

DEFORMATIONS OF RATIONAL SINGULARITIES AND THEIR RESOLUTIONS

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Let (X, x_0) be an isolated singularity of a complex space with resolution \tilde{X} and exceptional analytic subvariety $\tilde{E} \subset \tilde{X}$. If $\tilde{\pi}: \tilde{Z} \rightarrow S$ is a (flat) deformation of $\tilde{\pi}^{-1}(s_0) \cong \tilde{X}$, $s_0 \in S$ fixed, with nonsingular base space S , then there exists (locally with respect to s_0 and \tilde{E}) a *blowing down map*

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tau} & Z \\ \tilde{\pi} \searrow & & \swarrow \pi \\ & S & \end{array} ,$$

i.e., τ is proper and surjective, $\tau_{s_0} = \tau|_{\tilde{\pi}^{-1}(s_0)}$ maps $\tilde{E} \subset \tilde{\pi}^{-1}(s_0) = \tilde{X}$ onto a point $z_0 \in Z_{s_0} = \pi^{-1}(s_0)$ and $\tau_{s_0}: \tilde{X} - \tilde{E} \rightarrow Z_{s_0} - z_0$ is biholomorphic (Theorem 1). However, the isolated singularity (Z_{s_0}, z_0) is in general not isomorphic to the singularity (X, x_0) we started with (cf. [13], Bemerkung p. 242) and $\pi: Z \rightarrow S$ will not be a deformation of (X, x_0) .

For a *rational singularity*, (Z_{s_0}, z_0) is always equal to (X, x_0) . Hence, in that case, every deformation $\tilde{\pi}: \tilde{Z} \rightarrow S$ of a resolution \tilde{X} of (X, x_0) gives rise to a deformation $\pi: Z \rightarrow S$ of the rational singularity (X, x_0) itself. Moreover, all singularities in fibers $Z_s = \pi^{-1}(s)$ near s_0 are rational (Theorem 2).

In a second part of this note we apply Theorem 2 to special cases. Using the existence of a resolution for the versal family of a Kleinian singularity (Tjurina, Brieskorn), we obtain that every (local) deformation of a Kleinian singularity is again Kleinian, a Corollary which also follows from Brieskorn's deep theory about connections between subregular elements in simple complex Lie groups and versal deformations of the corresponding Kleinian singularities (cf. [6]). As a second application we blow down the complete family of a Hirzebruch manifold.

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§1. Proofs of Theorems 1 and 2

Theorem 1. Let $\tilde{\pi}: \tilde{Z} \rightarrow S$ be a regular family of complex manifolds with nonsingular base space S . Suppose there exists a point $s_0 \in S$ such that $\tilde{Z}_{s_0} = \tilde{\pi}^{-1}(s_0)$ has an exceptional subvariety \tilde{E}_{s_0} . Then there exist neighborhoods $\tilde{U} = \tilde{U}(\tilde{E}) \subset \tilde{Z}$ and $V = V(s_0) \subset S$ such that $\tilde{\pi}$ can be factorized as follows:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tau} & Z \\ & \searrow \tilde{\pi} & \swarrow \pi \\ & & V \end{array} .$$

In this diagram, Z is a normal complex space with a subvariety E finite over V , such that τ is proper, surjective and biholomorphic on $\tilde{U} - \tilde{E}$ where $\tilde{E} = \tau^{-1}(E)$ and $\tilde{E} \cap \tilde{\pi}^{-1}(s_0) = \tilde{E}_{s_0}$.

Remark. In case $\dim S = 1$ it can be easily shown that the family $\pi: Z \rightarrow V$ is flat. I do not know whether this is true in general.

