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The Formal Demography of Prospective Age: The Relationship Between the Old-Age Dependency Ratio and the Prospective Old-Age Dependency Ratio

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Abstract

BACKGROUND
Sanderson and Scherbov (2015) demonstrated that there was an inverse relation between the speeds of aging as measured by the conventional and the prospective old-age dependency ratios.

OBJECTIVE and METHODS
Here, we examine this counterintuitive finding analytically and with simulations. To this end, we decompose changes in mortality schedules into shift and compression processes. After studying the effects of these two processes on the dependency ratios analytically, we examine the effects empirically, using the HMD data.

RESULTS
Theory shows that the two mortality processes (of shift and compression) push the two old-age dependency ratios in opposite directions. Our formal results are supported by simulations that show a positive effect of a mortality shift on the conventional old-age dependency ratio and a negative effect of it on the prospective old-age dependency ratio. The effects are of opposite sign for the mortality compression. The analytical and empirical results also suggest the effect of the shift is stronger than the effect of compression. Hence, mortality declines, typically, imply increasing conventional but decreasing prospective old-age dependency ratios.

CONCLUSIONS
Our formal and empirical results taken together they suggest that the inverse relation between the conventional and prospective old-age dependency ratios is a universal feature of human mortality change.

CONTRIBUTION
This paper contributes to the literature on formal demographic analysis, particularly on the effects of mortality shifts and compression on dependency ratios. It demonstrates that the counter-intuitive finding in Sanderson and Scherbov (2015) is a fundamental feature of human mortality change. We also contribute in showing usefulness of the shift-compression decomposition model of mortality change developed in (Dalkhat M. Ediev 2013).

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The Formal Demography of Prospective Age: The Relationship Between the Old-Age Dependency Ratio and the Prospective Old-Age Dependency Ratio

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1. Introduction

Aging is a complex multidimensional phenomenon. In order to analyze multiple dimensions of aging within a unified framework Sanderson and Scherbov introduced and elaborated the concept of characteristic based ages called "alpha-ages" (Sanderson and Scherbov 2005, 2013). People who share an alpha-age also share the same level of some characteristic of interest. One especially important characteristic is remaining life expectancy. Alpha-ages that are based on remaining life expectancy are called prospective ages and indicators of aging that are based on prospective ages are called prospective measures of aging.

One especially important alpha-age–denoted here as $R(t)$, t being the time variable–is the one used to replace 65, the fixed age at which people become classified as being "old" in conventional measures such as the Old-Age Dependency Ratio (OADR)

$$OADR(t) = \frac{OAP(t)}{WAP(t)} = \frac{\int_{R_0}^{\omega} P(x,t)dx}{\int_{W}^{R_0} P(x,t)dx}. \quad (1)$$

Here, $OAP(t)$ is the old-age population at time t obtained by summing the population $P(x,t)$ over ages R_0 (the fixed definition of when the old-age starts, commonly 65 years) to ω (the maximum lifespan). The working-age population at time t, WAP(t), is the population between the ages W (the onset of the working age interval) and R_0. The OADR is often defined as the ratio of the population 65+ years old to those 20 to 64. To account for changing lifespans, we have introduced a varying alpha-age $R(t)$, the age at which the remaining life expectancy is time-invariant and equal to the remaining life expectancy at age R_0 in the base year t_0[1] (Sanderson and Scherbov 2005). When the varying alpha-age $R(t)$ is used instead of age R_0, a new ratio is obtained where the numerator is the number of people at or above $R(t)$ and the denominator is the number of people from age W to $R(t)$. This new measure is called the Prospective Old-Age Dependency Ratio (POADR) (Sanderson and Scherbov 2010):

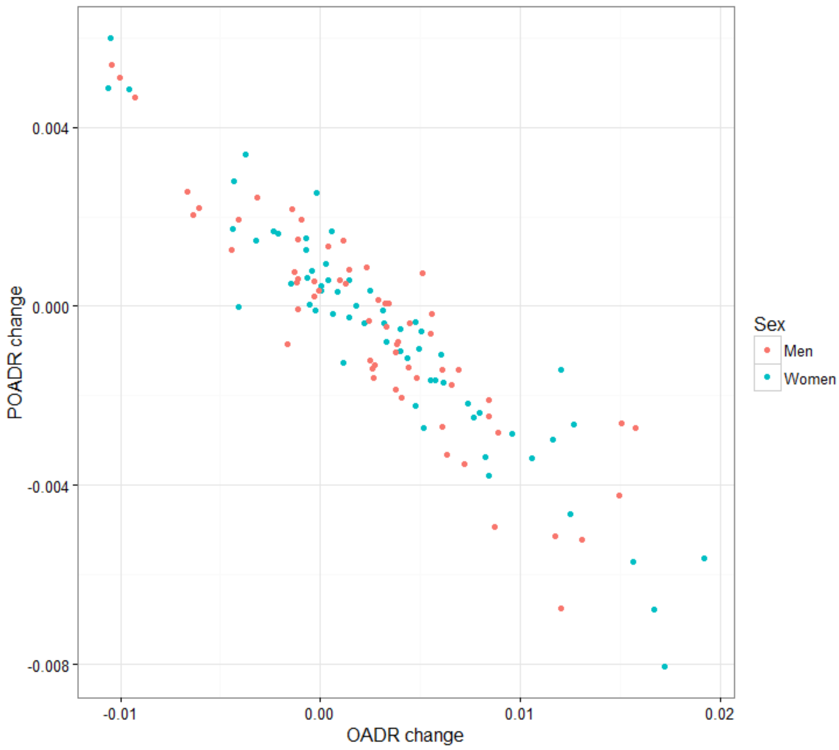
[1] t_0 is the base year against which we study changes in the dependency ratios. For numerical illustrations, we use a rolling base: for year t, we use year $t_0 = t - 1$ as the base year. Base-year relation $R(t_0) = R_0$ assures a similar 'start' for both the conventional and prospective indicators of population ageing.

$$POADR(t) = \frac{\int_{R(t)}^{\omega} P(x,t)dx}{\int_{W}^{R(t)} P(x,t)dx}. \tag{2}$$

Sanderson and Scherbov (2015) used population forecasts for European countries to demonstrate a counterintuitive result. When the speed of aging is measured using the increase in the POADR, faster increases in life expectancy lead to slower increases in aging. Figures 1 and 2 show the same phenomenon in a different context. Figure 1 shows the relationship between annual changes in OADRs and POADRs defined over French life table populations for the years 1950 to 2010. Figure 2 shows the same relationship plotted for all life table populations of the Human Mortality Database (HMD 2016) plotted separately by sex in currently low and high mortality countries[2], for all the years after 1950 that were available in HMD.

