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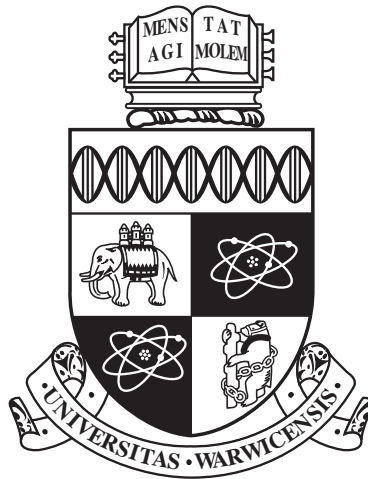
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# Universal Fréchet sets in Banach spaces

by

**Michael J. Doré**

**Thesis**

Submitted to the University of Warwick

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# Declarations

The first chapter is introductory and does not contain any original results. The second and third chapters draw heavily from methods due to Preiss in [23]. They are also closely related to results jointly proved by the author and Maleva in [8] and [10]. They lay the groundwork for chapter 4. The new constructions of universal Fréchet sets in Chapter 4.3-4.5 are joint work with Maleva. See [8], [9] and [10].

# Abstract

We define a universal Fréchet set  $S$  of a Banach space  $Y$  as a subset containing a point of Fréchet differentiability of every Lipschitz function  $g: Y \rightarrow \mathbb{R}$ . We prove a sufficient condition for  $S$  to be a universal Fréchet set and use this to construct new examples of such sets. The strongest such result says that in a non-zero Banach space  $Y$  with separable dual one can find a universal Fréchet set  $S \subseteq Y$  that is closed, bounded and has Hausdorff dimension one.

# Chapter 1

## Overview

### 1.1 Lipschitz functions and differentiability

We shall investigate Lipschitz functions defined on Banach spaces. Throughout, all Banach spaces will be over the field of real numbers. We shall denote the unit sphere of a Banach space  $X$  by  $S(X)$ .

**Definition 1.1.** *If  $X$  and  $Y$  are Banach spaces a mapping  $f: X \rightarrow Y$  is said to be Lipschitz if there exists  $L \geq 0$  such that*

$$\|f(x) - f(y)\|_Y \leq L\|x - y\|_X$$

for all  $x, y \in X$ . The smallest such constant  $L$  is denoted  $\text{Lip}(f)$ .

We are interested in investigating where such mappings are locally linear.

**Definition 1.2.** *If  $X$  and  $Y$  are Banach spaces a mapping  $f: X \rightarrow Y$  is said to be Fréchet differentiable at  $x \in X$  if we can find a bounded linear operator  $f'(x): X \rightarrow Y$  such that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that*

$$\|f(x+h) - f(x) - f'(x)(h)\| \leq \varepsilon\|h\|$$

for any  $h \in X$  with  $\|h\| \leq \delta$ .

We note that there is another common notion of differentiability that applies to mappings between Banach spaces. First another definition.

**Definition 1.3.** *If  $X$  and  $Y$  are Banach spaces and  $f: X \rightarrow Y$  then for  $x \in X$  we say that  $f$  is differentiable in the direction  $e \in X \setminus \{0\}$  if the limit*

$$f'(x, e) := \lim_{t \rightarrow 0} \frac{f(x + te) - f(x)}{t}$$



exists. If  $e \in S(X)$ , the number  $f'(x, e)$  is then called the directional derivative of  $f$  at  $x$  in the direction  $e$ .

We may now define the notion of Gâteaux differentiability.

**Definition 1.4.** *If  $X$  and  $Y$  are Banach spaces a mapping  $f: X \rightarrow Y$  is said to be Gâteaux differentiable at  $x \in X$  if we can find a bounded linear operator  $f'(x): X \rightarrow Y$  such that for each  $e \in S(X)$  the directional derivative  $f'(x, e)$  exists and is given by*

$$f'(x, e) = f'(x)e.$$

In this thesis we shall be exclusively concerned with Fréchet differentiability.

We remark that if  $f$  is Fréchet or Gâteaux differentiable then the operator  $f'(x)$  that appears in the respective definitions is unique and is called, respectively, the Fréchet or Gâteaux derivative of  $f$  at  $x$ .

Further, if a function  $f: X \rightarrow Y$  is Fréchet differentiable at a point  $x \in X$  then  $f$  is also Gâteaux differentiable at  $x$  and the two derivatives coincide. In general, however, Fréchet differentiability is a strictly stronger property than Gâteaux differentiability, even for Lipschitz functions.

On the other hand the two notions do in fact coincide for Lipschitz functions defined on a finite dimensional Banach space; see [5]. In this case we may simply speak of differentiability.

The question of whether a Lipschitz function is somewhere Fréchet differentiable is answered in the simplest possible case, in which  $X = Y = \mathbb{R}$ , by a classical theorem, which says that such a function is differentiable almost everywhere in the sense of Lebesgue measure.

**Theorem 1.5** (Lebesgue). *If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function that is not differentiable at any point in  $A \subseteq \mathbb{R}$  then  $A$  has Lebesgue measure zero.*

*Proof.* See for example [12]. □

What is less well known is that this statement has a converse.

**Theorem 1.6.** *If  $A \subseteq \mathbb{R}$  has Lebesgue measure zero, then there exists a Lipschitz function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is not differentiable at any point in  $A$ .*

*Proof.* We use the fact that a null set may be covered by a countable collection of open intervals whose union has arbitrarily small Lebesgue measure. Let  $I_0 = \mathbb{R}$ . Now for each  $n \geq 1$  find  $I_n$ , a countable union of open intervals, such that  $A \subseteq I_n \subseteq I_{n-1}$  and

$$m(I_n \cap I) \leq \frac{|I|}{2^n}$$

for any maximal interval  $I \subseteq I_{n-1}$ , where  $m(A)$  denotes the Lebesgue measure of the set  $A \subseteq \mathbb{R}$ .

Now if

$$x \in \mathbb{R} \setminus \bigcap_n I_n$$

let

$$g(x) = (-1)^{n_x}$$

where  $n_x$  is the maximal integer  $n \geq 0$  with  $x \in I_n$ , if such exists. We note that  $g$  is defined almost everywhere, is Borel measurable and is bounded by 1, so is locally integrable. Defining  $f: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \int_0^x g$$

we note that  $f$  is Lipschitz with  $\text{Lip}(f) \leq 1$ , as  $|g| \leq 1$ .

Suppose now that  $x \in A$  so that  $x \in I_n$  for every  $n$ . For each  $n$ , pick a maximal interval  $(a_n, b_n)$  with

$$x \in (a_n, b_n) \subseteq I_n.$$

We note that as the Lebesgue measure of  $I_{n+1} \cap (a_n, b_n)$  is less than or equal to  $(b_n - a_n)/2^{n+1}$  we have  $g(x) = (-1)^n$  on  $(a_n, b_n)$ , except on a set of measure at most  $(b_n - a_n)/2^{n+1}$ . Hence we may estimate the difference quotient

$$\begin{aligned} \left| \frac{f(b_n) - f(a_n)}{b_n - a_n} - (-1)^n \right| &\leq \frac{1}{b_n - a_n} \int_{a_n}^{b_n} |g - (-1)^n| \\ &\leq \frac{1}{b_n - a_n} \cdot \frac{b_n - a_n}{2^{n+1}} \cdot 2 \\ &= \frac{1}{2^n}. \end{aligned}$$

It follows that

$$\frac{f(b_n) - f(a_n)}{b_n - a_n}$$

does not converge as  $n \rightarrow \infty$ . As  $a_n, b_n \rightarrow x$  we see that  $f'(x)$  does not exist. That holds for all  $x \in A$ .  $\square$

In fact the exact characterization of the possible sets of non-differentiability of a Lipschitz function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is known. We first recall that a  $G_\delta$  subset of a topological space is a countable intersection of open sets and a  $G_{\delta\sigma}$  subset is a

countable union of  $G_\delta$  sets.

**Theorem 1.7** (Zahorski). *Given a set  $A \subseteq \mathbb{R}$ , there exists a Lipschitz function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$A = \{x \in \mathbb{R} \text{ such that } f'(x) \text{ does not exist}\}$$

*if and only if  $A$  is a  $G_{\delta\sigma}$  subset of  $\mathbb{R}$  with Lebesgue measure zero.*

*Proof.* See [27]. □

## 1.2 Universal sets

We now look at the case in which the domain has dimension greater than one. The following generalization of Lebesgue's theorem is also classical.

**Theorem 1.8** (Rademacher). *If  $m \geq 1$  and  $f: \mathbb{R}^m \rightarrow \mathbb{R}$  is a Lipschitz function that is not differentiable at any point in  $A \subseteq \mathbb{R}^m$  then  $A$  has Lebesgue measure zero.*

*Proof.* See [12]. □

Rather surprisingly, the converse to this statement turns out to be false for  $m \geq 2$ .

**Theorem 1.9** (Preiss). *If  $m \geq 2$  there exists a Lebesgue null set  $S$  such that if  $f: \mathbb{R}^m \rightarrow \mathbb{R}$  is Lipschitz then there exists  $x \in S$  such that  $f$  is differentiable at  $x$ .*

*Proof.* See [23, Corollary 6.5] and Theorem 4.11. □

In fact the proof of Preiss shows that under the assumptions of Theorem 1.9 one may take  $S$  to be any Lebesgue null  $G_\delta$  set that contains every line that passes through two points of  $\mathbb{R}^m$  whose coordinates are rational.

**Definition 1.10.** *If  $Y$  is a Banach space we call a set  $S \subseteq Y$  a universal Fréchet set if for every Lipschitz  $g: Y \rightarrow \mathbb{R}$  there exists  $y \in S$  such that  $g$  is Fréchet differentiable at  $y$ .*

Theorem 1.9 shows that any Euclidean space of dimension at least two has universal Fréchet sets with Lebesgue measure zero. Most optimistically we may ask the following.

**Question 1.11.** *If  $Y$  is a Banach space, is there a complete characterization of universal Fréchet sets in  $Y$ ?*

This question is wide open, for any  $Y$  of dimension at least two. However in this thesis we shall prove some partial results: we derive a sufficient condition for a subset  $S \subseteq Y$  to be a universal Fréchet set and then use that to construct some non-trivial examples of such sets.

The first question to ask is whether a given Banach space  $Y$  has the property that every Lipschitz  $f: Y \rightarrow \mathbb{R}$  has at least *one* point of Fréchet differentiability. We already know the answer is affirmative if  $Y$  is finite dimensional, by Rademacher's theorem. The next theorem will generalize this statement. First another definition.

**Definition 1.12.** *A Banach space  $Y$  is called Asplund if every separable subspace of  $Y$  has a separable dual.*

**Theorem 1.13** (Preiss). *If  $Y$  is an Asplund space,  $g: Y \rightarrow \mathbb{R}$  is a Lipschitz function and  $O \subseteq Y$  is a non-empty open subset of  $Y$  then there exists  $y \in O$  such that  $g$  is Fréchet differentiable at  $y$ .*

*Proof.* See [23]. □

**Remark 1.14.** *We make two quick remarks.*

1. *Unlike the classical Lebesgue and Rademacher theorems, Preiss's result is not an almost-everywhere result. Instead, a point of Fréchet differentiability is constructed explicitly, by an involved iteration argument.*
2. *The proof of Theorem 1.9 was in fact obtained as a byproduct in establishing Theorem 1.13.*

We now mention that the condition that  $Y$  is Asplund in Theorem 1.13 cannot be weakened. First we recall the following definition.

**Definition 1.15.** *If  $(Y, \|\cdot\|)$  is a Banach space then an equivalent norm on  $Y$  is a norm  $\|\cdot\|': Y \rightarrow \mathbb{R}$  on  $Y$  such that there exist  $A, B > 0$  with*

$$A\|y\| \leq \|y\|' \leq B\|y\|$$

*for all  $y \in Y$ .*

We now quote the following.

**Theorem 1.16** (Asplund). *If  $Y$  is not an Asplund space then there is an equivalent norm  $\|\cdot\|'$  on  $Y$  that is non-Fréchet differentiable at every point  $y \in Y$ .*

*Proof.* See [4]. □

As any equivalent norm is a Lipschitz function, any non-Asplund  $Y$  has a nowhere Fréchet-differentiable Lipschitz  $g: Y \rightarrow \mathbb{R}$ .

Therefore, it is sensible to investigate the collection of universal Fréchet sets in a Banach space  $Y$  if and only if  $Y$  is Asplund. If the latter is true, then the space  $Y$  itself is at least such a set; if the latter is false then  $Y$  has no such sets.

The first improvement on Theorem 1.9 we shall mention, also in the case in which  $Y$  is finite dimensional, was obtained by Maleva and the author in [8].

**Theorem 1.17.** *If  $n \geq 2$  then there exists a compact and null subset  $S \subseteq \mathbb{R}^n$  containing a differentiability point of every Lipschitz function.*

See Chapter 4.3 of this thesis.

Subsequently this result was improved further by obtaining a bound on the Hausdorff dimension of such a set.

**Theorem 1.18.** *If  $m \geq 1$  then there exists a compact set  $S \subseteq \mathbb{R}^m$  of Hausdorff dimension one containing a differentiability point of every Lipschitz function  $g: \mathbb{R}^m \rightarrow \mathbb{R}$ .*

See [9] and Chapter 4.4.

Finally, the result was generalized to include the case in which  $Y$  is an infinite dimensional Banach space.

**Theorem 1.19.** *If  $Y$  is a non-zero Banach space with separable dual then there exists a closed and bounded universal Fréchet set in  $Y$  of Hausdorff dimension one.*

See [10] and Chapter 4.5.

We note that the condition that  $Y$  has separable dual is exactly equivalent to demanding that  $Y$  is a separable Asplund space.

### 1.3 Sigma porosity

We now discuss an important necessary condition that  $S \subseteq Y$  must satisfy if  $S$  is a universal Fréchet set.

We start with the definition of porosity *at a point*.

**Definition 1.20.** *If  $A$  is a subset of a metric space  $Y$  then  $A$  is said to be porous at a point  $y \in Y$  if there exists  $\lambda > 0$  such that for all  $\delta > 0$  there exist  $y' \in B_\delta(y)$  and  $r \leq \delta$  such that  $r > \lambda d(y', y)$  and*

$$B_r(y') \cap A = \emptyset.$$

This allows one to define a porous set.

**Definition 1.21.** *A subset  $A$  of a metric space is said to be porous if the set  $A$  is porous at every  $x \in A$ .*

Finally we define a  $\sigma$ -porous set as follows.

**Definition 1.22.** *A subset  $A$  of a metric space is said to be  $\sigma$ -porous if  $A$  is a countable union of porous subsets of  $A$ .*

We remark that the family of  $\sigma$ -porous subsets of  $Y$  is a  $\sigma$ -ideal. More detailed information about  $\sigma$ -porous sets may be found in [28].

We can see the connection between porosity and differentiability immediately by proving a very simple observation.

**Lemma 1.23.** *If  $Y$  is a Banach space and  $A \subseteq Y$  is non-empty then the distance function*

$$f(y) = \|y - A\| := \inf_{a \in A} \|y - a\|$$

*is not Fréchet differentiable at any porosity point of the set  $A$ .*

*Proof.* For any  $y \in A$ , as  $f(y) = 0$  and  $f$  is non-negative function, the only possible value of the derivative  $f'(y)$  is zero. However, by the porosity property, there exists  $\lambda > 0$  such that for all  $\delta > 0$  we can find  $y' \in B_\delta(y)$  with

$$\|y' - A\| \geq \lambda \|y' - y\|.$$

Since then

$$\frac{f(y') - f(y)}{\|y' - y\|} \geq \lambda$$

for  $y'$  arbitrarily close to  $y$ , this quotient cannot converge to 0 as  $y' \rightarrow y$ . Hence the Fréchet derivative  $f'(y)$  does not exist.  $\square$

The following more general fact is slightly less trivial.

**Lemma 1.24** (Kirchheim, Preiss, Tišer). *If  $Y$  is a separable Banach space and  $A$  is a  $\sigma$ -porous subset of  $Y$ , then there exists a Lipschitz function  $f: Y \rightarrow \mathbb{R}$  such that  $f$  is not Fréchet differentiable at any point of  $A$ .*

This result, in the special case in which  $A$  is a countable union of closed porous subsets of  $Y$ , is due to Preiss and Tišer. A proof appears in [5]. Kirchheim subsequently observed that the condition of closedness is really not necessary so that the result holds for any  $\sigma$ -porous  $A$ .

Thus, a necessary condition for  $S$  to be a universal Fréchet set is that  $S$  is *not*  $\sigma$ -porous. To start our investigation we may ask whether there exists a compact, null subset of a finite dimensional Euclidean space that is *not*  $\sigma$ -porous. The answer is affirmative.

**Theorem 1.25.** *For  $n \geq 1$  there exists a compact and Lebesgue null  $\sigma$ -porous subset  $S \subseteq \mathbb{R}^n$  with no porosity points.*

*Proof.* See [29]. □

Hence, it is sensible to hope to find compact, null universal Fréchet sets in Euclidean spaces.

## 1.4 Outline of the approach

The obvious approach to constructing universal Fréchet sets in finite dimensions would be to construct a singular measure  $\mu$  such that every Lipschitz function is differentiable  $\mu$ -almost everywhere. Then any null set  $S$  with  $\mu(S) > 0$  would be an example of a universal Fréchet set. However the following result shows that such a measure cannot exist, at least in the plane.

**Theorem 1.26** (Alberti, Csörnyei, Preiss). *If  $S \subseteq \mathbb{R}^2$  is Lebesgue null then there exists a Lipschitz function  $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that is not differentiable at any point  $y \in S$ .*

*Proof.* See [1]. □

Equivalently, if  $S$  is a null subset of the plane then we can find a pair of real valued Lipschitz functions on the plane that have no common point of differentiability in  $S$ . If, on the other hand, there was a measure  $\mu$  with the property mentioned above then every pair - indeed every countable collection - of Lipschitz functions would have a common point of differentiability on any Lebesgue null subset  $S$  with  $\mu(S) > 0$ .

The following however is still open.

**Conjecture 1.27.** *There exists a singular measure  $\mu$  on  $\mathbb{R}^n$ , for  $n \geq 3$ , such that every Lipschitz  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable  $\mu$ -almost everywhere.*

As a measure approach does not seem to work, at least in an obvious way, our methods for constructing universal sets are much closer to those of Preiss in [23], in which a point of Fréchet differentiability is constructed by an explicit algorithm. In fact our methods draw heavily from the techniques of Preiss in [23].

The method used in [23] can be adapted to show the following.

**Theorem 1.28.** *Suppose  $Y$  has an equivalent norm that is Fréchet differentiable on  $Y \setminus \{0\}$  and  $O$  is a non-empty  $G_\delta$  subset of  $Y$  such that for every  $y \in O$  and  $\varepsilon > 0$  there exists*

$$\delta_0 = \delta_0(y, \varepsilon) > 0$$

*such that for any  $\delta \in (0, \delta_0)$  and any  $u, v \in B_\delta(y)$  there exists a line segment  $[u', v'] \subseteq O$  with  $\|u' - u\| < \varepsilon\delta$  and  $\|v' - v\| < \varepsilon\delta$ . Then  $O$  contains a point of Fréchet differentiability of every Lipschitz  $g: Y \rightarrow \mathbb{R}$ .*

However it is easy to check that any set  $O$  with this property is somewhere dense, so that its closure cannot be Lebesgue null if  $Y$  is finite dimensional.

**Lemma 1.29.** *If  $O$  has the properties of Theorem 1.28 then  $O$  is dense in some non-empty open subset of  $Y$ .*

*Proof.* For each  $n \geq 1$  let

$$U_n = \left\{ y \in O \text{ such that } \delta_0(y, 1/2) \geq \frac{1}{n} \right\}.$$

As  $O$  is  $G_\delta$  and  $\cup_n U_n = O$  there exists  $U_n$  such that  $U_n$  is somewhere dense in  $Y$ , by Baire's theorem. It quickly follows that one can find an open set  $U$  such that for any  $x \in O$  and any  $y \in U$  there exists  $x' \in O$  with

$$\|x' - y\| \leq \frac{\|x - y\|}{2}.$$

It follows that  $O$  is dense in  $U$ . □

Crucially, in [8], the single set  $O$  in Theorem 1.28 was replaced by a family of sets  $(S_i)_{i \in I}$  indexed by a partially ordered set  $I$ , with  $S_i \subseteq S_j$  whenever  $i \leq j$ . The condition, replacing the hypothesis of Theorem 1.28, is that for any  $y \in S_i$  and  $i < j$  one can find sufficiently many line segments in  $S_j$  nearby the point  $y$ .

In this thesis we use a slightly different approach; to prove  $S$  is universal we construct a bundle  $\pi: X \rightarrow S$  where  $X$  is a complete space. We then show that for  $x \in X$  one can find, near  $y := \pi x$ , line segments in  $Y$  that lie in  $\pi(N)$  where  $N$  is a small neighbourhood of  $x$  in  $X$ .

A brief outline of the proof that  $S$  is universal is as follows.

Given a Lipschitz function  $f: Y \rightarrow \mathbb{R}$ , we first find a point  $y \in Y$  and a direction  $e \in S(Y)$ , the unit sphere of  $Y$ , such that the directional derivative  $f'(y, e)$  exists and is in some sense locally maximal.



We then prove  $f$  is differentiable at  $y$  with derivative

$$f'(y, e)e^*$$

where  $e^*$  is the Fréchet derivative of the norm at  $e$ .

A heuristic outline goes as follows. Assuming, on the contrary, that we can find  $\eta > 0$  and a vector  $\lambda$  with small norm such that

$$|f(y + \lambda) - f(y) - f'(y, e)e^*(\lambda)| > \eta\|\lambda\|$$

then we construct an auxiliary point  $y + h$  lying near the line  $y + \mathbb{R}e$  and calculate the ratio

$$\frac{|f(y + \lambda) - f(y + h)|}{\|\lambda - h\|}.$$

We find that this is at least  $f'(y, e) + \varepsilon$  for some  $\varepsilon > 0$ . By using an appropriate mean value theorem it is possible to find a point  $y'$  on the line segment  $[y + h, y + \lambda]$  and a direction  $e' \in S(Y)$  such that  $f'(y', e') \geq f'(y, e) + \varepsilon$ . This contradicts the local maximality of  $f'(y, e)$  and so  $f$  is differentiable at  $y$ . The details are contained in chapters 2 and 3.

## 1.5 Comments on higher dimensional codomains

We have confined our attention so far to functions whose codomains have dimension one. This thesis is exclusively concerned with this case.

Only a few positive results are known about the case where the codomain is a space of dimension at least two.

One may conjecture the following.

**Conjecture 1.30.** *If  $n, m$  are positive integers then there is a null subset  $S \subseteq \mathbb{R}^n$  containing a point of differentiability of every Lipschitz  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  if and only if  $n > m$ .*

We have already mentioned that the case  $m = 1$  is known to be true. The case in which  $n > m = 2$  will be addressed in [11], building heavily on methods due to Lindenstrauss, Preiss and Tišer, who proved the following.

**Theorem 1.31** (Lindenstrauss, Preiss, Tišer). *If  $H$  is a separable Hilbert space then any Lipschitz function  $f: H \rightarrow \mathbb{R}^2$  has a point of Fréchet differentiability.*

See [19].

Equivalently, any pair of real valued Lipschitz functions defined on  $H$  has a common point of Fréchet differentiability.

The only significant partial result towards Conjecture 1.30 for  $n \geq 3$  uses a weaker notion of Fréchet differentiability, which is sometimes quite useful.

**Definition 1.32.** *If  $X, Y$  are Banach spaces and  $f: X \rightarrow Y$ ,  $\varepsilon > 0$  and  $x \in X$  then we say that  $f$  is  $\varepsilon$ -Fréchet differentiable at  $x$  if there exists  $\delta > 0$  and a bounded linear mapping  $f'(x)_\varepsilon: X \rightarrow Y$  such that*

$$\|f(x+h) - f(x) - f'(x)_\varepsilon h\| \leq \varepsilon \|h\|$$

for  $\|h\| \leq \delta$ .

**Remark 1.33.** *It is easy to check that a function  $f: X \rightarrow Y$  is Fréchet differentiable at  $x \in X$  if and only if  $f$  is  $\varepsilon$ -Fréchet differentiable at  $x$  for every  $\varepsilon > 0$ .*

**Theorem 1.34** (de Pauw, Huovinen). *If  $n \geq 3$  then there exists a Lebesgue null set  $S \subseteq \mathbb{R}^n$  such that if  $\varepsilon > 0$  and  $f: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  is Lipschitz, there exists  $x \in S$  such that  $f$  is  $\varepsilon$ -Fréchet differentiable at  $x$ .*

*Proof.* See [21]. □

In [14, 17] the notion of  $\varepsilon$ -Fréchet differentiability is studied in relation to Lipschitz mappings with the emphasis on the infinite dimensional case.

## Chapter 2

# A criterion for Fréchet differentiability

In this chapter we aim to prove that if  $f: Y \rightarrow \mathbb{R}$  is a Lipschitz function and

$$(y, e) \in Y \times S(Y)$$

is such that the directional derivative  $f'(y, e)$  is in some sense ‘almost locally maximal’ around  $y$ , then the function  $f$  is Fréchet differentiable at  $y$  with derivative

$$f'(y) = f'(y, e)e^*,$$

where  $e^*$  is the Fréchet derivative of the norm  $\|\cdot\|$  at  $e \in Y$ .

A similar result was proved by Preiss in [23, Theorem 4.1 and Theorem 6.3]. The method we use is the same, in outline, as that of Preiss. The result we prove requires slightly weaker conditions on the pair  $(y, e)$ ; given  $\nu > 0$  we only require that

$$f'(y', e') < f'(y, e) + \nu$$

holds for a collection of pairs  $(y', e')$  with  $y' \in F$ , where the set  $F$  is allowed to vary with  $\nu$ .

In fact, we prove the result of this chapter in more generality than we need to establish the main results of this thesis.

In all the examples we consider in Chapter 4, the set  $F$  we construct contains line segments, close to  $y$ , that point in a dense set of directions in  $Y$ . In Theorem 2.1 we only demand that the set  $F$  contains a large portion, in the sense of Lebesgue measure, of curves that belong to a collection that *closely approximates* a dense set

of line segments nearby  $y$ .

Further, we also give an explicit bound on the radius of  $\varepsilon$ -Fréchet differentiability of a function satisfying the conditions of Theorem 2.1. See Lemma 2.5.

Finally we note that a related result, proved by Maleva and the author, also appears in [8, Lemma 4.3], for the case in which  $Y$  is a Hilbert space.

To set the notation for this chapter, if  $Y$  is a Banach space we denote the unit sphere of  $Y$  by  $S(Y)$  and, given  $y \in Y$  and  $\delta > 0$ , we use  $B_\delta(y)$  to denote an open ball in  $Y$ , with centre  $y$  and radius  $\delta$ . We use  $m(A)$  to denote the Lebesgue measure of a measurable subset  $A$  of the real line.

The principal aim of this chapter is to establish Theorem 2.1.

