The Fibonacci family of non-equilibrium universality classes

Gunter M. Schütz

Institute of Complex Systems II, Forschungszentrum Jülich, 52425 Jülich, Germany

and

Interdisziplinäres Zentrum für Komplexe Systeme, Universität Bonn

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• Nonlinear fluctuating hydrodynamics
• Mode coupling theory
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1. Introduction

**Bulk-driven Particle systems with several conservation laws:**

- Interacting stochastic particle systems on lattice with biased hopping
  \[\Rightarrow\] Non-reversible Markovian dynamics

- A few (!) examples:
  1. Multi-species exclusion processes
  2. Bricklayer model
  3. Multilane exclusion processes

- Many Applications:
  1. Diffusion in carbon nano tubes
  2. Molecular Motors
  3. Automobile traffic flow

- Rich behaviour, e.g.,
  - Phase transitions (phase separation, spontaneous symmetry breaking)
  - Hydrodynamic equations sensitive to regularization
  - Intricate interplay of shocks and rarefaction waves
  - Universal fluctuations: \(z=2\) (Diffusive), \(z=3/2\) (KPZ) \(\Rightarrow\) Is that all?
Invariant measures for driven diffusive systems:

Assume translation invariance and ergodicity for fixed values of conserved particle numbers $N_\alpha$

$\Rightarrow$ "canonical" invariant measure $\mu$ is unique and translation invariant

Construct "grandcanonical" invariant measures with chemical potentials $\phi^\alpha$

- Stationary densities of particles of type $\alpha$: $\rho_\alpha(\{\phi\})$
- Stationary current of particle of type $\alpha$: $j_\alpha(\{\phi\})$

Onsager-type current symmetry \[ \frac{\partial j^\alpha}{\partial \phi^\beta} = \frac{\partial j^\beta}{\partial \phi^\alpha} \] [Grisi, GMS, 2011]

- Compressibility matrix $K$: $(K)_{\alpha\beta} = \frac{\partial \rho_\alpha}{\partial \phi_\beta} = 1/L \langle (N_\alpha - L \rho_\alpha)(N_\beta - L \rho_\beta) \rangle$

- Current Jacobian $J$: $(J)_{\alpha\beta} = \frac{\partial j_\alpha}{\partial \rho_\beta}$

Current symmetry $\Rightarrow$ $JK = (JK)^T$
Model: Interacting multi-lane TASEPs with densities $\rho_i$

For simplicity focus on two-lane model:

- $r_1 = 1 + \gamma n^{(2)/2}$, $r_2 = b + \gamma n^{(1)/2}$

Invariant measure: Canonical: Uniform $\Rightarrow$ Grandcanonical: Product

$$
j_1(\rho_1, \rho_2) = \rho_1(1 - \rho_1)(1 + \gamma \rho_2)$$

$$
j_2(\rho_1, \rho_2) = \rho_2(1 - \rho_2)(b + \gamma \rho_1)$$
2. Nonlinear fluctuating hydrodynamics

Study large-scale dynamics under Eulerian scaling \( x = ka, \ t = \tau a, \ a \to 0:\)

\[ \Rightarrow \text{Law large numbers: Occupation numbers on lattice } n_{\alpha k}(t) \to \rho_\alpha(x,t) \]  
\( \text{(Coarse-grained particle densities)} \)

\[ \Rightarrow \text{Local stationarity: Microscopic current } j^\alpha_k(t) \to j^\alpha(\{\rho\}): \text{Associated locally stationary currents} \]

\[ \Rightarrow \text{Lattice continuity equation } \to \text{System of hyperbolic conservation laws} \]

\[ \frac{\partial}{\partial t} \hat{\rho} + A \frac{\partial}{\partial x} \hat{\rho} = 0 \]

◆ Origin of hyperbolicity: Onsager-type symmetry

◆ General validity: Driven Diffusive Systems, Anharmonic chains, Hamiltonian dynamics, …

Introduce fluctuation fields \( u_i(x,t) = \rho_i(x,t) - \rho_i \) and expand in \( u_i \)
A) **Linear theory:**

- Diagonalize $A$: $RAR^{-1} = \text{diag}(v_i)$, Normalization $RCR^T = 1$

  $$\implies \text{Eigenmode equation for normal modes } \phi = R\ u: \ \partial_t \phi_i = -v_i \partial_x \phi_i$$

- Travelling waves (eigenmodes) $\phi_i(x,t) = \phi_i(x-v_it)$

- Characteristic speeds $v_1,2(\rho_1, \rho_2) = \text{eigenvalues of current Jacobian } A$

- Strict hyperbolicity for two-lane model: $v_1 \neq v_2 \ \forall (\rho_1, \rho_2) \in (0,1) \times (0,1)$

