SINGULAR KUMMER SURFACES AND HILBERT MODULAR FORMS*

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For my brother

A *Kummer surface* is a quartic surface in $\mathbf{P}_3$ which has the maximum possible number of point singularities, namely, sixteen. The first example of a surface of this type was published in 1821 by the physicist Fresnel [*13*] as a result of his investigations of the propagation of light in media having multiple optical axes and is generally referred to in optics as the *wave surface*. Another special form (the *tetrahedroid*) was investigated by Cayley [*2*] in 1846 and a surface birationally equivalent to a Kummer surface was studied by Weddle [*42*] (*Weddle's surface*) in 1850 prior to Kummer's introduction [*32*] (also [*33*], [*34*]) in 1864 of the general surface which bears his name. These surfaces could not in any event have remained unstudied for long because by the late 1860's the problem of the classification of quartic surfaces had engaged the attention of Cayley [*4*], [*5*], [*6*], [*7*] and later, of Rohn [*37*], [*38*].

For nearly half a century the properties of Kummer surfaces were unveiled in a steady stream of papers from the pens of Cayley, Klein, Weber, and numerous lesser luminaries, culminating in 1905 with the publication of Hudson's book *Kummer's Quartic Surface* [*16*] which succinctly organized most of what was by then known. Today, interest in Kummer surfaces lingers on, submerged to codimension 1 and sanitized by desingularization, in the study of $K-3$ surfaces; cf. [*39*].

Amongst the quartic surfaces the Kummer surfaces are distinguished because they can be uniformized by abelian functions of two variables, as Cayley [*9*] and Borchardt [*1*] simultaneously but independently discovered in 1877. Thus they are hyperelliptic surfaces in the sense of Picard (cf. [*18*], [*35*]), and their moduli space is intimately related to the Siegel upper half plane of degree two. Since the theory of Siegel modular functions of degree two has only recently been brought to a state of essential completion, primarily through Igusa's efforts [*24*], [*25*], [*26*], [*27*],

* Supported in part by grant NSF GP 19011.

one might suspect that there remain properties of *families* of Kummer surfaces yet to be revealed.

Consider, for instance, an irreducible algebraic curve γ that lies on a Kummer surface uniformized by abelian functions belonging to the period matrix $P = (\mathbf{1}, \tau)$; here $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$ is a complex matrix such that $i(\tau - \bar{\tau}) < 0$ and hence a point in the Siegel upper half plane of degree two. Then γ can be generically given as the zero set of a theta function associated with P but, as Humbert [*20*], [*21*], [*22*], has shown there are certain Kummer surfaces for which some curves of this type are reducible. Following Humbert, such surfaces are called *singular*; they occur in analytic families characterized by a positive integer called the *invariant* of the family, and singular Kummer surfaces belonging to the same family contain deformations of the curves belonging to any one of them. The irreducible components of the zero sets of generically irreducible theta functions are zero sets of intermediate functions which are factors of theta functions but are not theta functions themselves.

If the Kummer surfaces are normalized in the usual manner (cf. e.g., Hudson [*16*], Humbert [*18*]) so that their moduli space is dense in a fundamental domain for the Siegel modular group of degree two, then it is of interest to determine the subset of Siegel's space which corresponds to a family of singular Kummer surfaces of given invariant as well as the action on this subset of the appropriate modular group. The nature of the anticipated result is signalized by the fact that the values assumed by Humbert's invariant are precisely the discriminants of real quadratic number fields (including the denegerate quadratic fields, whose discriminants are squares); indeed, the modular group associated with singular surfaces of invariant Δ is just the Hilbert modular group associated with $\mathbf{Q}(\sqrt{\Delta})$. The subvariety of Siegel's space on which this group acts will be described later.

The general quartic surface may be made to pass through seven preassigned nodes, but if it is a Kummer surface, i.e., if it possesses sixteen nodes, then six (but not fewer) already uniquely determine the surface [*4*], [*16*], from which it can be concluded that the moduli space of Kummer surfaces is 18 dimensional. The group of projective automorphisms of $\mathbf{P}_3$ acts transitively on ordered quintuples of points, so, modulo a projective transformation, the Kummer surfaces are parametrized by three parameters.