**Theorem 2.1.** *Let  $(Y, \|\cdot\|)$  be a real Banach space and let  $f: Y \rightarrow \mathbb{R}$  be a Lipschitz function with  $\text{Lip}(f) \leq L$ , where  $L > 0$ . Suppose that  $(y, e) \in Y \times S(Y)$ , the directional derivative  $f'(y, e)$  exists and is non-negative and the norm  $\|\cdot\|$  is Fréchet differentiable at  $e$  with derivative  $e^* \in Y^*$ .*

*Suppose further that for each  $\nu, \eta, \mu > 0$  there exists a set  $F = F_{\nu, \eta, \mu} \subseteq Y$  and  $s_* = s_*(\nu, \eta, \mu) > 0$  with the following properties.*

1. *If  $s \in (0, s_*)$  and*

$$\|y_1 - y\| < s \text{ and } \|y_2 - y\| < s$$

*there exists an almost-everywhere-differentiable Lipschitz curve  $\gamma: [0, 1] \rightarrow Y$  with*

$$\|\gamma(0) - y_1\| \leq \eta s, \|\gamma(1) - y_2\| \leq \eta s \quad (2.1)$$

$$\|\gamma'(t) - (y_2 - y_1)\| \leq \eta s \text{ for almost all } t \in [0, 1] \quad (2.2)$$

$$m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin F_{\nu, \eta, \mu}\}) \leq \mu. \quad (2.3)$$

2. *If  $(y', e') \in F_{\nu, \eta, \mu} \times S(Y)$  is such that the directional derivative  $f'(y', e')$  exists,  $f'(y', e') \geq f'(y, e)$  and*

$$\begin{aligned} & |(f(y' + te) - f(y')) - (f(y + te) - f(y))| \\ & \leq 10^6 \sqrt{(f'(y', e') - f'(y, e))L} \cdot |t| \end{aligned} \quad (2.4)$$

*for every  $t \in \mathbb{R}$ , then*

$$f'(y', e') < f'(y, e) + \nu. \quad (2.5)$$

*Then  $f$  is Fréchet differentiable at  $y$  and its derivative  $f'(y)$  is given by the formula*

$$f'(y) = f'(y, e)e^*. \quad (2.6)$$

The following Lemma, due to Lindenstrauss and Preiss, can be understood as an improvement of the standard mean value theorem, on the real line. It is a consequence of the fact that the Hardy-Littlewood maximal operator is of weak type  $(1, 1)$ .

**Lemma 2.2.** *Suppose  $a < b$  are real numbers and  $h: [a, b] \rightarrow \mathbb{R}$  a Lipschitz function whose Lipschitz constant is  $\leq 1$ . Assume that*

$$\int_a^b |h'(t)| dt \geq 2|h(b) - h(a)|.$$

*Then there is a measurable set  $A \subseteq (a, b)$  so that*

1.  $m(A) \geq \frac{1}{16} \int_a^b |h'(t)| dt$ ,
2.  $h'(s) \geq \int_a^b |h'(t)| dt / 8(b-a)$  for every  $s \in A$ ,
3.  $|h(t) - h(s)| \leq 8\sqrt{h'(s)} \cdot |t - s|$  for every  $s \in A$  and  $t \in [a, b]$ .

*Proof.* See [15, Lemma 1]. □

**Lemma 2.3.** *Suppose that  $-s < \xi < s$ ,  $0 < \Delta < 1$  and  $\psi: \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function with  $\text{Lip}(\psi) \leq 3$ ,*

$$|\psi(\xi)| \geq \frac{2\Delta^2}{10^4} s \tag{2.7}$$

*and*

$$|\psi(t)| \leq \frac{\Delta^4}{10^5} |t| \text{ for all } s \leq |t| \leq \frac{2s}{\Delta}. \tag{2.8}$$

*Then there exists a measurable set  $B \subseteq (-s, s)$  with*

$$m(B) > 4\Delta^2 s / 10^6$$

*such that if  $\tau \in B$  then  $\psi'(\tau)$  exists,  $\psi'(\tau) \geq 2\Delta^2 / 10^6$  and*

$$|\psi(t + \tau) - \psi(\tau)| \leq 10^5 \sqrt{\psi'(\tau)} \cdot |t| \text{ for } |t| \leq s/\Delta \tag{2.9}$$

$$|\psi(t + \tau) - \psi(\tau) - \psi(t)| \leq 10^5 \sqrt{\psi'(\tau)} \cdot |t| \text{ for } |t| > s/\Delta. \tag{2.10}$$

*Proof.* Note, using (2.7) and (2.8), that

$$\begin{aligned}
\int_{-s}^s |\psi'(t)| &\geq |\psi(\xi) - \psi(-s)| + |\psi(s) - \psi(\xi)| \\
&\geq 2|\psi(\xi)| - |\psi(-s)| - |\psi(s)| \\
&\geq 4\Delta^2 s/10^4 - 2\Delta^4 s/10^5 \\
&\geq 3\Delta^2 s/10^4
\end{aligned}$$

and  $2|\psi(s) - \psi(-s)| \leq 4\Delta^4 s/10^5 < 3\Delta^2 s/10^4$ , by (2.8) once again. Hence, using Lemma 2.2 with  $(a, b) = (-s, s)$  and  $h = \psi/3$ , we can find a measurable set

$$A \subseteq (-s, s)$$

with

$$m(A) \geq \frac{1}{48} \int_{-s}^s |\psi'(t)| \geq \frac{1}{48} \frac{3\Delta^2 s}{10^4} > \frac{6\Delta^2 s}{10^6} \quad (2.11)$$

such that for all  $\tau \in A$  we have

$$\psi'(\tau) \geq \frac{1}{16s} \int_{-s}^s |\psi'(t)| \geq \frac{1}{16s} \frac{3\Delta^2 s}{10^4} > \frac{2\Delta^2}{10^6} \quad (2.12)$$

and, for any  $t \in [-s, s]$ ,

$$|\psi(t) - \psi(\tau)| \leq 14\sqrt{\psi'(\tau)} \cdot |t - \tau|. \quad (2.13)$$

Let  $B$  be the set of  $\tau \in A$  with  $|\tau| < (1 - \Delta^2/10^6)s$ . Note from (2.11) that

$$m(B) > \frac{6\Delta^2 s}{10^6} - \frac{2\Delta^2 s}{10^6} = \frac{4\Delta^2 s}{10^6}.$$

For any  $\tau \in B$  and  $s \leq t \leq 2s/\Delta$  we have, using (2.8) and (2.13),

$$\begin{aligned}
|\psi(t) - \psi(\tau)| &\leq |\psi(t) - \psi(s)| + |\psi(s) - \psi(\tau)| \\
&\leq \Delta^4(t+s)/10^5 + 14\sqrt{\psi'(\tau)} \cdot |s - \tau| \\
&\leq 4\Delta^3 s/10^5 + 14\sqrt{\psi'(\tau)} \cdot |s - \tau| \\
&\leq 40 \cdot \sqrt{\frac{1}{2}10^6 \psi'(\tau)} \cdot |t - \tau| + 14\sqrt{\psi'(\tau)} \cdot |t - \tau| \\
&\leq 10^5 \sqrt{\psi'(\tau)} \cdot |t - \tau|,
\end{aligned}$$

where, in the penultimate line, we used  $\Delta^2 s/10^6 \leq s - \tau \leq t - \tau$ , from the definition of  $B$ , and (2.12). An identical calculation proves this inequality for  $-2s/\Delta \leq t \leq -s$ .

Using (2.13) once more, along with the observation that if  $|t| \leq s/\Delta$  and  $\tau \in B$  then  $|t + \tau| \leq s/\Delta + s \leq 2s/\Delta$ , we may then deduce (2.9).

Finally to obtain (2.10) we let  $|t| > s/\Delta$  and use  $\text{Lip}(\psi) \leq 3$ ,  $|\tau| \leq s$  and (2.8):

$$\begin{aligned}
|\psi(t + \tau) - \psi(\tau) - \psi(t)| &\leq 3|\tau| + |\psi(\tau)| \\
&\leq 3|\tau| + 3|\tau - s| + |\psi(s)| \\
&\leq 3s + 6s + s = 10s \\
&< 10\Delta|t| \\
&\leq 20\Delta|t - \tau| \\
&\leq 20\sqrt{\frac{1}{2}10^6\psi'(\tau)} \cdot |t - \tau| \\
&\leq 10^5\sqrt{\psi'(\tau)} \cdot |t - \tau|,
\end{aligned}$$

by (2.12) again. □

**Lemma 2.4.** *Let  $Y$  be a real Banach space,  $f: Y \rightarrow \mathbb{R}$  be a Lipschitz function with  $\text{Lip}(f) \leq 1$  and let  $\Delta \in (0, 1)$  and  $M \in [0, 1]$ . Suppose  $y \in Y$ ,  $e \in S(Y)$  and  $s > 0$  are such that*

$$|f(y + te) - f(y) - Mt| \leq \frac{\Delta^4}{10^5}|t| \quad (2.14)$$

for  $|t| \leq 2s/\Delta$ . Suppose further that  $\xi \in (-s, s)$  and  $h \in Y$  satisfy

$$|f(y + h) - f(y) - M\xi| \geq 2\Delta^2s/10^4 \quad (2.15)$$

and  $\gamma: [-s, s] \rightarrow Y$  is an almost-everywhere-differentiable Lipschitz curve such that

$$\gamma(\pm s) = y \pm se \text{ and } \gamma(\xi) = y + h, \quad (2.16)$$

$$\|\gamma' - e\| \leq \Delta \text{ almost everywhere,} \quad (2.17)$$

$$m\left(\left\{\tau \in [-s, s] \text{ such that } \|\gamma'(\tau)\| > 1 + \frac{\Delta^2}{10^6}\right\}\right) \leq \frac{2\Delta^2}{10^6}s. \quad (2.18)$$

Then we can find a measurable set  $C \subseteq (-s, s)$  with

$$m(C) > 2\Delta^2s/10^6$$

such that for all  $\tau \in C$  then  $\gamma'(\tau)$  exists and is non-zero and, letting

$$y' = \gamma(\tau) \text{ and } e' = \frac{\gamma'(\tau)}{\|\gamma'(\tau)\|},$$

the directional derivative  $f'(y', e')$  exists with  $f'(y', e') \geq M + \Delta^2/10^6$  and

$$\begin{aligned} & |(f(y' + te) - f(y')) - (f(y + te) - f(y))| \\ & \leq 10^6 \sqrt{f'(y', e') - M} \cdot |t| \end{aligned} \quad (2.19)$$

for all  $t \in \mathbb{R}$ .

*Proof.* Extend  $\gamma$  to  $\mathbb{R}$  by  $\gamma(t) = y + te$  for  $|t| \geq s$ . Note that  $\text{Lip}(\gamma) \leq 2$  and

$$\|\gamma(\tau + t) - (\gamma(\tau) + te)\| \leq \Delta|t| \quad (2.20)$$

for any  $t, \tau \in \mathbb{R}$ , using (2.17), the fact  $\gamma$  is Lipschitz and  $\Delta \in (0, 1)$ .

Define  $\psi: \mathbb{R} \rightarrow Y$  by

$$\psi(t) = f(\gamma(t)) - f(y) - Mt. \quad (2.21)$$

Then  $\text{Lip}(\psi) \leq 3$ , from  $\text{Lip}(\gamma) \leq 2$ ,  $\text{Lip}(f) \leq 1$  and  $M \in [0, 1]$ .

We have  $|\psi(\xi)| \geq 2\Delta^2 s/10^4$  using (2.16) and (2.15).

For  $s \leq |t| \leq 2s/\Delta$  then

$$|\psi(t)| = |f(y + te) - f(y) - Mt| \leq \frac{\Delta^4}{10^5} |t| \quad (2.22)$$

by (2.14).

Hence by Lemma 2.3 we can find a measurable set  $B \in (-s, s)$  with

$$m(B) > 4\Delta^2 s/10^6 \quad (2.23)$$

such that for all  $\tau \in B$ ,  $\psi'(\tau)$  exists with

$$\psi'(\tau) \geq 2\Delta^2/10^6 \quad (2.24)$$

and (2.9) and (2.10) hold.

Let

$$C = \{\tau \in B \text{ such that } \gamma'(\tau) \text{ exists with } 0 < \|\gamma'(\tau)\| \leq 1 + \Delta^2/10^6\}. \quad (2.25)$$

Then  $C$  is measurable with

$$m(C) \geq m(B) - \frac{2\Delta^2}{10^6} s > \frac{2\Delta^2}{10^6} s$$



by (2.17) and (2.18); note that (2.17) implies that  $\gamma' \neq 0$  almost everywhere.

For any  $\tau \in C$  then as  $\gamma'(\tau)$  exists and is not equal to 0, we may define

$$y' = \gamma(\tau), e' = \frac{\gamma'(\tau)}{\|\gamma'(\tau)\|}.$$

We may note, from (2.21) and the fact that  $\psi'(\tau)$  exists and  $f$  is Lipschitz, that  $f'(y', e')$  exists with

$$\begin{aligned} \psi'(\tau) &= f'(\gamma(\tau), \gamma'(\tau)) - M \\ &= f'(y', e')\|\gamma'(\tau)\| - M. \end{aligned} \tag{2.26}$$

Now for  $|t| \leq s/\Delta$ ,

$$\begin{aligned} & |(f(\gamma(t+\tau)) - f(y')) - (f(y+te) - f(y))| \\ & \leq |f(\gamma(t+\tau)) - f(y') - Mt| + |f(y+te) - f(y) - Mt| \\ & = |\psi(t+\tau) - \psi(\tau)| + |f(y+te) - f(y) - Mt| \\ & \leq 10^5 \sqrt{\psi'(\tau)} \cdot |t| + \frac{\Delta^4}{10^5} \cdot |t| \end{aligned} \tag{2.27}$$

using (2.9) and (2.14).

For  $|t| > s/\Delta$  then as  $\gamma(t) = y + te$ ,

$$\begin{aligned} & |(f(\gamma(t+\tau)) - f(y')) - (f(y+te) - f(y))| \\ & = |(f(\gamma(t+\tau)) - f(\gamma(\tau))) - (f(\gamma(t)) - f(y))| \\ & = |\psi(t+\tau) - \psi(t) - \psi(\tau)| \\ & \leq 10^5 \sqrt{\psi'(\tau)} \cdot |t|. \end{aligned} \tag{2.28}$$

by (2.10).

Now using  $\text{Lip}(f) \leq 1$  and (2.20),

$$|f(y' + te) - f(\gamma(t + \tau))| \leq \Delta|t|$$

and so by (2.27), (2.28) and  $\psi'(\tau) \geq 2\Delta^2/10^6$ ,

$$|(f(y' + te) - f(y')) - (f(y+te) - f(y))| \leq 2 \cdot 10^5 \sqrt{\psi'(\tau)} \cdot |t| \tag{2.29}$$

for all  $t \in \mathbb{R}$ .

Recall that as  $\tau \in C \subseteq B$ ,  $\psi'(\tau)$  exists and we have  $\psi'(\tau) \geq 2\Delta^2/10^6$

from (2.24). Hence by (2.25),

$$\|\gamma'(\tau)\| \leq 1 + \frac{\Delta^2}{10^6} \leq 1 + \frac{1}{2}\psi'(\tau). \quad (2.30)$$

Now from (2.26),

$$\psi'(\tau) = f'(y', e')\|\gamma'(\tau)\| - M.$$

As  $\psi'(\tau) \geq 2\Delta^2/10^6 > 0$  and  $M \geq 0$  we deduce  $f'(y', e') \geq 0$ .

Hence from (2.30),

$$\begin{aligned} \psi'(\tau) &\leq f'(y', e')(1 + \psi'(\tau)/2) - M \\ &\leq f'(y', e') - M + \psi'(\tau)/2. \end{aligned}$$

using  $f'(y', e') \leq \text{Lip}(f) \leq 1$  in the final line.

Thus

$$f'(y', e') - M \geq \frac{\psi'(\tau)}{2} \geq \frac{\Delta^2}{10^6}$$

using (2.24) once again.

Finally from (2.29) we may now deduce (2.19).  $\square$

The following lemma gives an estimate on the size of the neighbourhood of  $\varepsilon$ -Fréchet differentiability of  $f$  at  $y \in Y$ .

**Lemma 2.5.** *Let  $\varepsilon \in (0, 1)$  and  $f: Y \rightarrow \mathbb{R}$  be a Lipschitz function with  $\text{Lip}(f) \leq 1$ . Suppose that  $(y, e) \in Y \times S(Y)$  and that  $e^* \in Y^*$ ,  $M \in [0, 1]$ ,  $\Delta \in (0, \varepsilon)$  and  $\delta > 0$  satisfy*

$$\|e + h\| \leq 1 + e^*(h) + \frac{\varepsilon}{10^3}\|h\| \text{ for } \|h\| \leq \frac{2\Delta^2}{10^4\varepsilon} \quad (2.31)$$

$$|f(y + te) - f(y) - Mt| \leq \frac{\Delta^4}{10^5}|t| \text{ for } |t| \leq \frac{10^4\varepsilon\delta}{\Delta^3}. \quad (2.32)$$

Suppose further that if

$$s \in \left(0, \frac{10^4\varepsilon\delta}{\Delta^2}\right) \quad (2.33)$$

and  $h \in Y$  with

$$\|h\| = \frac{2\Delta^2 s}{10^4\varepsilon} \quad (2.34)$$

then

1. for  $\pi = \pm 1$ , there exist almost-everywhere-differentiable Lipschitz curves

$$\gamma_\pi : [0, 1] \rightarrow Y$$

with  $\gamma_\pi(0) = y + h$ ,  $\gamma_\pi(1) = y + \pi se$  and

$$\|\gamma'_\pi - (\pi se - h)\| \leq \frac{\Delta}{8} \|\pi se - h\| \quad (2.35)$$

almost everywhere,

2. there exists a set  $D \subseteq [0, 1]$  with  $m(D) \leq \Delta^2/10^6$  such that if

$$v \in [0, 1] \setminus D$$

and  $\pi = \pm 1$  then  $\gamma'_\pi(v)$  exists,

$$0 < \|\gamma'_\pi(v)\| \leq \|\pi se - h\| \cdot \left(1 + \frac{\Delta^2}{10^7}\right) \quad (2.36)$$

and, defining

$$y' = \gamma_\pi(v) \text{ and } e' = \pi \frac{\gamma'_\pi(v)}{\|\gamma'_\pi(v)\|}, \quad (2.37)$$

if the directional derivative  $f'(y', e')$  exists,  $f'(y', e') \geq M$  and for any  $t \in \mathbb{R}$ ,

$$|(f(y' + te) - f(y')) - (f(y + te) - f(y))| \leq 10^6 \sqrt{f'(y', e') - M} \cdot |t| \quad (2.38)$$

then we have

$$f'(y', e') < M + \frac{\Delta^2}{10^6}. \quad (2.39)$$

Then  $f$  is  $\varepsilon$ -Fréchet differentiable at  $y$  with

$$|f(y + h) - f(y) - Me^*(h)| \leq \varepsilon \|h\| \quad (2.40)$$

for any  $h \in Y$  with

$$\|h\| \leq \delta.$$

*Proof.* We suppose, for a contradiction, we can find  $\|h\| \leq \delta$  such that

$$|f(y + h) - f(y) - Me^*(h)| > \varepsilon \|h\|. \quad (2.41)$$

We write  $h = ru$  where  $u \in S(Y)$  and  $r \geq 0$  and define

$$\xi = re^*(u)$$

and

$$s = \frac{10^4 \varepsilon r}{2\Delta^2}.$$

This equation, with  $r = \|h\|$ , implies (2.33) and (2.34). We let  $\gamma_\pi: [0, 1] \rightarrow Y$  and  $D \subseteq [0, 1]$  be given by (1) and (2) in the statement of the present lemma.

As  $\Delta < \varepsilon < 1$  we have  $|\xi| \leq r \leq s/2$ .

Further from

$$\frac{r}{s} = \frac{2\Delta^2}{10^4 \varepsilon}$$

we deduce by (2.31) that

$$\begin{aligned} \left\| e - \frac{\pi}{s} ru \right\| &\leq 1 - \frac{\pi}{s} re^*(u) + \frac{\varepsilon}{10^3} \cdot \frac{r}{s} \\ &= 1 - \frac{\pi}{s} re^*(u) + \frac{2\Delta^2}{10^7}. \end{aligned} \quad (2.42)$$

We may now note for  $t \in [0, 1] \setminus D$  and  $\pi = \pm 1$ , using (2.36), that

$$\begin{aligned} \|\gamma'_\pi(t)\| &= s \left\| e - \frac{\pi}{s} ru \right\| \cdot \left( 1 + \frac{\Delta^2}{10^7} \right) \\ &\leq s \left( 1 - \frac{\pi}{s} re^*(u) + \frac{2\Delta^2}{10^7} \right) \cdot \left( 1 + \frac{\Delta^2}{10^7} \right) \\ &= \left( s - \pi\xi + \frac{2\Delta^2}{10^7} s \right) \cdot \left( 1 + \frac{\Delta^2}{10^7} \right) \\ &\leq |s - \pi\xi| \cdot \left( 1 + \frac{4\Delta^2}{10^7} \right) \cdot \left( 1 + \frac{\Delta^2}{10^7} \right) \\ &\leq |s - \pi\xi| \cdot \left( 1 + \frac{\Delta^2}{10^6} \right), \end{aligned} \quad (2.43)$$

where, in the penultimate line, we have used  $|\xi| \leq r \leq s/2$  so that  $|s - \pi\xi| \geq s/2$  and, in the final line, we have used  $\Delta \in (0, 1)$ .

Now let  $\rho: [-s, s] \rightarrow [0, 1]$  be a function that is affine on  $[-s, \xi]$  and  $[\xi, s]$  with  $\rho(\xi) = 0$  and  $\rho(-s) = \rho(s) = 1$ .

Write

$$E = \rho^{-1}(D) \subseteq [-s, s] \quad (2.44)$$

and note that as the magnitude of the gradient of  $\rho$  is  $1/(\xi + s)$  on  $[-s, \xi]$  and

$1/(s - \xi)$  on  $[\xi, s]$  we have

$$\begin{aligned} m(E) &\leq (\xi + s) \cdot m(D) + (s - \xi) \cdot m(D) \\ &= 2s \cdot m(D) \leq \frac{2\Delta^2 s}{10^6}. \end{aligned} \quad (2.45)$$

Define  $\gamma: [-s, s] \rightarrow Y$  by  $\gamma(t) = (\gamma_{\pi_t} \circ \rho)(t)$  where

$$\begin{aligned} \pi_t &= -1 \text{ for } t \in [-s, \xi] \\ \pi_t &= +1 \text{ for } t \in [\xi, s]. \end{aligned} \quad (2.46)$$

We shall now verify the conditions of Lemma 2.4 for  $Y, \varepsilon, f, \Delta, M, y, e, s, \xi, h$  and  $\gamma$ .

We already know that  $f: Y \rightarrow \mathbb{R}$  is Lipschitz with  $\text{Lip}(f) \leq 1, \Delta < \varepsilon < 1, M \in [0, 1], (y, e) \in Y \times S(Y)$  and  $s > 0$ .

Since

$$\frac{2s}{\Delta} \leq \frac{10^4 \varepsilon \delta}{\Delta^3},$$

using  $r \leq \delta$ , we deduce (2.14) from (2.32).

As

$$\varepsilon \|h\| = \varepsilon r = \frac{2\Delta^2 s}{10^4},$$

we derive (2.15) from (2.41).

We know that  $\gamma: [-s, s] \rightarrow Y$  is Lipschitz and differentiable almost everywhere as  $\gamma_{\pi}$  and  $\rho$  are Lipschitz and differentiable almost everywhere.

We readily derive (2.16) from  $\gamma_{\pi}(0) = y + h, \gamma_{\pi}(1) = y + \pi se, \rho(\xi) = 0$  and  $\rho(-s) = \rho(s) = 1$ .

From (2.35) and the fact  $\|h\| = r \leq s/2$  we have for almost all  $t \in [0, 1]$  and  $\pi = \pm 1$  that  $\gamma'_{\pi}(t)$  exists and

$$\|\gamma'_{\pi}(t) - (\pi se - h)\| \leq \frac{\Delta}{4} s.$$

Therefore, by (2.46), for almost all  $t \in [-s, s]$ , we can find  $\pi \in \{-1, 1\}$  with

$$\begin{aligned}
\|\gamma'(t) - e\| &= \|(\gamma_\pi \circ \rho)'(t) - e\| \\
&= \|\gamma'_\pi(\rho(t)) \frac{1}{\pi s - \xi} - e\| \\
&= \frac{\|\gamma'_\pi(\rho(t)) - \pi s e + \xi e\|}{|\pi s - \xi|} \\
&\leq \frac{2}{s} \|\gamma'_\pi(\rho(t)) - \pi s e + \xi e\| \\
&\leq \frac{2}{s} (\|\gamma'_\pi(\rho(t)) - (\pi s e - h)\| + \|\xi e - h\|) \\
&\leq \frac{2\Delta}{s} + \frac{2}{s} \|\xi e - h\| \\
&\leq \frac{\Delta}{2} + \frac{4r}{s} \\
&\leq \Delta,
\end{aligned}$$

where, in the penultimate line, we have used  $|\xi| \leq \|h\| = r$  and, in the final line, we used

$$\frac{r}{s} = \frac{2\Delta^2}{10^4\varepsilon} < \frac{\Delta}{8},$$

by  $0 < \Delta < \varepsilon$ .

Thus we have verified (2.17).

Finally, if

$$t \in [-s, s] \setminus (E \cup \{\xi\})$$

then, as  $\rho(t) \in [0, 1] \setminus D$  from (2.44), we have for some  $\pi = \pm 1$ ,

$$\begin{aligned}
|\gamma'(t)| &= |\gamma'_\pi(\rho(t))| \cdot |\rho'(t)| \\
&\leq |\gamma'_\pi(\rho(t))| \cdot \frac{1}{|\pi s - \xi|} \\
&\leq 1 + \frac{\Delta^2}{10^6},
\end{aligned}$$

using (2.43). Hence from (2.45) we may derive (2.18).

We let the measurable set  $C \subseteq (-s, s)$  be given by Lemma 2.4. Note that as

$$m(C) > \frac{2\Delta^2 s}{10^6}$$

and

$$m(E) \leq \frac{2\Delta^2 s}{10^6}$$

we may pick

$$\tau \in C \setminus (E \cup \{\xi\}).$$

Then as  $\tau \in C$  we know  $\gamma'(\tau)$  exists,  $\gamma'(\tau) \neq 0$  and, defining

$$y' = \gamma(\tau) \text{ and } e' = \frac{\gamma'(\tau)}{\|\gamma'(\tau)\|},$$

then  $f'(y', e')$  exists with

$$f'(y', e') \geq M + \frac{\Delta^2}{10^6} \quad (2.47)$$

and such that (2.19) holds.

Letting  $v = \rho(\tau)$  then  $v \in [0, 1] \setminus D$  from (2.44). Further from (2.46),

$$y' = \gamma_\pi(v) \text{ and } e' = \pi \frac{\gamma'_\pi(v)}{\|\gamma'_\pi(v)\|}$$

for some  $\pi \in \{-1, 1\}$ .

Hence as  $f'(y', e')$  exists,  $f'(y', e') \geq M$  and (2.38), we deduce from (2.39) that

$$f'(y', e') < M + \frac{\Delta^2}{10^6}.$$

This contradicts (2.47). We are done.  $\square$

We require one more simple observation before proving Theorem 2.1.

