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# Digital Filtering of Band-limited Signals Using Periodic Nonuniform Sampling

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**Abstract**—We examine the problem of digital filtering of band-limited signals by means of a linear digital filter with one or more stopbands. The main target of the study is to filter the signals using lower than Landau sampling rates, where the Landau rate is defined as the total bandwidth of the input signal. In order to reach such low rates Periodic Nonuniform Sampling is employed. We derive necessary and sufficient conditions for perfect filtering, and propose a practical algorithm for constructing PNS grids that allow for sub-Landau sampling and filtering. Finally, we present a reconstruction system and provide a numerical result illustrating the proposed method.

## I. INTRODUCTION

The traditional DSP theory and applications are based on the classical Whittaker-Kotelnikov-Shannon theorem [1] which states that if a signal's spectrum is entirely placed inside the frequency interval  $F = [-f_0, f_0)$ , then the signal can be reconstructed from uniformly distributed samples taken at least at the rate  $f_s = 2f_0$ , which is known as the Nyquist rate. When lowpass signals are processed the minimum sampling rate required for signal reconstruction is equal to the total bandwidth of the signal which is  $B = 2f_0$ . If lowpass filtering of such signal is needed a sub-Nyquist uniform sampling rate could be used if the resulting aliasing is present only in the rejected stopbands. However, if highpass filtering is required then the minimum uniform sampling rate is the Nyquist rate, or  $B = 2f_0$ . In the case of bandpass and multiband signals, i.e. for which  $B < 2f_0$ , uniform sampling, reconstruction and filtering at the rate  $B$  is possible only if the aliases of the signal's spectrum  $X(f + nB)$ ,  $n \neq 0$  do not overlap with the passbands which are to be preserved. In that situation most often than not Nyquist sampling rate is the minimum one that can be achieved.

In order to reach lower than Nyquist sampling rates when processing bandpass and multiband signals, different sampling schemes were explored, amongst which was Periodic Nonuniform Sampling (PNS). Various authors ([2], [3], [4], and [5]) showed that PNS can be used for sampling and reconstruction of bandpass and multiband signals using lower than Nyquist rates. Landau [6] proved that the minimum sampling rate for arbitrary sampling and reconstruction of multiband signals asymptotically reaches the signal's total bandwidth  $B$  which may be significantly lower than the Nyquist rate. This result was exploited and several methods for

sampling and reconstruction of passband and multiband signals were introduced ([7], [8], [9], [10], and [11]).

However, all of the summarized work has focused primarily on the sampling and reconstruction of bandlimited signals. Digital filtering using lower than Nyquist, or even lower than Landau sampling rates, has not been studied widely. This work explores a practical topic – digital filtering of band-limited signals by means of a linear digital filter characterized by one or more stopbands. In this situation the output signal's bandwidth will be lower than the input's bandwidth which implies that lower than Landau sampling rate could possibly be used. The employed sampling scheme is PNS. We show that many digital filtering problems could be effectively tackled with by using the proposed method, which is most efficient when digital filtering of bandpass and multiband signals is required.

## II. PNS AND THE PROPOSED METHOD

PNS is a sampling scheme with sampling instants taken at times  $t_{l+qL} = t_l + qT$ , where  $l = 0, \dots, L-1$ . This sampling sequence is periodic with period  $T$ ,  $L$  denotes the number of sampling instants per period:  $0 = t_0 < t_1 < \dots < t_{L-1} < T$ , and  $q$  is an arbitrary integer indicating to which period the given sample time belongs. Let us assume that each sampling time  $t_n$ ,  $n = 0, \pm 1, \pm 2 \pm 3, \dots$  is a multiple of a short interval  $d$ :

$$t_n = c_n d, n \in \mathbf{I}. \quad (1)$$

A PNS scheme can be described as  $L$  parallel branches each using uniform sampling with period  $T$  but shifted in time with respect to each other. This is illustrated in Figure 1.