From the standpoint of the uniformization theory these parameters may be taken to be coordinates of a point in a fundamental domain for the Siegel modular group of degree two. This comes about as follows. Let $P = (\mathbf{1}, \tau)$ denote a period matrix as above, denote the $\mathbf{Z}$-module generated by the columns of P by $\mathbf{L}$ and let $\mathbf{T} = \mathbf{C}^2/\mathbf{L}$ stand for the torus associated with P. The involution $z \mapsto -z$ of $\mathbf{C}^2$ induces an involution on $\mathbf{T}$ whose fixed points are the images in $\mathbf{T}$ of the half-periods including 0. If f_1, f_2, f_3 denote three even meromorphic functions on $\mathbf{T}$, then the (affine) image $\mathbf{S}$ of $\mathbf{T} \ni z \mapsto (f_1(z), f_2(z), f_3(z))$ is a 16-nodal surface in $\mathbf{P}_3$. $\mathbf{S}$ is certainly algebraic since three abelian functions of two variables are algebraically dependent; if the dependence is of degree 4, then $\mathbf{S}$ is a Kummer surface.

In order to construct such a relation it is convenient to introduce the theta functions of two variables. Let $g, h \in \mathbf{Z}_2 \times \mathbf{Z}_2$ and $r \in \mathbf{Z}^+$. A theta function of order r and characteristic $[g, h]$ associated with the period matrix $P = (\mathbf{1}, \tau)$ is a holomorphic function $\Theta^{(r)}[g, h]$ defined on $\mathbf{C}^2$ which satisfies the relation

$$(1) \qquad \Theta^{(r)}[g,h](z + \mathbf{1}m + \tau n) = (-1)^{n'g + m'h} \exp\{-2\pi i r m'(z + \tfrac{1}{2}\tau m)\}\Theta^{(r)}[g,h](z)$$

for $m, n \in \mathbf{Z}^2$; "$'$" denotes matrix transposition. It follows that the product of theta functions of orders r_1 and r_2 and respective characteristics $[g_1, h_1]$ and $[g_2, h_2]$ is a theta function of order $r_1 + r_2$ and characteristic $[g_1 + g_2, h_1 + h_2]$, where addition of characteristics is performed in $\mathbf{Z}_2$. A fundamental classical result asserts that there are precisely r^2 linearly independent theta functions of order r and fixed characteristic.

For theta functions of two variables there are evidently sixteen characteristics. The characteristic $[g, h]$ is said to be *even* or *odd* according as $g'h$ is even or odd. Of the sixteen characteristics, ten are even and six odd. There are fifteen different ways to select four even characteristics $[g_i, h_i]$, $i = 1, \cdots, 4$ such that their sum is the zero characteristic. Let θ_i be corresponding theta functions of order 1; then

$$\theta_i^4, \ \theta_i^2\theta_j^2, \ \theta_1\theta_2\theta_3\theta_4, \quad 1 \leqq i < j \leqq 4$$

are eleven theta functions of order 4 and characteristic zero. However, if r is even, there are just $(r^2 + 4)/2$ linearly independent even theta functions (cf. [*30*]), ten in our case, whence the functions θ_i annihilate a quartic polynomial and

$$T \ni z \mapsto (\theta_1(z), \cdots, \theta_4(z))$$

is a uniformization of a Kummer surface.

Introduction of the explicit expressions

$$(2)\quad \theta[g,h](z) = \sum_{m \in \mathbf{Z}^2} \exp i\pi\{(m+\tfrac{1}{2}g)'\tau(m+\tfrac{1}{2}g)+(m+\tfrac{1}{2}g)'(2z+h)\}$$

and selection of the quadruple

$$(3)\qquad x=\theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\; y=\theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\; z=\theta\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix},\; w=\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

leads to an explicit homogeneous quartic relation; write $x(0)=\theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}(0)$, etc. The resulting *Göpel quartic relation* ([*14*], [*29*]) is