**Lemma 2.6.** *Suppose that  $Y$  is a Banach space,  $\eta, \mu > 0$  and  $\gamma: [0, 1] \rightarrow Y$  is an almost-everywhere-differentiable Lipschitz mapping with*

$$\begin{aligned} \|\gamma(0) - y_1\| &\leq \eta \\ \|\gamma(1) - y_2\| &\leq \eta \\ \|\gamma' - (y_2 - y_1)\| &\leq \eta \text{ almost everywhere.} \end{aligned}$$

*Then there exists an almost-everywhere-differentiable Lipschitz curve  $\bar{\gamma}: [0, 1] \rightarrow Y$  such that  $\bar{\gamma}(0) = y_1$ ,  $\bar{\gamma}(1) = y_2$ ,*

$$\|\bar{\gamma}' - (y_2 - y_1)\| \leq \frac{4\eta}{\mu} \text{ almost everywhere,} \quad (2.48)$$

and

$$m(\{t \in [0, 1] \text{ such that } \bar{\gamma}(t) \neq \gamma(t)\}) \leq \mu. \quad (2.49)$$

*Proof.* We can assume that  $\mu < 1$ ; otherwise we may simply take  $\bar{\gamma}$  to be an affine mapping with  $\bar{\gamma}(0) = y_1$  and  $\bar{\gamma}(1) = y_2$ .

Note first that by the mean value theorem,

$$\|\gamma(t) - (1-t)y_1 - ty_2\| \leq 2\eta \quad (2.50)$$

for any  $t \in [0, 1]$ .

We now let  $\bar{\gamma}(t) = \gamma(t)$  for  $\mu/2 \leq t \leq 1 - \mu/2$  and define  $\bar{\gamma}$  to be affine on  $[0, \mu/2]$  and  $[1 - \mu/2, 1]$  with  $\bar{\gamma}(0) = y_1$  and  $\bar{\gamma}(1) = y_2$ . We only need to check (2.48). Indeed for  $t \in (0, \mu/2)$ ,

$$\begin{aligned} \|\bar{\gamma}'(t) - (y_2 - y_1)\| &= \left\| \frac{\gamma(\mu/2) - y_1}{\mu/2} - (y_2 - y_1) \right\| \\ &= \frac{\|\gamma(\mu/2) - (\mu/2)y_2 - (1 - \mu/2)y_1\|}{\mu/2} \\ &\leq \frac{4\eta}{\mu}, \end{aligned}$$

using (2.50) in the final line. A similar proof shows that the bound also holds for  $t \in (1 - \mu/2, 1)$ . Finally the bound is also true for almost all  $t \in (\mu/2, 1 - \mu/2)$  because then  $\bar{\gamma}'(t) = \gamma'(t)$ .  $\square$

*Proof of Theorem 2.1.* We assume the conditions of Theorem 2.1. We may take  $L = 1$ , without loss of generality.

Let  $M = f'(y, e) \geq 0$  and  $\varepsilon \in (0, 1)$ . We shall verify the conditions of Lemma 2.5.

As the norm  $\|\cdot\|$  is Fréchet differentiable at  $e \in Y$  with derivative  $e^*$  we may pick  $\Delta \in (0, \varepsilon)$  such that (2.31) is satisfied.

Further, as the directional derivative  $f'(y, e)$  exists and equals  $M$ , there exists  $\delta > 0$  such that (2.32) holds. We may take

$$\delta < \frac{\Delta^2}{10^4 \varepsilon} s_*(\nu, \eta, \mu), \quad (2.51)$$

where

$$\nu = \frac{\Delta^2}{10^6}, \quad \eta = \frac{\Delta^4}{10^{15}} \text{ and } \mu = \frac{\Delta^2}{10^7}$$

and  $s_*$  is given by the conditions of Theorem 2.1.

If

$$s \in \left(0, \frac{10^4 \varepsilon \delta}{\Delta^2}\right)$$



and

$$\|h\| = \frac{2\Delta^2 s}{10^4 \varepsilon}$$

then we have, using  $0 < \Delta < \varepsilon < 1$ ,

$$2\|h\| \leq s < s_*(\nu, \eta, \mu). \quad (2.52)$$

Thus, for each  $\pi = \pm 1$ , we can find an almost-everywhere-differentiable Lipschitz curve  $\gamma: [0, 1] \rightarrow Y$  with

$$\begin{aligned} \|\gamma(0) - (y + h)\| &\leq \eta s \text{ and } \|\gamma(1) - (y + \pi s e)\| \leq \eta s \\ \|\gamma'(t) - (\pi s e - h)\| &\leq \eta s \text{ for almost all } t \in [0, 1] \end{aligned}$$

and such that

$$m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin F_{\nu, \eta, \mu}\}) \leq \mu. \quad (2.53)$$

By taking  $\gamma_\pi = \bar{\gamma}$ , as in Lemma 2.6 with

$$\begin{aligned} y_1 &= y + h \text{ and} \\ y_2 &= y + \pi s e, \end{aligned}$$

we have that  $\gamma_\pi$  is an almost-everywhere-differentiable Lipschitz mapping with  $\gamma_\pi(0) = y + h$ ,  $\gamma_\pi(1) = y + \pi s e$ ,

$$\begin{aligned} \|\gamma'_\pi(t) - (\pi s e - h)\| &\leq \frac{4\eta s}{\mu} \\ &\leq \frac{\Delta^2}{10^7} \cdot \frac{s}{2} \\ &\leq \frac{\Delta^2}{10^7} \|\pi s e - h\| \end{aligned} \quad (2.54)$$

for almost all  $t \in [0, 1]$  and

$$m(\{t \in [0, 1] \text{ such that } \gamma_\pi(t) \notin F_{\nu, \eta, \mu}, \text{ for some } \pi = \pm 1\}) \leq 4\mu \leq \frac{\Delta^2}{10^6} \quad (2.55)$$

by (2.53) and (2.49). As  $\Delta < 1$  we have verified (2.35) and hence (1) of Lemma 2.5.

To establish (2) of Lemma 2.5 we let  $D$  be the set of all  $t \in [0, 1]$  such that for some  $\pi = \pm 1$ ,

1.  $\gamma_\pi(t) \notin F_{\nu, \eta, \mu}$  or
2.  $\gamma'_\pi(t)$  does not exist or

3.  $\gamma'_\pi(t) = 0$  or

4.  $\|\gamma'_\pi(t) - (\pi se - h)\| > \frac{1}{10^7} \Delta^2 \|\pi se - h\|$ .

By (2.55) and the fact that  $\gamma'_\pi(t)$  exists and is non-zero for almost all  $t \in [0, 1]$ , with (2.54), we have

$$m(D) \leq \frac{\Delta^2}{10^6}.$$

It is immediate that for  $v \in [0, 1] \setminus D$  we have (2.36).

Finally if we assume (2.37) and (2.38), then as  $y' \in F_{\nu, \eta, \mu}$  we deduce (2.39) from (2.5). Hence (2) is satisfied.

Hence by Lemma 2.5, for  $h \in Y$  with  $\|h\| \leq \delta$ ,

$$|f(y + h) - f(y) - f'(y, e)e^*(h)| \leq \varepsilon \|h\|.$$

As  $\varepsilon \in (0, 1)$  was arbitrary, we are done. □

## Chapter 3

# An optimization algorithm

In this section, given a Banach space  $Y$ , we shall show how to find a point  $y \in Y$  and direction  $e$  in the unit sphere of  $Y$  with almost locally maximal directional derivative

$$f'(y, e),$$

where

$$f: Y \rightarrow \mathbb{R}$$

is a Lipschitz function. Eventually we shall combine this construction with the criterion proved in the previous chapter, to show that the point  $y$  we arrive at is a point of Fréchet differentiability of the function  $f$ ; see Theorem 4.2.

The basic idea behind the proof - to take a sequence of pairs  $(y_n, e_n)$  with the directional derivative  $f'(y_n, e_n)$  almost maximal, subject to some constraints, and to argue that  $(y_n, e_n) \rightarrow (y, e)$  for some  $(y, e)$  with the desired properties - goes back to Preiss; see [23].

The main difference between the algorithm presented in [23] and the one here is that we do not optimize over  $Y$  itself; instead we let  $\pi: X \rightarrow Y$  be a bundle over  $Y$ , where  $X$  is a complete space, and find a point  $x \in X$  and direction  $e \in Y$  such that the directional derivative  $f'(\pi x, e)$  is almost locally maximal, for  $x \in X$ .

A similar result was also proved, by the author and Maleva in [8, Theorem 3.1], for the case in which  $Y$  is a Hilbert space. Instead of working over a bundle, in this paper we optimize over a family  $(S_i)_{i \in I}$  of closed subsets of the Hilbert space, indexed by a partially ordered set  $I$ .

Further, the proof in [8] achieved the convergence of the direction vectors  $e_n$  by adjusting the Lipschitz function  $f$  by a small linear piece at each stage of the iteration. This method appears not to work for general Banach spaces; thus in this

chapter we instead achieve this convergence by making a small change to the norm  $\|\cdot\|$  of  $Y$  at each step of the construction. This latter idea is due to Preiss; see [23].

We recall that a complete space  $X$  is a topological space whose topology can be induced by a complete metric. Given a Banach space  $Y$  we let  $S(Y)$  be the unit sphere of  $Y$  and, given a metric space  $X$ ,  $x \in X$  and  $\delta > 0$ , we denote by  $B_\delta(x)$  and  $\overline{B}_\delta(x)$  the open and closed balls in  $X$ , respectively, with centre  $x$  and radius  $\delta$ .

**Theorem 3.1.** *Let*

- (a)  $X$  be a complete space,
- (b)  $(Y, \|\cdot\|)$  be a Banach space,
- (c)  $\pi: X \rightarrow Y$  be a continuous map,
- (d)  $\Theta: (0, \infty) \rightarrow \mathbb{R}$  be a real-valued function with  $\Theta(t) \rightarrow 0$  as  $t \rightarrow 0^+$  and
- (e)  $\mu > 1$ .

Suppose now that  $g: Y \rightarrow \mathbb{R}$  is a Lipschitz function such that the set

$$D := \{(x, e) \in X \times (Y \setminus \{0\}) \text{ such that } g'(\pi x, e) \text{ exists}\}$$

is non-empty.

Then there exist

- (1) a Lipschitz function  $f: Y \rightarrow \mathbb{R}$ , with  $f - g$  linear and  $\text{Lip}(f) \leq 3\text{Lip}(g)$ ,
- (2) a norm  $\|\cdot\|'$  on  $Y$ , with

$$\|y\| \leq \|y\|' \leq \mu\|y\|$$

for all  $y \in Y$ , and

- (3)  $(x, e) \in D$  with  $\|e\|' = 1$  such that the directional derivative  $f'(\pi x, e) \geq 0$  is almost locally maximal in the following sense. For any  $\varepsilon > 0$  there exists an open neighbourhood  $N_\varepsilon$  of  $x$  in  $X$  such that if  $(x', e') \in D$  with

(i)  $x' \in N_\varepsilon$ ,  $\|e'\|' = 1$  and

(ii) for any  $t \in \mathbb{R}$

$$\begin{aligned} & |(f(\pi x' + te) - f(\pi x')) - (f(\pi x + te) - f(\pi x))| \\ & \leq \Theta(f'(\pi x', e') - f'(\pi x, e))|t|, \end{aligned} \tag{3.1}$$

then we have  $f'(\pi x', e') < f'(\pi x, e) + \varepsilon$ .

Moreover, if the original norm  $\|\cdot\|$  is Fréchet differentiable on  $Y \setminus \{0\}$  then the norm  $\|\cdot\|'$  can be chosen with this property too.

The purpose of this chapter is to prove Theorem 3.1.

We let  $d$  be a complete metric on  $X$ . Without loss of generality we may take  $\text{Lip}(g) = 1/3$ ,  $\mu \leq 2$  and  $\Theta(t) > 0$  for all  $t > 0$ . Further since

$$\begin{aligned} & |(f(\pi x' + te) - f(\pi x')) - (f(\pi x + te) - f(\pi x))| \\ & \leq |f(\pi x' + te) - f(\pi x')| + |f(\pi x + te) - f(\pi x)| \leq 2|t| \end{aligned}$$

for all  $t \in \mathbb{R}$  whenever  $f: X \rightarrow R$  is a Lipschitz function with  $\text{Lip}(f) \leq 3\text{Lip}(g) = 1$  and  $x, x' \in X$ ,  $e \in Y$  with  $\|e\| \leq 1$ , we may assume that  $\Theta(t) \leq 2$  for all  $t > 0$ .

**Lemma 3.2.** *There exists a function  $\Omega: (0, \infty) \rightarrow (0, \infty)$  such that*

1.  $\Omega(t) \geq 2\Theta(t)$  for all  $t \in \mathbb{R}$ ,
2.  $\Omega(t) \rightarrow 0$  as  $t \rightarrow 0^+$ ,
3.  $\Omega$  is continuous and, for each  $a > 0$ ,  $\Omega$  is uniformly continuous on  $[a, \infty)$ ,
4. if  $A, B > 0$  then  $\Omega(A) + 2B \leq \Omega(A + B)$ .

*Proof.* For each  $n \in \mathbb{Z}$  let

$$\beta(2^n) = \sup_{0 < t' \leq 2^{n+1}} \Theta(t') \leq 2.$$

We may uniquely extend  $\beta$  to  $(0, \infty)$  by imposing the property that  $\beta$  is affine on each interval of the form  $[2^n, 2^{n+1}]$  for  $n \in \mathbb{Z}$ . Note that  $\beta$  is continuous, increasing and  $\beta(t) \geq \Theta(t)$  for every  $t > 0$  since, choosing  $n \in \mathbb{Z}$  with  $2^n \leq t < 2^{n+1}$ , we have  $\beta(t) \geq \beta(2^n) \geq \Theta(t)$ .

Further for  $t \leq 2^n$  where  $n \in \mathbb{Z}$ , we have

$$\beta(t) \leq \beta(2^n) = \sup_{0 < t' \leq 2^{n+1}} \Theta(t')$$

and as  $\Theta(t) \rightarrow 0$  as  $t \rightarrow 0^+$  we deduce that  $\beta(t) \rightarrow 0$  as  $t \rightarrow 0^+$ .

We now let  $\Omega(t) = 2\beta(t) + 2t$ . Then (1) and (2) are immediate as  $\beta(t) \geq \Theta(t)$  and  $\beta(t) \rightarrow 0$  as  $t \rightarrow 0^+$ . As  $\beta$  is continuous so is  $\Omega$ . But as  $\beta(t)$  is increasing with  $t$  and is bounded above by 2 it converges as  $t \rightarrow \infty$ . Hence  $\beta$  is uniformly continuous

on  $[a, \infty)$  for any  $a > 0$ ; therefore so is  $\Omega$ . Finally for (4) we may use the fact that  $\beta$  is increasing to deduce that for  $A, B > 0$ ,

$$\begin{aligned}\Omega(A + B) &= 2\beta(A + B) + 2A + 2B \\ &\geq 2\beta(A) + 2A + 2B \\ &= \Omega(A) + 2B.\end{aligned}\quad \square$$

We now begin the iteration procedure that will construct the required pair

$$(x, e) \in D.$$

As  $D \neq \emptyset$  we can pick  $(x_0, e_0) \in D$ . By rescaling  $e_0$  if necessary and by replacing  $e_0$  with  $-e_0$  if necessary we can assume that  $\|e_0\| = 1$  and  $g'(\pi x_0, e_0) \geq 0$ .

We let  $e_0^* \in Y^*$  with  $e_0^*(e_0) = 1$  and  $\|e_0^*\| = 1$ , and then we define  $f: Y \rightarrow \mathbb{R}$  by

$$f = g + \frac{2}{3}e_0^*. \quad (3.2)$$

Note that

$$\text{Lip}(f) \leq \text{Lip}(g) + 2/3 = 1. \quad (3.3)$$

As  $f - g$  is linear the set  $D$  is precisely the set of all  $(x, e) \in X \times Y \setminus \{0\}$  such that  $f'(\pi x, e)$  exists.

We can prove a very simple observation immediately.

**Lemma 3.3.** *If  $(x, e) \in D$  with  $f'(\pi x_0, e_0) \leq f'(\pi x, e)$  then  $e_0^*(e) \geq 1/2$ .*

*Proof.* We may write the inequality  $f'(\pi x_0, e_0) \leq f'(\pi x, e)$  as

$$g'(\pi x_0, e_0) + \frac{2}{3} \leq g'(\pi x, e) + \frac{2}{3}e_0^*(e).$$

As  $g'(\pi x_0, e_0) \geq 0$  and  $\text{Lip}(g) = 1/3$  we deduce

$$\frac{2}{3} \leq \frac{1}{3} + \frac{2}{3}e_0^*(e)$$

from which the result follows.  $\square$

Now define  $\sigma_0 = 16$ ,  $\delta_0 = 1$ ,  $t_0 \in (0, 1/2)$  with  $t_0^2 < \mu - 1$ , the norm  $p_0$  be the original norm  $\|\cdot\|$  of  $Y$  and  $w_0 = w_{p_0}$ , where the latter is defined as follows.

**Definition 3.4.** *If  $p$  is a norm on  $Y$  and  $(x, e) \in D$  then we write*

$$w_p(x, e) = \frac{f'(\pi x, e)}{p(e)}.$$

Further for  $\sigma \geq 0$  we let  $G_p(x, e, \sigma)$  be the set of all  $(x', e') \in D$  such that

$$w_p(x, e) \leq w_p(x', e')$$

and

$$\begin{aligned} & |(f(\pi x' + te) - f(\pi x')) - (f(\pi x + te) - f(\pi x))| \\ & \leq (\sigma + \Omega(w_p(x', e') - w_p(x, e))) |t| \end{aligned}$$

for all  $t \in \mathbb{R}$ , where the function  $\Omega$  is given by Lemma 3.2.

For  $n \geq 1$  we shall pick

$$p_n, w_n, \sigma_n, t_n, \varepsilon_n, D_n, x_n, e_n, \nu_n, \Delta_n, \delta_n$$

in that order where

- $p_n$  are norms on  $Y$ ,
- $w_n$  are weight functions defined on  $D$ ,
- $D_n$  are non-empty subsets of  $D \subseteq X \times Y \setminus \{0\}$ ,
- $(x_n, e_n) \in D_n$ ,
- $\sigma_n, t_n, \varepsilon_n, \nu_n, \Delta_n, \delta_n > 0$ .

After defining the set  $D_n$  and choosing  $\varepsilon_n > 0$  we shall pick  $(x_n, e_n) \in D_n \subseteq D$  such that weight  $w_n(x_n, e_n)$  is within  $\varepsilon_n$  of its supremum over  $D_n$ . We shall show that  $p_n \rightarrow p_\infty$  and  $(x_n, e_n) \rightarrow (x_\infty, e_\infty)$  for some norm  $p_\infty$  on  $Y$  and  $(x_\infty, e_\infty) \in D$ . The constants  $\delta_m > 0$  will be used to bound  $d(x_n, x_m)$  for  $n \geq m$  whereas  $\sigma_m > 0$  will bound  $\|e_n - e_m\|$  and  $t_m > 0$  will control the differences between the norms  $p_n$  and  $p_m$  for  $n \geq m$ .

Given  $y \in Y$  and  $e \in Y \setminus \{0\}$  we use the notation  $\|y - \mathbb{R}e\|$  to denote the distance between the point  $y$  and the one dimensional subspace of  $Y$  generated by  $e$ . This distance is calculated with the original norm  $\|\cdot\|$  on  $Y$ .

**Algorithm 3.5.** Given  $n \geq 1$  choose

$$(1) \ p_n(y)^2 = p_{n-1}(y)^2 + t_{n-1}^2 \|y - \mathbb{R}e_{n-1}\|^2 \text{ for all } y \in Y,$$

$$(2) \ w_n = w_{p_n} \text{ - see Definition 3.4,}$$

$$(3) \ \sigma_n \in (0, \sigma_{n-1}/16),$$

(4)  $t_n \in (0, t_{n-1}/2)$  with  $t_n^2 < \sigma_{n-1}/16$ ,

(5)  $\varepsilon_n \in (0, t_n^2 \sigma_n^2 / 2^{13})$ ,

(6)  $D_n$  to be the set of all pairs  $(x, e) \in D$  with  $d(x, x_{n-1}) < \delta_{n-1}$ ,  $\|e\| = 1$  and

$$(x, e) \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu)$$

for some  $\nu \in (0, \sigma_{n-1}/2)$ ,

(7)  $(x_n, e_n) \in D_n$  such that  $w_n(x, e) \leq w_n(x_n, e_n) + \varepsilon_n$  for every  $(x, e) \in D_n$ ,

(8)  $\nu_n \in (0, \sigma_{n-1}/2)$  such that  $(x_n, e_n) \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu_n)$ ,

(9)  $\Delta_n > 0$  such that

$$|f(\pi x_n + te_n) - f(\pi x_n) - f'(\pi x_n, e_n)t| \leq \sigma_{n-1}|t|/32 \quad (3.4)$$

$$|f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}) - f'(\pi x_{n-1}, e_{n-1})t| \leq \sigma_{n-1}|t|/32 \quad (3.5)$$

for all  $t$  with  $|t| \leq 4\Delta_n/\nu_n$ ,

(10)  $\delta_n \in (0, (\delta_{n-1} - d(x_n, x_{n-1}))/2)$  with  $\|\pi x - \pi x_n\| \leq \Delta_n$  whenever  $d(x, x_n) \leq \delta_n$ .

That  $p_n$  defines a norm on  $Y$  is almost immediate; the proof of the triangle inequality follows from the formula

$$p_n(y) = \sqrt{\|y\|^2 + \sum_{m=0}^{n-1} t_m^2 \|y - \mathbb{R}e_m\|^2}$$

and Minkowski's inequality, and the other properties are straightforward to check.

Note that (6) implies that  $(x_{n-1}, e_{n-1}) \in D_n$ , and so  $D_n \neq \emptyset$ ; further as  $\text{Lip}(f) \leq 1$  and  $p_n(e) \geq \|e\| = 1$ , for  $(x, e) \in D_n$ , we see

$$\sup_{(x,e) \in D_n} w_n(x, e) \leq 1.$$

Therefore we are able to pick  $(x_n, e_n) \in D_n$  with the property of (7).

The definition (6) of  $D_n$  then implies that we can find  $\nu_n$  with the property of (8). Further, we have  $d(x_n, x_{n-1}) < \delta_{n-1}$  by (6) and (7); as  $\pi$  is continuous at  $x_n$  we may choose  $\delta_n$  as in (10).

We collect some rather simple facts.

**Lemma 3.6.** *The following six statements hold.*



1. The sequences  $\sigma_n, t_n, \varepsilon_n, \delta_n, \nu_n$  all tend to 0 as  $n \rightarrow \infty$ .
2. If  $n \geq 1$  and  $(x, e) \in D$  then  $p_n(e) \geq p_{n-1}(e)$  and  $w_n(x, e) \leq w_{n-1}(x, e)$  with equality in both cases if  $(x, e) = (x_{n-1}, e_{n-1})$ .
3. For each  $n \geq 1$ ,  $\overline{B}_{\delta_n}(x_n) \subseteq B_{\delta_{n-1}}(x_{n-1})$ .
4. For each  $n \geq 1$ ,  $\|y\| \leq p_n(y) \leq \mu\|y\| \leq 2\|y\|$  for any  $y \in Y$ .
5. For each  $n \geq 1$  we have

$$w_n(x_n, e_n) \geq w_n(x_{n-1}, e_{n-1}) = w_{n-1}(x_{n-1}, e_{n-1}).$$

6. For each  $n \geq 1$  then if  $(x, e) \in D$ ,

$$|f'(\pi x, e) - f'(\pi x_{n-1}, e_{n-1})| \leq 2|w_n(x, e) - w_n(x_{n-1}, e_{n-1})| + 4\|e - e_{n-1}\|. \quad (3.6)$$

7. If  $(x, e) \in D_n$  where  $n \geq 1$  then  $e_0^*(e) \geq 1/2$ .

*Proof.* Item (1) is immediate from Algorithm 3.5(3),(4),(5),(8) and (10). Likewise items (2) follows from Algorithm 3.5(1),(2) and Definition 3.4. For item (3) we may note from Algorithm 3.5(10) that

$$\delta_n < \delta_{n-1} - d(x_n, x_{n-1}).$$

Hence if  $d(x, x_n) \leq \delta_n$  then

$$d(x, x_{n-1}) \leq \delta_n + d(x_n, x_{n-1}) < \delta_{n-1}.$$

For item (4) we can use Algorithm 3.5(1),(4) to deduce

$$\begin{aligned} p_n(y)^2 &= \|y\|^2 + \sum_{m=0}^{n-1} t_m^2 \|y - \mathbb{R}e_m\|^2 \\ &\leq \|y\|^2 + \sum_{m=0}^{n-1} \left(\frac{t_0}{2^m}\right)^2 \|y\|^2 \\ &\leq \|y\|^2 + 2t_0^2 \cdot \|y\|^2 \\ &\leq (2\mu - 1) \cdot \|y\|^2 \leq \mu^2 \|y\|^2 \end{aligned}$$

as  $t_0^2 \leq \mu - 1$ . Then we simply note that  $\mu \leq 2$ .

For item (5) we note that  $(x_n, e_n) \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu_n)$  by Algorithm 3.5(8). Hence Definition (3.4) implies  $w_n(x_{n-1}, e_{n-1}) \leq w_n(x_n, e_n)$ . We have already proved the equality  $w_n(x_{n-1}, e_{n-1}) = w_{n-1}(x_{n-1}, e_{n-1})$  in (2).

Next for item (6), using  $p_n(e_{n-1}) \leq 2$  from (4),

$$\begin{aligned}
& |f'(\pi x, e) - f'(\pi x_{n-1}, e_{n-1})| \\
& \leq 2 \frac{|f'(\pi x, e) - f'(\pi x_{n-1}, e_{n-1})|}{p_n(e_{n-1})} \\
& \leq 2 \left| \frac{f'(\pi x, e)}{p_n(e)} - \frac{f'(\pi x_{n-1}, e_{n-1})}{p_n(e_{n-1})} \right| + 2|f'(\pi x, e)| \left| \frac{1}{p_n(e_{n-1})} - \frac{1}{p_n(e)} \right| \\
& \leq 2|w_n(x, e) - w_n(x_{n-1}, e_{n-1})| + 2 \frac{\|e\|}{p_n(e_{n-1})p_n(e)} |p_n(e) - p_n(e_{n-1})| \\
& \leq 2|w_n(x, e) - w_n(x_{n-1}, e_{n-1})| + 4\|e - e_{n-1}\|,
\end{aligned}$$

where, in the penultimate line, we are using  $\text{Lip}(f) \leq 1$  and, in the final line,  $\|p_n(e_{n-1})\| \geq \|e_{n-1}\| = 1$ ,  $\|p_n(e)\| \geq \|e\|$  and the fact that

$$|p_n(e) - p_n(e_{n-1})| \leq p_n(e - e_{n-1}) \leq 2\|e - e_{n-1}\|,$$

using the triangle inequality and (4) again.

Finally for (7) we note that if  $(x, e) \in D_n$  then  $w_n(x_{n-1}, e_{n-1}) \leq w_n(x, e)$  so that by (4) and (5),  $w_0(x_0, e_0) \leq w_n(x, e) \leq f'(\pi x, e)$ . Hence by Lemma 3.3,  $e_0^*(e) \geq 1/2$ .  $\square$

We next require a slightly more complicated calculation.