We begin our analysis by dividing the frequency axis into an infinite number of frequency cells  $F_k$ :

$$F_k = \left[ \frac{k - 0.5}{T}, \frac{k + 0.5}{T} \right). \quad (2)$$

Each frequency cell has length  $|F_k| = 1/T$ . Here  $T$  is a long enough period such that its inverse  $f_T = 1/T$  provides satisfactory resolution in the frequency domain. We assume

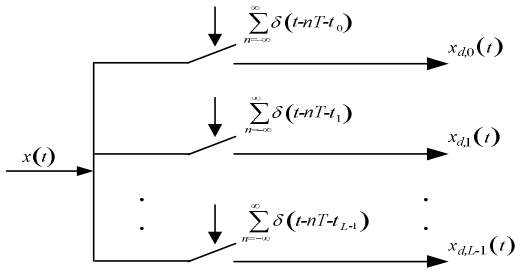


Figure 1. Periodic Nonuniform Sampling scheme.

that the signal occupies a finite number of not necessary adjacent cells  $F_{n_m} : F_{n_1}, F_{n_2}, \dots, F_{n_M}$ .  $M$  is the number of the occupied cells. The numbers  $n_M$  do not have to be put in a monotonic order. Let the first  $M_p$  cells represent the bands of the processed signal which will be preserved. For simplicity we assume that the frequency response of the filter at these bands is one. The remaining  $M_s$  frequency cells coincide with the stopbands of the filter; hence the filter's frequency response at these bands is zero. Here,  $M = P + S$ . We request that each cell  $F_{n_m}$  is fully occupied by either the passbands or the stopbands of the filter. (An example of the proposed analyzed scheme is illustrated in Figure 2.) The spectrum of the processed signal is thus:

$$X(f) = \sum_{m=1}^M X_{n_m} \left( f - \frac{n_m}{T} \right), \quad (3)$$

where the subspectra  $X_{n_m}(f)$  are defined by:

$$X_{n_m}(f) = \begin{cases} X \left( f + \frac{n_m}{T} \right), & f \in F_{n_m} \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

The discrete spectrum  $X_{d,l}(f)$  in the  $l$ -th branch of the PNS system is:

$$X_{d,l}(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X \left( f + \frac{k}{T} \right) e^{j2\pi \frac{k}{T} t_l}. \quad (5)$$

Substituting (3) and (4) in (5) we find the relation between the components  $X_{n_1}(f), \dots, X_{n_M}(f)$ , and the discrete-time spectra  $X_{d,0}(f), \dots, X_{d,L-1}(f)$ . Then we confine our analysis to the baseband frequency interval  $f \in [-0.5/T, 0.5/T]$ :

$$X_{d,l}(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \sum_{n_m=-\infty}^{\infty} X_{n_m} \left( f + \left( \frac{k - n_m}{T} \right) \right) \left( e^{j2\pi \frac{t_l}{T}} \right)^k. \quad (6)$$

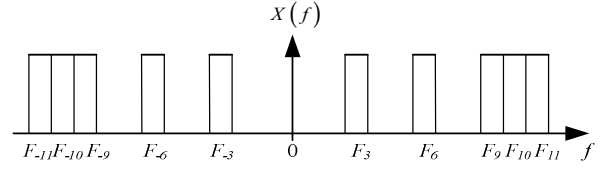


Figure 2. Frequency domain division of input signal's spectrum.

Considering the fact that if  $f \in [f_0, f_0 + 1/T]$ ,  $X_{n_m}(f - k/T)$  takes nonzero values inside this interval only if  $k = 0$ , we can simplify (6):

$$X_{d,l}(f) = \frac{1}{T} \sum_{m=1}^M X_{n_m}(f) z_l^{n_m}. \quad (7)$$

where  $z_l = e^{j2\pi \frac{t_l}{T}}$ . Note that  $z_0 = 1$ . Expanding (7) to all  $l$  branches of the PNS scheme and presenting it in the matrix form, we can write:

$$\begin{bmatrix} X_{d,0}(f) \\ X_{d,1}(f) \\ \dots \\ X_{d,L-1}(f) \end{bmatrix} = \frac{1}{T} \begin{bmatrix} 1 & \dots & 1 & | & 1 & \dots & 1 \\ z_1^{n_1} & \dots & z_1^{n_p} & | & z_1^{n_{p+1}} & \dots & z_1^{n_M} \\ \dots & \dots & \dots & | & \dots & \dots & \dots \\ z_{L-1}^{n_1} & \dots & z_{L-1}^{n_p} & | & z_{L-1}^{n_{p+1}} & \dots & z_{L-1}^{n_M} \end{bmatrix} \begin{bmatrix} \mathbf{X}_P \\ \dots \\ \mathbf{X}_S \end{bmatrix}, \quad (8)$$