Proof. All we have to show is that the map $\tilde{\pi}$ is 1-convex in the sense of Knorr-Schneider-Markoe-Rossi-Siu ([13], [14], [18]) in a neighborhood of \tilde{E}_{s_0} . That follows from the fact that \tilde{Z}_{s_0} carries a C^∞ function φ with the following properties (eventually after shrinking of \tilde{Z}):

- i) $\varphi \geq 0$, $\{\varphi = 0\} = \tilde{E}_{s_0}$,
- ii) φ is strongly plurisubharmonic on $\{\varphi > 0\}$,
- iii) $\{\varphi < c\} \subset \subset \tilde{Z}_{s_0}$ for all $c \in \mathbf{R}$,

an immediate consequence of \tilde{E}_{s_0} being exceptional, and Richberg's extension theorem for strongly plurisubharmonic functions:

Choose real numbers $0 = c_0 < c_1 < c_2 < c_3 < c_4 < c_5 < c_6$, and denote by \tilde{X}_j^k the set $\{c_j < \varphi < c_k\} \subset \tilde{Z}_{s_0}$ for $0 \leq j < k \leq 6$. Since a regular holomorphic family of manifolds is locally trivial there exists a neighborhood $\tilde{U}_6 = \tilde{U}(\tilde{E}_{s_0})$, a neighborhood $V = V(s_0)$ and a diffeomorphism $\delta: \tilde{U}_6 \rightarrow \tilde{X}_0^6 \times V$ which preserves the fibration of \tilde{U}_6 ; here $\tilde{X}^k = \tilde{X}_0^k \cup \tilde{E}_{s_0} = \{\varphi < c_k\}$ (for a proof cf. [1], Remark 2, p. 220). By Richberg's result ([15], Satz 3.3) φ can be extended to a function $\tilde{\varphi}$ which is strongly plurisubharmonic on a neighborhood of \tilde{X}_0^6 . Hence, if V is small enough, $\tilde{\varphi}$ is strongly plurisubharmonic on $\delta^{-1}(\tilde{X}_1^5 \times V)$. If we extend $\tilde{\varphi}$ arbitrarily

into $\delta^{-1}(\tilde{X}^2 \times V)$ and glue both extensions together, we may assume that φ has a C^∞ extension $\tilde{\varphi}$ into $\tilde{U}_5 = \delta^{-1}(\tilde{X}^5 \times V)$ such that $\tilde{\varphi}$ is strongly plurisubharmonic on $\delta^{-1}(\tilde{X}_2^5 \times V)$. After shrinking V again we can achieve that $\tilde{\varphi}$ is strongly plurisubharmonic on $\{\tilde{z} \in \tilde{U}_5: \tilde{\varphi}(\tilde{z}) > c_3\}$ and $\tilde{\pi} | \{\tilde{z} \in \tilde{U}_5: \tilde{\varphi}(\tilde{z}) \leq c_3\}$ is proper for every $c \leq c_4$. This means that $\tilde{\pi}$ restricted to $\tilde{U} = \{\tilde{\varphi}(\tilde{z}) < c_4\}$ is a 1-convex holomorphic map (with exhaustion function $\tilde{\psi} = 1/(c_4 - \tilde{\varphi})$ and convexity bound $(c_4 - c_3)^{-1}$).

1-convex mappings are holomorphically convex; hence \tilde{U} may be assumed to be holomorphically convex (if V is Stein). The canonical map τ of \tilde{U} onto its Remmert-quotient $Q(\tilde{U}) = Z$ has the properties stated in the theorem. (The Remmert-quotient $Q(\tilde{U})$ of a holomorphically convex complex space \tilde{U} is the [uniquely determined] Stein space $Q(\tilde{U})$ together with a proper, surjective holomorphic map $\tau: \tilde{U} \rightarrow Q(\tilde{U})$ such that $R^0\tau_*\mathcal{O}_{\tilde{U}} = \mathcal{O}_{Q(\tilde{U})}$. The last property implies that $Q(\tilde{U})$ is normal if \tilde{U} is normal; in our case \tilde{U} is even nonsingular.) Q.E.D.