**Lemma 3.7.** *The following three statements hold.*

(i) *If  $n \geq 1$  and  $(x, e) \in D_{n+1}$ , then  $(x, e) \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu_n/2)$ .*

(ii) *If  $n \geq 1$  then  $D_{n+1} \subseteq D_n$ .*

(iii) *If  $n \geq 0$  and  $(x, e) \in D_{n+1}$ , then  $\|e - e_n\| \leq \sigma_n/8$ .*

*Proof.* For  $n = 0$ , condition (iii) is satisfied by  $\|e\| = \|e_n\| = 1$  and  $\sigma_0 = 16$ . It suffices now to check that if  $n \geq 1$  and condition (iii) is satisfied for  $n - 1$ , i.e.  $\|e' - e_{n-1}\| \leq \sigma_{n-1}/8$  for all  $(x', e') \in D_n$ , then (i)-(iii) are satisfied for  $n$ ; the lemma will then follow by induction on  $n$ .

Thus we may let  $n \geq 1$  and assume that

$$\|e_n - e_{n-1}\| \leq \frac{\sigma_{n-1}}{8} \tag{3.7}$$

as  $(x_n, e_n) \in D_n$ .

To establish (i) we let  $(x, e) \in D_{n+1}$ . We have

$$d(x, x_n) < \delta_n \quad (3.8)$$

and

$$(x, e) \in G_{p_{n+1}}(x_n, e_n, \sigma_n - \nu) \quad (3.9)$$

by Algorithm 3.5(6), for some  $\nu \in (0, \sigma_n/2)$ , and we wish to verify

$$(x, e) \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu_n/2). \quad (3.10)$$

First note that as  $w_{n+1}(x, e) \geq w_{n+1}(x_n, e_n)$ , from (3.9), we have

$$w_n(x, e) \geq w_{n+1}(x, e) \geq w_{n+1}(x_n, e_n) = w_n(x_n, e_n) \geq w_n(x_{n-1}, e_{n-1}) \quad (3.11)$$

using Lemma 3.6(2) and (5).

Now for  $|t| < 4\Delta_n/\nu_n$ , using first (3.4), (3.5) and then  $\text{Lip}(f) \leq 1$ ,

$$\begin{aligned} & |(f(\pi x + te_{n-1}) - f(\pi x)) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq |(f(\pi x + te_{n-1}) - f(\pi x)) - (f(\pi x_n + te_n) - f(\pi x_n))| \\ & \quad + |f'(\pi x_n, e_n) - f'(\pi x_{n-1}, e_{n-1})| \cdot |t| + \frac{1}{16}\sigma_{n-1}|t| \\ & \leq |(f(\pi x + te_n) - f(\pi x)) - (f(\pi x_n + te_n) - f(\pi x_n))| + \|e_n - e_{n-1}\| \cdot |t| \\ & \quad + |f'(\pi x_n, e_n) - f'(\pi x_{n-1}, e_{n-1})| \cdot |t| + \frac{1}{16}\sigma_{n-1}|t|. \end{aligned}$$

We may now apply (3.7) and (3.9) to deduce

$$\begin{aligned} & |(f(\pi x + te_{n-1}) - f(\pi x)) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq \left( \sigma_n - \nu + \Omega(w_{n+1}(x, e) - w_{n+1}(x_n, e_n)) + \frac{3}{16}\sigma_{n-1} \right) |t| \\ & \quad + |f'(\pi x_n, e_n) - f'(\pi x_{n-1}, e_{n-1})| \cdot |t| \\ & \leq \left( \frac{1}{4}\sigma_{n-1} + \Omega(w_n(x, e) - w_n(x_n, e_n)) \right) |t| \\ & \quad + (2(w_n(x_n, e_n) - w_n(x_{n-1}, e_{n-1})) + 4\|e_n - e_{n-1}\|) |t| \\ & \leq \left( \frac{3}{4}\sigma_{n-1} + \Omega(w_n(x, e) - w_n(x_n, e_n)) + 2(w_n(x_n, e_n) - w_n(x_{n-1}, e_{n-1})) \right) |t| \\ & \leq (\sigma_{n-1} - \nu_n/2 + \Omega(w_n(x, e) - w_n(x_{n-1}, e_{n-1}))) |t|, \quad (3.12) \end{aligned}$$

for  $|t| < 4\Delta/\nu_n$ , where we have used  $\sigma_n \in (0, \sigma_{n-1}/16)$ , from Algorithm 3.5(3), and then Lemma 3.6(2),(5),(6), Lemma 3.2(4) and  $\nu_n \in (0, \sigma_{n-1}/2)$ , from Algorithm 3.5(8).

Now we consider the case  $|t| \geq 4\Delta/\nu_n$ . As  $d(x, x_n) < \delta_n$  from (3.8) we have

$$\|\pi x - \pi x_n\| \leq \Delta_n \leq \nu_n |t|/4$$

from Algorithm 3.5(10), so we get

$$\begin{aligned} & |(f(\pi x + te_{n-1}) - f(\pi x)) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq 2\|\pi x - \pi x_n\| + |(f(\pi x_n + te_{n-1}) - f(\pi x_n)) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq \nu_n |t|/2 + (\sigma_{n-1} - \nu_n + \Omega(w_n(x_n, e_n) - w_n(x_{n-1}, e_{n-1}))) |t| \\ & \leq (\sigma_{n-1} - \nu_n/2 + \Omega(w_n(x, e) - w_n(x_{n-1}, e_{n-1}))) |t|, \end{aligned}$$

using  $(x_n, e_n) \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu_n)$ , from Algorithm 3.5(8),  $w_n(x_n, e_n) \leq w_n(x, e)$ , from (3.11), and the fact  $\Omega$  is increasing, from Lemma 3.2(4).

Then (3.12) together with this last inequality verifies (i).

Further, for  $(x, e) \in D_{n+1}$  we have  $\|e\| = 1$  and  $d(x, x_{n-1}) < \delta_{n-1}$ , using  $d(x, x_n) < \delta_n$  and Lemma 3.6(3). From (i) we see then that  $(x, e) \in D_n$ ; hence (ii).

Finally to see (iii), let  $(x, e) \in D_{n+1}$  with  $n \geq 1$  and recall that

$$w_n(x_n, e_n) = w_{n+1}(x_n, e_n) \leq w_{n+1}(x, e).$$

We deduce from Definition 3.4,

$$\begin{aligned} w_n(x_n, e_n) & \leq \frac{p_n(e)}{p_{n+1}(e)} w_n(x, e) \\ & \leq \frac{p_n(e)}{p_{n+1}(e)} (w_n(x_n, e_n) + \varepsilon_n) \end{aligned} \tag{3.13}$$

using  $(x, e) \in D_n$ , from (ii), and Algorithm 3.5(7).

Writing  $d := \|e - \mathbb{R}e_n\| \leq 1$  we note that  $t_n < t_0 < 1/2$  so that

$$\frac{p_n(e)}{p_{n+1}(e)} = \frac{1}{\sqrt{1 + t_n^2 d^2 / p_n(e)^2}} \leq 1 - \frac{t_n^2 d^2}{4p_n(e)^2}$$

using the fact that  $1/\sqrt{1+x} \leq 1 - x/4$  for  $0 \leq x \leq 1$ .

Substituting this inequality into (3.13) we obtain

$$\frac{t_n^2 d^2}{4p_n(e)^2} w_n(x_n, e_n) \leq \varepsilon_n \left( 1 - \frac{t_n^2 d^2}{4p_n(e)^2} \right) \leq \varepsilon_n.$$

But  $p_n(e) \leq 2$  and

$$w_n(x_n, e_n) \geq w_0(x_0, e_0) = g'(\pi x_0, e_0) + \frac{2}{3} > \frac{1}{2},$$

using Lemma 3.6(5) and  $g'(\pi x_0, e_0) \geq 0$ , so that, using Algorithm 3.5(5), we have  $d^2 < \sigma_n^2/2^8$  and so

$$\|e - te_n\| \leq \frac{\sigma_n}{16} \tag{3.14}$$

for some  $t \in \mathbb{R}$ .

Now  $|e_0^*(e - te_n)| \leq \sigma_n/16 \leq 1/2$ , for  $n \geq 1$ , and  $e_0^*(e) \geq 1/2$ ,  $e_0^*(e_n) \geq 1/2$  by Lemma 3.6(7). Hence  $t \geq 0$ .

Then from (3.14) and  $\|e_n\| = \|e\| = 1$  we get that

$$|1 - t| \leq \frac{\sigma_n}{16}$$

and so

$$\|e - e_n\| \leq \frac{\sigma_n}{8}$$

using (3.14) once again. □

We now show that the sequences  $x_n$ ,  $e_n$  and  $p_n$  converge and establish some properties of the limits. We first quote a lemma for determining the Fréchet differentiability of the norm  $p_\infty$ .

**Lemma 3.8.** *If the norm of a Banach space  $Y$  is Fréchet differentiable on  $Y \setminus \{0\}$ , if  $e_m \in Y$  and  $t_m \geq 0$  with  $\sum t_m^2 < \infty$ , then the function  $p: Y \rightarrow \mathbb{R}$  defined by the formula*

$$p(y) := \sqrt{\|y\|^2 + \sum_{m=1}^{\infty} t_m^2 \|y - \mathbb{R}e_m\|^2}$$

*is an equivalent norm on  $Y$  that is Fréchet differentiable on  $Y \setminus \{0\}$ .*

*Proof.* See [23, Lemma 4.3]. □

**Lemma 3.9.** *We have  $x_m \rightarrow x_\infty \in X$ ,  $e_m \rightarrow e_\infty \in S(Y)$  and  $p_m \rightarrow p_\infty$  where*

- (i)  $d(x_\infty, x_m) < \delta_m$  and  $\|e_\infty - e_m\| \leq \sigma_m/8$  for all  $m \geq 0$ ,

(ii)  $p_\infty$  is a norm on  $Y$  with

$$p_m(y) \leq p_\infty(y) \leq (1 + t_m^2)p_m(y) \leq \mu\|y\|$$

for all  $y \in Y$  and  $m \geq 0$ ; further  $p_m$  is Fréchet differentiable on  $Y \setminus \{0\}$  provided this is true of the original norm  $\|\cdot\|$  of  $Y$ ,

(iii) If  $(x, e) \in D$  and  $n \geq 1$  then  $|w_n(x, e) - w_\infty(x, e)| \leq t_n^2$  where  $w_\infty := w_{p_\infty}$ ,

(iv)  $(x_\infty, e_\infty) \in D$  with  $w_\infty(x_\infty, e_\infty) > 0$  and  $w_m(x_m, e_m) \nearrow w_\infty(x_\infty, e_\infty)$ ,

(v)  $(x_\infty, e_\infty) \in G_{p_m}(x_{m-1}, e_{m-1}, \sigma_{m-1} - \nu_m/2)$  and

(vi)  $(x_\infty, e_\infty) \in D_m$  for all  $m \geq 1$ .

*Proof.* For  $n \geq m \geq 1$ , by parts (ii) and (iii) of Lemma 3.7 we have

$$(x_n, e_n) \in D_{n+1} \subseteq D_{m+1}$$

and  $\|e_n - e_m\| \leq \sigma_m/8$ . Hence  $d(x_n, x_m) < \delta_m$  from the definition of  $D_{m+1}$  in Algorithm 3.5(6).

As  $(X, d)$  and  $(Y, \|\cdot\|)$  are complete and  $\delta_m, \sigma_m \rightarrow 0$  we have  $x_m \rightarrow x_\infty \in X$  and  $e_m \rightarrow e_\infty \in Y$  where  $d(x_\infty, x_m) \leq \delta_m$  and  $\|e_\infty - e_m\| \leq \sigma_m/8$ . The former implies

$$x_\infty \in \overline{B}_{\delta_m}(x_m) \subseteq B_{\delta_{m-1}}(x_{m-1})$$

for all  $m \geq 1$ , using Lemma 3.6(3). Finally as  $e_m \in S(Y)$  for all  $m$  we have  $e_\infty \in S(Y)$  too.

For item (ii) we note by Lemma 3.8 that

$$p_\infty(y) := \sqrt{\|y\|^2 + \sum_{m=1}^{\infty} t_m^2 \|y - \mathbb{R}e_m\|^2}$$

is a norm on  $Y$  that is Fréchet differentiable on  $Y \setminus \{0\}$  provided this is true of  $\|\cdot\|$ ; further from  $t_{m+1} \in (0, t_m/2)$ ,

$$\begin{aligned} p_\infty(y)^2 &= p_n(y)^2 + \sum_{m=n}^{\infty} t_m^2 \|y - \mathbb{R}e_m\|^2 \\ &\leq p_n(y)^2 + 2t_n^2 \cdot \|y\|^2 \\ &\leq (1 + 2t_n^2)p_n(y)^2 \\ &\leq (1 + t_n^2)^2 p_n(y)^2. \end{aligned}$$

We then may note that  $1 + t_n^2 \leq 1 + t_0^2 \leq \mu$ .

For (iii),

$$\begin{aligned}
& |w_n(x, e) - w_\infty(x, e)| \\
&= \left| f'(x, e) \left( \frac{1}{p_n(e)} - \frac{1}{p_\infty(e)} \right) \right| \\
&= |f'(x, e)| \cdot \frac{|p_\infty(e) - p_n(e)|}{p_n(e)p_\infty(e)} \\
&\leq \|e\| \frac{t_n^2 \cdot p_n(e)}{p_n(e)p_\infty(e)} \\
&\leq t_n^2
\end{aligned}$$

using  $\text{Lip}(f) \leq 1$ ,  $|p_\infty(e) - p_n(e)| \leq t_n^2 p_n(e)$  from part (ii) and  $p_\infty(e) \geq \|e\|$ .

We now show that the directional derivative  $f'(\pi x_\infty, e_\infty)$  exists.

For  $n \geq m$  we have  $(x_n, e_n) \in D_{m+1}$ ; therefore by part (i) of Lemma 3.7 we know

$$(x_n, e_n) \in G_{p_m}(x_{m-1}, e_{m-1}, \sigma_{m-1} - \nu_m/2). \quad (3.15)$$

Now from Lemma 3.6(5),  $w_0(x_0, e_0) > 0$  and  $\text{Lip}(f) \leq 1$  we have

$$w_n(x_n, e_n) \nearrow L$$

for some  $L \in (0, 1]$ . By part (ii) of the present lemma and  $t_n \rightarrow 0$  we also have  $w_\infty(x_n, e_n) \rightarrow L$  and  $w_{n+1}(x_n, e_n) \rightarrow L$ . Note then that for each fixed  $m$ ,

$$w_m(x_n, e_n) - w_m(x_{m-1}, e_{m-1}) \xrightarrow{n \rightarrow \infty} s_m,$$

where

$$s_m := \frac{p_\infty(e_\infty)}{p_m(e_\infty)} L - w_m(x_{m-1}, e_{m-1}) \xrightarrow{m \rightarrow \infty} 0. \quad (3.16)$$

As  $w_m(x_n, e_n) \geq w_m(x_{m-1}, e_{m-1})$  from (3.15) we have  $s_m \geq 0$  for each  $m$ . Taking  $n \rightarrow \infty$  in (3.15) we obtain

$$|(f(\pi x_\infty + t e_{m-1}) - f(\pi x_\infty)) - (f(\pi x_{m-1} + t e_{m-1}) - f(\pi x_{m-1}))| \leq r_m |t| \quad (3.17)$$

for any  $t \in \mathbb{R}$ , where

$$r_m := \sigma_{m-1} - \nu_m/2 + \Omega(s_m) \rightarrow 0 \quad (3.18)$$

by Lemma 3.2(2). Using  $\|e_\infty - e_{m-1}\| \leq \sigma_{m-1}$  and  $\text{Lip}(f) \leq 1$ :

$$|(f(\pi x_\infty + te_\infty) - f(\pi x_\infty)) - (f(\pi x_{m-1} + te_{m-1}) - f(\pi x_{m-1}))| \leq (r_m + \sigma_{m-1})|t|. \quad (3.19)$$

Let  $\varepsilon > 0$ . Note that as

$$f'(\pi x_{m-1}, e_{m-1}) = p_{m-1}(e_{m-1})w_{m-1}(x_{m-1}, e_{m-1}) \rightarrow p_\infty(e_\infty)L$$

we may pick  $m$  such that

$$r_m + \sigma_{m-1} \leq \varepsilon/3 \text{ and } |f'(\pi x_{m-1}, e_{m-1}) - p_\infty(e_\infty)L| \leq \varepsilon/3 \quad (3.20)$$

and then  $\delta > 0$  with

$$|f(\pi x_{m-1} + te_{m-1}) - f(\pi x_{m-1}) - f'(\pi x_{m-1}, e_{m-1})t| \leq \varepsilon|t|/3 \quad (3.21)$$

for all  $t$  with  $|t| \leq \delta$ . Combining (3.19), (3.20) and (3.21) we obtain

$$|f(\pi x_\infty + te_\infty) - f(\pi x_\infty) - p_\infty(e_\infty)Lt| \leq \varepsilon|t|$$

for  $|t| \leq \delta$ . Hence the directional derivative  $f'(\pi x_\infty, e_\infty)$  exists and equals  $p_\infty(e_\infty)L$ . Therefore  $(x_\infty, e_\infty) \in D$  with

$$w_\infty(x_\infty, e_\infty) = L.$$

As  $L > 0$  and  $w_n(x_n, e_n) \nearrow L$ , we obtain (iv).

From (3.16) we may now deduce

$$s_m = w_m(x_\infty, e_\infty) - w_m(x_{m-1}, e_{m-1}).$$

As  $s_m \geq 0$  for all  $m$ , we have  $w_m(x_\infty, e_\infty) \geq w_m(x_{m-1}, e_{m-1})$  for all  $m$ . Further from (3.17) and (3.18),

$$\begin{aligned} & |(f(\pi x_\infty + te_{m-1}) - f(\pi x_\infty)) - (f(\pi x_{m-1} + te_{m-1}) - f(\pi x_{m-1}))| \\ & \leq \sigma_{m-1} - \nu_m/2 + \Omega(w_m(x_\infty, e_\infty) - w_m(x_{m-1}, e_{m-1}))|t| \end{aligned}$$

for any  $t$ . Hence

$$(x_\infty, e_\infty) \in G_{p_m}(x_{m-1}, e_{m-1}, \sigma_{m-1} - \nu_m/2).$$



This establishes (v).

Finally (vi) follows immediately from (i),(iv),(v) and  $\|e_\infty\| = 1$ .  $\square$

We are left needing to verify that the weight  $w_\infty(x_\infty, e_\infty)$  is almost locally maximal in the sense of Theorem 3.1.

**Lemma 3.10.** *If  $\varepsilon > 0$  then there exists  $\delta > 0$  such that whenever*

$$(x', e') \in G_{p_\infty}(x_\infty, e_\infty, 0)$$

with  $d(x', x_\infty) \leq \delta$  then we have  $w_\infty(x', e') < w_\infty(x_\infty, e_\infty) + \varepsilon$ .

*Proof.* Find  $n \geq 1$  with  $\varepsilon_n + 2t_n^2 < \varepsilon$  and then pick  $\Delta > 0$  such that for  $|t| < 8\Delta/\nu_n$ ,

$$|f(\pi x_\infty + te_\infty) - f(\pi x_\infty) - f'(\pi x_\infty, e_\infty)t| \leq \frac{1}{16}\sigma_{n-1}|t| \quad (3.22)$$

$$|f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}) - f'(\pi x_{n-1}, e_{n-1})t| \leq \frac{1}{16}\sigma_{n-1}|t|. \quad (3.23)$$

Using Lemma 3.9(i) and the continuity of  $\pi$  we can find

$$\delta \in (0, \delta_{n-1} - d(x_\infty, x_{n-1})) \quad (3.24)$$

such that

$$\|\pi x' - \pi x_\infty\| \leq \Delta \quad (3.25)$$

whenever  $d(x', x_\infty) \leq \delta$ .

We now suppose, for a contradiction, that

$$\begin{cases} (x', e') \in G_{p_\infty}(x_\infty, e_\infty, 0) \\ d(x', x_\infty) \leq \delta \\ w_\infty(x', e') \geq w_\infty(x_\infty, e_\infty) + \varepsilon. \end{cases} \quad (3.26)$$

As  $w_\infty(x', e')$  is invariant if we scale  $e'$  by a positive factor, as is the membership relation  $(x', e') \in G_{p_\infty}(x_\infty, e_\infty, 0)$ , we may assume that  $\|e'\| = 1$ .

First we shall show that  $(x', e') \in D_n$ .

Since (3.24) and  $d(x', x_\infty) \leq \delta$  imply  $d(x', x_{n-1}) < \delta_{n-1}$ , to prove  $(x', e') \in D_n$  it is enough to show that

$$(x', e') \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu_n/4); \quad (3.27)$$

see (6) in Algorithm 3.5.

First note that as  $w_{n+1}(x, e) \geq w_{n+1}(x_n, e_n)$ , from (3.9), we have

$$w_n(x, e) \geq w_{n+1}(x, e) \geq w_{n+1}(x_n, e_n) = w_n(x_n, e_n) \geq w_n(x_{n-1}, e_{n-1}) \quad (3.28)$$

using Lemma 3.6(2) and (5).

Then for  $|t| < 8\Delta/\nu_n$ , using first (3.22), (3.23) and then  $\text{Lip}(f) \leq 1$ ,

$$\begin{aligned} & |(f(\pi x' + te_{n-1}) - f(\pi x')) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq |(f(\pi x' + te_{n-1}) - f(\pi x')) - (f(\pi x_\infty + te_\infty) - f(\pi x_\infty))| \\ & \quad + |f'(\pi x_\infty, e_\infty) - f'(\pi x_{n-1}, e_{n-1})| \cdot |t| + \frac{1}{8}\sigma_{n-1}|t| \\ & \leq |(f(\pi x' + te_\infty) - f(\pi x')) - (f(\pi x_\infty + te_\infty) - f(\pi x_\infty))| + \|e_\infty - e_{n-1}\| \cdot |t| \\ & \quad + |f'(\pi x_\infty, e_\infty) - f'(\pi x_{n-1}, e_{n-1})| \cdot |t| + \frac{1}{8}\sigma_{n-1}|t|. \end{aligned}$$

Note now that  $\|e_\infty - e_{n-1}\| \leq \sigma_{n-1}/8$  from Lemma 3.9(i). Further, using

$$(x', e') \in G_{p_\infty}(x_\infty, e_\infty, 0)$$

we have

$$\begin{aligned} & |(f(\pi x' + te_\infty) - f(\pi x')) - (f(\pi x_\infty + te_\infty) - f(\pi x_\infty))| \\ & \leq \Omega(w_\infty(x', e') - w_\infty(x_\infty, e_\infty)) \cdot |t| \end{aligned}$$

for all  $t \in \mathbb{R}$ . Hence

$$\begin{aligned} & |(f(\pi x' + te_{n-1}) - f(\pi x')) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq \left( \frac{1}{4}\sigma_{n-1} + \Omega(w_\infty(x', e') - w_\infty(x_\infty, e_\infty)) \right) |t| \\ & \quad + |f'(\pi x_\infty, e_\infty) - f'(\pi x_{n-1}, e_{n-1})| \cdot |t| \\ & \leq \left( \frac{1}{4}\sigma_{n-1} + \Omega(w_\infty(x', e') - w_\infty(x_\infty, e_\infty)) \right) |t| \\ & \quad + (2(w_n(x_\infty, e_\infty) - w_n(x_{n-1}, e_{n-1})) + 4\|e_\infty - e_{n-1}\|) |t| \\ & \leq \left( \frac{3}{4}\sigma_{n-1} + \Omega(w_\infty(x', e') - w_\infty(x_\infty, e_\infty)) \right) |t| \\ & \quad + 2(w_n(x_\infty, e_\infty) - w_n(x_{n-1}, e_{n-1}))|t| \\ & \leq \left( \frac{7}{8}\sigma_{n-1} + \Omega(w_n(x', e') - w_\infty(x_\infty, e_\infty)) \right) |t| \\ & \quad + 2(w_\infty(x_\infty, e_\infty) - w_n(x_{n-1}, e_{n-1}))|t| \end{aligned}$$

for  $|t| < 8\Delta/\nu_n$ , where we have used Algorithm 3.5(3) and then Lemma 3.6(6) with  $(x, e) = (x_\infty, e_\infty)$ . In the final line we note from Lemma 3.9(ii),(iii) and Algorithm 3.5(4) that  $w_\infty(x', e') \leq w_n(x', e')$  and

$$|w_n(x_\infty, e_\infty) - w_\infty(x_\infty, e_\infty)| \leq t_n^2 < \sigma_{n-1}/16.$$

Finally by Lemma 3.2(4) and  $\nu_n \in (0, \sigma_{n-1}/2)$ ,

$$\begin{aligned} & |(f(\pi x' + te_{n-1}) - f(\pi x')) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq (\sigma_{n-1} - \nu_n/4 + \Omega(w_n(x', e') - w_n(x_{n-1}, e_{n-1}))) |t|. \end{aligned} \quad (3.29)$$

Now we consider the case  $|t| \geq 8\Delta/\nu_n$ . From  $d(x', x_\infty) \leq \delta$  and (3.25) we have

$$\|\pi x' - \pi x_\infty\| \leq \Delta \leq \nu_n |t|/8$$

so we get, using  $(x_\infty, e_\infty) \in G_{p_n}(x_{n-1}, e_{n-1}, \sigma_{n-1} - \nu_n/2)$  from Lemma 3.9(v),

$$\begin{aligned} & |(f(\pi x' + te_{n-1}) - f(\pi x')) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \leq |(f(\pi x_\infty + te_{n-1}) - f(\pi x_\infty)) - (f(\pi x_{n-1} + te_{n-1}) - f(\pi x_{n-1}))| \\ & \quad + 2\|\pi x' - \pi x_\infty\| \\ & \leq (\sigma_{n-1} - \nu_n/2 + \Omega(w_n(x_\infty, e_\infty) - w_n(x_{n-1}, e_{n-1}))) |t| + \nu_n |t|/4 \\ & \leq (\sigma_{n-1} - \nu_n/4 + \Omega(w_n(x', e') - w_n(x_{n-1}, e_{n-1}))) |t|, \end{aligned} \quad (3.30)$$

where, in the final line, we have noted from Lemma 3.9(iii) and

$$w_\infty(x', e') \geq w_\infty(x_\infty, e_\infty) + \varepsilon \geq w_\infty(x_\infty, e_\infty) + \varepsilon_n + 2t_n^2$$

that

$$w_n(x', e') \geq w_n(x_\infty, e_\infty).$$

From (3.29) and (3.30) we deduce  $(x', e') \in D_n$ . It follows from Algorithm 3.5(7) that

$$w_n(x', e') \leq w_n(x_n, e_n) + \varepsilon_n \leq w_\infty(x_\infty, e_\infty) + \varepsilon_n,$$

by Lemma 3.9(iv).

Using Lemma 3.9(iii) once more,

$$w_\infty(x', e') \leq w_\infty(x_\infty, e_\infty) + \varepsilon_n + t_n^2 < w_\infty(x_\infty, e_\infty) + \varepsilon.$$

This contradicts our assumption

$$w_\infty(x', e') \geq w_\infty(x_\infty, e_\infty) + \varepsilon,$$

completing the proof.  $\square$

*Proof of Theorem 3.1.* We verify the conclusions of the theorem.