or in short:

$$\mathbf{X}_d = \frac{1}{T} (\mathbf{Z}_P \mathbf{X}_P + \mathbf{Z}_S \mathbf{X}_S), \quad (9)$$

where

$$\mathbf{X}_P = \begin{bmatrix} X_{n_1}(f) \\ \dots \\ X_{n_p}(f) \end{bmatrix}, \quad \mathbf{X}_S = \begin{bmatrix} X_{n_{p+1}}(f) \\ \dots \\ X_{n_M}(f) \end{bmatrix},$$

$$\mathbf{Z}_P = \begin{bmatrix} 1 & \dots & 1 \\ z_1^{n_1} & \dots & z_1^{n_p} \\ \dots & \dots & \dots \\ z_{L-1}^{n_1} & \dots & z_{L-1}^{n_p} \end{bmatrix}, \quad \text{and} \quad \mathbf{Z}_S = \begin{bmatrix} 1 & \dots & 1 \\ z_1^{n_{p+1}} & \dots & z_1^{n_M} \\ \dots & \dots & \dots \\ z_{L-1}^{n_{p+1}} & \dots & z_{L-1}^{n_M} \end{bmatrix}.$$

The solution to (9) with respect to  $\mathbf{X}_P$  will obtain the spectral components  $X_{n_1}(f), \dots, X_{n_M}(f)$ , thus allowing for reconstruction of the passband frequencies. However, the main problems we address here are:

1) Knowing the vector  $\mathbf{X}_d$ , and the matrices  $\mathbf{Z}_P$  and  $\mathbf{Z}_S$ , under what conditions can we find  $\mathbf{X}_P$ ?

2) What is the smallest number  $L$  of sampling instants inside one period  $T$  for which all solutions to (9) give correct results for  $\mathbf{X}_p$ ?

### III. CONDITIONS FOR PERFECT FILTERING

The following theorem summarizes the necessary and sufficient conditions required for the recovery of  $\mathbf{X}_p$ .

*Theorem:* If a band-limited signal  $x(t)$ , described in the context of (3) and (4), is sampled using Periodic Nonuniform Sampling scheme, the necessary and sufficient conditions for its perfect filtering with respect to  $\mathbf{X}_p$  are:

$$\text{rank} [\mathbf{X}_d, \mathbf{Z}_p, \mathbf{Z}_S] = \text{rank} [\mathbf{Z}_p, \mathbf{Z}_S],$$

$$\text{rank} [\mathbf{Z}_p] = P,$$

$$\text{rank} [\mathbf{Z}_p | \mathbf{Z}_S] = \text{rank} [\mathbf{Z}_p] + \text{rank} [\mathbf{Z}_S].$$

(The proof of the theorem is presented in the Appendix.) Resorting on Theorem 1, we can find  $\mathbf{X}_p$  by:

$$\mathbf{X}_p = T\Phi_p \mathbf{X}_d, \quad (10)$$

where:

$$\Phi_p = \mathbf{Z}_p^T (\mathbf{Z}_p \mathbf{Z}_p^T + \mathbf{Z}_S \mathbf{Z}_S^T)^{-1}. \quad (11)$$

Taking the inverse Fourier transform of  $\mathbf{X}_p$ , we derive:

$$\mathbf{x}_p = T\Phi_p \mathbf{x}_d, \quad (12)$$

where  $\mathbf{x}_p = [x_{n_1}(t), \dots, x_{n_p}(t)]'$  contains the samples carrying information of each passband component, and  $\mathbf{x}_d = [x_{d0}(t), \dots, x_{dL-1}(t)]'$ .

### IV. DESIGN OF PNS SEQUENCES

In this section we propose an algorithm that aims at reducing the values of  $L$  for which all solutions to (9) still give correct results for  $\mathbf{X}_p$ .