Recall the definition of a *rational singularity* ([2]): That is a 2-dimensional normal singularity (X, x_0) such that for a certain resolution $\rho: \tilde{X} \rightarrow X$ the first direct image $R^1\rho_*\mathcal{O}_{\tilde{X}}$ vanishes on X . This property is known to be independent of the resolution $\rho: \tilde{X} \rightarrow X$. We now prove:

Theorem 2. *If, under the assumptions of Theorem 1, the exceptional subvariety \tilde{E}_{s_0} in \tilde{Z}_{s_0} can be blown down to a rational singularity (X, x_0) , then $\pi: Z \rightarrow V$ is a (flat) deformation of $\pi^{-1}(s_0) \cong (X, x_0)$. All singularities of the fibers Z_s, s near s_0 , are rational.*

Proof. Let us assume that \tilde{Z} is holomorphically convex and

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tau} & Z \\ & \searrow \tilde{\pi} & \swarrow \pi \\ & & S \end{array}$$

is the factorization of $\tilde{\pi}$ as in Theorem 1. By assumption there exists a proper map $\rho: \tilde{Z}_{s_0} \rightarrow X$, where X is Stein and $R^1\rho_*\mathcal{O}_{\tilde{Z}_{s_0}} = 0$. This implies $H^1(\tilde{Z}_{s_0}, \mathcal{O}_{\tilde{Z}_{s_0}}) = \Gamma(X, R^1\rho_*\mathcal{O}_{\tilde{Z}_{s_0}}) = 0$. Since $\tilde{\pi}$ is 1-convex and $\mathcal{O}_{\tilde{Z}}$ is flat over \mathcal{O}_S the function

$$s \mapsto \dim_{\mathbb{C}} H^1(\tilde{Z}_s, \mathcal{O}_{\tilde{Z}_s}), \quad \mathcal{O}_{\tilde{Z}_s} = \mathcal{O}_{\tilde{Z}}/\mathfrak{m}_s\mathcal{O}_{\tilde{Z}},$$

is upper semicontinuous on S ([16], [19]); \mathfrak{m}_s denotes the maximal ideal

sheaf of local holomorphic functions on S vanishing at s . Hence we may assume that

$$H^1(\tilde{Z}_s, \mathcal{O}_{Z_s}) = 0$$

for all $s \in S$.

For every Stein open subset $V \subset S$ and every coherent analytic sheaf $\tilde{\mathcal{F}}$ on \tilde{Z} one has canonical isomorphisms (cf. for instance [18]):

$$H^1(\tilde{\pi}^{-1}(V), \tilde{\mathcal{F}}) = \Gamma(V, R^1\tilde{\pi}_*\tilde{\mathcal{F}}) = \Gamma(\pi^{-1}(V) \cap E, R^1\tau_*\tilde{\mathcal{F}}).$$

Taking the inductive limit over all Stein open neighborhoods of a point $s \in S$ we obtain for $\tilde{\mathcal{F}} = \mathcal{O}_{Z_s}$:

$$\bigoplus_{z \in \pi^{-1}(s) \cap E} (R^1\tau_{s*}\mathcal{O}_{Z_s})_z = H^1(\tilde{Z}_s, \mathcal{O}_{Z_s}) = 0,$$

where τ_s is the restriction of τ to \tilde{Z}_s . By construction τ_s is a resolution of the singularities $\pi^{-1}(s) \cap E$ of $Z_s = \pi^{-1}(s)$, and we proved $R^1\tau_{s*}\mathcal{O}_{Z_s} = 0$. The proof is completed if we show that each fiber Z_s is the Remmert-quotient of \tilde{Z}_s , since then $Z_{s_0} = Q(\tilde{Z}_{s_0}) = X$ and all singularities of Z_s , $s \in S$, are in particular normal and hence rational.

Take $z \in \pi^{-1}(s) \cap E$ and assume without loss of generality that $\pi^{-1}(s) \cap E = \{z\}$. $\pi: Z \rightarrow S$ is a 1-complete (i.e., a Stein) mapping and possesses a strongly plurisubharmonic nonnegative exhaustion function ψ with $\psi(z) = 0$. The lifted function $\tilde{\psi} = \psi \circ \tau$ is an exhaustion function for the 1-convex mapping $\tilde{\pi}: \tilde{Z} \rightarrow S$. Define $\tau_c = \tau|_{\{\tilde{\psi} < c\}}$, $c \in \mathbf{R}$. We claim then that the canonical restriction mapping