- Microscopic: Stationary center of mass motion of localized perturbation
  [Popkov, GMS (2003)]
B) Nonlinear fluctuating theory

- Expand to second order, add phenomenological diffusion term and noise [Spohn]

\[ \partial_t \phi_i = -\partial_x [c_i \phi_i + \langle \hat{\phi}, G^{(i)} \hat{\phi} \rangle - \partial_x (D \phi)_i + (B \xi)_i] \]

Diffusion = regularization, noise B and diffusion matrix D related by FDT

- Mode coupling coefficients for eigenmodes

\[ G^{(i)} = (1/2) \sum_j R_{ij} (R^{-1})^T H^{(j)} R^{-1} \]

- Hessian \( H^{(\gamma)} \) with matrix elements \( \partial^2 j_\gamma / (\partial \rho_\alpha \partial \rho_\beta) \)
One component: \( \partial_t \phi = - \partial_x [c \phi + g \phi^2 - D \partial_x \phi + B \xi] \) (KPZ equation, \( g = j''/2 \))

Two components ==> Two coupled KPZ equations

Remarks:

1) Higher order terms irrelevant in RG sense (if second order non-zero)

2) Offdiagonal terms negligible for strictly hyperbolic systems (no overlap between modes)

3) Self-coupling terms \( G^{(\alpha)}_{\alpha\alpha} \) leading, other diagonal terms \( G^{(\alpha)}_{\beta\beta} \) subleading

\[
\begin{align*}
\partial_t \phi_1 &= - \partial_x [c_1 \phi_1 + G^{(1)}_{11} (\phi_1)^2 + G^{(1)}_{22} (\phi_2)^2 + \text{diff.} + \text{noise}] \\
\partial_t \phi_2 &= - \partial_x [c_2 \phi_2 + G^{(2)}_{11} (\phi_1)^2 + G^{(2)}_{22} (\phi_2)^2 + \text{diff.} + \text{noise}]
\end{align*}
\]
3. Mode coupling theory

Go beyond LLN and study fluctuations:

- Dynamical structure function (lattice)

\[ S_{\alpha\beta}(p,t) = \sum_k e^{ikp} \langle (\xi_k^\alpha(t) - \rho_\alpha)(\xi_0^\beta(0) - \rho_\beta) \rangle = \langle u_\alpha(p,t) u_\beta(-p,t) \rangle \]

where \( u_\alpha(p,t) \) = Fourier transform of locally conserved quantity \( \xi_k^\alpha(t) - \rho_\alpha \)

- One conservation law: Scaling form \( S(p,t) = F(p^z t) \)

- KPZ universality class \( z=3/2 \), universal scaling function \( F \) [Praehofer, Spohn (2002)]

- Several conservation laws: Different universality classes in the same DDS

- Known cases for two-component DDS: (a) Both KPZ (generic) (b) KPZ and Diffusive (\( z=2 \)) [Das et al (2001), Rakos, GMS (2005)]

\[ \Rightarrow \text{Is that all there is?} \]
Mode coupling scenarios [van Beijeren (2012), Spohn (2013), Popkov, Schmidt, GMS (2014)]

Some scenarios:

A) Both self-coupling coefficients nonzero:   \( G^{(1)}_{11} \neq 0, G^{(2)}_{22} \neq 0 \)

\( \Rightarrow \) two KPZ modes (\( z_1 = 3/2, z_2 = 3/2 \))

B) One self-coupling coefficient nonzero, all other diagonal terms of mode-coupling matrices 0, e.g., \( G^{(1)}_{11} \neq 0, G^{(1)}_{22} = G^{(2)}_{22} = G^{(2)}_{11} = 0 \)

\( \Rightarrow \) one KPZ mode, one diffusive mode (\( z_1 = 3/2, z_2 = 2 \))

C) One self-coupling coefficient nonzero, subleading diagonal of other mode-coupling matrix 0, e.g., \( G^{(1)}_{11} \neq 0, G^{(2)}_{11} \neq 0, G^{(2)}_{22} = 0 \)

\( \Rightarrow \) one KPZ mode, second non-KPZ superdiffusive mode (\( z_1 = 3/2, z_2 = 5/3 \))

Remark: Heat mode with \( z = 5/3 \), two KPZ sound modes in Hamiltonian dynamics with three conservation laws [van Beijeren (2012)]
General solution of mode coupling equations (cont’)

Scaling ansatz, with $\tilde{\omega}_\alpha := \omega + iv_\alpha p \; \zeta_\alpha = \tilde{\omega}_\alpha |p|^{-z_\alpha}$

$$\tilde{S}_\alpha(p, \tilde{\omega}_\alpha) = p^{-z_\alpha} g_\alpha(\zeta_\alpha)$$