The third condition of Theorem 1 states that  $\text{rank} [\mathbf{Z}_p | \mathbf{Z}_S] = \text{rank} [\mathbf{Z}_p] + \text{rank} [\mathbf{Z}_S]$ . Hence:

$$P+1 \leq L \leq P + \text{rank} [\mathbf{Z}_S]. \quad (13)$$

To reduce the value of  $L$ , we have to find sampling instants  $t_l$  such that  $\text{rank} [\mathbf{Z}_S]$  is minimum, and  $\text{rank} [\mathbf{Z}_p] = P$ . Or, in other words, find sampling instants  $t_l$  such that  $\mathbf{Z}_S$  has as many pairs of linearly dependent columns as possible, while all columns of  $\mathbf{Z}_S$  are linearly independent from each other and the columns of  $\mathbf{Z}_p$ .  $\mathbf{Z}_S$  can be presented as:

$$\mathbf{Z}_S = \begin{bmatrix} 1 & \dots & 1 \\ e^{j2\pi t_1 n_{p+1}} & \dots & e^{j2\pi t_1 n_M} \\ \dots & \dots & \dots \\ e^{j2\pi t_{L-1} n_{p+1}} & \dots & e^{j2\pi t_{L-1} n_M} \end{bmatrix}. \quad (14)$$

Since the first element of each column of  $\mathbf{Z}_S$  is one, then the pairs of linearly dependent columns of  $\mathbf{Z}_S$  have to be identical. Comparing the first two columns of  $\mathbf{Z}_S$ , we have:

$$\begin{aligned} t_1 n_{p+1} &= t_1 n_{p+2} + k_1 \\ &\vdots \\ t_{L-1} n_{p+1} &= t_{L-1} n_{p+2} + k_{L-1} \end{aligned} \quad (15)$$

Denoting

$$\Delta_{p+1, p+2} = n_{p+1} - n_{p+2}, \quad (16)$$

we write

$$\Delta_{p+1, p+2} t_l \in \mathbf{I}, \quad l = 0, \dots, L-1. \quad (17)$$

Similarly,  $\Delta_{p+2, p+3} t_l \in \mathbf{I}$ ,  $\Delta_{p+3, p+4} t_l \in \mathbf{I}$ , etc. Having as many  $\Delta_{i,j} t_l \in \mathbf{I}$ ,  $i, j = p+1, \dots, M$  as possible will minimize  $\text{rank} [\mathbf{Z}_S]$ , and hence  $L$ . Referring back to (1), we have  $\Delta_{i,j} t_l = \Delta_{i,j} c_l d \in \mathbf{I}$ ,  $c_l \in \mathbf{I}$ . There is no known analytical expression for finding the optimum  $d$  which will guarantee finding the minimum  $L$ , and thus reaching the minimum average sampling rate  $f_{S_{ave}} = L/T$ . However, an exhaustive search for  $d$  would be confined to a finite number of iterations since  $t_l$  and  $T$  are multiples of  $d$ . Based on the conclusions drawn above, we propose the following algorithm for optimal PNS sequence design:

*Step 1:* Choice of period  $T$ :  $T$  must be a long enough period which will allow for dividing the input and output signals' spectra in the context of (3) and (4). A good choice for an initial  $T$  can be the least common multiple of the spectral support functions' frequencies.

*Step 2:* Choice of  $d$  and  $L$ : choose  $d$  such that  $d = T/N$ ,  $N \in \mathbf{I}$ . A practical guide to finding an optimal  $d$  is to check if after uniform sampling of the signal with period  $d$  the passbands should be alias-free. The aim is to create a "constructive aliasing" which concentrates in the rejected bands, or the stopbands of the filter. The total bandwidth of all spectral components inside the interval  $f \in [-0.5/d, 0.5/d]$  will present the minimum number of sampling instants  $L$  inside one period  $T$ , and hence the minimum average sampling rate  $f_{S_{ave}} = L/T$ .

*Step 3:* Choice of  $t_l$ : choose  $t_l$  such that  $t_l = c_l d$ ,  $c_l \in \mathbf{I}$ .

*Step 4:* Necessary and sufficient conditions check : check if the matrices  $\mathbf{Z}_p$  and  $\mathbf{Z}_s$ , constructed with the chosen  $T$ ,  $d$  and  $t_l$ , meet the conditions from the Theorem. If not, restart the algorithm from Step 2 by choosing a different  $d$ , or from Step 1 by choosing a different period  $T$ .