$$(*) \quad (R^0\tau_{c*}\mathcal{O}_Z)_s \rightarrow (R^0\tau_{c*}\mathcal{O}_{Z_s})_s$$

is surjective for every $c > 0$. Fix $c > 0$; then there exists a Stein open neighborhood $V = V(s) \subset S$ such that $E \cap \pi^{-1}(V) \subset \{\psi < c_0 = c/2\}$ and $\tilde{\psi}$ is an exhaustion function on $\tilde{\pi}^{-1}(V)$ with convexity bound c_0 . By [18, Proposition 1.2] there exists a constant $k \in \mathbf{N}$ such that

$$\text{im}((R^0\tau_{c*}\mathcal{O}_Z)_s \rightarrow (R^0\tau_{c*}\mathcal{O}_{Z_s})_s) = \text{im}((R^0\tau_{c*}\mathcal{O}_Z/\mathfrak{m}_s^k\mathcal{O}_Z)_s \rightarrow (R^0\tau_{c*}\mathcal{O}_{Z_s})_s).$$

If we shrink Z once more taking $Z = \{\psi < c\}$ and $\tilde{Z} = \{\tilde{\psi} < c\}$, it is enough to prove that for all $k > 1$ the canonical mapping

$$H^0(\tilde{Z}_s, \mathcal{O}_Z/\mathfrak{m}_s^k\mathcal{O}_Z) \rightarrow H^0(\tilde{Z}_s, \mathcal{O}_Z/\mathfrak{m}_s^{k-1}\mathcal{O}_Z)$$

is surjective. This will be derived from the exact sequence

$$0 \rightarrow \mathfrak{m}_s^{k-1}\mathcal{O}_Z/\mathfrak{m}_s^k\mathcal{O}_Z \rightarrow \mathcal{O}_Z/\mathfrak{m}_s^k\mathcal{O}_Z \rightarrow \mathcal{O}_Z/\mathfrak{m}_s^{k-1}\mathcal{O}_Z \rightarrow 0$$

and the canonical isomorphism

$$m_s^{k-1} \mathcal{O}_Z / m_s^k \mathcal{O}_Z \cong \mathcal{O}_{Z_s} \otimes_{\mathbb{C}} (m_s^{k-1} / m_s^k)$$

which is a consequence of the flatness of \mathcal{O}_Z over \mathcal{O}_S . Applying the cohomology sequence we get exactness of

$$H^0(\tilde{Z}_s, \mathcal{O}_Z / m_s^k \mathcal{O}_Z) \rightarrow H^0(\tilde{Z}_s, \mathcal{O}_Z / m_s^{k-1} \mathcal{O}_Z) \rightarrow H^1(\tilde{Z}_s, \mathcal{O}_{Z_s} \otimes_{\mathbb{C}} (m_s^{k-1} / m_s^k)).$$

But

$$H^1(\tilde{Z}_s, \mathcal{O}_{Z_s} \otimes_{\mathbb{C}} (m_s^{k-1} / m_s^k)) = H^1(\tilde{Z}_s, \mathcal{O}_{Z_s}) \otimes_{\mathbb{C}} (m_s^{k-1} / m_s^k) = 0$$

and (*) is established.

The proof of $Z_s = Q(\tilde{Z}_s)$ is now trivial. We only have to show that $R^0 \tau_{s*} \mathcal{O}_{\tilde{Z}_s} = \mathcal{O}_{Z_s}$. Let \tilde{f}_s be a holomorphic function on \tilde{Z}_s in a neighborhood \tilde{U}_s of $\tau^{-1}(z)$; assume $\tilde{U}_s = \{\tilde{\psi} < c\} = \tilde{Z}_s$ without loss of generality. Since the mapping (*) is surjective, \tilde{f}_s can be holomorphically extended to a neighborhood of \tilde{U}_s in \tilde{Z} . Let \tilde{f} be such an extension, then $\tilde{f} = f \circ \tau$ with a holomorphic function f on a neighborhood of Z_s in Z because of $Z = Q(\tilde{Z})$. This implies $\tilde{f}_s = f_s \circ \tau_s, f_s = f|_{Z_s} \in \Gamma(Z_s, \mathcal{O}_{Z_s})$.

