of $\tilde{f}_0$.

Proof

This is analogous to Lemma 8.4.

$\bowtie$

We saw at the end of chapter 5 that if h is a multigerm in the standard form of the real version of Theorem 4.28 for which both f and g are of type iii) then given $\tilde{f}$ and $\tilde{g}$, there are two possibilities for h; one given by the formula of Theorem 4.28 and the other by replacing f in this formula by $(\lambda, x, \mu) \mapsto (\lambda, f_{-\mu}(x), \mu)$. Each of these two real h's has two disentanglements and if we work through Lemma 8.7 making the usual changes from complex to real, the result we get is:

Lemma 8.18

Suppose that h is a multigerm in the standard form of the real version of Theorem 4.28 for which both f and g are of type iii), then the four possible disentanglements of h for this $\tilde{f}$ and $\tilde{g}$ are:

$$\begin{array}{lcl} S(X^+(\tilde{F},\tilde{f}) * X^+(\tilde{G},\tilde{g})) & & S(X^-(\tilde{F},\tilde{f}) * X^+(\tilde{G},\tilde{g})) \\ S(X^+(\tilde{F},\tilde{f}) * X^-(\tilde{G},\tilde{g})) & \text{and} & S(X^-(\tilde{F},\tilde{f}) * X^-(\tilde{G},\tilde{g})). \end{array}$$

Proof

This is analogous to Lemma 8.7.

$\bowtie$

We would like to show that each of the complex maps that we classified in Theorem 7.1 has a real form with image homotopy equivalent to an n-sphere. Unfortunately there is some difficulty with this (see remark 2.3 of [15]). In [15] a real stable perturbation is called a *good perturbation* if the rank of the n^{th} homology group of its image is equal to the rank of the n^{th} homology group of the image of the corresponding complex map. We will show that each of the maps of Theorem 7.1 has a real form with a good perturbation. We will need two results from [15].

Lemma 8.19 [2.1 of [15]]

Suppose that $f\colon \mathbf{R}^n, S \to \mathbf{R}^p, \{0\}$ (with $n \geq p-1$) is a real analytic map germ of finite $\mathcal{A}_e$-codimension and that the complexification $f_{\mathbf{C},t}$ of the real perturbation f_t of f defines, on the domain $U \subseteq \mathbf{C}^n$ a stable perturbation of the complexification $f_{\mathbf{C}}$ of f. Let $D(f_t)$ and $D(f_{\mathbf{C},t})$ be the discriminants of the mappings f_t and $f_{\mathbf{C},t}$. Then the inclusion $D(f_t) \subseteq D(f_{\mathbf{C},t}) \cap \mathbf{R}^p$ induces an isomorphism of $p-1$'st homology groups.

$\bowtie$

Lemma 8.20 [5.1 of [15]]

If $f: \mathbf{C}^n, S \to \mathbf{C}^p, \{0\}$ $(p \leq n+1)$ is a codimension one germ with Milnor number one then the real part of the complex discriminant of a disentanglement has the homotopy type of a sphere S^{k-1} for some k with $1 \leq k \leq p$.

$\bowtie$

Lemma 8.21

Suppose that $k \in \mathbf{N}_0$ and that the map $f_\lambda: \mathbf{R}^{2k+1}, \{0\} \to \mathbf{R}^{2k+2}, \{0\}$ sends $(x_1, \ldots, x_{2k}, y)$ to

$$(x_1, \ldots, x_{2k}, \sum_{i=1}^{k} x_{2i-1} y^i + y^{k+2}, \sum_{i=1}^{k} x_{2i} y^i + y^{k+3}).$$

Then there is a disentanglement of f for which the real part of the complex image has the homotopy type of a $2k+1$-sphere.

Proof

If we follow through the proof of Corollary 8.9 swapping $\mathbf{R}$ with $\mathbf{C}$ then everything works except that in the proof of Proposition 8.8, $(p_1^2 - p_2 + \lambda, p_1)$ only defines a k-sphere if $\lambda < 0$ but this is sufficient to show that there is a disentanglement of f satisfying the conditions of Corollary 8.9 (i.e. with the right homology groups). Now the result follows from Lemma 8.19 and Lemma 8.20.

$\bowtie$

Theorem 8.22

If $h: \mathbf{C}^n, S \to \mathbf{C}^{n+1}, \{0\}$ is a codimension one germ each of whose components has corank at most one then h has a good real form.

Proof

Firstly we notice that each of Proposition 8.15, Lemma 8.16, Lemma 8.17 and Lemma 8.18 are all still true if we replace *image* with *real part of the complex image* in their statements; the proofs are analogous. When we want to refer to the modified results we shall append a prime to the name of the result to get Proposition 8.15' for example. Now we shall use the inductive framework of Theorem 7.1 to prove that the real part of the complex image of a disentanglement of some real form of h has the homotopy type of an n-sphere.

Proposition 8.15' allows us to reduce to the case where h is primitive. Now we treat the case where all the components of h are immersions by induction on the number t of these immersions. If $t = 2$ then the result is Lemma 8.16'. If $t > 2$ then let g be the last component of h and let f be made up all the others. Then as in Theorem 7.1 we may apply Theorem 4.28; the inductive step follows by Lemma 8.17'.

Now we treat the case where the number s of non-immersive components of h is non-zero by induction on $s+t$ the total number of components. If $s+t=1$ then $s=1$ and $t=0$ and so the result follows from Lemma 8.21. If $s+t\geq 2$ then we let g be the last component and let f be made up of the other components. Again we can apply Theorem 4.28 but this time g is of type iii). The inductive step in this case follows from Lemma 8.17' or Lemma 8.18' depending on whether f is of type i) or iii).

Now the result follows from Lemma 8.19.

$\bowtie$

We remark that we have shown that the multigerms classified in Theorem 7.1 all satisfy the hypothesis of 5.2 of [15] (the condition that the quasihomogeneous monodromy be real follows from the fact that the unfolding parameter has weight two).

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