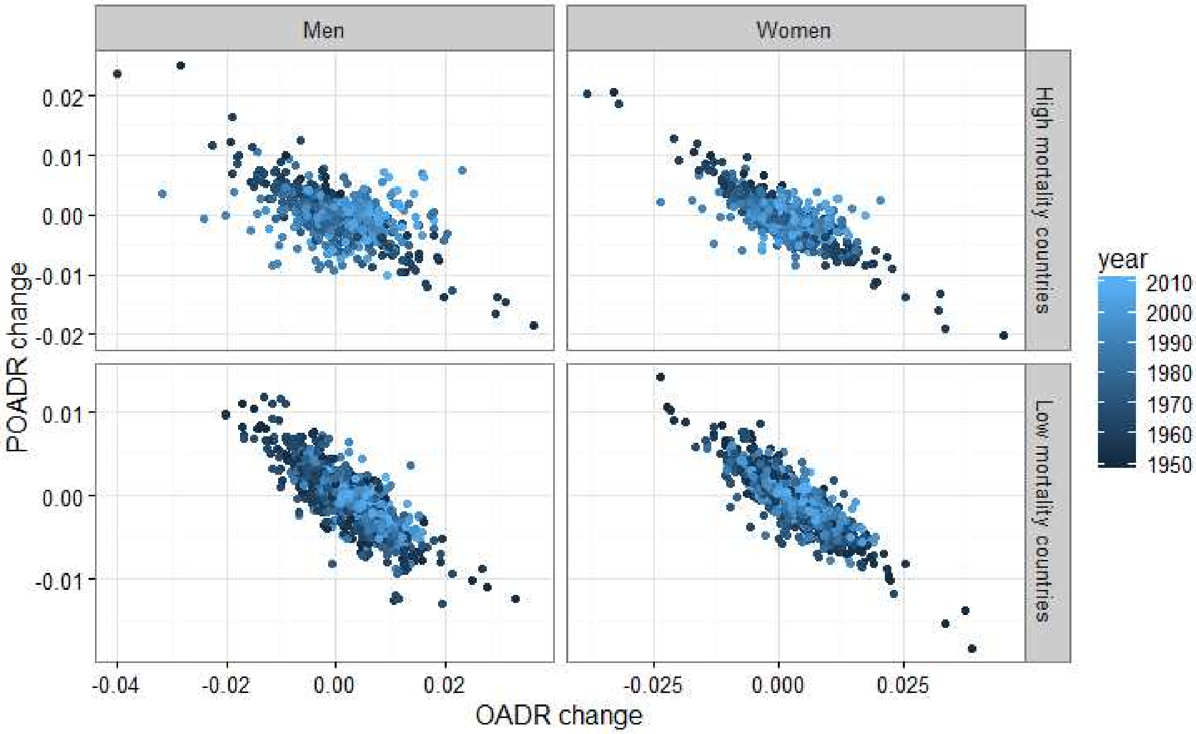
In both Figures, we can see a strong negative relationship between the annual changes in OADRs and POADRs. The object of this paper is to explain analytically why this counterintuitive relationship exists. We focus on the effects of mortality change alone, to which end we consider OADRs in life table (stationary) populations $P(x,t) = l(x,t)$, where $l(x,t)$ is the life table proportion surviving to age x in the period life table for the year t.

Figure 1: Annual change of life-table POADR vs the conventional OADR, France, both sexes; period life tables for the years 1950-2010.



[2] The list of currently 'Low-mortality' countries is obtained by excluding Eastern-European countries from the HMD. It includes: Australia, Austria, Belgium, Canada, Chile, Denmark, England and Wales, Finland, France, Total
Population, Germany, Ireland, Israel, Italy, Japan, Netherlands, New Zealand (Non-Maori), Northern Ireland,
Norway, Portugal, Scotland, Spain, Sweden, Switzerland, Taiwan, USA, UK, West Germany. We also exclude some small-size populations (Iceland, New Zealand--Maori, and Luxemburg). "High-mortality" countries include the Eastern-European countries.

Figure 2: Annual change of life-table POADR vs the conventional OADR, entire HMD: men, women, currently High- and Low-mortality countries.



The paper is organized as follows. The analytic framework that we use here is the shift-compression model of the age distribution of adult deaths in Ediev (2013). This model is presented in Section 2. In Section 3, we show that the shift-compression model produces a closed form solution for the alpha-age $R(t)$. Section 4 presents the analytic relationship between shifts and compressions of mortality schedules and changes in OADRs and POADRs. In Section 5, we quantify shifts and compressions in mortality schedules using data from the HMD and show that the observed shifts and compressions produce the results seen in Figures 1 and 2. Section 6 contains our concluding discussion. In the appendices we provide details of formal derivations and summarize notations used in the paper.

2. The mortality model

We use the shift-compression/expansion model of the age distribution of adult deaths (Dalkhat M. Ediev 2013). Here the age distribution of adult deaths is expressed in relation to a baseline age distribution. The age distribution of adult deaths can be expressed as:

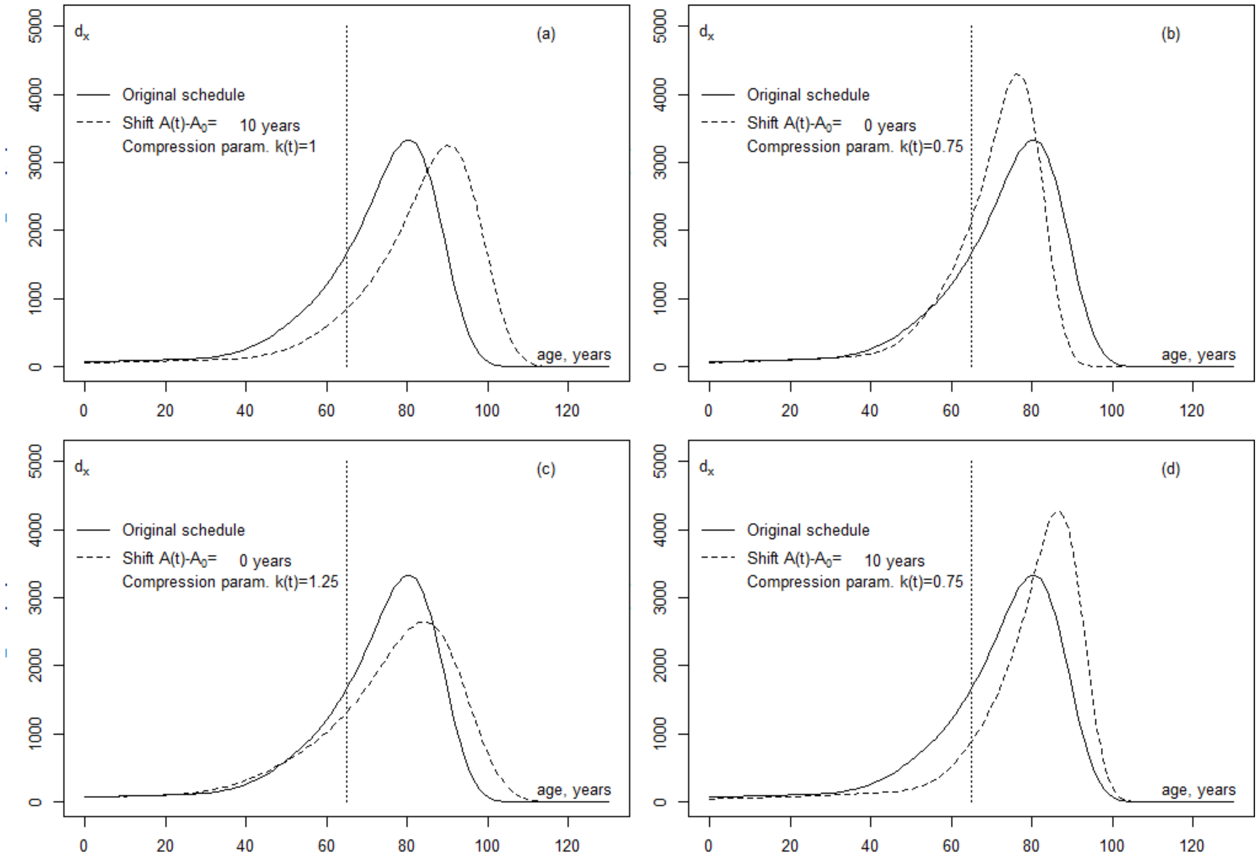
$$d(x,t) \propto d\left(A_0 - \frac{A(t)-x}{k(t)}, t_0\right) \tag{3}$$