Subballistic scaling $z > 1$ (short range interactions) and strict hyperbolicity:

$$g_\alpha(\zeta_\alpha) = \lim_{k \to 0} \left[ \zeta_\alpha + D_\alpha |p|^{2-z_\alpha} + Q_{\alpha\alpha} \zeta_\alpha^{2-z_\alpha} - \frac{1}{z_\alpha} |p|^{3-2z_\alpha} + \sum_{\beta \neq \alpha} Q_{\alpha\beta} \left(-iv_p^\alpha \beta\right)^{\frac{1}{z_\beta} - 1} |p|^{1+\frac{1}{z_\beta} - z_\alpha} \right]^{-1}$$

with $v_p^\alpha \beta := |v_\alpha - v_\beta| \text{sgn}[p(v_\alpha - v_\beta)]$

$$Q_{\alpha\beta} = 2(G^\alpha_{\beta\beta})^2 \Gamma \left(1 - \frac{1}{z_\beta}\right) \Omega[\hat{S}_\beta] \geq 0$$

$$\Omega[\hat{f}] = \int_{-\infty}^{\infty} dp \; \hat{f}(p) \hat{f}(-p) \quad \text{(Integral over square of real-space scaling function)}$$
General solution of mode coupling equations (cont’)

Power counting:

\[ z_\alpha = \begin{cases} 
2 & \text{if } \mathbb{I}_\alpha = \emptyset \\
3/2 & \text{if } \alpha \in \mathbb{I}_\alpha \\
\min_{\beta \in \mathbb{I}_\alpha} \left(1 + \frac{1}{z_\beta}\right) & \text{else}
\end{cases} \]

where \[ \mathbb{I}_\alpha := \{ \beta : G_{\beta \beta}^\alpha \neq 0 \} \]

Structure of diagonal elements of mode coupling matrix \( \rightarrow \) universality classes

1) All diagonal elements of mode \( \alpha \) vanish: Diffusive mode \( (z=2) \)

2) Self coupling \( G_{\alpha \alpha}^\alpha \neq 0 \): KPZ or modified KPZ (for non-zero coupling to diffusive mode [Stoltz, Spohn (2015)])

3) Otherwise: ?
General solution of mode coupling equations (Case 3)

A) Coupling to either Diffusive or KPZ-type mode:

(i) Consider sequential coupling with KPZ mode 1:

- Recursion for dynamical exponents

\[ z_\alpha = 1 + \frac{1}{z_\alpha - 1} \]

Solution: Kepler ratios

\[ z_\alpha = \frac{F_{\alpha+3}}{F_{\alpha+2}} \]

of neighbouring Fibonacci numbers 1, 1, 2, 3, 5, 8,…

\[ z = 3/2, 5/3, 8/5, \ldots \rightarrow \varphi = (1+\sqrt{5})/2 \approx 1.618 \text{ (golden mean)} \]

Scaling functions: \( z \)-stable Levy distributions

\[ \hat{S}_\alpha(p, t) = \frac{1}{\sqrt{2\pi}} e^{-iv_\alpha pt - E_\alpha |p|z_\alpha t} \left( 1 - i\sigma_\alpha \tan \left( \frac{\pi z_\alpha}{2} \right) \right) \]

Time scales and asymmetries given recursively in terms of mode coupling coefficients and (numerically known) integral over square of KPZ scaling function
General solution of mode coupling equations (Case 3 cont’)

(ii) Consider sequential coupling to \textit{diffusive} mode 1:

\[ \Rightarrow \text{Same recursion for dynamical exponents, but shifted initial value} \]

Solution: Shifted Kepler ratios \( z = 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots \)

Scaling functions: Levy for \( z < 2 \), time scales contain diffusion coefficient instead of KPZ-integral

(iii) Non-sequential coupling: Still valid for large general class of mode-coupling matrices (work in progress)

B) No coupling to either diffusive or KPZ-type mode

\[ \Rightarrow \text{Golden mean universality class} \]
Universality classes for two-component systems

[Popkov, Schmidt, GMS (2015); Spohn, Stoltz (2015)]

<table>
<thead>
<tr>
<th>$G^1$</th>
<th>$G^2$</th>
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<td>($\frac{5}{3}$L, KPZ)</td>
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4. Simulation results

Measure dynamical exponents $z_i$

1) Monte Carlo random sequential update

2) Excite modes at site $k=L/2$ at $t=0$ and measure dynamical structure function of each mode

3) Compute center of mass motion $\langle X_i(t) \rangle$ of excitation $\Rightarrow v_i$ ✔

4) Measure amplitudes $A_i(t)$ at maximum: Mass conservation $A_i(t) \sim 1/t^{1/z_i} \Rightarrow z_i$