First, item (1) is immediate from (3.2) and (3.3). For item (2) we define

$$\|\cdot\|' = p_\infty$$

and then the inequality  $\|y\| \leq \|y\|' \leq \mu\|y\|$  holds for all  $y \in Y$  by Lemma 3.9(ii).

For (3) we define

$$\begin{aligned} x &= x_\infty \\ e &= e_\infty / \|e_\infty\|'. \end{aligned}$$

Then Lemma 3.9(iv) shows that  $(x, e) \in D$ . That  $\|e\|' = 1$  is immediate from the definition of  $e$ .

Now given any  $\varepsilon > 0$  we choose  $\delta > 0$  as in Lemma 3.10 and then define the open neighbourhood  $N_\varepsilon$  of  $x$  in  $X$  by

$$N_\varepsilon = B_\delta(x).$$

If  $(x', e') \in D$  with  $x' \in N_\varepsilon$  and  $\|e'\|' = 1$  then from Definition 3.4 we have

$$\begin{aligned} w_\infty(x_\infty, e_\infty) &= f'(\pi x, e) \\ w_\infty(x', e') &= f'(\pi x', e'). \end{aligned}$$

Subsequently if (3.1) is satisfied then as  $2\Theta \leq \Omega$ , from Lemma 3.2(1), and

$$\|e_\infty\|' \leq \mu\|e_\infty\| \leq 2$$

we have

$$(x', e') \in G_{p_\infty}(x_\infty, e_\infty, 0).$$

Then by Lemma 3.10, as  $x' \in B_\delta(x)$ ,

$$w_\infty(x', e') < w_\infty(x_\infty, e_\infty) + \varepsilon,$$

and so

$$f'(\pi x', e') < f'(\pi x, e) + \varepsilon.$$

Finally, by Lemma 3.9(ii), if the norm  $\|\cdot\|$  is Fréchet differentiable on  $Y \setminus \{0\}$  then so is the norm  $\|\cdot\|'$ . □

## Chapter 4

# Universal Fréchet sets

### 4.1 A criterion for universality

**Definition 4.1.** *Given a Banach space  $Y$  we say that a subset  $S \subseteq Y$  is a universal Fréchet set if for every Lipschitz function  $f: Y \rightarrow \mathbb{R}$  there exists  $y \in S$  such that  $f$  is Fréchet differentiable at  $x$ .*

We remark that if  $Y$  is an Asplund space then, by Theorem 1.13, any non-empty open subset of  $Y$  is a universal Fréchet set. If, on the other hand,  $Y$  is not Asplund then by Theorem 1.16, there exists a nowhere Fréchet-differentiable Lipschitz function  $f: Y \rightarrow \mathbb{R}$ ; in fact  $f$  may be taken to be equal to a certain equivalent norm on  $Y$ . It follows that any non-Asplund  $Y$  has no universal Fréchet sets.

We now prove a sufficient condition for a set  $S \subseteq Y$  to be a universal Fréchet set. We use the notation  $B_\delta(y)$  and  $\overline{B}_\delta(y)$  to denote open and closed balls in  $Y$ , respectively, with centre  $y$  and radius  $\delta$ . As usual we shall use  $S(Y)$  to denote the unit sphere of a Banach space  $Y$ .

**Theorem 4.2.** *Let  $X$  be a non-empty complete space,  $Y$  be a Banach space with an equivalent norm that is Fréchet differentiable on  $Y \setminus \{0\}$  and  $\pi: X \rightarrow Y$  be a continuous function.*

*Suppose that for every  $\eta, \mu > 0$ ,  $x \in X$  and every open neighbourhood  $N$  of  $x$  in  $X$  there exists  $\delta_0 = \delta_0(x, N, \eta, \mu) > 0$  such that for any  $\delta \in (0, \delta_0)$  and  $y_1, y_2 \in B_\delta(y)$  where  $y = \pi x$ , there exists an almost-everywhere-differentiable Lips-*

chitz  $\gamma: [0, 1] \rightarrow Y$  with

$$\|\gamma(0) - y_1\| \leq \eta\delta, \|\gamma(1) - y_2\| \leq \eta\delta \quad (4.1)$$

$$\|\gamma'(t) - (y_2 - y_1)\| \leq \eta\delta \text{ for almost all } t \in [0, 1] \quad (4.2)$$

$$m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin \pi(N)\}) \leq \mu. \quad (4.3)$$

Then the image  $\pi(X)$  is a universal Fréchet set in  $Y$ . In fact, writing  $D_g \subseteq Y$  for the set of points of Fréchet differentiability of a Lipschitz function  $g: Y \rightarrow \mathbb{R}$ , the intersection  $D_g \cap \pi(X)$  is a dense subset of  $\pi(X)$ .

*Proof.* As the conditions of the theorem are invariant under a renorming of  $Y$  and as the set of points of Fréchet differentiability of any Lipschitz function is unchanged under such a renorming, we may assume that the original norm  $\|\cdot\|$  of  $Y$  is Fréchet differentiable on  $Y \setminus \{0\}$ .

Let  $x \in X$  and  $y = \pi x \in Y$  and

$$\delta \in (0, \delta_0(x, X, 1/2, 1/2)).$$

Pick  $y_1, y_2 \in B_\delta(y)$  with  $\|y_2 - y_1\| \geq \delta$ . We may find a Lipschitz curve  $\gamma: [0, 1] \rightarrow Y$  with

$$\|\gamma'(t) - (y_2 - y_1)\| \leq \delta/2 \quad (4.4)$$

for almost all  $t \in [0, 1]$  and such that

$$m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin \pi(X)\}) \leq 1/2. \quad (4.5)$$

Let now  $g: Y \rightarrow \mathbb{R}$  be Lipschitz with  $\text{Lip}(g) \leq 1$ . Note then that

$$g \circ \gamma: [0, 1] \rightarrow \mathbb{R}$$

is a Lipschitz function so by Lebesgue's theorem it is differentiable almost everywhere.

Using (4.4) and (4.5) we may pick  $t \in [0, 1]$  such that  $\gamma(t) \in \pi(X)$ ,  $\gamma'(t)$  exists and is non-zero and  $(g \circ \gamma)'(t)$  exists. Write  $\gamma(t) = \pi x$  where  $x \in X$ . As  $g$  is Lipschitz we deduce that  $g'(\gamma(t), \gamma'(t))$  exists. Therefore the set

$$D := \{(x, e) \in X \times (Y \setminus \{0\}) \text{ such that } g'(\pi x, e) \text{ exists}\}$$

is non-empty because it contains  $(x, \gamma'(t))$ .

Define  $\Theta: (0, \infty) \rightarrow \mathbb{R}$  by

$$\Theta(s) = 10^6 \sqrt{3s},$$

and  $\mu = 2$ . Then conditions (a)-(e) of Theorem 3.1 are readily verified and we have already established that  $D \neq \emptyset$ .

Let the Lipschitz function  $f: Y \rightarrow \mathbb{R}$ , the norm  $\|\cdot\|'$  on  $Y$ ,  $(x, e) \in D$  and, for each  $\nu > 0$ , the open neighbourhood  $N_\nu$  of  $x \in X$  be given by the conclusion of Theorem 3.1. Note that  $\text{Lip}(f) \leq 3$ , that we may take  $\|\cdot\|'$  to be Fréchet differentiable on  $Y \setminus \{0\}$  and finally

$$\|z\| \leq \|z\|' \leq 2\|z\| \tag{4.6}$$

for all  $z \in Y$ .

We claim that  $y = \pi x$  is a point of Fréchet differentiability of  $f$ . As Fréchet differentiability in  $(Y, \|\cdot\|)$  is equivalent to Fréchet differentiability in  $(Y, \|\cdot\|')$ , since the two norms are equivalent, it suffices to verify the conditions of Theorem 2.1 for  $(Y, \|\cdot\|')$ , applied to the Lipschitz function  $f$ ,  $L = 3$  and the pair  $(y, e)$ . To accomplish this, we let  $\nu, \eta, \mu > 0$  and prove the existence of

$$\begin{aligned} F &= F_{\nu, \eta, \mu} \\ s_* &= s_*(\nu, \eta, \mu) \end{aligned}$$

such that (1) and (2) of Theorem 2.1 hold, with the norm  $\|\cdot\|$  replaced by  $\|\cdot\|'$ . We shall take

$$\begin{aligned} F_{\nu, \eta, \mu} &= \pi(N_\nu) \\ s_*(\nu, \eta, \mu) &= \delta_0(x, N_\nu, \eta/2, \mu). \end{aligned}$$

Suppose  $s \in (0, s_*)$ ,  $\|y_1 - y\|' < s$  and  $\|y_2 - y\|' < s$ . Then

$$s \in (0, \delta_0(x, N_\nu, \eta/2, \mu))$$

and, from (4.6), we have  $\|y_1 - y\| < s$ ,  $\|y_2 - y\| < s$ .

Thus there exists an almost-everywhere-differentiable Lipschitz curve

$$\gamma: [0, 1] \rightarrow Y$$



such that

$$\begin{aligned} \|\gamma(0) - y_1\| &\leq \eta s/2, \quad \|\gamma(1) - y_2\| \leq \eta s/2 \\ \|\gamma'(t) - (y_2 - y_1)\| &\leq \eta s/2 \text{ for almost all } t \in [0, 1] \\ m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin \pi(N_\nu)\}) &\leq \mu. \end{aligned}$$

Thus by (4.6),

$$\begin{aligned} \|\gamma(0) - y_1\|' &\leq \eta s, \quad \|\gamma(1) - y_2\|' \leq \eta s \\ \|\gamma'(t) - (y_2 - y_1)\|' &\leq \eta s \text{ for almost all } t \in [0, 1]. \end{aligned}$$

This verifies condition (1) of Theorem 2.1.

For condition (2) we can note that if  $y' \in F_{\nu, \eta, \mu}$ ,  $\|e'\|' = 1$  and

$$\begin{aligned} |(f(y' + te) - f(y')) - (f(y + te) - f(y))| \\ \leq 10^6 \sqrt{(f'(y', e') - f'(y, e))L} \cdot |t| \end{aligned}$$

for all  $t \in \mathbb{R}$ , then as  $F_{\nu, \eta, \mu} = \pi(N_\nu)$  we may write  $y' = \pi x'$  where  $x' \in N_\nu$ . As  $L = 3$  and  $\Theta(s) = 10^6 \sqrt{3s}$ , the conditions (3)(i) and (3)(ii) of Theorem 3.1 are satisfied, with  $\varepsilon$  replaced with  $\nu$ , so we deduce that

$$f'(\pi x', e') < f'(\pi x, e) + \nu,$$

which verifies (2.5).

As all the conditions of Theorem 2.1 are satisfied we deduce that  $f$  is Fréchet differentiable at  $y = \pi x \in \pi(X)$ . As  $f - g$  is linear,  $g$  is also Fréchet differentiable at  $y \in \pi(X)$ . Hence  $\pi(X)$  is indeed a universal Fréchet set in  $Y$ .

To verify the last observation of the present theorem, it suffices to note that if  $y \in \pi(X)$  and  $\varepsilon > 0$  then by the continuity of  $\pi$  we may pick a non-empty open set  $N \subseteq X$  such that

$$\pi(N) \subseteq B_\varepsilon(y).$$

Then as the restriction bundle  $\pi|_N: N \rightarrow Y$  satisfies the conditions of the present theorem, as can easily be checked, any Lipschitz  $g: Y \rightarrow \mathbb{R}$  contains a point of Fréchet differentiability in

$$\pi(N) \subseteq \pi(X) \cap B_\varepsilon(y). \quad \square$$

It is convenient to reformulate Theorem 4.2 for the case in which the set we

wish to prove universal is a closed subset of the Banach space  $Y$ . This reformulation is closer in spirit to the approach in [8].

**Corollary 4.3.** *Let  $Y$  be a Banach space with an equivalent norm that is Fréchet differentiable on  $Y \setminus \{0\}$  and  $(F_\lambda)_{0 < \lambda < 1}$  be a family of non-empty, closed subsets of  $Y$  with  $F_\lambda \subseteq F_{\lambda'}$  whenever  $\lambda \leq \lambda'$ .*

*Suppose that for every  $\eta, \mu > 0$ ,  $0 < \lambda < 1$  and*

$$y \in \bigcup_{0 < \lambda' < \lambda} F_{\lambda'}$$

*there exists  $\delta_1 = \delta_1(y, \lambda, \eta, \mu) > 0$  such that, for any  $\delta \in (0, \delta_1)$  and  $y_1, y_2 \in B_\delta(y)$ , there exists an almost-everywhere-differentiable Lipschitz curve  $\gamma: [0, 1] \rightarrow Y$  with*

$$\|\gamma(0) - y_1\| \leq \eta\delta, \quad \|\gamma(1) - y_2\| \leq \eta\delta \quad (4.7)$$

$$\|\gamma'(t) - (y_2 - y_1)\| \leq \eta\delta \text{ for almost all } t \in [0, 1] \quad (4.8)$$

$$m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin F_\lambda\}) \leq \mu. \quad (4.9)$$

*Then for each  $\lambda \in (0, 1)$  the set  $F_\lambda$  is a universal Fréchet set. In fact, writing  $D_g \subseteq Y$  for the set of points of Fréchet differentiability of a Lipschitz function  $g: Y \rightarrow \mathbb{R}$ , the intersection  $D_g \cap \bigcup_{0 < \lambda < t} F_\lambda$  is a dense subset of  $\bigcup_{0 < \lambda < t} F_\lambda$ , for any  $t \in (0, 1)$ .*

*Proof.* For each  $\lambda \in (0, 1)$  we let

$$F'_\lambda := \bigcap_{\lambda' > \lambda} F_{\lambda'}.$$

Fix  $t \in (0, 1)$  and define  $X \subseteq (0, t) \times Y$  by

$$X = \{(\lambda, y) \text{ such that } y \in F'_\lambda\}.$$

Note that  $X \neq \emptyset$ . We claim that  $X$  is a complete space; indeed if we take  $\Delta$  to be a complete metric on  $(0, t)$  then we may check that the metric

$$d((\lambda', y'), (\lambda, y)) = \max(\Delta(\lambda', \lambda), \|y' - y\|) \quad (4.10)$$

is a complete metric on  $X$  as follows.

Suppose that  $(\lambda_n, y_n)$  is a Cauchy sequence in  $(X, d)$ . As  $((0, t), \Delta)$  and  $(Y, \|\cdot\|)$  are complete we deduce that  $\lambda_n \rightarrow \lambda \in (0, t)$  and  $y_n \rightarrow y \in Y$ . We wish to show that  $(\lambda, y) \in X$ . Note that if  $\lambda' \in (\lambda, 1)$  then as  $\lambda_n < \lambda'$  for sufficiently

high  $n$  we have  $y_n \in F'_{\lambda_n} \subseteq F_{\lambda'}$ . As the latter set is closed we deduce that  $y \in F_{\lambda'}$ . This holds for every  $\lambda' \in (\lambda, 1)$  so that  $y \in F'_\lambda$ . Hence  $(\lambda, y) \in X$ , proving that  $X$  is indeed complete.

The map  $\pi: X \rightarrow Y$  given by

$$\pi(\lambda, y) = y$$

is of course continuous.

We shall now verify the conditions of Theorem 4.2. Given  $\eta, \mu > 0$ ,  $x = (\lambda, y) \in X$  and an open neighbourhood  $N$  of  $x$  in  $X$ , to show the existence of  $\delta_0(x, N, \eta, \mu)$  we may assume, without loss of generality, that  $N$  is an open ball

$$N = \{x' \in X \text{ such that } d(x', x) < \psi\}$$

in  $X$  and  $\eta \in (0, 1)$ .

We then pick  $\lambda' \in (\lambda, 1)$  with  $\Delta(\lambda', \lambda) < \psi$  and let

$$\delta_0(x, N, \eta, \mu) = \min \left( \delta_1(y, \lambda', \eta, \mu), \frac{\psi}{3} \right).$$

If  $\delta \in (0, \delta_0)$  then as  $\delta \in (0, \delta_1(y, \lambda', \eta, \mu))$ , for any  $y_1, y_2 \in B_\delta(y)$  there exists an almost-everywhere-differentiable Lipschitz curve  $\gamma: [0, 1] \rightarrow Y$  with

$$\|\gamma(0) - y_1\| \leq \eta\delta, \quad \|\gamma(1) - y_2\| \leq \eta\delta \quad (4.11)$$

$$\|\gamma'(t) - (y_2 - y_1)\| \leq \eta\delta \text{ for almost all } t \in [0, 1] \quad (4.12)$$

$$m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin F_{\lambda'}\}) \leq \mu. \quad (4.13)$$

Using  $y_1, y_2 \in B_\delta(y)$ , (4.11), (4.12),  $\eta < 1$  and  $\delta < \delta_0 \leq \psi/3$  we deduce that

$$\gamma(t) \in B_{3\delta}(y) \subseteq B_\psi(y)$$

for all  $t \in [0, 1]$ .

Thus if  $\gamma(t) \in F_{\lambda'}$  then as

$$d((\lambda', \gamma(t)), x) = d((\lambda', \gamma(t)), (\lambda, y)) < \psi,$$

by (4.10), we have  $(\lambda', \gamma(t)) \in N$  and so  $\gamma(t) \in \pi(N)$ . Therefore

$$m(\{t \in [0, 1] \text{ such that } \gamma(t) \notin \pi(N)\}) \leq \mu$$

by (4.13).

Hence by Theorem 4.2, we deduce that  $\pi(X) \subseteq F_\lambda$  is a universal Fréchet set. This holds for every  $\lambda \in (0, 1)$ .

The final observation again follows by noting that if  $y \in F_{t'}$  for some  $t' \in (0, t)$  and  $\varepsilon > 0$ , then the collection

$$(F_{\lambda t + (1-\lambda)t'} \cap B_\varepsilon(y))_{0 < \lambda < 1}$$

still satisfies the conditions of the present theorem, so contains, for each  $\lambda$ , a point of Fréchet differentiability of any Lipschitz function  $g: Y \rightarrow \mathbb{R}$ . Therefore so does

$$\left( \bigcup_{0 < \lambda < t} F_\lambda \right) \cap B_\varepsilon(y). \quad \square$$

**Remark 4.4.** *We make two short remarks.*

1. *The condition that  $Y$  has an equivalent norm that is Fréchet differentiable on  $Y \setminus \{0\}$  is always satisfied if  $Y^*$  is separable - that is if  $Y$  is a separable Asplund space. See [4]. We shall make a brief remark about the case in which  $Y$  is non-separable in the final section of the thesis.*
2. *If  $Y$  is a Banach space then we say that a function  $g: Y \rightarrow \mathbb{R}$  is locally Lipschitz if for every  $y \in Y$  there exists an open neighbourhood  $U$  of  $y$  in  $Y$  and a constant  $L_U > 0$  such that for every  $z, w \in U$ ,*

$$|g(z) - g(w)| \leq L_U \|z - w\|.$$

*It is easy to show that if  $X, Y$  and  $\pi: X \rightarrow Y$  satisfy the conditions of Theorem 4.2 then the set  $\pi(X)$  contains a Fréchet differentiability point of every locally Lipschitz  $g: Y \rightarrow \mathbb{R}$ . Let  $x \in X$ ,  $y = \pi x$  and let  $U$  be an open neighbourhood of  $y \in Y$  as above. Then one can extend  $g|_U$  to a Lipschitz function  $\bar{g}: Y \rightarrow \mathbb{R}$ . As the points of Fréchet differentiability of  $\bar{g}$  are dense in  $\pi(X)$  we can find such a point  $y \in \pi(X) \cap U$ ; then  $y$  is a point of Fréchet differentiability of  $g$ . Similarly, if  $(F_\lambda)_{0 < \lambda < 1}$  satisfies the conditions of Corollary 4.3 then  $\bigcup_{0 < \lambda < 1} F_\lambda$  contains a point of Fréchet differentiability of every locally Lipschitz  $g: Y \rightarrow \mathbb{R}$ .*

## 4.2 The original example of Preiss revisited

By Rademacher's theorem, every subset of a Euclidean space with positive Lebesgue measure is trivially seen to be a universal Fréchet set. The converse however is false in any Euclidean space of dimension at least two. The original proof of this statement is due to Preiss - see [23] - and in this section we shall revisit his example.

We first recall the elementary fact that any  $G_\delta$  subset of a complete metric space is a complete topological space.

**Lemma 4.5.** *If  $(Y, \Delta)$  is a complete metric space and  $O = \bigcap_{n \geq 1} O_n$ , where the sets  $O_n$  are open, then the subspace topology on  $O$  is induced by a complete metric  $d$  where*

$$d(x, y) = \Delta(x, y) + \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left( \left| \frac{1}{\Delta(y, O_n^c)} - \frac{1}{\Delta(x, O_n^c)} \right|, 1 \right).$$

*Proof.* It is straightforward to check this formula defines a metric on  $O$ . That  $d$  induces the subspace topology on  $O$ , considered as a subset of  $Y$ , follows from the inequality  $\Delta(x, y) \leq d(x, y)$  and the fact that  $d$  is a continuous function on  $O \times O$ , with respect to the metric  $\Delta$ ; the latter holds as  $d$  is a uniform limit of continuous functions.

Finally if  $(y_m)_{m \geq 1}$  is a Cauchy sequence in  $O$  then as  $\Delta(y_m, y_n) \leq d(y_m, y_n)$  and  $(Y, \Delta)$  is complete we have  $\Delta(y_m, y) \rightarrow 0$  for some  $y \in Y$ . Further, for each  $n \geq 1$  the sequence

$$\left( \frac{1}{\Delta(y_m, O_n^c)} \right)_{m \geq 1}$$

is Cauchy; therefore it is a bounded sequence. It quickly follows that  $\Delta(y, O_n^c) > 0$  for all  $n \geq 1$  so that  $y \in O$ . Hence  $d$  is complete.  $\square$

**Theorem 4.6.** *Suppose that  $Y$  is a Banach space with separable dual,  $T$  is a dense subset of  $Y$  and  $S \subseteq Y$  is a  $G_\delta$  subset of  $Y$  containing every line segment joining two points of  $T$ . Then  $S$  is a universal Fréchet set in  $Y$ .*

*Proof.* We let the space  $X = S$  with the topology inherited from  $Y$ ; as  $S$  is  $G_\delta$ ,  $X$  is a complete space. Let  $\pi: X \rightarrow Y$  be the inclusion map; clearly  $\pi$  is continuous. Note that  $Y$  has an equivalent norm that is Fréchet differentiable on  $Y \setminus \{0\}$  by Theorem 1.16. The conditions (4.1)-(4.3) of Theorem 4.2 can be satisfied for any  $\delta_0 > 0$  by taking  $\gamma$  to be an affine map and using the density of  $T$ . Therefore  $\pi(X) = X = S$  is a universal Fréchet set.  $\square$

Instead of proving directly that  $S$  may be taken to be Lebesgue null if  $Y = \mathbb{R}^M$  for  $M \geq 2$ , we shall show the stronger fact that  $S$  may be taken to have

Hausdorff dimension at most one for any Banach space  $Y$  with separable dual.

We first recall the definition of Hausdorff dimension.

**Definition 4.7.** *If  $S$  is a metric space then we call a sequence  $(C_i)_{i \geq 1}$  of subsets of  $S$  a countable cover of  $S$  if  $S = \cup_{i \geq 1} C_i$ . We then define, for any non-negative real number  $d$ ,*

$$C_H^d(S) = \inf \sum_{i \geq 1} \text{diameter}(C_i)^d$$

and then define the Hausdorff dimension

$$d_H(S) = \inf \{d > 0 \text{ such that } C_H^d(S) = 0\}.$$

We may consider any subset  $S$  of a Banach space  $Y$  as a metric space in its own right and apply this definition to calculate its Hausdorff dimension.

**Lemma 4.8.** *If  $Y$  is a Banach space and  $L$  is a countable union of line segments in  $Y$  then there exists a  $G_\delta$  subset  $O$  of  $Y$  with  $L \subseteq O$  and such that the Hausdorff dimension of  $O$  is less than or equal to one.*

*Proof.* We first note that it is easily checked that  $C_H^d$  is countably subadditive, when considered as a set function defined on subsets of a metric space, and that if  $l$  is a line segment of length at most one,  $d > 0$  and  $k$  is a positive integer then

$$C_H^d(B_{1/k}(l)) \leq k \cdot \left(\frac{4}{k}\right)^d = 4^d \cdot k^{-(d-1)}, \quad (4.14)$$

as we may cover  $B_{1/k}(l)$  with  $k$  open balls of radius  $2/k$ . Here, as usual,  $B_r(S)$  denotes the open  $r$ -neighbourhood of  $S$

$$B_r(S) := \bigcup_{y \in S} B_r(y)$$

for any  $S \subseteq Y$  and  $r > 0$ .

Now let  $L$  be a countable union of line segments in a Banach space  $Y$ . One may write

$$L = \bigcup_{m \geq 1} L_m$$

where each  $L_m$  is a line segment in  $Y$  of length at most one.

We let

$$O_n = \bigcup_{m=1}^{\infty} B_{1/2^{m+n}}(L_m)$$

and

$$O = \bigcap_{n=1}^{\infty} O_n.$$

Note that  $O$  is a  $G_\delta$  subset of  $Y$ , containing  $L$ . We must verify that the Hausdorff dimension of  $O$  is no greater than one.

Let  $d > 1$ . It suffices, by Definition 4.7, to show that

$$C_H^d(O) = 0.$$

But for each  $n$ ,

$$\begin{aligned} C_H^d(O) &\leq C_H^d(O_n) \\ &\leq \sum_{m=1}^{\infty} C_H^d(B_{1/2^{m+n}}(L_m)) \\ &\leq \sum_{m=1}^{\infty} 4^d \cdot 2^{-(m+n)(d-1)} \\ &= 4^d \cdot \frac{2^{-(n+1)(d-1)}}{1 - 2^{-(d-1)}} \end{aligned}$$

using the countable subadditivity of  $C_H^d$ , (4.14) and  $d > 1$ . The right hand side tends to 0 as  $n \rightarrow \infty$  so we deduce that

$$C_H^d(O) = 0$$

for all  $d > 1$ , as required.  $\square$

**Corollary 4.9.** *Any Banach space  $Y$  with separable dual has a universal Fréchet set  $S \subseteq Y$  with Hausdorff dimension at most one.*

*Proof.* This is immediate by combining Theorem 4.6 with Lemma 4.8.  $\square$

We may remark that, for non-trivial  $Y$ , the Hausdorff dimension of a such a set  $S$  must in fact be exactly one.

**Lemma 4.10.** *If  $Y$  is a Banach space with  $Y \neq \{0\}$  and  $S \subseteq Y$  has Hausdorff dimension less than one then there exists a Lipschitz function  $g: Y \rightarrow \mathbb{R}$  that is not Fréchet differentiable at any  $y \in S$ .*

*Proof.* Let  $\pi \in Y^*$  with  $\|\pi\| = 1$ . As  $\pi$  is Lipschitz and Hausdorff dimension is non-increasing under taking the Lipschitz image, the set

$$\pi(S) \subseteq \mathbb{R}$$

has Hausdorff dimension less than one. Hence  $\pi(S)$  has Lebesgue measure zero. By Lemma 1.6 we can therefore find a Lipschitz function  $f: \mathbb{R} \rightarrow \mathbb{R}$  that is nowhere differentiable on  $\pi(S)$ . We let  $g: Y \rightarrow \mathbb{R}$  be given by

$$g = f \circ \pi.$$

We note that  $g$  is Lipschitz. Let  $e \in S(Y)$  with  $\pi e \neq 0$ . For any  $y \in S$  and  $t \in \mathbb{R}$ ,

$$g(y + te) - g(y) = f(\pi y + (\pi e)t) - f(\pi y).$$

As  $f$  is not differentiable at  $\pi y \in \pi(S)$ , we deduce that the directional derivative

$$g'(y, e)$$

does not exist for any  $y \in S$ ; hence  $g: Y \rightarrow \mathbb{R}$  is not Fréchet differentiable at any  $y \in S$ .  $\square$

Finally, we use the elementary fact that any subset of  $Y = \mathbb{R}^M$  with Hausdorff dimension strictly less than  $M$  has Lebesgue measure zero to deduce the following, from Corollary 4.9.