## V. IMPLEMENTATION OF THE DIGITAL FILTER

We propose and compare two filter implementation systems. The first realization is described as follows. After sampling the input signal using PNS with period  $T$  and sampling instants  $t_l$ , derived using the algorithm, we upsample the signals  $x_{dl}(t)$  to a sampling rate  $f_d = 1/d = R/T$ , where  $d$  is defined by (1) and found with the use of the outlined algorithm. The reconstruction matrix is calculated using (11) and each upsampled  $x_{dl}(t)$  is multiplied by the corresponding element of  $\Phi_p$ . Subsequently, each  $x_p(t)$  is lowpass filtered in the baseband frequency interval  $f \in [-0.5/T, 0.5/T)$ , and frequency-shifted to the original band position. The reconstruction formula is thus:

$$x(t) = T \sum_{q=-\infty}^{\infty} \sum_{l=0}^{L-1} \sum_{p=1}^P [\Phi_p]_{pl} x_{dl}(t_l + qT) \phi_p(t - t_l - qT), \quad (18)$$

where:

$$\phi_p(t) = e^{j2\pi \frac{n_p}{T} t} \frac{R}{T} \text{sinc}\left(\frac{t}{T}\right). \quad (19)$$

We will denote this filter implementation scheme as Filter A which is shown in Figure 3.

The second realization of the filter is a direct extension of the reconstruction method for PNS [10]. If the implementation is performed along the branches of PNS, that is there is a synthesis filter  $\Psi_l(f)$  after each upsampler, the implementation scheme can be described as:

$$x(t) = T \sum_{q=-\infty}^{\infty} \sum_{l=0}^{L-1} \sum_{p=1}^P [\Phi_p]_{pl} x_{dl}(t_l + qT) \psi_l(t - t_l - qT), \quad (20)$$

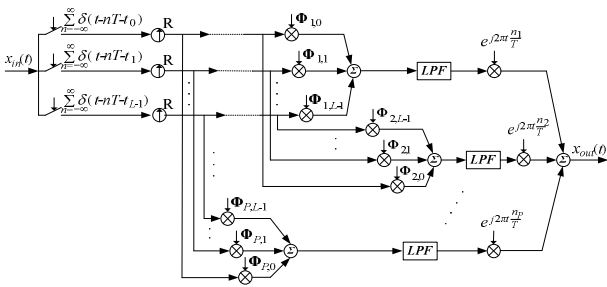


Figure 3. Digital filter implementation – Filter A.

where the filters  $\psi_l(t)$  are described by:

$$\Psi_l(f) = \begin{cases} \frac{R}{T} [\Phi_p]_{pl} e^{j2\pi \frac{n_p}{T} t}, & f \in F_{n_p}, \\ 0, & \text{otherwise} \end{cases}, \quad (21)$$

and  $\Psi_l(f)$  is the Fourier transform of  $\psi_l(t)$ . The latter reconstruction scheme is shown in Figure 4, and will be denoted as Filter B. The two filter implementation schemes are equivalent because they produce identical results, as will be later confirmed by a numerical example. The difference is in the filters that are used. Filter A scheme uses one lowpass filter for each passband, whereas Filter B scheme uses  $L$  passband filters with piece-wise constant frequency response. In practice causal, linear-phase FIR or IIR filters could be used, which introduce some delay and distortion. Therefore, additional care must be taken that the group delays in each branch of the filter structure are equalized.

## VI. A NUMERICAL EXAMPLE

Consider a random test signal which has the following spectral support (in normalized frequencies):

$$SSF_{in} = (-7.5, -5.5) \cup (-2.5, -1.5) \cup (-0.5, 0.5) \cup (1.5, 2.5) \cup (5.5, 7.5).$$

Let the filter passbands be:

$$SSF_{out} = (-6.5, -5.5) \cup (-2.5, -1.5) \cup (1.5, 2.5) \cup (5.5, 6.5).$$

Figure 5 shows the power spectrum density of the input and output signals estimated from a periodogram taken over a window  $[0, 108T]$ . The Nyquist uniform sampling rate for this signal is  $f_{Nyquist} = 2f_{max} = 2 * 7.5 = 15$ . The Landau rate, or the minimum average sampling rate for sampling and reconstruction of the input signal, is  $f_{Landau} = 7$ . Initiating the algorithm described in section IV, we choose  $T = 1$ . Hence, the input and output spectral supports can be represented as:  $F_{in} = [-7 \ -6 \ -2 \ 0 \ 2 \ 6 \ 7]$ , and  $F_{out} = [-6 \ -2 \ 2 \ 6]$ . We choose  $d = 1/f_{Landau} = 1/7$ , as uniform sampling of the input signal with rate  $f_d = 1/d = 7$  does not induce aliasing in the passbands. Also, uniform sampling of the input signal with such rate shows that the total bandwidth of the spectral components in the interval  $[-7/2, 7/2)$  is 5. Hence, we choose  $L = 5$ . The vector  $t_l$  of the sampling instants inside a