It remains to prove that $\pi: Z \rightarrow S$ is flat. This follows from [II, Satz I] since π is open (because $\tilde{\pi}$ is open) and $Z \rightarrow S$ is a reduced family (the fibers being even normal). Q.E.D.

§2. Application to special singularities

Let $\pi: Z \rightarrow T$ be a flat morphism in the category of complex spaces. A resolution of π is a commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tau} & Z \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ S & \xrightarrow{\sigma} & T \end{array}$$

where $\tau: \tilde{Z} \rightarrow Z$ is proper and surjective, $\sigma: S \rightarrow T$ is finite and surjective, $\tilde{\pi}: \tilde{Z} \rightarrow S$ is a regular family of complex manifolds and $\tau_s: \tilde{Z}_s \rightarrow Z_{\sigma(s)}$ is a resolution of singularities for all $s \in S$.

We first prove

Theorem 3. *Let $\pi: Z \rightarrow T$ be a deformation of a rational singularity $\pi^{-1}(t_0) = (X, z_0)$. If π admits a resolution with nonsingular base S , then there exists a neighborhood $V = V(t_0) \subset T$ such that Z_t has only rational singularities for all $t \in V$.*

Proof. By lifting Z to S we may assume that $S = T$ and $\sigma = id$ since σ is an open mapping as a ramified surjective covering. We shall show that the diagram

$$\begin{array}{ccc}
 \tilde{Z} & \xrightarrow{\tau} & Z \\
 \tilde{\pi} \searrow & & \swarrow \pi \\
 & S &
 \end{array}$$

is that of Theorem 1; our assertion is then an immediate Corollary of Theorem 2.

We assume moreover that Z and S are Stein and \tilde{Z} is holomorphically convex. Hence we only have to prove that Z is the Remmert-quotient of \tilde{Z} , i.e., $R^0\tau_*\mathcal{O}_{\tilde{Z}} = \mathcal{O}_Z$, a trivial consequence of the following statements:

i) Z is normal

ii) there exists a 2-codimensional analytic subvariety $E \subset Z$ such that $\tau^{-1}(E) = \tilde{E}$ is nowhere dense in \tilde{Z} and τ is biholomorphic on $\tilde{Z} - \tilde{E}$.

To prove i) remark that $\pi^{-1}(s_0)$ is normal and we have more generally:

If $\pi: Z \rightarrow S$ is a flat morphism with normal fiber Z_{s_0} and nonsingular base S then Z is normal (in a neighborhood of Z_{s_0}).

This is proved by induction over $m = \dim S$, $m = 1$ being Corollaire (5.12.7) in EGA_{IV} [9]. If $m > 1$, let s_1, \dots, s_m be a coordinate system of S about s_0 , let i be the embedding $\{s_m = 0\} = S_{m-1} \hookrightarrow S$ and p the projection $S \rightarrow S_1 = \{s_1 = \dots = s_{m-1} = 0\}$. $Z_{m-1} = Z \times_S S_{m-1}$ is a flat family over S_{m-1} with normal fiber Z_{s_0} ; hence Z_{m-1} is normal by induction hypothesis. Since p is flat, $p \circ \pi$ is a flat family of complex spaces over S with normal fiber Z_{m-1} . By the case $m = 1$, Z is normal.

To prove ii), define E as the singular set of π , i.e., the set of points $z \in Z$ which are singular in $Z_{\pi(z)}$. This is an analytic subvariety of Z such that $\pi|_E$ is discrete at s_0 . Thus E is finite over S and hence at least 2-codimensional in Z (since all fibers Z_s are 2-dimensional). Obviously all fibers \tilde{E}_s are 1-codimensional in \tilde{Z}_s such that \tilde{E} is nowhere dense in \tilde{Z} .