**Corollary 4.11.** *If  $M \geq 2$  then  $\mathbb{R}^M$  has a universal Fréchet set of Lebesgue measure zero.*

See [23, Corollary 6.5], and the subsequent remark, for the original presentation of this result.

### 4.3 A compact null universal Fréchet set in Euclidean space

Although the universal sets we constructed in the previous section can be taken to be Lebesgue null, in finite dimensional Euclidean space, their closure necessarily has positive measure. In this section, for our second non-trivial example of a universal Fréchet set, we shall re-derive the result, due to Maleva and the author [8], that in any Euclidean space of dimension at least two, such a set may be taken to be *compact* as well as Lebesgue null.

To demonstrate this, we shall construct a family of compact and Lebesgue null sets  $(F_\lambda)_{0 < \lambda < 1}$  that satisfy the conditions of Corollary 4.3.

We shall work in  $\mathbb{R}^M$  where  $M \geq 2$ . We shall denote the standard inner product on  $\mathbb{R}^M$  by  $\langle \cdot, \cdot \rangle$ , the Euclidean norm on  $\mathbb{R}^M$  by  $\|\cdot\|$  and we shall respectively



denote open and closed balls, in the Euclidean norm, by  $B_r(x)$  and  $\overline{B}_r(x)$ . It will also be convenient to use the supremum norm  $\|\cdot\|_\infty$  on  $\mathbb{R}^M$ ; we shall denote balls in this norm by  $B_\infty(a, r)$  and  $\overline{B}_\infty(a, r)$ .

The construction we give here is very similar to the one in [8]; in fact the example of a compact null universal Fréchet set coincides exactly in the case  $M = 2$ . We shall construct a set system  $(F_\lambda)_{0 < \lambda < 1}$  such that if  $\lambda' < \lambda$  and  $x \in F_{\lambda'}$  then in  $F_\lambda$  one can find, nearby  $x$ , pieces of a dense set of *hyperplanes*, with co-dimension one in  $\mathbb{R}^M$ . In fact hyperplanes are not strictly necessary; to apply Corollary 4.3 we only require that  $F_\lambda$  contains sufficiently many *line segments* nearby  $x$ .

**Definition 4.12.** *We define the following.*

1. Let  $(N_n)_{n \geq 1}$  be a sequence of odd integers with  $N_n > 1$  such that  $N_n \rightarrow \infty$  and  $\sum_n \frac{1}{N_n^M} = \infty$ .
2. For  $\lambda \in [0, 1]$  put  $d_0 = d_0(\lambda) = 1$  and, for each  $n \geq 1$ , set

$$d_n(\lambda) = \frac{1}{N_1 N_2 \dots N_{n-1} N_n^\lambda}$$

and  $d_n = d_n(1) = \frac{1}{N_1 N_2 \dots N_{n-1} N_n}$ .

3. For each  $n \geq 1$  define the lattice  $C_n \subseteq \mathbb{R}^M$  by

$$C_n = d_{n-1} \left( \left( \frac{1}{2}, \dots, \frac{1}{2} \right) + \mathbb{Z}^M \right). \quad (4.15)$$

4. Given  $\lambda \in [0, 1]$  define the set  $T_n(\lambda) \subseteq \mathbb{R}^M$  by

$$T_n(\lambda) = \bigcup_{c \in C_n} B_\infty \left( c, \frac{1}{2} d_n(\lambda) \right). \quad (4.16)$$

5. Given  $\lambda \in [0, 1]$  define the set  $F_\lambda \subseteq \mathbb{R}^M$  by

$$F_\lambda = \mathbb{R}^M \setminus \bigcup_{n=1}^{\infty} T_n(\lambda). \quad (4.17)$$

We note that for  $\lambda' \leq \lambda$  we have  $T_n(\lambda') \supseteq T_n(\lambda)$  so that  $F_{\lambda'} \subseteq F_\lambda$ .

**Lemma 4.13.** *For each  $\lambda \in [0, 1]$  the set  $F_\lambda$  is a closed, non-empty subset of  $\mathbb{R}^M$  with Lebesgue measure zero.*

*Proof.* The set  $F_\lambda$  is closed as each  $T_n(\lambda)$  is open and it is non-empty as it contains 0. We now verify that  $F_\lambda$  has Lebesgue measure zero.

For each  $n \geq 0$  we define sets  $D_n$  and  $R_n$  of disjoint open  $M$ -dimensional hypercubes as follows. Let  $D_0 = \emptyset$  and  $R_0 = \{(0,1)^M\}$ . Given  $n \geq 1$  divide each hypercube in the set  $R_{n-1}$  into  $N_n^M$  equal open cubes. Let  $D_n$  comprise the central cubes of each such division and let  $R_n$  comprise all the remaining cubes. By induction each cube in  $D_n$  and  $R_n$  has side  $d_n$  and the centres of the cubes in  $D_n$  belong to the lattice  $C_n$ .

Now for each  $m \geq 1$ ,

$$F_\lambda \subseteq \mathbb{R}^M \setminus \left( \bigcup_{n=1}^m \bigcup_{c \in C_n} B_\infty \left( c, \frac{1}{2} d_n \right) \right)$$

so that

$$F_\lambda \cap [0,1]^M \subseteq [0,1]^M \setminus \bigcup_{n=1}^m D_n = \overline{\bigcup_{n=1}^m R_n},$$

and as  $|R_m| = (N_1^M - 1) \dots (N_m^M - 1)$  we can now estimate the Lebesgue measure of  $F_\lambda \cap [0,1]^M$  as follows:

$$m(F_\lambda \cap [0,1]^M) \leq |R_m| d_m^M = \left(1 - \frac{1}{N_1^M}\right) \dots \left(1 - \frac{1}{N_m^M}\right).$$

This tends to 0 as  $m \rightarrow \infty$ , because  $\sum \frac{1}{N_r^M} = \infty$ . Therefore the Lebesgue measure

$$m(F_\lambda \cap [0,1]^M) = 0.$$

Furthermore, from (4.15), (4.16) and (4.17),  $F_\lambda$  is invariant under translations by the lattice  $\mathbb{Z}^M$ . Hence  $m(F_\lambda) = 0$  for every  $\lambda \in [0,1]$ . □

Given a non-zero vector  $a \in \mathbb{R}^M$  we use the notation  $a^\perp$  to denote the hyperplane of all vectors perpendicular to  $a$ :

$$a^\perp := \{x \in \mathbb{R}^M \text{ such that } \langle x, a \rangle = 0\}.$$

If  $a \in \mathbb{Z}^M \setminus \{0\}$  we call the set  $a^\perp$  a *rational hyperplane*.

The following is a simple observation.

**Lemma 4.14.** *If  $y, c, a \in \mathbb{R}^M$  and  $d > 0$  with  $a \neq 0$  and the affine hyperplane  $y + a^\perp$*

intersects the hypercube  $B_\infty(c, d)$  then we have

$$|\langle y - c, a \rangle| < dM^{1/2}\|a\|. \quad (4.18)$$

*Proof.* Letting  $y'$  be a point of intersection, we have  $\langle y' - y, a \rangle = 0$  and  $\|y' - c\|_\infty < d$ . Hence

$$\begin{aligned} |\langle y - c, a \rangle| &= |\langle y' - c, a \rangle| \\ &\leq \|y' - c\| \cdot \|a\| \\ &\leq M^{1/2} \cdot \|y' - c\|_\infty \cdot \|a\| \\ &< M^{1/2} \cdot d \cdot \|a\|. \end{aligned}$$

□

The next couple of lemmas show that we can shift any rational hyperplane slightly to make it avoid  $T_n(\lambda)$  for large values of  $n$ .

**Lemma 4.15.** *If  $\lambda \in [0, 1]$ ,  $a \in \mathbb{Z}^M \setminus \{0\}$ ,  $x \in \mathbb{R}^M$  and  $I \subseteq \mathbb{R}$  is a closed interval of length at least  $3L$  where*

$$L := \frac{M^{1/2}d_n(\lambda)}{\|a\|}$$

and  $n \geq 1$  is such that

$$N_n^\lambda \geq 4M^{1/2}\|a\|$$

then we may find a closed subinterval  $I' \subseteq I$  of length at least  $L$  such that the affine hyperplane  $x + \mu a + a^\perp$  does not intersect  $T_n(\lambda)$  for any  $\mu \in I'$ .

*Proof.* We may assume that  $I = [t, t + 3L]$  for some  $t \in \mathbb{R}$ . We claim we may either take  $I' = [t, t + L]$  or  $I' = [t + 2L, t + 3L]$ .

Assuming, for a contradiction, that there exists

$$\mu_1 \in [t, t + L] \text{ and } \mu_2 \in [t + 2L, t + 3L]$$

with

$$(x + \mu_i a + a^\perp) \cap T_n(\lambda) \neq \emptyset$$

for  $i = 1, 2$ , then we may find  $c_1, c_2 \in C_n$  with

$$(x + \mu_i a + a^\perp) \cap B_\infty\left(c_i, \frac{1}{2}d_n(\lambda)\right) \neq \emptyset.$$

Then

$$|\langle x + \mu_i a - c_i, a \rangle| < \frac{d_n(\lambda)M^{1/2}}{2}\|a\| \quad (4.19)$$

for  $i = 1, 2$ , by Lemma 4.14. We also note that

$$L \leq |\mu_2 - \mu_1| \leq 3L. \quad (4.20)$$

Using the triangle inequality on (4.19) we obtain

$$|\langle (c_2 - c_1) - (\mu_2 - \mu_1)a, a \rangle| < d_n(\lambda)M^{1/2}\|a\| = L\|a\|^2. \quad (4.21)$$

If  $\langle c_2 - c_1, a \rangle = 0$  then

$$|\mu_2 - \mu_1| \cdot \|a\|^2 < L\|a\|^2$$

contradicting (4.20).

If  $\langle c_2 - c_1, a \rangle \neq 0$  then, as  $c_2 - c_1 = d_{n-1}l$  for some  $l \in \mathbb{Z}^M$ , we have

$$|\langle c_2 - c_1, a \rangle| = d_{n-1}|\langle l, a \rangle| \geq d_{n-1},$$

since  $a \in \mathbb{Z}^M$ . Hence from (4.21) and (4.20),

$$d_{n-1} - 3L\|a\|^2 < L\|a\|^2.$$

This can be re-written

$$d_{n-1} < 4d_n(\lambda)M^{1/2}\|a\|$$

i.e.

$$N_n^\lambda < 4M^{1/2}\|a\|,$$

which contradicts the condition given in the statement.

Either way we have a contradiction.  $\square$

**Lemma 4.16.** *If  $\lambda \in [0, 1]$ ,  $a \in \mathbb{Z}^M \setminus \{0\}$  and  $n \geq 1$  are such that*

$$N_m^\lambda \geq 4M^{1/2}\|a\|$$

*for all  $m \geq n$ , then for any  $y \in \mathbb{R}^M$  there exists  $y' \in \mathbb{R}^M$  with*

$$\|y' - y\| \leq 3M^{1/2}d_n(\lambda)$$

*such that the affine hyperplane  $y' + a^\perp$  does not intersect  $T_m(\lambda)$  for any  $m \geq n$ .*

*Proof.* Let

$$L_m(\lambda) = \frac{M^{1/2}d_m(\lambda)}{\|a\|}$$

for  $m \geq n$ . Using Lemma 4.15 and  $L_{m+1}(\lambda) \leq L_m(\lambda)/3$  for  $m \geq n$ , from

$$d_{m+1}(\lambda) = \frac{d_m(\lambda)}{N_m^{1-\lambda}N_{m+1}^\lambda} \leq \frac{d_m(\lambda)}{3},$$

we may inductively construct a sequence of nested closed intervals

$$[0, 3L_n(\lambda)] = I_n \supseteq I_{n+1} \supseteq \dots$$

with  $|I_m| = 3L_m(\lambda)$  for each  $m \geq n$  such that for any  $m \geq n$  and  $\mu \in I_m$  the affine hyperplane  $x + \mu a + a^\perp$  does not intersect  $T_m(\lambda)$ .

Taking

$$\mu \in \bigcap_{m=n}^{\infty} I_m$$

we may set  $y' = y + \mu a$  and note that as  $\mu \in I_n$  we have

$$\|y' - y\| = \mu\|a\| \leq 3L_n(\lambda)\|a\| = 3M^{1/2}d_n(\lambda).$$

□

We now show how to avoid  $T_n(\lambda')$  for low values of  $n$  and some value  $\lambda' \in (0, 1)$ .

**Lemma 4.17.** *If  $n \geq 1$ ,  $0 \leq \lambda < \lambda + \psi \leq 1$ ,  $x \in \mathbb{R}^M \setminus T_n(\lambda)$  and*

$$0 < \alpha < 1 - \frac{1}{N_n^\psi}$$

*then*

$$B_{\alpha d_n(\lambda)/2}(x) \cap T_n(\lambda + \psi) = \emptyset.$$

*Proof.* If  $x \notin T_n(\lambda)$  then for every  $c \in C_n$  we have

$$\|x - c\|_\infty \geq \frac{1}{2}d_n(\lambda).$$

Then for any  $x' \in B_{\alpha d_n/2}(x)$ ,

$$\begin{aligned}
\|x' - c\|_\infty &\geq \|x - c\|_\infty - \|x' - x\|_\infty \\
&\geq \frac{1}{2}d_n(\lambda) - \|x' - x\| \\
&\geq \frac{1}{2}d_n(\lambda) - \alpha \frac{d_n(\lambda)}{2} \\
&\geq \frac{1}{2}d_n(\lambda) - \left(1 - \frac{1}{N_n^\psi}\right) \frac{d_n(\lambda)}{2} \\
&= \frac{1}{2}d_n(\lambda + \psi).
\end{aligned}$$

This holds for every  $c \in C_n$  so that

$$x' \notin T_n(\lambda + \psi).$$

□

Combining Lemma 4.16 and Lemma 4.17 we obtain the following.

**Theorem 4.18.** *If  $\varepsilon > 0$ ,  $\psi > 0$  and  $a \in \mathbb{Z}^M \setminus \{0\}$ , there exists*

$$\delta_2 = \delta_2(\varepsilon, \psi, a)$$

*such that for any  $\delta \in (0, \delta_2)$ ,  $\lambda \in (0, 1)$  with  $0 \leq \lambda < \lambda + \psi \leq 1$ ,  $x \in F_\lambda$  and  $y \in \mathbb{R}^M$  there exists  $y' \in B_{\varepsilon\delta}(y)$  such that*

$$(y' + a^\perp) \cap B_\delta(x) \subseteq F_{\lambda+\psi}.$$

*Proof.* Pick  $\alpha > 0$  with  $\alpha < 1 - N_n^{-\psi}$  for all  $n \geq 1$ . Find  $n_0 \geq 1$  such that

$$6M^{1/2} < \varepsilon\alpha N_n^\psi$$

and

$$N_n^\psi \geq 4M^{1/2}\|a\|$$

for  $n > n_0$ . Choose  $\delta_2 > 0$  such that  $2\delta_2 < d_n\alpha$  for  $n \leq n_0$ . Let  $\delta \in (0, \delta_2)$ . Pick a minimal  $n$  such that  $d_n(\lambda)\alpha < 2\delta$ . Note that  $n > n_0$  so that  $6M^{1/2} < \varepsilon\alpha N_n^\psi$ . Given  $y \in \mathbb{R}^M$ , by Lemma 4.16 we can find  $y' \in \mathbb{R}^M$  with

$$(y' + a^\perp) \cap T_m(\lambda + \psi) = \emptyset \tag{4.22}$$

for all  $m \geq n$  where

$$\begin{aligned} \|y' - y\| &\leq 3M^{1/2}d_n(\lambda + \psi) \\ &= 3M^{1/2}N_n^{-\psi} \cdot d_n(\lambda) \\ &\leq \frac{\varepsilon\alpha}{2} \cdot \frac{2\delta}{\alpha} \\ &= \varepsilon\delta. \end{aligned}$$

Now for  $m < n$  we have  $d_n(\lambda)\alpha \geq 2\delta$ , by the minimality of  $n$ , so that

$$B_\delta(x) \cap T_m(\lambda + \psi) = \emptyset \tag{4.23}$$

by Lemma 4.17.

Combining 4.22 and 4.23 we deduce that for all  $m \geq 1$ ,

$$(y' + a^\perp) \cap B_\delta(x) \cap T_m(\lambda + \psi) = \emptyset$$

so that

$$(y' + a^\perp) \cap B_\delta(x) \subseteq F_{\lambda+\psi}$$

as required. □

We now wish to replace the condition that  $a \in \mathbb{Z}^M \setminus \{0\}$  with  $a \in S(\mathbb{R}^M)$ , the unit sphere of  $\mathbb{R}^M$ , and to obtain a uniform estimate over  $a$  belonging to the latter. To accomplish this we note that the set of scalar multiples of elements of  $\mathbb{Z}^M$  is dense in the unit sphere and then use the fact that the unit sphere is totally bounded.

**Corollary 4.19.** *If  $\varepsilon, \psi > 0$  there exists*

$$\delta_3 = \delta_3(\varepsilon, \psi)$$

*such that if  $a \in S(\mathbb{R}^M)$ ,  $0 \leq \lambda < \lambda + \psi \leq 1$ ,  $\delta \in (0, \delta_3)$ ,  $x \in F_\lambda$  and  $y \in B_\delta(x)$ , then there exists  $y' \in \mathbb{R}^M$  with  $\|y' - y\| < \varepsilon\delta$  and  $a' \in S(\mathbb{R}^M)$  with  $\|a' - a\| < \varepsilon$  such that*

$$(y' + a'^\perp) \cap B_\delta(x) \subseteq F_{\lambda+\psi}.$$

*Proof.* Find  $a_1, \dots, a_n \in \mathbb{Z}^M \setminus \{0\}$  such that

$$S(\mathbb{R}^M) \subseteq \bigcup_{1 \leq i \leq n} B_\varepsilon \left( \frac{a_i}{\|a_i\|} \right)$$

and set

$$\delta_3 = \min_{1 \leq i \leq n} \delta_2(\varepsilon, \psi, a_i).$$

□

It follows immediately that the set system  $(F_\lambda)_{0 < \lambda < 1}$  satisfies the conditions of Corollary 4.3. Hence we deduce the following.

**Corollary 4.20.** *For  $M \geq 2$  there exists a compact, null universal Fréchet set in  $\mathbb{R}^M$ .*

**Remark 4.21.** *We make two quick remarks.*

1. *As any universal Fréchet set is non- $\sigma$ -porous, the sets we have constructed are examples of closed, null, non- $\sigma$ -porous subsets of  $\mathbb{R}^M$ , for  $M \geq 2$ . Such sets were originally constructed by Zajíček in [29]. In fact one of the sets constructed in the aforementioned paper is precisely our set  $F_1$ .*
2. *Since the set system we construct has the strong property that one can find pieces of hyperplanes of codimension one, nearby a suitable set of points, it is natural to conjecture that the sets we have constructed should contain a point of Fréchet differentiability of every Lipschitz*

$$f: \mathbb{R}^M \rightarrow \mathbb{R}^{M-1}$$

*at least for the case  $M = 3$ . This will be investigated in [11], building on work of Lindenstrauss, Preiss and Tišer; see [19].*

## 4.4 A compact universal Fréchet set of Hausdorff dimension one in Euclidean space

So far we have shown, in finite dimensional Euclidean space, that a universal Fréchet set may be constructed with Hausdorff dimension one. We have also shown, separately, that such a set may be constructed to be compact and Lebesgue null. The set the former construction gives, however, is necessarily somewhere dense, so that



its closure has positive Lebesgue measure. It can be checked that the Hausdorff dimension of the set given by the latter construction is equal to the dimension of the underlying space. One may ask, therefore, if we may combine the features of these two examples and construct, in Euclidean space, a universal Fréchet set that is compact and has Hausdorff dimension one. We shall answer this question affirmatively by proving the following.

**Theorem 4.22.** *If  $M \geq 1$  then there is a compact universal Fréchet set  $S \subseteq \mathbb{R}^M$  with Hausdorff dimension one.*

See also [9].

We shall start by working in a more general setting. First, we fix some notation. If  $(Y, d)$  is a metric space then, as is standard, we use  $d(y, S)$  to denote the distance from a point  $y$  to a set  $S$

$$d(y, S) := \inf_{z \in S} d(y, z)$$

and given  $r > 0$  we define the open and closed  $r$ -neighbourhoods of  $S$  by

$$\begin{aligned} B_r(S) &= \{y \in Y \text{ such that } d(y, S) < r\} \\ \overline{B}_r(S) &= \{y \in Y \text{ such that } d(y, S) \leq r\}. \end{aligned}$$

As before we use  $B_r(y) := B_r(\{y\})$  and  $\overline{B}_r(y) := \overline{B}_r(\{y\})$  to respectively denote the open and closed balls of radius  $r$  centred at a point  $y \in Y$ .

**Theorem 4.23.** *We fix the following data. Let*

- $(Y, d)$  be a metric space,
- $(K_r)_{r \in R}$  be a collection of compact subsets of  $Y$  indexed by a metric space  $(R, \gamma)$ , with the property that if  $\gamma(s, r) \leq \delta$  then

$$K_s \subseteq \overline{B}_\delta(K_r), \tag{4.24}$$

*i.e.  $\mathcal{H}(K_r, K_s) \leq \gamma(r, s)$  for every  $r, s \in R$  where  $\mathcal{H}$  is the Hausdorff metric,*

- $O$  be a  $G_\delta$  subset of  $Y$  containing a  $\gamma$ -dense subset of  $(K_r)_{r \in R}$ , *i.e.*

$$\forall r \in R, \varepsilon > 0 \exists s \in R \text{ such that } K_s \subseteq O \text{ and } \gamma(s, r) < \varepsilon, \tag{4.25}$$

- $(A_n)_{n \geq 1}$  be a strictly increasing sequence of positive integers and the metric space  $(I, \Delta)$  comprise all  $(\lambda, T)$  such that  $\lambda \in [0, 1]$  and  $T = (T_n)_{n \geq 1}$  is a

strictly increasing sequence of positive integers with  $\sup_n \frac{T_n}{A_n} < \infty$ , with the metric  $\Delta$  given by

$$\Delta((\lambda', T'), (\lambda, T)) = \max \left( |\lambda' - \lambda|, \sup_n \frac{|T'_n - T_n|}{A_n} \right).$$

If the space  $(R, \gamma)$  is totally bounded - that is for every  $\varepsilon > 0$  there exists a finite subset  $R(\varepsilon) \subseteq R$  such that for every  $r \in R$  there exists  $s \in R(\varepsilon)$  with  $\gamma(s, r) < \varepsilon$  - then for any  $r_0 \in R$  with  $K_{r_0} \subseteq O$  we can find a collection  $(S_i)_{i \in I}$  of subsets of  $Y$  with

$$K_{r_0} \subseteq S_i \subseteq O \text{ for every } i \in I, \quad (4.26)$$

such that the following hold.

(1) For each  $\varepsilon, \psi > 0$  and  $i = (\lambda, T) \in I$  with  $\lambda < 1$  there exists

$$\delta_4 = \delta_4(\varepsilon, \psi, i) > 0$$

such that the following property holds. If  $\delta \in (0, \delta_4)$ ,  $y \in S_i$  and  $s \in R$  with  $K_s \subseteq \overline{B}_\delta(y)$  there exists  $t \in R$  and  $j \in I$  such that  $K_t \subseteq S_j$ ,  $\gamma(t, s) < \varepsilon \delta$  and  $\Delta(j, i) \leq \psi$ .

(2) The set  $X$  is a closed subset of  $I \times Y$  where

$$X := \{(i, y) \text{ such that } y \in S_i\}.$$

(3) For every  $i \in I$  and  $\psi > 0$  then  $F = F(i, \psi)$  is a closed subset of  $Y$  where

$$F := \{y \in Y \text{ such that } y \in S_j \text{ for some } j \in I \text{ with } \Delta(j, i) \leq \psi\}.$$

*Proof.* Using (4.25) and the fact that  $R$  is totally bounded, for each  $\varepsilon > 0$  we can find a finite set  $R(\varepsilon) \subseteq R$  with  $K_r \subseteq O$  for every  $r \in R(\varepsilon)$ , such that for every  $r \in R$  there exists  $s \in R(\varepsilon)$  with  $\gamma(s, r) < \varepsilon$ .

We write  $O = \bigcap_{k=1}^{\infty} O_k$  where  $O_k$  are open subsets of  $Y$  with  $O_{k+1} \subseteq O_k$  for each  $k \geq 1$ .

Let  $R_0 = \{r_0\}$  and  $w_0 = 1$ .

For each  $k \geq 1$  we define a non-empty finite set  $R_k \subseteq R$  and  $w_k > 0$ , with  $K_r \subseteq O$  for every  $r \in R_k$ , by the recursion

- $R_k = R_{k-1} \cup R(w_{k-1}/k)$

- $w_k \in (0, 1/k)$  is such that  $\overline{B}_{w_k}(K_r) \subseteq O_k$  for every  $r \in R_k$ .

Such a  $w_k$  exists because for every  $r \in R_k$  we know  $K_r \subseteq O \subseteq O_k$ ,  $K_r$  is compact,  $O_k$  is open and the set  $R_k$  is finite.

Given  $k \geq 1$  and  $\lambda \in [0, 1]$  we put

$$M_k(\lambda) = \bigcup_{r \in R_k} \overline{B}_{\lambda w_k}(K_r) \quad (4.27)$$

and then define  $S_i$  for each  $i = (\lambda, T) \in I$  by

$$S_i = \bigcap_{n=1}^{\infty} M_{T_n}(\lambda). \quad (4.28)$$

Note that as  $M_k(\lambda) \subseteq O_k$  we have  $S_i \subseteq O_{T_n}$  for each  $n \geq 1$  and hence  $S_i \subseteq O$  since  $T_n \rightarrow \infty$ . Further, as  $r_0 \in R_0 \subseteq R_k$  we have  $K_{r_0} \subseteq M_k(\lambda)$  for all  $k \geq 1$  and  $\lambda \in [0, 1]$ , so that  $K_{r_0} \subseteq S_i$  for every  $i \in I$ . Hence (4.26).

We now verify (1). First note that we may assume that  $\varepsilon < 1$  and that  $\psi \leq 1 - \lambda$  where  $i = (\lambda, T)$ .

As  $T_n \rightarrow \infty$  and  $A_n \rightarrow \infty$  we may pick  $k_0$  such that  $T_{k_0} > 2/\psi\varepsilon$  and  $A_{k_0} > 1/\psi$ . Let  $\delta_4 > 0$  be such that  $2\delta_4 \leq \psi w_{T_n}$  for all  $n < k_0$ .