$$(4)\qquad w^4+x^4+y^4+z^4+2Awxyz-B(w^2x^2+y^2z^2)$$
$$-C(w^2y^2+x^2z^2)-D(w^2z^2+x^2y^2)=0$$

with the abbreviations

$$A=\left\{\frac{wxyz\prod_{\varepsilon i}(w^2+\varepsilon_1x^2+\varepsilon_2y^2+\varepsilon_1\varepsilon_2z^2)}{(w^2x^2-y^2z^2)(w^2y^2-x^2z^2)(w^2z^2-x^2y^2)}\right\}(0),\quad \varepsilon_i=\pm 1;$$

$$B=\left\{\frac{w^4+x^4-y^4-z^4}{w^2x^2-y^2z^2}\right\}(0);$$

$$C=\left\{\frac{w^4-x^4+y^4-z^4}{w^2y^2-x^2z^2}\right\}(0);$$

$$D=\left\{\frac{w^4-x^4-y^4+z^4}{w^2z^2-x^2y^2}\right\}(0).$$

Using the irreducibility of this equation, one readily verifies that $(w,x,y,z)(0)$ is a node.

As τ varies in a fundamental domain for the Siegel modular group, that is, through representatives of equivalence classes of complex analytic tori, (4) runs through representatives of the equivalence classes of Kummer surfaces with respect to the action of the group of projective automorphisms of $\mathbf{P}_3$. Thus uniformization of the general Kummer surface is obtained by setting

$$(t_1,t_2,t_3,t_4)M=(x,y,z,w)$$

where the t_i are homogeneous coordinates of a point in $\mathbf{P}_3$, $M\in \mathrm{GL}(4,\mathbf{C})$, and x,y,z,w are given by (3).

One is naturally led to the notion of the singular Kummer surfaces by the theory of complex multiplication. The period matrix P is said to admit a *complex multiplication* if there are matrices $q \in \mathrm{GL}(2, \mathbf{C})$, $M \in \mathrm{GL}(4, \mathbf{Z})$ such that $qP = PM$.

Write

$$M = \begin{pmatrix} b & d \\ a & c \end{pmatrix}$$

and recall that $P = (\mathbf{1}, \tau)$. The condition that P admit a complex multiplication is therefore that

$$q = \tau a + b$$
$$q\tau = \tau c + d,$$

which is equivalent to the quadratic equation in τ

$$\tau a \tau + b\tau - \tau c - d = 0 \tag{5}$$

since q is nonsingular. Subtraction of the transposed equation from (5) yields

$$\tau(a - a')\tau + (b + c')\tau - \tau(c + b') + (d' - d) = 0; \tag{6}$$

set

$$a - a' = \alpha = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}, \ d' - d = \gamma = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \tag{7}$$
$$b + c' = \beta = \begin{pmatrix} B + B_2 & C \\ -A & B_2 \end{pmatrix}.$$

Then (6) reduces to the single relation

$$A\tau_1 + B\tau_3 + C\tau_2 + D\left|\tau\right| + E = 0 \tag{8}$$

where $\left|\tau\right| = \tau_1\tau_2 - \tau_3^2$ and $A, B, C, D, E \in \mathbf{Z}$.

Following Humbert [*20*], an equation of the form (8) is called a *singular relation* and the corresponding Kummer surfaces are also called *singular*. Conversely, if τ satisfies a singular relation, then, as Humbert proved, P admits a complex multiplication.

The *invariant* of the singular relation (8) is (Humbert [*20*]):

$$\Delta = B^2 - 4AC - 4DE. \tag{9}$$

In terms of the abbreviations introduced in (7), the singular relation (8) can be written

$$(\tau, \mathbf{1}) \begin{pmatrix} \alpha & -\beta' \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} \tau \\ \mathbf{1} \end{pmatrix} = 0.$$