where $d(x,t)$ is the age distribution of life table adult deaths at time t, t_0 is the base year against which we study changes in the dependency ratios. The relationship between the age distribution of deaths in year t and year t_0 depends on two parameters $A(t)$ and $k(t)$. $A(t) - A_0$ is the amount of age shift of the distribution at the arbitrarily chosen pivotal age $A_0 = A(t_0)$ of the original distribution of the life table deaths. The parameter $k(t)$ describes the amount of compression or expansion of the distribution ($k(t_0) = 1$ by definition). When $A(t) - A_0 > 0$, the age distribution of deaths in year t is shifted rightwards relative to the distribution in year t_0, i.e. towards longer lifespans. When

$k(t) = 1$, the mortality shift occurs without a change in the shape of the distribution of ages at death. When $k(t) > 1$, mortality expands, i.e., the deaths stretch over more years of life. When $k(t) < 1$, the distribution of age at death is compressed. .

In Figure 3, we present four schematic scenarios for shift and compression or expansion with the pivotal age $A_0 = 65$. In panel (a), the original schedule is shifted rightwards by 10 years without compression or expansion. In panel (b), compression ($k(t) = 0.75$) occurs without shift. The transformed schedule is also scaled upwards so as to sum up to the same total number of deaths as in the original schedule. In panel (c), the distribution expands ($k(t) = 1.25$) without shift. And in panel (d), we present the more realistic scenario of shift and compression. Typically, period mortality decline manifests itself in positive shifts combined with compression, although the cohort mortality decline may also manifest itself without compression or even with expansion (Dalkhat M Ediev 2013; Ediev 2011).

Figure 3: Illustrative scenarios of the mortality model. The vertical broken line indicates the pivotal age A_0 =65. (a) The original distribution of life table deaths is shifted by 10 years. (b) The original distribution of life table deaths is compressed by 25%. (c) The original distribution of life table deaths is expanded by 25%. (d) The original distribution of life table deaths is shifted by 10 years and compressed by 25%.



Before proceeding to the implications of the mortality model, it may be useful to develop a better intuition for how the model works. Importantly, the choice of the pivotal age does not affect the fit of the model to empirical data and, in that sense, is arbitrary. For any pivotal age, it is always possible to set the model parameters so that the model produces exact the same distribution of deaths. This choice makes difference, however, for definition and interpretation of the shift parameter. Unlike the compression/expansion parameter $k(t)$ that has no link to any particular age, the mortality shift may only be defined with respect to a given age. Indeed, if mortality would be showing a universal shift, equal for all ages, there could have been no compression or expansion of the mortality distribution: all parts of the curve would be shifting without changing the distance between them. Such a pure shift, however, is a rare case; a more plausible scenario involves the age-specific shifts. In our model, the variety of age-specific shifts, say, $\Delta(x,t)$ at some age x by time t and $\Delta(y,t)$ at another age y, are linked to each other through the compression/expansion parameter $k(t)$: $\big(y+\Delta(y,t)\big)-\big(x+\Delta(x,t)\big)=k(t)(y-x)$. This implies:

$$\Delta(y,t)=\Delta(x,t)+\ (k(t)-1)(y-x). \qquad (4)$$

Therefore, it becomes possible to describe the whole set of age-specific shifts of the deaths distribution curve by the shift at one (arbitrarily chosen) age A_0 and by the compression/expansion coefficient $k(t)$. The choice of the pivotal age is arbitrary and has no consequences for the kinds of changes the model describes. If one wishes to opt for a different pivotal age, it is only necessary to recalculate the amount of shift at new pivotal age according to (4). When making statements about the shift, however, one must always be clear about to which age the shift applies (for the ease of interpretation, we set $A_0=R_0=65$ in all numerical and empirical illustrations throughout the paper). Indeed, the model simplifies the analysis of real-life mortality change as it reduces the change to two parameters by assuming a universal compression/expansion of the age distribution of deaths. A limitation of this simplification becomes evident if one notices that the model produces no shift at the age ${A_0}^*=A_0+\frac{A(t)-A_0}{k(t)-1}$. At ages below and above the age ${A_0}^*$, the distribution of deaths is shifted in different directions. In reality, adult mortality, typically, changes in the same direction across all ages. That means that the model assumption of no shift at age ${A_0}^*$ might be not realistic. However, for a typical mortality change, the age ${A_0}^*$ is beyond the range where the most of mortality change occurs. In our data, the mean annual shift was about 0.2 years at age 65 and the mean annual compression $(k(t)-1)$ was 0.003. With these parameters, the model would produce non-positive shifts, on average, only at ages above ${A_0}^*=65+\frac{0.2}{0.003}\approx 132$ years. It may also be noticed that, when we conducted tests of fit, not shown here, we found that the model performs marginally better than the Gompertz model (Gompertz 1825; Thatcher, Kannisto, and Vaupel 1998). The biggest advantage of the model for our study, however, comes from the convenience and accuracy it offers in deriving formal relations for indicators of population ageing.

3. The analytic expression for the alpha-age $R(t)$

In the prospective definition of who is old adopted here (Sanderson and Scherbov 2005, 2013), the alpha-age $R(t)$ defines age groups with a fixed remaining life expectancy. Given our consistency constraint $R(t_0)=R_0$ assuring comparability of the estimates for the conventional and prospective indicators of ageing, the alpha age $R(t)$ must follow the identity:

$$e(R(t),t)\equiv e(R_0,t_0), \qquad (5)$$

where $e(x,t)$ is life expectancy at age x in year t. To explore solutions of this identity, note that the shift-compression/expansion mortality model (3) implies (Dalkhat M. Ediev 2013)

$$e(x,t) = k(t)e\left(A_0 - \frac{A(t)-x}{k(t)}, t_0\right). \tag{6}$$

Combining this with (5), we obtain the following main equation for $R(t)$:

$$e(R(t),t) = k(t)e\left(A_0 - \frac{A(t)-R(t)}{k(t)}, t_0\right) = e(R_0,t_0). \tag{7}$$

The second equality in (7) is resolved by

$$R(t) = A(t) + \ k(t)\left[e^{-1}\left(\frac{e(R_0,t_0)}{k(t)}, t_0\right) - A_0\right], \tag{8}$$

where $e^{-1}(z,t_0)$ is the inverse function of the baseline remaining life expectancy by age $e(x,t_0)$. In the pure mortality shift scenario, $k(t) = 1$, Equation (8) produces the intuitively obvious solution for the alpha-age:

$$R(t) = R_0 + A(t) - A_0 \tag{9}$$

Under pure mortality shift scenario, the alpha-age shifts by exactly the same amount of years as the age distribution of deaths. A pure mortality shift, therefore, implies a time-invariant number of life-table person years at old-age and an increasing number of life-table person years in the working ages defined here as the ages 20 (the fixed onset of the working age interval) to the alpha-age $R(t)$. Hence, that shift would produce a fall in the POADR.