Suppose that  $\delta \in (0, \delta_4)$ . As  $w_{T_n} \rightarrow 0$  we may pick a minimal  $k$  with  $\psi w_{T_k} < 2\delta$ . Note that  $k \geq k_0$ . Put  $j = (\lambda', T') \in I$  where  $\lambda' = \lambda + \psi \in [0, 1]$  and  $T'$  is given by

$$\begin{aligned} T'_l &= T_l \text{ for } l < k \\ T'_l &= T_l + 1 \text{ for } l \geq k. \end{aligned}$$

We have  $\Delta(j, i) = \max(|\lambda' - \lambda|, \sup_{l \geq k} 1/A_l) = \psi$  as  $\lambda' - \lambda = \psi$  and  $A_k \geq A_{k_0} > \psi^{-1}$ .

Now let  $y \in S_i$  and  $K_s \subseteq \overline{B}_\delta(y)$ . Pick  $t \in R(w_{T_k}/(T_k + 1)) \subseteq R_{T_k+1}$  with

$$\begin{aligned} \gamma(t, s) &< w_{T_k}/(T_k + 1) \\ &< \psi w_{T_k}/\psi T_k \\ &< 2\delta/\psi T_{k_0} < \varepsilon\delta, \end{aligned}$$

where we have used  $T_{k_0} > 2/\psi\varepsilon$  and  $\psi w_{T_k} < 2\delta$ .

Note that if  $l \geq k$  then  $T'_l \geq T_k + 1$  so that  $t \in R_{T_k+1} \subseteq R_{T'_l}$ . Hence

$K_t \subseteq M_{T_l'}(\lambda')$  for  $l \geq k$  by (4.27). However for  $l < k$  we have

$$w_{T_l'}\lambda' - w_{T_l'}\lambda = w_{T_l'}\psi \geq 2\delta \quad (4.29)$$

using the minimality of  $k$  with  $w_{T_k}\psi < 2\delta$ . As  $y \in M_{T_l'}(\lambda)$  we conclude  $\overline{B}_{2\delta}(y) \subseteq M_{T_l'}(\lambda')$  using the definition (4.27) and (4.29). Then from  $\gamma(t, s) < \delta e \leq \delta$ , (4.24) and  $K_s \in \overline{B}_\delta(y)$ ,

$$K_t \subseteq \overline{B}_\delta(K_s) \subseteq \overline{B}_{2\delta}(y) \subseteq M_{T_l'}(\lambda').$$

Hence  $K_t \subseteq M_{T_l'}(\lambda')$  for all  $l \geq 1$  so that, using (4.28),  $K_t \subseteq S_j$  as required.

This establishes (1). Before turning to (2) we note that for any  $k \geq 1$  the set

$$H_k := \{(y, \lambda) \text{ such that } y \in M_k(\lambda)\} \quad (4.30)$$

is a closed subset of  $Y \times [0, 1]$ . Indeed if  $(y^d, \lambda^d) \in H_k$  with

$$(y^d, \lambda^d) \rightarrow (y, \lambda) \in Y \times [0, 1]$$

then using (4.27) we have  $d(y^d, K_{r^d}) \leq \lambda^d w_k$  where  $r^d \in R_k$ . But as  $R_k$  is finite we may assume, passing to a subsequence if necessary, that  $r^d = r$  is a constant sequence. Then taking the  $d \rightarrow \infty$  limit we obtain, using  $y^d \rightarrow y$  and  $\lambda^d \rightarrow \lambda$ , that  $d(y, K_r) \leq \lambda w_k$  where  $r \in R_k$ . Hence  $y \in M_k(\lambda)$  and  $(y, \lambda) \in H_k$  so that  $H_k$  is indeed closed.

To establish (2), suppose that  $(i^d, y^d) \in X$  with

$$(i^d, y^d) \rightarrow (i, y) \in I \times Y.$$

Note that  $y^d \in S_{i^d}$ . Write  $i^d = (\lambda^d, T^d)$  and  $i = (\lambda, T)$ . For each fixed  $n$  then for sufficiently high  $d$  we have  $T_n^d = T_n$  so that  $y^d \in M_{T_n}(\lambda^d)$  and  $(y^d, \lambda^d) \in H_{T_n}$ ; see (4.28) and (4.30). As  $H_{T_n}$  is closed we deduce that  $(y, \lambda) \in H_{T_n}$ ; hence  $y \in M_{T_n}(\lambda)$  for each  $n \geq 1$ . We conclude  $y \in S_i$  and  $(i, y) \in X$ . Hence (2).

Finally for (3) we suppose that  $y^d \in F$  and  $y^d \rightarrow y \in Y$ . We aim to show  $y \in F$ . Find  $j^d = (\lambda^d, T^d) \in I$  with  $y^d \in S_{j^d}$  and  $\Delta(j^d, i) \leq \psi$  for all  $d$ . Write  $i = (\lambda, T)$ .

By passing to a subsequence if necessary we may assume that  $\lambda^d \rightarrow \lambda' \in [0, 1]$  where  $|\lambda' - \lambda| \leq \psi$ . Further as  $|T_n^d - T_n| \leq \psi A_n$  we may assume, after passing to another subsequence if necessary, that for each fixed  $n$  we have  $T_n^d = T_n'$  for sufficiently high  $d$ , where  $|T_n' - T_n| \leq \psi A_n$  and  $T_n' < T_{n+1}'$ . Note that  $j := (\lambda', T')$  is an element of  $I$  with  $\Delta(j, i) \leq \psi$ . It is enough, therefore, to prove that  $y \in S_j$ .

Fixing  $n \geq 1$  we have, using  $y^d \in S_{j^d}$  and (4.28),  $y^d \in M_{T_n^d}(\lambda^m)$  so that for sufficiently high  $d$ ,  $y^d \in M_{T_n^d}(\lambda^d)$  and so  $(y^d, \lambda^d) \in H_{T_n^d}$ . As the latter set is closed we deduce that  $(y, \lambda') \in H_{T_n^d}$  and  $y \in M_{T_n^d}(\lambda')$ .

This holds for every  $n \geq 1$ ; hence  $y \in S_j$ , establishing (3).  $\square$

We note that the metric space  $(I, \Delta)$  used in Theorem 4.23 is complete; in fact it is isometric to a closed subset of  $l_\infty(\mathbb{R})$  by the mapping  $(\lambda, T) \mapsto (y_n)_{n \geq 0}$  where  $y_0 = \lambda$  and  $y_n = \frac{T_n}{A_n}$  for  $n \geq 1$ . We also note that  $(\lambda, A) \in I$  for any  $\lambda \in [0, 1]$  so  $I \neq \emptyset$ .

**Theorem 4.24.** *If  $M \geq 1$ ,  $T$  is a dense subset of the open unit ball  $B_0(1) \subseteq \mathbb{R}^M$  and  $O$  is a  $G_\delta$  set containing every line segment  $[u, v]$  with  $u, v \in T$  then there exists a closed subset  $F \subseteq O$  such that  $F$  is a universal Fréchet set.*

*Proof.* Let  $Y = \mathbb{R}^M$  with the Euclidean metric and let  $(R, \gamma)$  be the collection of line segments with endpoints in  $B_1(0)$ , with the metric

$$\gamma([u, v], [u', v']) = \max(\|u' - u\|, \|v' - v\|).$$

We note that  $(R, \gamma)$  is totally bounded and that the other hypotheses of Theorem 4.23 are readily verified. We may then let  $(S_i)_{i \in I}$  be the collection of subsets of  $O$  given by the conclusion of Theorem 4.23.

Let  $i_0 \in I$ . We note that as

$$X = \{(i, y) \text{ such that } y \in S_i \text{ for some } i \in I \text{ with } \Delta(i, i_0) < 1\}$$

is a  $G_\delta$  subset of  $I \times Y$ , using Theorem 4.23(2), it is a complete topological space. We also note that  $X \neq \emptyset$ .

The map  $\pi: X \rightarrow Y$  given by  $\pi(i, y) = y$  is continuous. To complete the verification of the conditions of Theorem 4.2 we note that if  $x = (i, y) \in X$  and  $N$  is an open neighbourhood of  $x$  in  $X$  then we may pick  $\psi, r > 0$  with

$$\{j \in I \text{ such that } \Delta(j, i) < \psi\} \times B_r(y) \subseteq N \tag{4.31}$$

and

$$B_r(y) \subseteq B_1(0), \tag{4.32}$$

and then, for any  $\eta \in (0, 1)$  and  $\mu > 0$ , set

$$\delta_0(x, N, \eta, \mu) = \min\left(\frac{1}{2}r, \delta_4\left(\frac{\eta}{2}, \psi, i\right)\right).$$

Then if  $\delta \in (0, \delta_0)$  and  $y_1, y_2 \in B_\delta(y)$  we may find a segment  $[u, v] \subseteq S_j$  with  $\|u - y_1\| \leq \eta\delta/2$  and  $\|v - y_2\| \leq \eta\delta/2$ , so that

$$\|(v - u) - (y_2 - y_1)\| \leq \eta\delta,$$

where  $j \in I$  with  $\Delta(j, i) < \psi$ . This verifies (4.1) and (4.2). Note now that as  $\eta \in (0, 1)$  we have  $u, v \in B_{2\delta}(y) \subseteq B_r(y)$ . Hence  $u, v \in B_1(0)$  by (4.32) and further, as  $\Delta(j, i) < \psi$ , we have  $\{j\} \times [u, v] \subseteq N$ , using (4.31), and therefore  $[u, v] \subseteq \pi(N)$ . Hence (4.3), and so the hypotheses of Theorem 4.2 are satisfied.

We conclude, from Theorem 4.2, that  $\pi(X)$  is a universal Fréchet set. Hence, so is the set

$$F = \{y \in Y \text{ such that } y \in S_i \text{ for some } i \in I \text{ with } \Delta(i, i_0) \leq 1\}.$$

But, by Theorem 4.23(3),  $F$  is a closed subset of  $Y = \mathbb{R}^M$ . Further, as  $S_i \subseteq O$  for all  $i \in I$  we have  $F \subseteq O$ . We're done.  $\square$

**Corollary 4.25.** *If  $M \geq 1$  then there exists a universal Fréchet set  $S \subseteq \mathbb{R}^M$  such that  $S$  is a compact set of Hausdorff dimension one.*

*Proof.* This is now immediate from Theorem 4.24, Lemma 4.8 and Lemma 4.10.  $\square$

## 4.5 A closed and bounded universal set of Hausdorff dimension one in Banach spaces with separable dual

In this section we wish to establish the following.

**Theorem 4.26.** *If  $Y$  is a non-zero Banach space with separable dual there exists a universal Fréchet set  $S \subseteq Y$  such that  $S$  is closed, bounded and with Hausdorff dimension equal to one.*

See also [10].

As we have already proved this result for  $Y = \mathbb{R}^M$  where  $M \geq 1$ , we may take  $Y$  to be infinite dimensional. We note that in this case, one cannot demand that  $S$  is compact since, as the following simple result shows, any compact subset of an infinite dimensional Banach space is porous. As in the previous section, we use the notation  $d(y, S)$  to denote the distance between a point  $y \in Y$  and a set  $S \subseteq Y$  in any metric space  $Y$ .

**Lemma 4.27.** *If  $Y$  is an infinite dimensional Banach space and  $K \subseteq Y$  is compact then for every  $\varepsilon > 0$  there exists  $y \in Y$  with  $\|y\| = \varepsilon$  and  $d(y, K) > \varepsilon/3$ .*

*Proof.* We may assume that  $\varepsilon = 1$ . It is well known that one may find an infinite collection  $(e_n)_{n \in \mathbb{N}}$  in  $Y$  with  $\|e_n\| = 1$  and  $\|e_n - e_m\| \geq 1$  for  $m \neq n$ . Assuming, for a contradiction, that we cannot find  $n$  with  $d(e_n, K) > 1/3$  then we can pick  $k_n \in K_n$  for each  $n$  with  $\|k_n - e_n\| \leq 1/3$ . It then follows that  $\|k_n - k_m\| \geq 1/3$  for all  $m \neq n$ , contradicting the compactness of  $K$ .  $\square$

To establish Theorem 4.26 we shall, as in the previous section, start by working in greater generality. As before we use  $B_r(S)$  and  $\overline{B}_r(S)$  to respectively denote open and closed  $r$ -neighbourhoods of a set  $S \subseteq Y$  and  $B_r(y)$  and  $\overline{B}_r(y)$  to respectively denote open and closed balls of radius  $r$  and centre  $y \in Y$ .

**Theorem 4.28.** *Let*

- $(Y, d)$  be a metric space,
- $(K_r)_{r \in R}$  be a collection of compact subsets of  $Y$  indexed by a metric space  $(R, \gamma)$ , with the property that if  $\gamma(s, r) \leq \delta$  then

$$K_s \subseteq \overline{B}_\delta(K_r), \quad (4.33)$$

*i.e.*  $\mathcal{H}(K_r, K_s) \leq \gamma(r, s)$  for every  $r, s \in R$  where  $\mathcal{H}$  is the Hausdorff metric,

- $O$  be a  $G_\delta$  subset of  $Y$  containing a  $\gamma$ -dense subset of  $(K_r)_{r \in R}$ , *i.e.*

$$\forall r \in R, \varepsilon > 0 \exists s \in R \text{ such that } K_s \subseteq O \text{ and } \gamma(s, r) < \varepsilon, \quad (4.34)$$

- $(A_n)_{n \geq 1}$  be a strictly increasing sequence of positive integers and the metric space  $(I, \Delta)$  comprise all  $(\lambda, T)$  such that  $\lambda \in [0, 1]$  and  $T = (T_n)_{n \geq 1}$  is a strictly increasing sequence of positive integers with  $\sup_n \frac{T_n}{A_n} < \infty$ , with the metric  $\Delta$  given by

$$\Delta((\lambda', T'), (\lambda, T)) = \max \left( |\lambda' - \lambda|, \sup_n \frac{|T'_n - T_n|}{A_n} \right).$$

Suppose that the following condition is satisfied for some constants  $\rho, \varepsilon_0 > 0$ . For every  $\varepsilon \in (0, \varepsilon_0)$  there exists a set  $R(\varepsilon) \subseteq R$  such that

- for all  $s \in R$  there exists  $t \in R(\varepsilon)$  with  $\gamma(t, s) < \varepsilon$ ,
- every set in  $Y$  of diameter at most  $\rho\varepsilon$  only intersects  $K_r$  for finitely many  $r \in R(\varepsilon)$ .

Then, given  $r_0 \in R$  with  $K_{r_0} \subseteq O$  and  $\alpha_0 > 0$ , we can find a collection  $(S_i)_{i \in I}$  of subsets of  $Y$  with

$$K_{r_0} \subseteq S_i \subseteq O \cap \overline{B}_{\alpha_0}(K_{r_0}), \quad (4.35)$$

for every  $i \in I$ , such that the following hold.

(1) For each  $\varepsilon, \psi > 0$ ,  $i = (\lambda, T) \in I$  with  $\lambda < 1$  and  $y \in S_i$  there exists

$$\delta_5 = \delta_5(\varepsilon, \psi, i, y) > 0$$

such that the following property holds. If  $\delta \in (0, \delta_5)$  and  $s \in R$  with  $K_s \subseteq \overline{B}_\delta(y)$  there exists  $t \in R$  and  $j \in I$  such that  $K_t \subseteq S_j$ ,  $\gamma(t, s) < \varepsilon\delta$  and  $\Delta(j, i) \leq \psi$ .

(2) The set  $X$  is a closed subset of  $I \times Y$  where

$$X := \{(i, y) \text{ such that } y \in S_i\}.$$

(3) For every  $i \in I$  and  $\psi > 0$  then  $F = F(i, \psi)$  is a closed subset of  $Y$  where

$$F := \{y \in Y \text{ such that } y \in S_j \text{ for some } j \in I \text{ with } \Delta(j, i) \leq \psi\}.$$

To prove this result, we may assume  $\rho \in (0, 1)$ . Write  $O = \bigcap_{n=1}^{\infty} O_n$  where  $O_n$  are open subsets of  $Y$  with  $O_{n+1} \subseteq O_n$  for each  $n \geq 1$ .

We first observe that we may replace the sets  $R(\varepsilon)$  with sets  $R'(\varepsilon)$  that have the additional property that if  $r \in R'(\varepsilon)$  then  $K_r \subseteq O$ .

**Lemma 4.29.** For every  $\varepsilon \in (0, \varepsilon_0)$  we can find  $R'(\varepsilon) \subseteq R$  such that  $K_r \subseteq O$  for all  $r \in R'(\varepsilon)$  and

- for all  $r \in R$  there exists  $t \in R'(\varepsilon)$  with  $\gamma(t, s) < \varepsilon$ ,
- if  $B \subseteq Y$  has diameter at most  $\frac{4}{5}\rho\varepsilon$  then the set  $F^B(\varepsilon)$  is finite where

$$F^B(\varepsilon) := \{t \in R'(\varepsilon) \text{ with } K_t \cap B \neq \emptyset\}. \quad (4.36)$$

*Proof.* For each  $s \in R$  take  $t_s \in R$  to be such that  $K_{t_s} \subseteq O$  and  $\gamma(t_s, s) < \rho\varepsilon/10$ , using (4.34). Then set

$$R'(\varepsilon) = \{t_s \text{ for all } s \in R(4\varepsilon/5)\}.$$

It is clear that  $K_r \subseteq O$  for every  $r \in R'(\varepsilon)$  and that for every  $r \in R$  we can find  $t \in R'(\varepsilon)$  with  $\gamma(t, r) < 4\varepsilon/5 + \rho\varepsilon/10 < \varepsilon$ .



Now if  $t \in F^B(\varepsilon)$  then, writing  $t = t_s$  with  $s \in R(4\varepsilon/5)$ , we see that from  $\gamma(t_s, s) < \rho\varepsilon/10$  and (4.33) that  $K_s$  intersects  $\overline{B}_{\rho\varepsilon/10}(B)$ ; this set has diameter at most  $\rho\varepsilon$  so the set  $F^B(\varepsilon)$  is finite.  $\square$

We now define the set

$$\mathcal{T} = \{(r, w, \alpha) \in R \times (0, \varepsilon_0) \times (0, \infty) \text{ such that } K_r \subseteq O \text{ and } w \leq \alpha\}. \quad (4.37)$$

Note that as  $K_{r_0} \subseteq O$  and  $\alpha_0 > 0$  we can find  $w_0 > 0$  with  $(r_0, w_0, \alpha_0) \in \mathcal{T}$ . Let

$$R_0 = \{(r_0, w_0, \alpha_0)\}. \quad (4.38)$$

We shall now construct, for each  $k \geq 1$ , a set  $R_k \subseteq \mathcal{T}$  inductively by adding, for every  $(r, w, \alpha) \in R_l$  where  $l < k$ , a collection  $R_{k,l} = R_{k,l}(r, w, \alpha)$  of elements  $(t, v, \beta)$  of  $\mathcal{T}$  with  $\overline{B}_v(K_t) \subseteq O_k$  such that the set of  $K_t$  well approximates the collection  $(K_s)_{s \in R}$  in  $\overline{B}_\alpha(K_r)$ .

First let

$$r_{k,l} \in (0, \rho/10) \quad (4.39)$$

for each  $0 \leq l < k$ .

**Lemma 4.30.** *If  $0 \leq l < k$  and  $(r, w, \alpha) \in \mathcal{T}$  then there is a set*

$$R_{k,l} = R_{k,l}(r, w, \alpha) \subseteq \mathcal{T}$$

such that

1. for every  $s \in R$  with  $K_s \subseteq \overline{B}_\alpha(K_r)$  there exists  $(t, v, \beta) \in R_{k,l}$  such that

$$\gamma(t, s) \leq \frac{10}{\rho} r_{k,l} w,$$

2. if  $(t, v, \beta) \in R_{k,l}$  then  $\beta = r_{k,l} w < \alpha/10$  and  $v < \varepsilon_0/k$ ,
3. if  $(t, v, \beta) \in R_{k,l}$  then  $\overline{B}_v(K_t) \subseteq O_k$  and  $K_t \subseteq \overline{B}_{2\alpha}(K_r)$ ,
4. if  $B \subseteq Y$  has diameter at most  $8r_{k,l} w$  then the set  $F = F_{k,l}^B(r, w, \alpha)$  of all  $(t, v, \beta) \in R_{k,l}$  such that  $K_t$  intersects  $B$  is finite,
5. there exists  $v > 0$  such that  $(r, v, r_{k,l} w) \in R_{k,l}$ .

*Proof.* For each  $t \in R$  with  $K_t \subseteq O$  we can pick  $v_t \in (0, \varepsilon_0/k)$  such that  $v_t \leq r_{k,l} w$  and

$$\overline{B}_{v_t}(K_t) \subseteq O_k,$$

as  $K_t \subseteq O \subseteq O_k$ ,  $K_t$  is compact and  $O_k$  is open. Now let

$$\varepsilon = \frac{10}{\rho} r_{k,l} w.$$

Note that  $\varepsilon < w < \varepsilon_0$  from (4.39) and (4.37) and that for any  $t \in R'(\varepsilon) \cup \{r\}$  we have  $K_t \subseteq O$ . So we may set

$$R_{k,l} = \{(t, v_t, r_{k,l} w) \text{ for any } t \in R'(\varepsilon) \cup \{r\} \text{ with } K_t \subseteq \overline{B}_{2\alpha}(K_r)\}.$$

Observe that  $R_{k,l} \subseteq \mathcal{T}$ , using the definition of  $v_t$ .

To see item (1), for  $s \in R$  with  $K_s \subseteq \overline{B}_\alpha(K_r)$  then pick  $t \in R'(\varepsilon)$  with  $\gamma(t, s) < \varepsilon$ . Then  $\gamma(t, s) \leq w \leq \alpha$  so that  $K_t \subseteq \overline{B}_\alpha(K_s)$  using (4.33). It follows that  $K_t \subseteq \overline{B}_{2\alpha}(K_r)$  so that  $(t, v_t, r_{k,l} w) \in R_{k,l}$ .

Items (2) and (3) are immediate.

For (4) note that if  $(t, v_t, r_{k,l} w) \in F$  then as  $t \in R'(\varepsilon) \cup \{r\}$  and  $B$  has diameter at most  $\frac{4}{5}\rho\varepsilon$  we have

$$t \in F^B(\varepsilon) \cup \{r\};$$

see (4.36). As this set is finite then so is  $F$ .

Finally item (5) is immediate with  $v = v_r$ . □

Recall from (4.38) that we have defined  $R_0 \subseteq \mathcal{T}$ . Now for  $k \geq 1$  define  $R_k \subseteq \mathcal{T}$  by the recursion

$$R_k = \bigcup_{l=0}^{k-1} \bigcup_{(r,w,\alpha) \in R_l} R_{k,l}(r, w, \alpha). \quad (4.40)$$

Note that for any  $(t, v, \beta) \in R_k$  we have

$$K_t \subseteq O \text{ and } \overline{B}_v(K_t) \subseteq O_k \quad (4.41)$$

and

$$0 < v \leq \min\left(\beta, \frac{\varepsilon_0}{k}\right) \quad (4.42)$$

using (4.37) and Lemma 4.30(2)-(3).

**Lemma 4.31.** *If  $y \in Y$  and  $k \geq 0$  there exists  $\delta_k = \delta_k(y) > 0$  such that the set  $F_k = F_k(y)$  is finite where*

$$F_k := \{(r, w, \alpha) \in R_k \text{ such that } d(y, K_r) \leq \delta_k + 3\alpha\}.$$

*Proof.* Let  $y \in Y$ . For any  $\delta_0 > 0$  we pick, the set  $F_0 \subseteq R_0$  will be finite. Suppose now that  $k \geq 1$  and we have picked  $\delta_l > 0$  for every  $0 \leq l < k$  such that  $F_l$  is finite.

Pick  $\delta_k > 0$  such that for every  $l < k$  we have  $\delta_k < \delta_l$  and, for any  $(r, w, \alpha) \in F_l$ ,  $\delta_k < r_{k,l}w$ . We shall show that  $F_k$  is finite.

Suppose that  $(t, v, \beta) \in F_k$ . We may write  $(t, v, \beta) \in R_{k,l}(r, w, \alpha)$  where  $l < k$  and  $(r, w, \alpha) \in R_l$ , using (4.40). Note that  $K_t \subseteq \overline{B_{2\alpha}(K_r)}$  by Lemma 4.30(3). Hence

$$\begin{aligned} d(y, K_r) &\leq d(y, K_t) + 2\alpha \\ &\leq \delta_k + 3\beta + 2\alpha \\ &\leq \delta_l + 3\alpha \end{aligned}$$

using  $\delta_k < \delta_l$  and  $\beta = r_{k,l}w < \alpha/10$  from Lemma 4.30(2). Hence  $(r, w, \alpha) \in F_l$  and so  $\delta_k < r_{k,l}w$ . We get  $d(y, K_t) \leq \delta_k + 3\beta < 4r_{k,l}w$  so that

$$K_t \cap \overline{B_{4r_{k,l}w}(y)} \neq \emptyset$$

and

$$(t, v, \beta) \in F_{k,l}^{\overline{B_{4r_{k,l}w}(y)}}(r, w, \alpha);$$

see Lemma 4.30(4).

We conclude that

$$F_k \subseteq \bigcup_{l=0}^{k-1} \bigcup_{(r,w,\alpha) \in F_l} F_{k,l}^{\overline{B_{4r_{k,l}w}(y)}}(r, w, \alpha),$$

which is finite. □

**Definition 4.32.** If  $k \geq 1$ ,  $\lambda \in [0, 1]$  and  $w > 0$  we define  $M_k(\lambda, w)$  to be the set of  $y \in Y$  such that there exist integers  $n \geq 1$ ,

$$0 = l_0 < l_1 < \dots < l_n = k$$

and  $(r_m, w_m, \alpha_m) \in R_{l_m}$  for  $0 \leq m \leq n$  with

1.  $(r_m, w_m, \alpha_m) \in R_{l_m, l_{m-1}}(r_{m-1}, w_{m-1}, \alpha_{m-1})$  for  $1 \leq m \leq n$
2.  $d(y, K_{r_m}) \leq \lambda \alpha_m$  for  $0 \leq m \leq n$
3.  $d(y, K_{r_n}) \leq \lambda w_n$
4.  $w_n = w$ .

We then let

$$M_k(\lambda) = \bigcup_{w>0} M_k(\lambda, w). \quad (4.43)$$

Note that using Definition 4.32(3) and (4.41) we have

$$M_k(\lambda) \subseteq \bigcup_{(r,w,\alpha) \in R_k} \overline{B}_{\lambda w}(K_r) \subseteq O_k. \quad (4.44)$$

Further from (4.38), (4.40), Lemma 4.30(5) and Definition (4.32)(2),

$$K_{r_0} \subseteq M_k(\lambda) \subseteq \overline{B}_{\alpha_0}(K_{r_0}) \quad (4.45)$$

for all  $k \geq 1$  and  $\lambda \in [0, 1]$ . Finally if  $M_k(\lambda, w) \neq \emptyset$  then by Lemma 4.30(2),

$$w < \varepsilon_0/k. \quad (4.46)$$

**Lemma 4.33.** *For any  $k \geq 1$  the set*

$$H_k := \{(y, \lambda) \text{ such that } y \in M_k(\lambda)\}$$

*is a closed subset of  $Y \times [0, 1]$ .*

*Proof.* Suppose that  $(y^d, \lambda^d) \in H_k$  with  $(y^d, \lambda^d) \rightarrow (y, \lambda) \in Y \times [0, 1]$ . It suffices to show that  $(y, \lambda) \in H_k$ .