Put

$$R = \begin{pmatrix} \alpha & -\beta' \\ \beta & \gamma \end{pmatrix}$$

and denote the trace of a matrix by σ. Then

$$|R| = (|\beta| + DE)^2$$

and using (9) one verifies that the invariant can be expressed as

$$\Delta = \Delta(R) = \sigma(\beta)^2 - 4|R|^{\frac{1}{2}}. \tag{10}$$

Now suppose that the Siegel modular group acts on the Siegel upper half space in the usual way: to a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}\,(4, \mathbf{Z})$$

corresponds the biholomorphism

$$\tau \mapsto (a\tau + b)(c\tau + d)^{-1}.$$

The singular relation (8) can be rewritten as $P'RP = 0$ and under the action of M it is converted into the singular relation $P'R^*P = 0$ where $R^* = M'RM$ with corresponding invariant

$$\Delta(R^*) = \sigma(\beta^*)^2 - 4|R^*|^{\frac{1}{2}}.$$

Now $M \in \mathrm{Sp}(4, \mathbf{Z})$ if

$$M' \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

which entails the relations

$$|M| = 1, \quad a'd - b'c = \mathbf{1}, \tag{11}$$
$$ba' = (ba')', \quad dc' = (dc')'.$$

The first condition implies $|R^*| = |R|$. Regarding β^*, one readily calculates

$$\beta^* = a'\alpha b + c'\gamma d + c'\beta b - a'\beta' d$$

from which

$$\sigma(\beta^*) = \sigma(\alpha b a') + \sigma(\gamma d c') + \sigma(\beta\{ad' - bc'\}).$$

Conditions (11) show that ba' and dc' are symmetric whereas α and γ are skew. Therefore, $\sigma(\alpha ba') = \sigma(\gamma dc') = 0$ and $\sigma(\beta\{ad' - bc'\}) = \sigma(\beta)$ which proves Humbert's result that if $M \in \mathrm{Sp}\,(4, \mathbf{Z})$, then $\Delta(M'RM) = \Delta(R)$. That is, the invariant of a singular relation is invariant under the action of Siegel's modular group.

It is easy to verify that the invariant of a singular relation is a strictly positive integer which can assume all values $\equiv 0, 1 \bmod 4$.

The action of $\mathrm{Sp}\,(4, \mathbf{Z})$ can be utilized to reduce the singular relation (8) to a canonical form without, according to the proof above, changing the value of the invariant. Humbert gave several useful canonical forms of which the most convenient for our purposes is

$$\Pi\colon n\tau_1 - m\tau_2 - u\tau_3 = 0, \quad m, n, u \in \mathbf{Z}, \tag{12}$$

with invariant

$$\Delta = u^2 + 4mn > 0. \tag{13}$$

Eq. (12) defines an analytic hyperplane whose intersection with the Siegel upper half plane is a parameter space for singular Kummer surfaces of invariant Δ. Hecke [*17*] proved that the modular group associated with the Kummer surfaces of invariant Δ is isomorphic to Hilbert's modular group associated with $\mathbf{Q}(\sqrt{\Delta})$. We will show how this group acts on Π. The homogeneous Hilbert modular group associated with the real quadratic number field k is the group

$$\Gamma_k = \left\{ M\colon M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad |M| = 1, \ \alpha, \beta, \gamma, \delta \text{ integral in } k \right\};$$

the Hilbert modular group is the quotient of Γ_k by its center. Let σ_i denote the distinct embeddings of k in $\mathbf{R}$ and introduce the notation $\alpha^{(i)} = \sigma_i(\alpha)$ for the conjugates of $\alpha \in k$. Γ_k acts on $H = \{(\zeta_1, \zeta_2) \in \mathbf{C}^2\colon \mathrm{Im}\zeta_i > 0\}$ by

$$\zeta = (\zeta_1, \zeta_2) \mapsto \langle M\zeta \rangle = \left(\frac{\alpha^{(1)}\zeta_1 + \beta^{(1)}}{\gamma^{(1)}\zeta_1 + \delta^{(1)}}, \frac{\alpha^{(2)}\zeta_2 + \beta^{(2)}}{\gamma^{(2)}\zeta_2 + \delta^{(2)}} \right).$$

Any holomorphic mapping of H onto Π will automatically carry Γ_k onto a group of biholomorphic mappings of Π, but these mappings may not be the restrictions to Π of biholomorphic automorphisms of the Siegel upper half space. We will introduce a mapping which will not only extend to the Siegel space but will also carry Γ_k onto a certain arithmetic subgroup of $\mathrm{Sp}\,(4, \mathbf{R})$.