Intuitively less obvious and analytically more challenging is the general case of shift combined with compression or expansion, which is considered next.

4. The analytic relationship between the OADR and the POADR under general mortality change

In the general case, the following first-order approximation may be derived for the alpha-age (see Appendix B for details of this and following derivations):

$$R(t) \approx A(t) + k(t)(R_0 - A_0) - \left(1 - k(t)\right)\frac{e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}. \tag{10}$$

Here, $\mu(x,t)$ is the death rate (the force of mortality) at age x, time t. The first two terms in (10) describe shift of the deaths distribution at the age corresponding to R_0 in the original distribution. The last term shows the additional correction to the old-age threshold due to mortality compression/expansion. In the case of mortality expansion, the alpha-age $R(t)$ increases by more than the pure shift. In the case of compression, it increases by less.

Taking the changing alpha-age into account, we show in Appendix B that the life table population above the alpha-age ("prospective old-age population", POAP) changes in line with the mortality compression/expansion.

$$POAP(t) = \int_{R(t)}^{\omega} l(x,t)dx \approx POAP(t_0)\left[1 + \left(1 - k(t)\right)\frac{\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}\right]. \tag{11}$$

This means, in particular, that the life table old-age population in the prospective definition is not subject to effects of mortality shift and only influenced by the compression/expansion. It will increase with mortality compression and decrease with expansion.

Another consequence of (11) is that the POADR will fall in case of mortality decline accompanied by expansion or pure shift. For the more realistic scenario of shifting and compressing period mortality, POADR may change in a more complex way:

$$POADR(t) \approx POADR(t_0)\left[1-(A(t)-A_0)\frac{l(W,t_0)}{WAP(t_0)}+\left(1-k(t)\right)\left\{1-\frac{l(W,t_0)}{WAP(t_0)}(A_0-W)+\frac{OADR(t_0)+\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}\right\}\right]. \quad (12)$$

The effect of a positive shift in (12) is always negative, while the sign of the effect of compression may depend on the pivotal age A_0. When A_0 is set equal to R_0 (as we do here), the effect of the compression in (12) is typically positive, because the ratio $\frac{WAP(t_0)}{l(W,t_0)}$ (the average duration of life at ages W to R_0 for people surviving to age W) does not differ principally from the upper limit $R_0 - W$. For example, in the case of French females, with $W = 20$ and $R_0 = 65$, the expression $1-\frac{l(W,t_0)}{WAP(t_0)}(R_0-W)$ was -0.23 in 1900, -0.07 in 1950 and only -0.02 in 2013. These numbers were small as compared to the magnitude of the last summand in the expression for the compression effect in (12) where $\frac{OADR(t_0)+\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}$ was 1.04 in 1900, 0.78 in 1950, and 0.71 in 2013. As a result, compression effect is positive when $A_0 = R_0$[3].

Typically, the effect of a shift (at age $A_0 = R_0$) in (12) dominates the effect of a compression and the overall effect of mortality decrease is negative. To develop the intuition for this observation, consider the limit case of a population with negligible mortality at ages between W and R_0. In that population, (12) simplifies to:

$$POADR(t) \approx POADR(t_0)\left[1-(A(t)-A_0)\frac{1}{R_0-W}+\left(1-k(t)\right)OADR(t_0)\right], \quad (13)$$

and the effect of a shift will be about $-\frac{1}{65-20} \approx -0.02$ when the mortality shift is one year, while the effect of a compression will be $0.01 \cdot OADR(t_0)$, i.e., about 0.003 per one percent of compression (k=0.99) assuming a typical $OADR(t_0) \approx 0.3$. To have an idea of the relative contributions of the shift (at age R_0 =65 years, as we consider here) and compression, consider a case when mortality compresses to age $A_0^* = 100$ years. In such a scenario, a one percent compression will result in a shift of $0.01(100-65) = 0.35$ years at age 65. Given the shift and compression effect coefficients above, the scenario will result in change in POADR by $-0.02 \cdot 0.35 = -0.007$ due to the shift and by 0.003 due to the compression effect.

Empirically, the relative effect of the shift is even higher because mortality declines even at age 100. The average annual mortality shift at age 65 in low-mortality HMD countries after 1950 was about 0.2 and the annual compression was about 0.3%. Therefore, the effect of the shift on the POADR has been stronger than the effect of compression by about four times ($0.02 \cdot 0.2/(0.003 \cdot 0.3) \approx 4.4$). Hence, the POADR, typically, decreases, as mortality declines.

By comparison, first-order approximations to the conventional OADR with a fixed old-age threshold $R(t) \equiv R$, lead to (see the Appendix A for details):

[3] One may find it counterintuitive that the compression effect depends on the choice of the parameter A_0 describing the mortality shift. To explain this finding, note that parameter A_0 determines the age to which the mortality shift refers. Shifts of the mortality curve at all other ages are described by the combination of the shift and the compression parameters. The higher the age A_0, the more the compression effect in (12) is 'contaminated' by the effects of shifts at younger ages. Yet, our numerical assessments, not presented here in detail, suggest the compression effect remains positive at all A_0 below age 90. One important implication of this observation is that the effect of compression with respect to the modal age, often used to describe the mortality shift (Canudas-Romo 2008; Kannisto 2001), is also positive.

$$OADR(t) \approx OADR(t_0)\left[1 + (A(t) - A_0)\left(\frac{l(R_0,t_0)}{OAP(t_0)} - \frac{l(W,t_0) - l(R_0,t_0)}{WAP(t_0)}\right) - \left(1 - k(t)\right)\left\{\frac{l(R_0,t_0)(R_0 - A_0)}{OAP(t_0)} + \frac{l(R_0,t_0)(R_0 - A_0) + l(W,t_0)(A_0 - W)}{WAP(t_0)}\right\}\right]. \quad (14)$$