5) Fit predicted scaling functions
Choose equal densities $\rho_1 = \rho_2 = \rho$, and symmetric lanes $b=1$

$$G^1 = -4A_0(1 + \gamma \rho) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G^2 = -4A_0 \begin{pmatrix} 1 + \gamma(1 - \rho) & 0 \\ 0 & 1 - \gamma(1 - 3\rho) \end{pmatrix}$$

$\gamma = 1/(1-3\rho)$: Diffusive and $3/2$-Fibonacci mode

$$S_1(x, t) = \frac{1}{\sqrt{4\pi D_1 t}} e^{-\frac{(x-v_1 t)^2}{4D_1 t}}$$

$$\hat{S}_2(p, t) = \frac{1}{\sqrt{2\pi}} \exp\left(-iv_2 pt - C_0|p|^{3/2} t \left[1 - i \text{sgn}(p(v_1 - v_2))\right]\right)$$

$$C_0 = \frac{(G_{11}^2)^2}{2\sqrt{D_1|v_2 - v_1|}}$$
Monte-Carlo simulations

3/2L Fibonacci mode: $\rho_1 = \rho_2 = 0.2$, $\gamma = 2.5$, $b = 1$ (symmetry between lanes)

Theory: $v_2 = 1.3$, $2/z = 1.333$, $\beta = -1$

$V_2 = 1.30(1)$ ✔

Variance: $t^{2/z}$
$z = 1.52$ ✓

$\beta = -0.692$ ✖
B) Golden mean universality class

\[ G^{(1)}_{11} = G^{(2)}_{22} = 0, \quad G^{(1)}_{22} \neq 0; \quad G^{(2)}_{11} \neq 0; \]

Dynamical structure function:

\[ \hat{S}_\pm(p, t) = \frac{1}{\sqrt{2\pi}} \exp \left( -iv_\pm pt - C_\pm |p|^\varphi t \left[ 1 \pm \text{isgn}(p(v_+ - v_-)) \tan \left( \frac{\pi \varphi}{2} \right) \right] \right) \]

\[ \varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618 \quad \text{(golden mean)} \]

\[ C_\pm = \frac{1}{2} |v_+ - v_-|^{1 - \frac{\varphi}{2}} \left( \frac{2G^{1}_{22}G^{2}_{11}}{\varphi \sin \left( \frac{\pi \varphi}{2} \right) \left( \frac{G^{1}_{22}}{G^{2}_{11}} \right)^{\pm(1+\varphi)}} \right) \]

⇒ All parameters given by J and K!

(No free fitting parameters)
Choose manifold:

\[ \rho_1 = \frac{1-b}{3\gamma}, \quad \rho_2 = \frac{\gamma - 1}{3\gamma} \]

\[ v_\pm = (1 + \gamma \rho_2) (1 - 2 \rho_1) \pm \gamma \sqrt{\rho_1 (1 - \rho_1) \rho_2 (1 - \rho_2)} \]

\[ \rho_1 = 0.25, \quad \rho_2 = 0.20, \quad \gamma = 2.5, \quad b = 0.625 \rightarrow v_- = 0.317, v_+ = 1.183 \]

\[
\begin{pmatrix}
0 & -0.406416 \\
-0.406416 & -0.105726
\end{pmatrix}, \quad
\begin{pmatrix}
-0.812833 & -0.052863 \\
-0.052863 & 0
\end{pmatrix}
\]
Simulation results (cont’)

Measurement of center of mass: Error << 1%

Asymmetry:  + mode: $\beta \approx -1$ for $t=600$;
- mode: $|\beta| < 1$ for all measured $t$ (small coupling constant)

Amplitude at maximum:

$\Rightarrow z = 1.618(5)$

Scaling plot                       Fit with max. asym. $\varphi$-Levy
Three lane model

New Fibonacci universality class: $z = 2, 3/2, 5/3, 8/5$

Mode 1: 8/5-Fibonacci, Mode 2: 5/3-Fibonacci, Mode 3: 3/2-KPZ.
Simulation results for 8/5 Fibonacci and GM

Choose large coupling constants ➔ maximal asymmetry at finite t

8/5 Fibonacci mode at t=1000:
Fit with max. asym. 8/5–Levy

Golden mean mode at t=3000:
Fit with max. asym. φ–Levy
5. Conclusions

- Mode coupling theory gives infinite discrete family of non-equilibrium universality classes for fluctuating hydrodynamics for hyperbolic systems.

- Dynamical exponents are Kepler ratios of consecutive Fibonacci numbers or the golden mean limit.

- Universality classes completely fixed by macroscopic stationary current-density relation.

- Scaling functions completely fixed by current-density relation and macroscopic stationary compressibility matrix.

- Stunning agreement of scaling functions with simulation data.