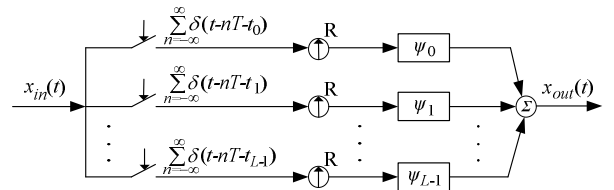


Figure 4. Digital filter implementation – Filter B.

period  $T$  is then selected to be  $t_l = [0 \ 2d \ 3d \ 5d \ 6d]$ . Step 4 of the proposed algorithm confirms the proper choice of  $T$ ,  $d$ , and  $t_l$ . Note that  $\mathbf{Z}_S$  is thus:

$$\mathbf{Z}_S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

or in other words  $\text{rank}[\mathbf{Z}_S]=1$ , which confirms that the achieved average sampling rate of  $f_{S_{av}} = L/T = 5$  is the optimal rate, a 29% lower sampling rate than the Landau rate for signal reconstruction. The results presented below confirm the feasibility of the proposed approach. Figure 6 shows that when both implementation schemes, Filter A and Filter B, are realized with ideal filters, they produce identical outputs. Figure 7 shows the magnitude of the spectrum of filtering error when Filter A scheme has been constructed using a lowpass equiripple FIR filter [12] of order 221, which has a transition band of  $0.25/T$ , 0.3 dB passband ripple and -48.99 dB stopband attenuation. Figure 8 shows the magnitude of the spectrum of filtering error when Filter B scheme has been realized, using multiband Chebyshev FIR filters [13] of order 221, with transition band of  $0.25/T$ , 0.3 dB passband ripple and -57.8 dB stopband attenuation. Figure 9 shows the magnitude of the spectrum of the filtering error when Filter A scheme has been constructed using a lowpass IIR filter of order 10, acquired from the lowpass FIR filter used above through balanced model reduction [14]. The IIR filter has the same bandwidth and transition band as the lowpass FIR filter, and has 0.5 dB passband ripple and -47 dB stopband attenuation. All three realizations produced almost identical results. (Note that the peak errors are due to the wide transition bands of the filters.) However, Filter A scheme has the advantage of using one lowpass filter, whereas Filter B scheme uses three different multiband filters (two pairs of the branches of Filter B structure happen to use the same filters).

VII. CONCLUSIONS

We have discussed the possibility of perfect filtering of band-limited signals using sub-Landau average sampling rates. We have shown that many signals can be filtered at such rates,

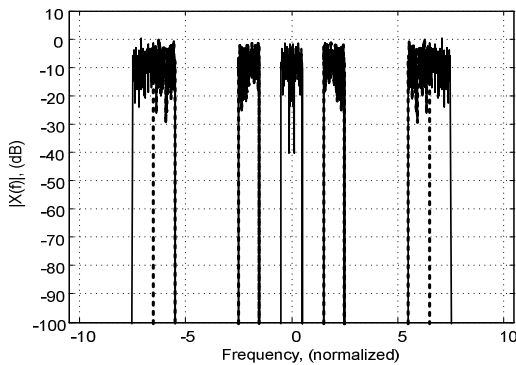


Figure 5. Power spectrum density of input and output signals: a) solid line – input signal; b) dotted line – output signal.

specifically signals with sparse passband or multiband spectra used in high frequency applications, and in the cases when it is possible, necessary and sufficient conditions have been derived. A procedure for designing PNS schemes which facilitates the proposed approach has been proposed. Two different filter realizations have been developed and compared – the first, Filter A, uses one lowpass filter, whereas the second, Filter B, uses up to  $L$  different passband or multiband filters. A numerical result has confirmed the feasibility of the proposed method, suggesting that the quality of filtering highly depends on the quality of the real FIR or IIR filters that are used in practice.