Let integers m, n, u be given such that $\Delta = u^2 + 4mn > 0$. Introduce

$v = gcd(m,n)$, $\omega_{\pm} = \frac{1}{2}(1 \pm u/\sqrt{\Delta})$, and consider the map of H into Siegel's upper half space defined by

$$(14) \qquad \begin{cases} \dfrac{n}{v}\tau_1 = \omega_+\zeta_1 + \omega_-\zeta_2, \\[2ex] \dfrac{m}{v}\tau_2 = \omega_-\zeta_1 + \omega_+\zeta_2, \\[2ex] \dfrac{\sqrt{\Delta}}{v}\tau_3 = \zeta_1 - \zeta_2. \end{cases}$$

The image of H is Π because the map is linear and H and Π are open subsets of $\mathbf{C}$-linear spaces of the same dimension.

We readily check that (14) can be written in the form

$$\tau_1 = a^2\zeta_1 + b^2\zeta_2$$
$$\tau_2 = c^2\zeta_1 + d^2\zeta_2$$
$$\tau_3 = ac\zeta_1 + bd\zeta_2$$

with

$$(15) \qquad \begin{aligned} a &= \sqrt{\frac{v}{n}}\cos\phi, \quad & b &= -\sqrt{\frac{v}{n}}\sin\phi \\ c &= \sqrt{\frac{v}{m}}\sin\phi, \quad & d &= \sqrt{\frac{v}{m}}\cos\phi \end{aligned}$$

where ϕ is defined by

$$(16) \qquad \phi = \tfrac{1}{2}\cos^{-1}\frac{u}{\sqrt{\Delta}}, \quad \text{i.e., } \cos^2\phi = \omega_+,$$

and consequently (14) is the same as

$$(17) \qquad \zeta \mapsto \tau = A\begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} A'$$

with the entries of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

given by (15). The map (17) is a *modular embedding* in the sense of Freitag and Schneider [*12*] if there is an integral matrix

$$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

such that $AA'T$ is unimodular and $A \begin{bmatrix} \frac{\Delta + \sqrt{\Delta}}{2} & 0 \\ 0 & \frac{\Delta - \sqrt{\Delta}}{2} \end{bmatrix} A^{-1}$ is integral, and in this event, the modular embedding (17) carries H onto Π and the Hilbert modular group Γ_k onto a subgroup of

$$\Gamma(T) = \left\{ \mathfrak{M} \in \mathrm{Sp}\,(4, \mathbf{R}) \colon \mathfrak{T}^{-1}\mathfrak{M}\mathfrak{T} \text{ is integral}, \ \mathfrak{T} = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \right\}.$$

We will show that (17) is a modular embedding into $\Gamma\begin{pmatrix} n/v & 0 \\ 0 & m/v \end{pmatrix}$. Indeed, using (15), we find that

$$AA'T = \begin{pmatrix} t_1 v/n & 0 \\ 0 & t_2 v/m \end{pmatrix},$$

so $t_1 = n/v$, $t_2 = m/v$ implies $AA'T$ is unimodular. Regarding the second condition, set $\eta_{\pm} = \frac{\Delta \pm \sqrt{\Delta}}{2}$; then $A\begin{pmatrix} \eta_+ & 0 \\ 0 & \eta_- \end{pmatrix}A^{-1}$ is equivalent to

$$ab\sqrt{\Delta t}, \ cd\sqrt{\Delta t}, \ \frac{(ad + bc)}{2}\sqrt{\Delta t} \in \mathbf{Z}$$

where we have written $t = t_1 t_2 = mn/v^2$. Substitution from (15) and (16) yields

$$ab\sqrt{\Delta t} = -\frac{v\sqrt{m}}{\sqrt{\Delta n}} \cdot \frac{\sqrt{\Delta mn}}{v} = -m,$$

$$cd\sqrt{\Delta t} = \frac{v\sqrt{n}}{\sqrt{\Delta m}} \cdot \frac{\sqrt{\Delta mn}}{v} = n,$$

$$(ad + bc)\sqrt{\Delta t} = \frac{vu}{\sqrt{\Delta mn}} \cdot \frac{\sqrt{\Delta mn}}{v} = u,$$

which completes the verification.