For each  $d \geq 1$  then as  $y^d \in M_k(\lambda^d)$  we can find  $n^d \geq 1$ ,  $0 = l_0^d < \dots < l_{n^d}^d = k$  and  $(r_m^d, w_m^d, \alpha_m^d) \in R_{l_m^d}$  for  $0 \leq m \leq n^d$  such that

$$\bullet (r_m^d, w_m^d, \alpha_m^d) \in R_{l_m^d, l_{m-1}^d}(r_{m-1}^d, w_{m-1}^d, \alpha_{m-1}^d) \text{ for } 1 \leq m \leq n^d \quad (4.47)$$

$$\bullet d(y^d, K_{r_m^d}) \leq \lambda^d \alpha_m^d \text{ for } 0 \leq m \leq n^d \quad (4.48)$$

$$\bullet d(y^d, K_{r_{n^d}^d}) \leq \lambda^d w_{n^d}^d. \quad (4.49)$$

As  $1 \leq n^d \leq k$  we may assume, passing to a subsequence if necessary, that  $n^d = n$  is constant. But then as  $0 \leq l_m^d \leq k$  we may assume, passing to another subsequence, that  $l_m^d = l_m$  is constant for each  $0 \leq m \leq n$  with  $0 = l_0 < l_1 < \dots < l_n = k$ .

Fixing  $m$  then as  $d(y, y^d) \rightarrow 0$ ,  $\lambda^d \leq 1$  and

$$d(y, K_{r_m^d}) \leq d(y, y^d) + \lambda^d \alpha_m^d,$$

from (4.48), we have  $(r_m^d, w_m^d, \alpha_m^d) \in F_{l_m}(y)$  for  $d$  sufficiently high; see Lemma 4.31.

As this set is finite we can assume, passing to another subsequence, that

$$(r_m^d, w_m^d, \alpha_m^d) = (r_m, w_m, \alpha_m)$$

is constant for each  $0 \leq m \leq n$ , with  $(r_m, w_m, \alpha_m) \in R_{l_m}$ . Further from (4.47)-(4.49) we have

- $(r_m, w_m, \alpha_m) \in R_{l_m, l_{m-1}}(r_{m-1}, w_{m-1}, \alpha_{m-1})$  for  $1 \leq m \leq n$
- $d(y^d, K_{r_m}) \leq \lambda^d \alpha_m$  for  $0 \leq m \leq n$
- $d(y^d, K_{r_n}) \leq \lambda^d w_n$ ;

taking the  $d \rightarrow \infty$  limit and using  $y^d \rightarrow y$ ,  $\lambda^d \rightarrow \lambda$  we obtain

- $(r_m, w_m, \alpha_m) \in R_{l_m, l_{m-1}}(r_{m-1}, w_{m-1}, \alpha_{m-1})$  for  $1 \leq m \leq n$
- $d(y, K_{r_m}) \leq \lambda \alpha_m$  for  $0 \leq m \leq n$
- $d(y, K_{r_n}) \leq \lambda w_n$ ,

so that  $y \in M_k(\lambda)$  and  $(y, \lambda) \in H_k$ . □

Up to this point we have let  $r_{k,l} \in (0, \rho/10)$  be arbitrary; see (4.39). We now further stipulate that if  $0 \leq l < l' \leq k$  then we have

$$r_{k+1,k} \leq \frac{1}{k} \text{ and } r_{k+1,l} \leq \frac{1}{k} r_{l',l}. \quad (4.50)$$

**Lemma 4.34.** *Suppose  $k \geq 1$ ,  $0 \leq \lambda < \lambda + \psi \leq 1$ ,  $w > 0$  and  $y \in M_k(\lambda, w)$ . Then*

1.  $\overline{B}_{\psi w}(y) \subseteq M_k(\lambda + \psi, w)$ ,
2. if  $2\delta \in (\psi w, \psi \alpha_0)$  and  $\varepsilon \in (20/\rho\psi k, 1)$  then for all  $s \in R$  with  $K_s \subseteq \overline{B}_\delta(y)$  there exists  $t \in R$  with  $\gamma(t, s) < \varepsilon\delta$  and  $K_t \subseteq M_{k+l}(\lambda + \psi)$  for all  $l \geq 1$ .

*Proof.* From Definition 4.32 we can find integers  $n \geq 1$ ,

$$0 = l_0 < l_1 < \dots < l_n = k$$

and  $(r_m, w_m, \alpha_m) \in R_{l_m}$  for  $0 \leq m \leq n$  with

$$\bullet (r_m, w_m, \alpha_m) \in R_{l_m, l_{m-1}}(r_{m-1}, w_{m-1}, \alpha_{m-1}) \text{ for } 1 \leq m \leq n \quad (4.51)$$

$$\bullet d(y, K_{r_m}) \leq \lambda \alpha_m \text{ for } 0 \leq m \leq n \quad (4.52)$$

$$\bullet d(y, K_{r_n}) \leq \lambda w_n \quad (4.53)$$

$$\bullet w_n = w. \quad (4.54)$$

Note that

$$\alpha_m = r_{l_m, l_{m-1}} w_{m-1} < \alpha_{m-1} \quad (4.55)$$

for each  $1 \leq m \leq n$  by Lemma 4.30(2).

To establish (1), suppose  $d(y', y) \leq \psi w$ ; then from (4.52) and (4.53),

$$d(y', K_{r_m}) \leq \lambda \alpha_m + \psi w \text{ for } 0 \leq m \leq n$$

$$d(y', K_{r_n}) \leq \lambda w_n + \psi w.$$

Using (4.42), (4.54) and (4.55) we have  $w = w_n \leq \alpha_n \leq \alpha_m$  so that

$$d(y', K_{r_m}) \leq (\lambda + \psi) \alpha_m \text{ for } 0 \leq m \leq n$$

$$d(y', K_{r_n}) \leq (\lambda + \psi) w_n;$$

combining these with (4.51) and (4.54) we get  $y' \in M_k(\lambda + \psi, w)$ , as required.

We now turn to (2). We claim that we can find  $m$  with  $0 \leq m \leq n$  and

$$(t, w, \alpha) \in R_{k+1, l_m}(r_m, w_m, \alpha_m) \quad (4.56)$$

where  $2\delta \leq \psi \alpha_m$  and  $\gamma(t, s) < \varepsilon \delta$ .

To see this suffices, note first that from (4.33) and  $\varepsilon \leq 1$  we have

$$K_t \subseteq \overline{B}_\delta(K_s) \subseteq \overline{B}_{2\delta}(y) \subseteq \overline{B}_{\psi \alpha_m}(y). \quad (4.57)$$

Let  $l'_l = l_l$  and  $(r'_m, w'_m, \alpha'_m) = (r_m, w_m, \alpha_m)$  for  $l \leq m$  and  $l'_{m+l} = k + l$  for  $l \geq 1$  and, using (4.56) and Lemma 4.30(5), pick inductively

$$(r'_{m+l}, w'_{m+l}, \alpha'_{m+l}) \in R_{l'_{m+l}, l'_{m+l-1}}(r'_{m+l-1}, w'_{m+l-1}, \alpha'_{m+l-1})$$

for each  $l \geq 1$ , with  $r'_{m+l} = t$ ; then for any  $y' \in K_t$ , as

$$d(y', K_{r_l}) \leq d(y, K_{r_l}) + \psi \alpha_m \leq (\lambda + \psi) \alpha'_l$$

for  $l \leq m$ , using (4.55) and (4.57), while  $d(y', K_{r_{m+l}}) = 0$  for  $l \geq 1$ , we have  $y' \in M_{k+l}(\lambda + \psi)$  for  $l \geq 1$  as required.

We now establish the claim. Suppose first that  $2\delta \leq \psi\alpha_n$ . Then as

$$K_s \subseteq \overline{B}_\delta(y) \subseteq \overline{B}_{\lambda\alpha_n + \delta}(K_{r_n}) \subseteq \overline{B}_{\alpha_n}(K_{r_n}),$$

using (4.52), we may pick, by Lemma 4.30(1),  $(t, w, \alpha) \in R_{k+1,k}(r_n, w_n, \alpha_n)$  with

$$\gamma(t, s) \leq \frac{10}{\rho} r_{k+1,k} w_n \leq \frac{10}{\rho} \frac{1}{k} \frac{2\delta}{\psi} < \varepsilon\delta$$

using (4.50) and  $2\delta \in (\psi w_n, \psi)$ . Thus we can satisfy the claim with  $m = n$ .

Suppose instead that  $\psi\alpha_n < 2\delta$ . As  $2\delta \leq \psi\alpha_0$  we can find  $m$  with

$$\psi\alpha_{m+1} < 2\delta \leq \psi\alpha_m \tag{4.58}$$

where  $0 \leq m \leq n - 1$ . Then as

$$K_s \subseteq \overline{B}_\delta(y) \subseteq \overline{B}_{\lambda\alpha_m + \delta}(K_{r_m}) \subseteq \overline{B}_{\alpha_m}(K_{r_m}),$$

we may pick, by Lemma 4.30(1),  $(t, w, \alpha) \in R_{k+1,l_m}(r_m, w_m, \alpha_m)$  with

$$\gamma(t, s) \leq \frac{10}{\rho} r_{k+1,l_m} w_m \leq \frac{10}{\rho} \frac{1}{k} r_{l_{m+1},l_m} w_m = \frac{10}{\rho} \frac{1}{k} \alpha_{m+1} < \frac{10}{\rho} \frac{1}{k} \frac{2\delta}{\psi} < \varepsilon\delta$$

using (4.50), (4.55) and (4.58). Thus the claim is satisfied.  $\square$

*Proof of Theorem 4.28* For each  $i = (\lambda, T) \in I$  we define the set  $S_i$  by

$$S_i = \bigcap_{n=1}^{\infty} M_{T_n}(\lambda). \tag{4.59}$$

Note that as  $M_k(\lambda) \subseteq O_k$  by (4.44) we have  $S_i \subseteq O_{T_n}$  for each  $n \geq 1$  and hence  $S_i \subseteq O$  since  $T_n \rightarrow \infty$ . Further as  $(r_0, w_0, \alpha_0) \in R_0 \subseteq R_k$  we have  $K_{r_0} \subseteq M_k(\lambda) \subseteq \overline{B}_{\alpha_0}(K_{r_0})$  for every  $k \geq 1$  and  $\lambda \in [0, 1]$  by (4.45); hence (4.35) holds for every  $i \in I$ .

We now verify (1). First note that we may assume that  $\varepsilon < 1$  and that  $\psi \leq 1 - \lambda$  where  $i = (\lambda, T)$ . Using (4.43) we find  $w_n > 0$  with  $y \in M_{T_n}(\lambda, w_n)$ . As  $T_n \rightarrow \infty$  and  $A_n \rightarrow \infty$  we may pick  $k_0$  such that  $T_{k_0} > 20/\rho\psi\varepsilon$  and  $A_{k_0} > 1/\psi$ . Let  $\delta_5 \in (0, \psi\alpha_0/2)$  be such that  $2\delta_5 \leq \psi w_{T_n}$  for all  $n < k_0$ .

Suppose that  $\delta \in (0, \delta_5)$ . As  $w_{T_n} \rightarrow 0$  by (4.46) we may pick a minimal  $k \geq 1$  with  $\psi w_{T_k} < 2\delta$ . Note that  $k \geq k_0$ . Put  $j = (\lambda', T') \in I$  where  $\lambda' = \lambda + \psi \in [0, 1]$

and  $T'$  is given by

$$\begin{aligned} T'_l &= T_l \text{ for } l < k \\ T'_l &= T_l + 1 \text{ for } l \geq k. \end{aligned}$$

We have  $\Delta(j, i) = \max(|\lambda' - \lambda|, \sup_{l \geq k} 1/A_l) = \psi$  as  $\lambda' - \lambda = \psi$  and  $A_k \geq A_{k_0} > \psi^{-1}$ .

Now let  $K_s \subseteq \overline{B}_\delta(y)$ . By Lemma 4.34(2) since  $y \in M_{T_k}(\lambda, w_{T_k})$ ,  $2\delta \in (\psi w_{T_k}, \psi \alpha_0)$  and  $\varepsilon \in (20/\rho \psi T_k, 1)$ , from  $\delta < \delta_5$  and  $k \geq k_0$ , we can find  $t \in R$  with  $\gamma(t, s) < \varepsilon \delta$  and

$$K_t \subseteq M_{T_k+t}(\lambda')$$

for  $l \geq 1$ . But for  $l < k$  we have  $2\delta \leq \psi w_{T_l}$ , using the minimality of  $k$ , and  $\overline{B}_{\psi w_{T_l}}(y) \subseteq M_{T_l}(\lambda', w)$  by Lemma 4.34(1) so that as  $K_t \subseteq \overline{B}_{2\delta}(y)$ , from  $\gamma(t, s) < \varepsilon \delta \leq \delta$  and (4.33), we deduce finally that  $K_t \subseteq M_{T_l}(\lambda', w)$ .

Hence  $K_t \subseteq M_{T'_l}(\lambda')$  for all  $l \geq 1$  so that, using (4.59),  $K_t \subseteq S_j$  as required.

To establish (2), suppose that  $(i^d, y^d) \in X$  with

$$(i^d, y^d) \rightarrow (i, y) \in I \times Y.$$

Note that  $y^d \in S_{i^d}$ . Write  $i^d = (\lambda^d, T^d)$  and  $i = (\lambda, T)$ . For each fixed  $n$  then for sufficiently high  $d$  we have  $T_n^d = T_n$  so that  $y^d \in M_{T_n}(\lambda^d)$  and  $(y^d, \lambda^d) \in H_{T_n}$ ; see (4.59) and Lemma 4.33. As  $H_{T_n}$  is closed and  $y^d \rightarrow y$ ,  $\lambda^d \rightarrow \lambda$  we deduce that  $(y, \lambda) \in H_{T_n}$ ; hence  $y \in M_{T_n}(\lambda)$  for each  $n \geq 1$ . We conclude  $y \in S_i$  and  $(i, y) \in X$ . Hence (2).

Finally for (3) we suppose that  $y^d \in F$  and  $y^d \rightarrow y \in Y$ . We aim to show  $y \in F$ . Find  $j^d = (\lambda^d, T^d) \in I$  with  $y^d \in S_{j^d}$  and  $\Delta(j^d, i) \leq \psi$  for all  $d$ . Write  $i = (\lambda, T)$ .

By passing to a subsequence if necessary we may assume that  $\lambda^d \rightarrow \lambda' \in [0, 1]$  where  $|\lambda' - \lambda| \leq \psi$ . Further as  $|T_n^d - T_n| \leq \psi A_n$  we may assume, after passing to another subsequence if necessary, that for each fixed  $n$  we have  $T_n^d = T'_n$  for sufficiently high  $d$ , where  $|T'_n - T_n| \leq \psi A_n$  and  $T'_n < T'_{n+1}$ . Note that  $j := (\lambda', T')$  is an element of  $I$  with  $\Delta(j, i) \leq \psi$ . It is enough, therefore, to prove that  $y \in S_j$ .

Fixing  $n \geq 1$  we have, using  $y^d \in S_{j^d}$  and (4.59),  $y^d \in M_{T_n^d}(\lambda^d)$  so that for sufficiently high  $d$ ,  $y^d \in M_{T'_n}(\lambda^d)$  and so  $(y^d, \lambda^d) \in H_{T'_n}$ . As the latter set is closed we deduce that  $(y, \lambda') \in H_{T'_n}$  and  $y \in M_{T'_n}(\lambda')$ .

This holds for every  $n \geq 1$ ; hence  $y \in S_j$ , establishing (3).  $\square$

We shall now show that the existence of  $R(\varepsilon)$  is guaranteed whenever  $(Y, d)$



is an infinite dimensional Banach space and  $(R, \gamma)$  satisfies some further mild conditions.

**Lemma 4.35.** *Suppose  $(Y, d)$  is an infinite dimensional Banach space,  $(R, \gamma)$  is separable and has the property that whenever  $r \in R$  and  $y \in Y$  then  $K_s = y + K_r$  for some  $s \in R$  with*

$$\gamma(s, r) \leq \frac{1}{4\rho} \|y\|. \quad (4.60)$$

*Then for every  $\varepsilon > 0$  there exists a set  $R(\varepsilon) \subseteq R$  such that*

1. *for all  $r \in R$  there exists  $s \in R(\varepsilon)$  with  $\gamma(s, r) < \varepsilon$ ,*
2. *if  $r, s$  are distinct elements of  $R(\varepsilon)$  then  $d(K_r, K_s) > \rho\varepsilon$ .*

Most natural choices of  $(R, \gamma)$  in  $Y$ , an infinite dimensional separable Banach space, satisfy the conditions of Lemma 4.35 with  $\rho = 1/4$ . Note that the conclusion (2) implies that any set of diameter at most  $\rho\varepsilon$  can only intersect  $K_r$  for at most *one* element  $r \in R(\varepsilon)$ , so that we may apply Theorem 4.28.

To establish Lemma 4.35 we first prove the following.

**Lemma 4.36.** *If  $Y$  is an infinite dimensional Banach space and  $(K_n)_{n \geq 1}$  are compact subsets of  $Y$  then for any  $\varepsilon > 0$  we can find  $y_n \in Y$  with  $\|y_n\| = \varepsilon$  for each  $n \geq 1$  such that  $K'_n := y_n + K_n$  satisfy  $d(K'_n, K'_m) > \varepsilon/3$  for  $n \neq m$ .*

*Proof.* Suppose  $n \geq 1$  and we have chosen  $(y_m)_{1 \leq m < n}$  such that  $d(K'_m, K'_l) > \varepsilon/3$  for  $1 \leq l < m < n$ . It suffices to pick  $y_n$  such that  $d(K'_n, K'_m) > \varepsilon/3$  for  $1 \leq m < n$ .

The difference set

$$K := K_n - \cup_{1 \leq m < n} K'_m = \{k - k' \text{ where } k \in K_n, k' \in K_m \text{ for some } m < n\}$$

is compact so that we may find  $y \in Y$  with  $\|y\| = \varepsilon$  and  $d(y, K) > \varepsilon/3$ , using Lemma 4.27. Then  $d(0, -y + K) > \varepsilon/3$  so that, choosing  $y_n = -y$ ,

$$d(0, K'_n - \cup_{1 \leq m < n} K'_m) > \varepsilon/3.$$

□

*Proof of Lemma 4.35* Let  $(r_n)_{n \geq 1}$  be a dense sequence in  $R$ . By Lemma 4.36 we can find  $y_n \in Y$  with  $\|y_n\| = 3\rho\varepsilon$  such that  $K'_n := y_n + K_{r_n}$  satisfy  $d(K'_n, K'_m) > \rho\varepsilon$  for  $n \neq m$ . Now we may pick  $r'_n$  with  $K_{r'_n} = y_n + K_{r_n} = K'_n$  and

$$\gamma(r'_n, r_n) \leq \frac{1}{4\rho} \|y_n\| = \frac{3}{4}\varepsilon$$

using (4.60). Setting  $R(\varepsilon) = \{r'_n \text{ where } n \in \mathbb{N}\}$  we are home.  $\square$

**Corollary 4.37.** *If  $Y$  is an infinite dimensional Banach space with separable dual,  $T$  is a dense subset of  $Y$  and  $O$  is a  $G_\delta$  set containing every line segment  $[u, v]$  with  $u, v \in T$  then there exists a closed and bounded subset  $G \subseteq O$  such that  $F$  is a universal Fréchet set.*

*Proof.* We may view  $Y$  as a metric space. Let  $(R, \gamma)$  be the collection of line segments in  $Y$  with the metric

$$\gamma([u, v], [u', v']) = \max(\|u' - u\|, \|v' - v\|).$$

We note that  $(R, \gamma)$  satisfies the conditions of Theorem 4.28 by Lemma 4.35 with  $\rho = 1/4$ . Let  $(S_i)_{i \in I}$  be the collection of subsets of  $O$  given by the conclusion of Theorem 4.28.

Let  $i_0 \in I$ . We note that as

$$X = \{(i, y) \text{ such that } y \in S_i \text{ for some } i \in I \text{ with } \Delta(i, i_0) < 1\}$$

is a  $G_\delta$  subset of  $I \times Y$ , using Theorem 4.28(2), it is a complete topological space. We also note that  $X \neq \emptyset$ .

The map  $\pi: X \rightarrow Y$  given by  $\pi(i, y) = y$  is continuous. To complete the verification of the conditions of Theorem 4.2 we note that if  $x = (i, y) \in X$  and  $N$  is an open neighbourhood of  $x$  in  $X$  then we may pick  $\psi, r > 0$  with

$$\{j \in I \text{ such that } \Delta(j, i) < \psi\} \times B_r(y) \subseteq N \quad (4.61)$$

and then, for any  $\eta \in (0, 1)$  and  $\mu > 0$ , set

$$\delta_0(x, N, \eta, \mu) = \min\left(\frac{1}{2}r, \delta_5\left(\frac{\eta}{2}, \psi, i\right)\right).$$

Then if  $\delta \in (0, \delta_0)$  and  $y_1, y_2 \in B_\delta(y)$  we may find a segment  $[u, v] \subseteq S_j$  with  $\|u - y_1\| \leq \eta\delta/2$  and  $\|v - y_2\| \leq \eta\delta/2$ , so that

$$\|(v - u) - (y_2 - y_1)\| \leq \eta\delta,$$

where  $j \in I$  with  $\Delta(j, i) < \psi$ . This verifies (4.1) and (4.2). Note now that as  $\eta \in (0, 1)$  we have  $u, v \in B_{2\delta}(y) \subseteq B_r(y)$ . Hence, from  $\Delta(j, i) < \psi$ , we have  $\{j\} \times [u, v] \subseteq N$ , using (4.61), and therefore  $[u, v] \subseteq \pi(N)$ . Hence (4.3), and so the hypotheses of Theorem 4.2 are satisfied.

We conclude, from Theorem 4.2, that  $\pi(X)$  is a universal Fréchet set and that the set of points  $D_g$  of Fréchet differentiability of any Lipschitz function  $g: \rightarrow \mathbb{R}$  has dense intersection with  $\pi(X)$  in  $\pi(X)$ . Hence

$$G := \{y \in \overline{B}_r(y_0) \text{ such that } y \in S_i \text{ for some } i \in I \text{ with } \Delta(i, i_0) \leq 1\}$$

is also universal, for any  $(i_0, y_0) \in X$ . But, by Theorem 4.23(3),  $G = F \cap \overline{B}_r(y_0)$  where  $F$  is a closed subset of  $Y$ . Hence  $G$  is a closed and bounded subset of  $Y$ . Further, as  $S_i \subseteq O$  for all  $i \in I$  we have  $G \subseteq O$ . We're done.  $\square$

**Corollary 4.38.** *If  $Y$  is a Banach space with separable dual then there exists a universal Fréchet set  $S \subseteq Y$  such that  $S$  is closed, bounded and has Hausdorff dimension one.*

*Proof.* This is now immediate from Theorem 4.37, Lemma 4.8 and Lemma 4.10.  $\square$

**Remark 4.39.** *Since any totally bounded metric space  $(R, \gamma)$  clearly satisfies the condition in Theorem 4.28 that  $R(\varepsilon)$  exists for all  $\varepsilon > 0$ , one does not really need to separate the finite and infinite dimensional cases as we have done here. A more unified approach, relying just on Theorem 4.28, will be taken in [10].*

## 4.6 The non-separable case

We have proved the existence of a closed and bounded universal Fréchet set of Hausdorff dimension one in any Banach space with separable dual - that is, in any separable Asplund space. We already know that any non-Asplund space has no universal Fréchet sets. In this final section, we make a few quick remarks about the remaining case, in which the Banach space is a non-separable Asplund space.

We already know that such a space does indeed have universal Fréchet sets; by Theorem 1.13 any non-empty open set is an example. We may do slightly better and construct a universal Fréchet set in  $Y$  that is nowhere dense. First we need to quote a useful result, due to Preiss, that allows us to reduce the task of proving the existence of a point of Fréchet differentiability on a non-separable space to proving it for the restriction of the function to a particular separable subspace, which may depend on the function.

**Theorem 4.40** (Separable reduction). *If  $Y$  is a Banach space,  $Z$  is a separable subspace of  $Y$  and  $g: Y \rightarrow \mathbb{R}$  is a Lipschitz function then there exists a separable subspace  $W$  of  $Y$  with  $Z \subseteq W$  such that  $g$  is Fréchet differentiable at every point  $y \in Y$  such that  $g|_W$  is Fréchet differentiable.*

*Proof.* See [24] and [15]. □

**Corollary 4.41.** *If  $Y$  is a non-zero Asplund space then there exists a closed universal Fréchet set  $S \subseteq Y$  whose complement is dense.*

*Proof.* If  $Y$  is one dimensional we may take, by Lebesgue's theorem,  $S$  to be any closed and bounded subset with dense complement whose Lebesgue measure is positive. Suppose now that  $Y$  has dimension at least two. Let  $Z$  be a two dimensional subspace of  $Y$ . Let

$$\pi: Y \rightarrow Z$$

be a continuous linear map with  $\|\pi\| = 1$  and  $\pi(Z) = Z$ . Let  $(F_\lambda)_{0 < \lambda < 1} \subseteq Z$  be any collection of subsets satisfying the conditions of Corollary 4.3.

Set

$$F'_\lambda = \pi^{-1}(F_\lambda) \subseteq Y.$$

Let  $g: Y \rightarrow \mathbb{R}$  be Lipschitz. Find  $W$  as in Theorem 4.40 with  $Z \subseteq W$ . Note that as  $W$  is a separable subspace of an Asplund space  $Y$  it has separable dual, so has an equivalent norm that is Fréchet differentiable on  $W \setminus \{0\}$ . It is readily verified that  $(F'_\lambda \cap W)_{0 < \lambda < 1}$  still satisfies the conditions of Corollary 4.3. Therefore each set  $F'_\lambda \cap W$  contains a point of Fréchet differentiability of  $g|_W$ . Therefore  $F'_\lambda$  contains a point of Fréchet differentiability of  $g$ .

This holds for every Lipschitz  $g: Y \rightarrow \mathbb{R}$  so that  $F'_\lambda$  is a universal Fréchet set in  $Y$ .

But using chapter 4.3 we may take the sets  $F_\lambda \subseteq Z$  to be compact and null and, therefore, to have dense complement in  $Z$ . It follows immediately that the sets  $F'_\lambda$  are closed and have dense complement in  $Y$ . □

**Remark 4.42.** *As a final comment, however, we note that a universal Fréchet set  $S$  in a non-separable Banach space  $Y$  cannot have Hausdorff dimension one - or indeed finite Hausdorff dimension. If  $D_H(S) < \infty$  then we can find  $d > 0$  with  $C_H^d(S) = 0$ ; see Definition 4.7. Hence for every  $\delta > 0$  there must exist a cover of  $S$  by a countable collection of sets, each of which has diameter less than  $\delta$ . It follows that there exists a countable subset of  $Y$  whose closure contains  $S$ , so that  $S$  lies in a separable subspace of  $Y$  and  $S$  is therefore porous. Hence, using Lemma 1.23,  $S$  is not a universal Fréchet set.*

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