The effect of the mortality shift in (14) will be positive for human mortality change, because $\frac{l(R_0,t_0)}{OAP(t_0)} - \frac{l(W,t_0)-l(R_0,t_0)}{WAP(t_0)} = \frac{1}{e(R_0,t_0)} - \frac{l(W,t_0)-l(R_0,t_0)}{l(W,t_0)e(W,t_0)-l(R_0,t_0)e(R_0,t_0)} = \frac{1}{e(R_0,t_0)}\left(1 - \frac{l(W,t_0)-l(R_0,t_0)}{\frac{e(W,t_0)}{e(R_0,t_0)}l(W,t_0)-l(R_0,t_0)}\right) > 0$ assuming $\frac{e(W,t_0)}{e(R_0,t_0)} > 1$. The effect of the compression, on the other hand, will be negative for all pivotal ages $A_0 \le R_0$. Hence, the effects of both shift and compression on the change of the POADR are typically opposite in sign to the effects on the conventional OADR. Furthermore, setting $A_0 = R_0$, as in our previous assessments for the POADR and the empirical assessments in the next section, and assuming the limiting case of negligible low mortality at the old-age threshold and at working ages, yields

$$OADR(t) \approx OADR(t_0)\left[1 + (A(t) - A_0)\frac{1}{OAP(t_0)} - \left(1 - k(t)\right)\right]. \quad (15)$$

Comparing this to (13), we see that the change in POADR will, approximately, be $-\mathrm{OADR}(\mathrm{t}_0)$ (i.e., about -0.3) times the change in the conventional OADR for a population with negligibly low mortality at working ages. Although realistically, one may not assume negligible mortality at age 65 years, the following empirical section shows that our approximation of the link between changes in OADR's is not all that far from reality.

One may naturally be interested in whether our conclusion about oppositely signed effects of shift and compression on OADR's, with the effect of a shift dominating over the effect of a compression, is more generally applicable to cases where R_0 is higher than 65 (a plausible scenario given the trend towards increasing retirement ages). We did not aim to study this aspect in detail, but numerical analysis of (12) and (14) suggests that our conclusions about the effects' signs remain valid at all R_0's up to age 90 years and the effect of the shift remains dominant at R_0's up to age 75.

Although our focus here is on OADR's and not the other measures of population ageing, it is worthwhile noting that setting $W = 0$ in the above equations, similar relations also apply to the prospective ($PPO(t)$) and conventional ($PO(t)$) proportions of the population who are old. In particular, when mortality is negligibly low at the threshold and younger ages, the change in the PPO is, approximately, $-\frac{PO(t_0)}{1-PO(t_0)}$ (i.e., about -0.2) times the change in the conventional $PO(t_0)$.

5. Empirical assessment

In this section, we examine the accuracy of the first-order approximations presented above based on empirical data from the Human Mortality Database (University of California, Berkeley and Max Planck Institute for Demographic Research 2014) that covers 31 currently low-mortality populations and 12 currently higher mortality populations (the latter include Eastern-European countries that experienced periods of mortality increase during the economic transition).

In our empirical study, we set $R_0 = 65$ years, for consistency with the common old-age definition. We also define $A_0 = R_0 = 65$ as the shift in the base year old-age threshold age to simplify the interpretation of the shift parameter. To estimate the amounts of mortality shift and compression $k(t)$, we used Eq. (6) and a similar equation for the standard deviation of ages at death above certain age x (Dalkhat M. Ediev 2013):

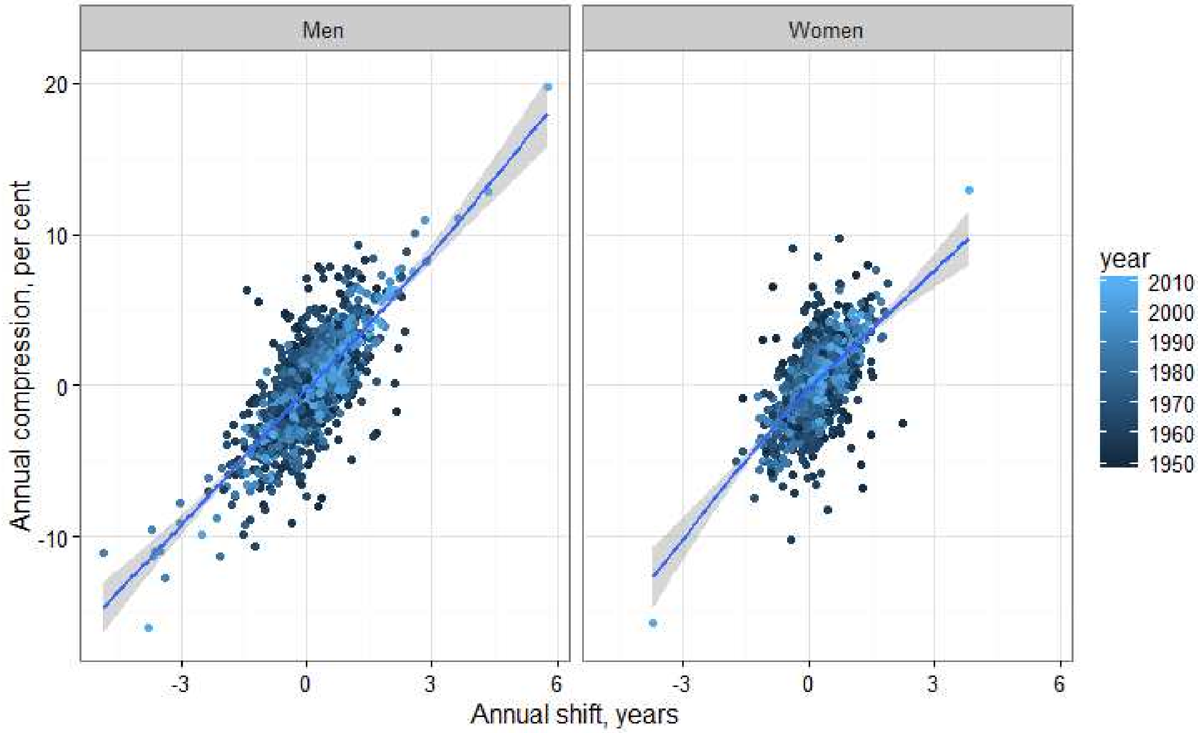
$$\sigma(x,t) = k(t)\sigma\left(A_0 - \frac{A(t)-x}{k(t)}, t_0\right). \quad (16)$$

We apply both equations to the pivotal age $A(t)$ to obtain the following relations for the shift and compression:

$$\frac{\sigma(A(t),t)}{\sigma(A_0,t_0)} = \frac{e(A(t),t)}{e(A_0,t_0)} = k(t) \quad (17)$$

We estimate the annual shift $\delta(t) = A(t) - A(t-1)$ by setting $t_0 = t - 1$ and solving numerically the first equality in (17) and the expansion coefficient $k(t)$ from the second equality.

Figure 4: Estimates of annual shifts and compressions of the life table distributions of deaths, entire HMD: men, women



In Figure 4, we show estimation results for the shift and compression in HMD countries, for men and women separately. In agreement with the previous empirical studies and theories (Cheung et al. 2005; Dalkhat M Ediev 2013; Dalkhat M. Ediev 2013; Ediev 2011; Fries 1980; Kannisto 2001; Keyfitz and Golini 1975; Thatcher et al. 2010; Tuljapurkar and Edwards 2011; Wilmoth and Horiuchi 1999), a positive period mortality shift is, typically, accompanied by the compression of the deaths distribution. Negative shifts, on the other hand, come with mortality expansion, also in agreement with theory (Ediev 2011) Our database covers a wide range of the shift and compression values, and of their combinations. The large amount of negative shifts in low mortality countries seen in the figure is not a surprise, as the annual mortality changes are rather volatile and may easily go against the main trend. On average, though, mortality was shifting rightwards with a slight compression. The annual mortality shift in low-mortality HMD countries after 1950 was about 0.2 years and the annual compression was about 0.3% (k=0.997).