ACKNOWLEDGMENT

The authors would like to thank Prof. J. Samsonowicz for his invaluable suggestions and discussions.

APPENDIX

*Proof of Theorem:* We remark that because of the structure of the vector  $\mathbf{X}_d$  and the matrices  $\mathbf{Z}_p$  and  $\mathbf{Z}_S$ , the equation

$$\mathbf{X}_d = \mathbf{Z}_p \hat{\mathbf{X}}_p + \mathbf{Z}_S \hat{\mathbf{X}}_S, \tag{A.1}$$

can always be solved with respect to  $\hat{\mathbf{X}}_p$  and  $\hat{\mathbf{X}}_S$ . We only need to choose  $\hat{\mathbf{X}}_p = \mathbf{X}_p$  and  $\hat{\mathbf{X}}_S = \mathbf{X}_S$ . We therefore expect that:

$$\text{rank}[\mathbf{X}_d, \mathbf{Z}_p, \mathbf{Z}_S] = \text{rank}[\mathbf{Z}_p, \mathbf{Z}_S]. \tag{A.2}$$

Next, the conditions under which all solutions to (A.1) are such that:

$$\hat{\mathbf{X}}_p = \mathbf{X}_p, \tag{A.3}$$

have to be found. Then we will be able to reconstruct the filtered signal. Let us denote  $\Delta\mathbf{X}_p = \hat{\mathbf{X}}_p - \mathbf{X}_p$ , and  $\Delta\mathbf{X}_S = \hat{\mathbf{X}}_S - \mathbf{X}_S$ . We prove that all solutions to (A.1) satisfy (A.3) if and only if:

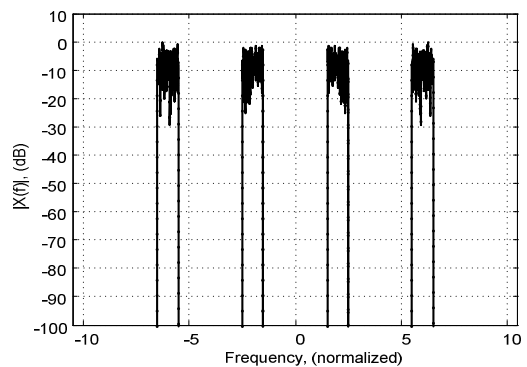


Figure 6. Power spectrum density of filtered signals using ideal filters: a) solid line – ideal output; b) dash-dot line – Filter A output; c) dotted line – Filter B output.

- 1) The only solution of  $\mathbf{Z}_p \Delta \mathbf{X}_p = 0$  is  $\Delta \mathbf{X}_p = 0$ ;
- 2) All solutions to  $\mathbf{Z}_p \Delta \mathbf{X}_p = -\mathbf{Z}_s \Delta \mathbf{X}_s$  satisfy  $\mathbf{Z}_p \Delta \mathbf{X}_p = -\mathbf{Z}_s \Delta \mathbf{X}_s = 0$ .

Both conditions are necessary. If the first condition is not satisfied then we could find nonzero  $\Delta \mathbf{X}_p$  such that  $\mathbf{Z}_p \Delta \mathbf{X}_p = 0$  and construct the following solution (A.1):  $\hat{\mathbf{X}}_p = \mathbf{X}_p + \Delta \mathbf{X}_p$ ,  $\hat{\mathbf{X}}_s = \mathbf{X}_s$ , which does not satisfy (A.3). Similarly, if the second condition is not satisfied, we could find a solution to  $\mathbf{Z}_p \Delta \mathbf{X}_p = -\mathbf{Z}_s \Delta \mathbf{X}_s$  such that  $\mathbf{Z}_p \Delta \mathbf{X}_p = -\mathbf{Z}_s \Delta \mathbf{X}_s \neq 0$ . By choosing  $\hat{\mathbf{X}}_p = \mathbf{X}_p + \Delta \mathbf{X}_p$  and  $\hat{\mathbf{X}}_s = \mathbf{X}_s + \Delta \mathbf{X}_s$  we get a solution to (A.1) that does not comply with (A.3).