If $v = m = n$, the embedding (14) reduces to an orthogonal modular embedding into $\Gamma(\mathbf{1})$, originally studied by Hammond [*15*]. Following

[*12*], let us say that a modular embedding of $\Gamma_k \to \Gamma(T)$ is *orthogonal* if $A'AT = \mathbf{1}$. Then $AT^{\frac{1}{2}}$ is an orthogonal matrix, hence A is of the form

$$A = \begin{bmatrix} \dfrac{\cos\theta}{\sqrt{t_1}} & \dfrac{\sin\theta}{\sqrt{t_2}} \\ \dfrac{-\sin\theta}{\sqrt{t_1}} & \dfrac{\cos\theta}{\sqrt{t_2}} \end{bmatrix};$$

the second condition for a modular embedding, that $A \begin{pmatrix} \eta_+ & 0 \\ 0 & \eta_- \end{pmatrix} A^{-1}$ be integral, implies

$$v = \frac{\sin 2\theta}{2\sqrt{t}}\sqrt{t\Delta}, \quad u = \frac{\cos 2\theta}{\sqrt{t}}\sqrt{t\Delta} \in \mathbf{Z},$$

where t has the same meaning as above, and the relation of Pythagoras consequently entails

$$\Delta = u^2 + 4v^2. \tag{18}$$

Retracing the constructive argument given above with $m = n = v$, we see that Γ_k admits an orthogonal modular embedding into $\Gamma(T)$ if and only if the discriminant of k is of the form (18), and in this case we may take $T = \mathbf{1}$. This amends [*12*], Satz 4.

Let us return to our principal theme. If P does not admit a complex multiplication, then every intermediate, or *Jacobi*, function (see [*11*], [*30*], [*31*]) is a theta function and consequently every algebraic curve on the corresponding Kummer surface is the zero set of a theta function. If, however, P admits a complex multiplication, there will be Jacobi functions which are not theta functions, and the algebraic curves on the corresponding Kummer surface will be the zero sets of the Jacobi functions [*20*], [*21*]. It is easy to show that for every Jacobi function there exists another such that their product is a theta function. Hence, if that theta function, considered as a joint function of $z \in \mathbf{C}^2$ and τ, defines an irreducible algebraic curve on the Kummer surfaces corresponding to non-singular τ, then the deformation of that curve over the analytic plane Π defined by (12) (and over its images under the action of $\mathrm{Sp}(4,\mathbf{Z})$) will be *reducible*. In particular, irreducible generic algebraic curves on Kummer surfaces are of even order, but these can split into a union of curves some of which are of odd order on singular Kummer surfaces. These remarks illustrate the nature of the geometric significance that the theory of complex multiplication has for Kummer surfaces.

There are analytical implications as well. Denote the graded ring of modular forms for the group Γ by $\mathscr{R}(\Gamma)$. Igusa [27] proved that $\mathscr{R}(\Gamma(\mathbf{1}))$ is generated by appropriate polynomials in the theta "Nullwerte." If k is a real quadratic field whose discriminant is of the form (18), then $\mathscr{R}(\Gamma(\mathbf{1}))\,|\,\Pi \subset \mathscr{R}(\Gamma_k)$, but the simplest cases already show that this restriction map is not surjective. It is an interesting mystery to uncover the source of the forms in $\mathscr{R}(\Gamma_k)$ which do not extend to elements of $\mathscr{R}(\Gamma(\mathbf{1}))$. From what we have already said, the "Nullwerte" of singular Jacobi functions emerge as likely candidates, for suitable products of these functions are indeed theta "Nullwerte" polynomials and consequently extend to $\mathscr{R}(\Gamma(\mathbf{1}))$. In a forthcoming paper we show that this is precisely what occurs if $k = Q(\sqrt{5})$.

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