Using the empirical data, we ran regressions (linear model, no intercept) of annual changes in the conventional and prospective OADRs on estimated annual shifts and compressions (Table 1. We pooled together results for all countries, except for countries with small populations, because the results did not differ substantially for different types of populations). All effects are highly significant (p-values, not shown in the table, are all negligible). As expected from our formal inquiry, effects of shift and compression are of opposite sign; and the effects change sign when switching from the conventional to the prospective OADR. Our empirical estimates (effects of about. -0.006 for the one-year shift and 0.2 for one-percent compression on POADR change) are somewhat different from the rough assessments above based on the first-order approximations (-0.02 and 0.3, respectively). The empirical relation between the changes of the prospective and conventional OADRs, however (POADR change being approximately -0.38 times the change in the conventional OADR) is very close to our first-order approximation (-0.3 times).

Table 1: Regression results for the shift and compression effects (linear model, no intercept) on the annual changes of the conventional OADR and POADR: entire HMD data, men (m), women (f), excluding small-size populations (Iceland, New Zealand--Maori, and Luxemburg)

Indicator	Sex	Shift effect	Compression effect	R^2
OADR	**f**	0.0166 (0.0001)	-0.350 (0.001)	0.958
OADR	**m**	0.0136 (0.0001)	-0.289 (0.002)	0.929
POADR	**f**	-0.0064 (0.0001)	0.173 (0.001)	0.865
POADR	**m**	-0.0060 (0.0001)	0.161 (0.002)	0.754

Note: numbers in the parentheses indicate standard errors of regression coefficients

6. Discussion

The OADR is one of the most widely used measures of population aging. It assumes that the old-age threshold is fixed regardless of time or place. There is no perfect way to categorize people. Nevertheless, in a world where life expectancy is increasing, where people are often healthier at given ages than they were in the past, where age-specific cognitive functioning is improving, where older people are now more educated than they were in the past, and where people in OECD countries will generally be facing higher normal pension ages, another measure of aging, consistent with these improvements, seems appropriate. The POADR is such a measure.

Population aging, viewed from the perspective of the POADR looks very different from the picture provided by the OADR. Sanderson and Scherbov (Sanderson and Scherbov 2015) showed that faster increases in life expectancy lead to slower rates of population aging when measured by the percentage increase in the POADR while the faster increases in life expectancy lead to faster rates of population aging when measured by the percentage increase in the OADR.

In this paper, we show why those differences were observed and, indeed, that they were predictable given the sorts of shifts and compressions that have been observed. In this paper, we have shown that in a wide variety of life table populations, annual changes in OADRs and POADRs move in opposite directions. Ediev's (2013) shift-compression model provides an analytic two-parameter specification of the age distribution of adult deaths. We used that model to provide analytic expressions for both the OADR and the

POADR in terms of the shift and compression parameters. The theoretical expressions that we obtained predicted that the observed negative relationship between annual changes in OADRs and POADRs is exactly what we should expect to see. In the empirical analysis, we estimated the shift and compression parameters from the countries in the HMD database. The data showed that the change in the POADR was around -0.38 times the change in the OADR. Our theoretical approximation predicted that it would be around -0.3 times the change in the OADR.

The POADR provides an alternative to assuming that the characteristics of people do not change over time or differ from place to place. Population aging looks very different when viewed from its perspective. Decreasing period mortality in developed countries is typically accompanied by mortality compression that has been gradually slowing. Here, we show that such mortality dynamics will result in increasing numbers of people considered old in the prospective definition. That number will stop increasing, however, if mortality compression stalls (Bongaarts 2005; Canudas-Romo 2008) and gives way to a pure mortality shift. In either scenario, be it mortality compression or shift, our results show that, in life table populations, the number of people below the prospective old-age threshold will grow faster than the number of people old, so that POADR will fall as lifespans increase. In non-stationary populations, there are factors, other than mortality, that influence age structure. An earlier study (Sanderson and Scherbov 2015) of OADR and POADR changes in observed populations suggests, however, that the effects of fertility and migration on the changes in those ageing indicators might be secondary compared to the effects of mortality that we have elucidated here.

7. References

Cheung, S.L.K., Robine, J.-M., Tu, E.J.-C., and Caselli, G. (2005). Three dimensions of the survival curve: horizontalization, verticalization, and longevity extension. Demography 42(2):243–258.

Ediev, D.M. (2011). Life expectancy in developed countries is higher than conventionally estimated. Implications from improved measurement of human longevity. Journal of Population Ageing 4(1–2):5–32. doi:10.1007/s12062-011-9040-x.

Ediev, D.M. (2013a). Decompression of period old-age mortality: when adjusted for bias, the variance in the ages at death shows compression. Mathematical Population Studies 20(3):137–154. doi:10.1080/08898480.2013.816218.

Ediev, D.M. (2013b). Mortality compression in period life tables hides decompression in birth cohorts in low-mortality countries. Genus 69(2):53–84. doi:10.4402/genus-451.

Fries, J.F. (1980). Aging, natural death, and the compression of morbidity. The New England Journal of Medicine 303(3):130–135. doi:10.1056/NEJM198007173030304.

Kannisto, V. (2001). Mode and dispersion of the length of life. Population: An English Selection 13(1):159–171.

Keyfitz, N. and Golini, A. (1975). Mortality comparisons: The male-female ratio. Genus 31(1/4):1–34. doi:10.1007/978-1-4757-1879-9_3.

Ryder, N.B. (1975). Notes on stationary populations. Population Index 41(1):3–28. doi:10.2307/2734140.

Sanderson, W.C. and Scherbov, S. (2005). Average remaining lifetimes can increase as human populations age. Nature 435(7043):811–813. doi:10.1038/nature03593.

Sanderson, W.C. and Scherbov, S. (2010). Remeasuring Aging. Science 329(5997):1287–1288 doi:10.1126/science.1193647.

Sanderson, W.C. and Scherbov, S. (2013). The characteristics approach to the measurement of population aging. Population and Development Review 39(4):673–685. doi:10.1111/j.1728-4457.2013.00633.x.

Sanderson, W.C. and Scherbov, S. (2015). Faster increases in human life expectancy could lead to slower population aging. PLoS ONE 10(4):e0121922. doi:10.1371/journal.pone.0121922.

Thatcher, A.R., Cheung, S.L.K., Horiuchi, S., and Robine, J.-M. (2010). The compression of deaths above the mode. Demographic Research 22(17):505–538. doi:10.4054/DemRes.2010.22.17.