The above conditions are also sufficient. We demonstrate this by proving that if there exists a solution to (A.1) that contradicts (A.3), then at least one of the above conditions is violated. Let  $\mathbf{X}_d = \mathbf{Z}_p \hat{\mathbf{X}}_p + \mathbf{Z}_s \hat{\mathbf{X}}_s$ , and  $\hat{\mathbf{X}}_p \neq \mathbf{X}_p$ . Subtracting (A.3) from (A.1) gives  $\mathbf{Z}_p (\mathbf{X}_p - \hat{\mathbf{X}}_p) = -\mathbf{Z}_s (\mathbf{X}_s - \hat{\mathbf{X}}_s)$ . If the second condition is to be satisfied, then  $\mathbf{Z}_p (\mathbf{X}_p - \hat{\mathbf{X}}_p) = 0$ . Consequently, if the first condition is satisfied, we get  $\hat{\mathbf{X}}_p = \mathbf{X}_p$ , which contradicts our earlier assumption that  $\hat{\mathbf{X}}_p \neq \mathbf{X}_p$ . We note that it is necessary that the matrix  $\mathbf{Z}_p$  has full column rank. If this was not satisfied then there would be nonzero values  $\Delta \mathbf{X}_p$  such that  $\mathbf{Z}_p \Delta \mathbf{X}_p = 0$ . In such cases, if  $\mathbf{X}_p$  and  $\mathbf{X}_s$  solve (9) then  $\mathbf{X}_d = \mathbf{Z}_p (\hat{\mathbf{X}}_p - \Delta \mathbf{X}_p) + \mathbf{Z}_s \mathbf{X}_s$ . Having  $\mathbf{Z}_p \Delta \mathbf{X}_p = 0$ , we get  $\mathbf{X}_d = \mathbf{Z}_p \hat{\mathbf{X}}_p + \mathbf{Z}_s \mathbf{X}_s$ , which contradicts (9) and (A.1).

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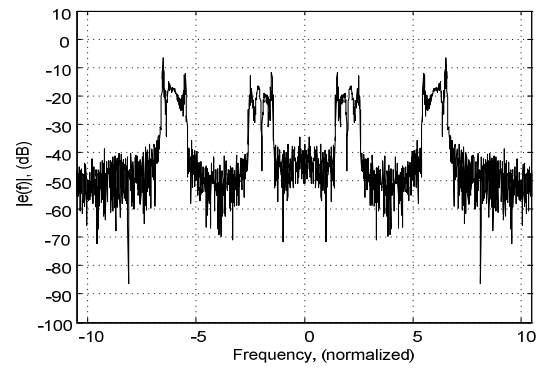


Figure 7. Magnitude of the spectrum of filtering error – Filter A scheme using lowpass FIR filter.

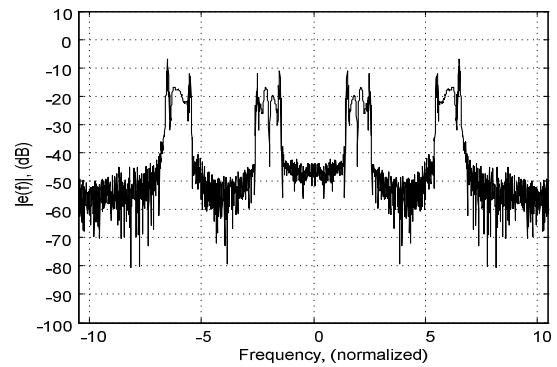


Figure 8. Magnitude of the spectrum of filtering error – Filter B scheme using multiband FIR filters.

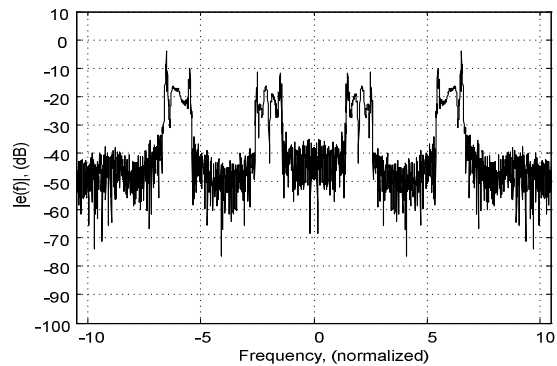


Figure 9. Magnitude of the spectrum of filtering error – Filter A scheme using lowpass IIR filter.

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