Tuljapurkar, S. and Edwards, R.D. (2011). Variance in death and its implications for modeling and forecasting mortality. Demographic Research 24:497–526. doi:10.4054/DemRes.2011.24.21.

University of California, Berkeley and Max Planck Institute for Demographic Research (2014). Human Mortality Database [electronic resource]. http://www.mortality.org.

Wilmoth, J.R. and Horiuchi, S. (1999). Rectangularization revisited: variability of age at death within human populations. Demography 36(4):475–495. doi:10.2307/2648085.

Appendix A: First-order approximation to the change of the conventional OADR

In this paper, we study relations between the OADR's based on their values computed for stationary, i.e., life table, populations. In particular, the conventional life table OADR may be written as

$$OADR(t) = \frac{OAP(t)}{WAP(t)} = \frac{\int_{R_0}^{\omega} l(x,t)dx}{\int_{W}^{R_0} l(x,t)dx}, \tag{A1}$$

here, $l(x,t)$ is the life table proportion surviving to age x in the period life table for the year t; $OAP(t) = \int_{R_0}^{\omega} l(x,t)dx$ is the old-age life table population, where R_0 is the conventional (time-invariant) age when the 'old age' starts; and $WAP(t) = \int_{W}^{R_0} l(x,t)dx$ is the working-age life table population, where W is the (time-invariant) age when the 'working age' starts.

Under model (3), assuming a fixed proportion of adult and old-age life-table deaths subject to the model (or, in other words, neglecting the change of the share of infant and child mortality),

$$l(x,t) = l\left(A_0 - \frac{A(t)-x}{k(t)}, t_0\right). \tag{A2}$$

Therefore,

$$OAP(t) = \int_{R_0}^{\omega} l(x,t)dx = \int_{R_0}^{\omega} l\left(A_0 - \frac{A(t)-x}{k(t)}, t_0\right)dx = k(t)\int_{A_0-\frac{A(t)-R_0}{k(t)}}^{\omega} l(z,t_0)dz \approx k(t)OAP(t_0) - k(t)l(R_0,t_0)\left(A_0 - \frac{A(t)-R_0}{k(t)} - R_0\right) \approx OAP(t_0)\left[1 + (A(t) - A_0)\frac{l(R_0,t_0)}{OAP(t_0)} - (1-k(t))\left(1 + \frac{l(R_0,t_0)}{OAP(t_0)}(R_0 - A_0)\right)\right]. \tag{A3}$$

$$WAP(t) = \int_{W}^{R_0} l(x,t)dx = \int_{W}^{R_0} l\left(A_0 - \frac{A(t)-x}{k(t)}, t_0\right)dx = k(t)\int_{A_0-\frac{A(t)-W}{k(t)}}^{A_0-\frac{A(t)-R_0}{k(t)}} l(z,t_0)dz \approx$$

$$\approx k(t)WAP(t_0) + (A(t) - A_0)\big(l(W,t_0) - l(R_0,t_0)\big) - (1-k(t))\big(l(R_0,t_0)(R - A_0) - l(W,t_0)(A_0 - W)\big) = WAP(t_0)\left[1 + (A(t) - A_0)\frac{l(W,t_0)-l(R_0,t_0)}{WAP(t_0)} - (1-k(t))\left(1 + \frac{l(R_0,t_0)(R-A_0)-l(W,t_0)(A_0-W)}{WAP(t_0)}\right)\right]. \tag{A4}$$

Combining this with (A1), (A3):

$$OADR(t) \approx OADR(t_0)\frac{1+(A(t)-A_0)\frac{l(R_0,t_0)}{OAP(t_0)}-(1-k(t))\left(1+\frac{l(R_0,t_0)}{OAP(t_0)}(R_0-A_0)\right)}{1+(A(t)-A_0)\frac{l(W,t_0)-l(R_0,t_0)}{WAP(t_0)}-(1-k(t))\left(1+\frac{l(R_0,t_0)(R-A_0)-l(W,t_0)(A_0-W)}{WAP(t_0)}\right)} \approx$$

$$\approx OADR(t_0)\left[1 + (A(t) - A_0)\left(\frac{l(R_0,t_0)}{OAP(t_0)} - \frac{l(W,t_0)-l(R_0,t_0)}{WAP(t_0)}\right) - (1-k(t))\left\{\frac{l(R_0,t_0)(R_0-A_0)}{OAP(t_0)} + \frac{l(R_0,t_0)(R_0-A_0)+l(W,t_0)(A_0-W)}{WAP(t_0)}\right\}\right]. \tag{A5}$$

Appendix B: First-order approximation to the change of the prospective OADR

To distinguish ageing indicators assuming the prospective definition of who is old, we add letter 'P' to the notations of the conventional indicators and use time-dependent function $R(t)$ to denote the time-varying threshold age defining the stating age of the prospective old-age interval. Hence, the life table prospective OADR (POADR) is

$$POADR(t) = \frac{POAP(t)}{PWAP(t)} = \frac{\int_{R(t)}^{\omega} l(x,t)dx}{\int_{W}^{R(t)} l(x,t)dx}, \quad \text{(B1)}$$

where $POAP(t) = \int_{R(t)}^{\omega} l(x,t)dx$ is the prospective old-age life table population and $PWAP(t) = \int_{W}^{R(t)} l(x,t)dx$ is the working-age life table population (with prospective definition of who is old). Note that the two sets of ageing indicators are identical at the base year t_0,so that $R(t_0) = R_0$, $POAP(t_0) = OAP(t_0)$, $PWAP(t_0) = WAP(t_0)$, and $POADR(t_0) = OADR(t_0)$.

To assess the POADR, we first need to estimate the threshold age $R(t)$. To this end, we derive the first-order approximation for the inverse function in Eq. (8) (noticing that $k(t_0) = 1$ by definition):

$$e^{-1}\left(\frac{e(R_0,t_0)}{k(t)}, t_0\right) \approx e^{-1}(e(R_0,t_0),t_0) + \frac{1}{e'_x(R_0,t_0)}\left(\frac{e(R_0,t_0)}{k(t)} - e(R_0,t_0)\right) = R_0 + \frac{1}{e'_x(R_0,t_0)}\left(\frac{e(R_0,t_0)}{k(t)} - e(R_0,t_0)\right), \quad \text{(B2)}$$

where, $e'_x(R_0,t_0)$ is the partial derivative with respect to age of $e(x,t)$ at $x = R_0$ and $t = t_0$. Substituting (B2) into (8), we get the desired relation for the threshold age:

$$R(t) \approx A(t) + k(t)\left[R_0 + \frac{1}{e'_x(R_0,t_0)}\left(\frac{e(R_0,t_0)}{k(t)} - e(R_0,t_0)\right) - A_0\right] = A(t) + k(t)(R_0 - A_0) + \big(1-k(t)\big)\frac{e(R_0,t_0)}{e'_x(R_0,t_0)}, \quad \text{(B3)}$$

To describe the threshold age (B3) in more common demographic functions, one may note that $e'_x(R_0,t_0) = -1 + \mu(R_0,t_0)e(R_0,t_0)$ and rewrite Eq. (B3):

$$R(t) \approx A(t) + k(t)(R_0 - A_0) - \big(1-k(t)\big)\frac{e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}. \quad \text{(B4)}$$

Having obtained the change in the threshold age (B4), we move on to estimate the corresponding change in the survival function that is necessary to assess the old-age population. Substituting $x = R(t)$ in (A2),

$$l(R(t),t) = l\left(A_0 - \frac{A(t)-R(t)}{k(t)}, t_0\right). \quad \text{(B5)}$$

Combining this with (B4) and using first-order approximations leads to

$$l(R(t),t) \approx l\left(R_0 - \frac{1}{k(t)}\frac{\big(1-k(t)\big)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}, t_0\right) \approx l(R_0,t_0) - l'_x(R_0,t_0)\frac{1}{k(t)}\frac{\big(1-k(t)\big)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)} = l(R_0,t_0) + \frac{\mu(R_0,t_0)l(R_0,t_0)}{k(t)}\frac{\big(1-k(t)\big)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)} \approx l(R_0,t_0)\left[1 + \big(1-k(t)\big)\frac{\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}\right]. \quad \text{(B6)}$$

The life-table proportion surviving to the prospective old-age threshold is not influenced by the mortality shift. It is only affected by the mortality compression or decompression. It increases with compression and decreases with mortality expansion.

Because the life-table population at old ages is the product of the number surviving to the old-age threshold multiplied by the remaining life expectancy at that age and because the latter is, by definition, time constant in the definition of the prospective old age, a relation similar to (B6) applies to the life table prospective old-age population:

$$POAP(t) = \int_{R(t)}^{\omega} l(x,t)dx \approx POAP(t_0)\left[1 + \left(1 - k(t)\right)\frac{\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}\right], \qquad \text{(B7)}$$

where, ω stands for the maximum possible age.

The denominator in (B1) may also be approximated using the relations for the survival function:

$$PWAP(t) = \int_{W}^{R(t)} l\left(A_0 - \frac{A(t)-x}{k(t)}, t_0\right)dx = k(t)\int_{A_0-\frac{A(t)-W}{k(t)}}^{A_0-\frac{A(t)-R(t)}{k(t)}} l(z,t_0)dz \approx k(t)PWAP(t_0) - k(t)l(W,t_0)\left(A_0 - \frac{A(t)-W}{k(t)} - W\right) + k(t)l(R_0,t_0)\left(A_0 - \frac{A(t)-R(t)}{k(t)} - R_0\right). \qquad \text{(B8)}$$

Substituting $R(t)$ from (B3), rearranging and keeping only the first-order terms:

$$PWAP(t) \approx k(t)PWAP(t_0) - l(W,t_0)\left(k(t)A_0 - A(t) + \left(1-k(t)\right)W\right) - l(R_0,t_0)\left(1-k(t)\right)\frac{e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)} \approx k(t)PWAP(t_0) + l(W,t_0)\left(A(t) - A_0 + \left(1-k(t)\right)(A_0 - W)\right) - \left(1-k(t)\right)\frac{OAP(t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}. \qquad \text{(B9)}$$

Finally, combining this with (B7), (B1) and noticing the base-year equalities between the conventional and prospective indicators, $POAP(t_0) = OAP(t_0)$, $PWAP(t_0) = WAP(t_0)$, we obtain:

$$POADR(t) = \frac{POAP(t)}{PWAP(t)} \approx \frac{OAP(t_0)\left[1+\left(1-k(t)\right)\frac{\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}\right]}{k(t)WAP(t_0)+l(W,t_0)\left(A(t)-A_0+\left(1-k(t)\right)(A_0-W)\right)-\left(1-k(t)\right)\frac{OAP(t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}} \approx POADR(t_0)\left[1 + \left(1-k(t)\right)\frac{\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)} + \left(1-k(t)\right) - \frac{l(W,t_0)}{WAP(t_0)}\left(A(t) - A_0 + \left(1-k(t)\right)(A_0 - W)\right) + \left(1-k(t)\right)\frac{OAP(t_0)}{WAP(t_0)\left(1-\mu(R_0,t_0)e(R_0,t_0)\right)}\right] = POADR(t_0)\left[1 - (A(t) - A_0)\frac{l(W,t_0)}{WAP(t_0)} + \left(1-k(t)\right)\left\{1 - \frac{l(W,t_0)}{PWAP(t_0)}(A_0 - W) + \frac{OADR(t_0)+\mu(R_0,t_0)e(R_0,t_0)}{1-\mu(R_0,t_0)e(R_0,t_0)}\right\}\right]. \qquad \text{(B10)}$$

Appendix C: Notations and basic relations used in the paper

Notation	Description
t	time variable
t_0	base year used as a reference in computing prospective indicators of ageing
x	age variable
$P(x,t)$	population density at age x, time t
ω	the maximum lifespan
W	the starting age of the working age interval
R_0	the time-fixed starting age of the conventional old age interval
$R(t)$	the time-varying threshold age, the starting age of the prospective old age interval at time t
$OAP(t) = \int_{R_0}^{\omega} P(x,t)dx$	conventional old-age population
$WAP(t) = \int_{W}^{R_0} P(x,t)dx$	working-age population with the conventional definition of old age
$POAP(t) = \int_{R(t)}^{\omega} P(x,t)dx$	prospective old-age population above time-dependent threshold age
$PWAP(t) = \int_{W}^{R(t)} P(x,t)dx$	working-age population with the prospective definition of old age
$OADR(t) = \frac{OAP(t)}{WAP(t)}$	conventional old-age dependency ratio
$POADR(t) = \frac{POAP(t)}{PWAP(t)}$	prospective old-age dependency ratio
$PO(t) = \frac{OAP(t)}{\int_0^{\omega} P(x,t)dx}$	proportion of the population who are old (conventional definition)
$PPO(t) = \frac{POAP(t)}{\int_0^{\omega} P(x,t)dx}$	proportion of the population who are old (prospective definition)
$d(x,t)$	the age distribution of life table adult deaths at time t
A_0	mortality curve's "pivotal age" used to mark the amount of mortality shift
$A(t)$	mortality model's parameter used to describe the amount of shift $(A(t) - A_0)$ at the pivotal age. $A(t_0) = A_0$ by definition.
$k(t)$	mortality model's parameter used to describe the amount of compression or expansion of the deaths' distribution
${A_0}^* = A_0 + \frac{A(t) - A_0}{k(t) - 1}$	age at which the mortality model produces no shift
$\Delta(x,t)$	mortality shift at age x by time period t
$\delta(t) = A(t) - A(t-1)$	the annual mortality shift
$e(x,t)$	remaining life expectancy at age x at time t

$\mu(x,t)$	the death rate (the force of mortality) at age x, time t
$l(x,t)$	the life table proportion surviving to age x in the period life table for the year t
$\sigma(x,t)$	the standard deviation of ages at death above age x
Base year correspondence between the conventional and prospective indicators:	$R(t_0) = R_0$ $POAP(t_0) = OAP(t_0)$ $PWAP(t_0) = WAP(t_0)$ $POADR(t_0) = OADR(t_0)$ $PPO(t_0) = PO(t_0)$