Outside \tilde{E} the map τ is biholomorphic on the fibers and respects the local product structure of \tilde{Z} and Z . This implies that τ is locally biholomorphic on $\tilde{Z} - \tilde{E}$. Since $\tau: \tilde{Z} - \tilde{E} \rightarrow Z - E$ is bijective and $\tilde{Z} - \tilde{E}$ and $Z - E$ are complex manifolds, $\tau|_{\tilde{Z} - \tilde{E}}$ is biholomorphic. Q.E.D.

It is known that the versal family $Z \rightarrow T$ of a Kleinian singularity (i.e., a rational double point) admits a resolution with nonsingular base S ([21], [6], [10]; cf. also [3], [4]). By Theorem 3 any (local) deformation of such a singularity (X, x_0) is rational. Since (X, x_0) has embedding dimension 3, any deformation has embedding dimension ≤ 3 (that follows easily from [7, Satz 2.2]). Using Brieskorn's classification [3] we obtain

Corollary. Any (local) deformation of a Kleinian singularity is Kleinian (or nonsingular).

To get more information about all possible deformations of a Kleinian singularity we prove the following upper semicontinuity theorem:

Lemma. Let $\pi: Z \rightarrow S$ be a reduced family of complex spaces with nonsingular base space and isolated hypersurface singularity z_0 over $s_0 \in S$. Then

$$\sum_{z \in \text{sing } Z_s} \dim_{\mathbb{C}} \text{Ext}_z^1(Z_s, \Omega(Z_s)) \leq \dim_{\mathbb{C}} \text{Ext}_{z_0}^1(Z_{s_0}, \Omega(Z_{s_0})),$$

where $\text{sing } Z_s$ denotes the singular set of $Z_s = \pi^{-1}(s)$ and $\Omega(Z_s)$ is the sheaf of germs of holomorphic 1-forms on Z_s .

Proof. The set $E = \bigcup_{s \in S} \text{sing } Z_s$ is analytic and finite over a neighborhood of s_0 (proof of Theorem 3). Suppose $(S, s_0) = (\mathbb{C}^d, 0)$ with coordinates t_1, \dots, t_d about 0 and that $Z_{s_0} = (X, z_0)$ is described by $f(z_1, \dots, z_r) = 0$, $z_0 = 0 \in \mathbb{C}^r$. Then Z can be realized by a holomorphic equation

$$F(z, t) = f(z_1, \dots, z_r) + \sum_{j=1}^d t_j g_j(z_1, \dots, z_r) = 0$$

in $\mathbb{C}^r \times \mathbb{C}^d$ and $\pi: Z \rightarrow S$ is induced by the canonical projection $\mathbb{C}^r \times \mathbb{C}^d \rightarrow \mathbb{C}^d$ (cf. [7]). Define now

$$\mathfrak{F} = \mathcal{O}_{\mathbb{C}^r \times \mathbb{C}^d} \left(F, \frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_r} \right).$$

\mathfrak{F} is a coherent analytic sheaf with $\text{supp } \mathfrak{F} \subset E$, hence the direct image $R^0 \pi_* \mathfrak{F}$ is coherent on S , and we have

$$\text{cg}(R^0 \pi_* \mathfrak{F})_s \leq \text{cg}(R^0 \pi_* \mathfrak{F})_{s_0}$$

in a neighborhood of s_0 ($\text{cg}(R^0 \pi_* \mathfrak{F})_s$ denotes the minimal number of generators of $(R^0 \pi_* \mathfrak{F})_s$ over $\mathcal{O}_{\mathbb{C}^d, s}$).

But obviously

$$\begin{aligned} \text{cg}(R^0 \pi_* \mathfrak{F})_s &= \dim_{\mathbb{C}} (R^0 \pi_* \mathfrak{F})_s / \mathfrak{m}_s (R^0 \pi_* \mathfrak{F})_s \\ &= \sum_{z \in \pi^{-1}(s) \cap E} \dim_{\mathbb{C}} \mathfrak{F}_z / \mathfrak{m}_z \mathfrak{F}_z \end{aligned}$$

and $\mathfrak{F}_z / \mathfrak{m}_z \mathfrak{F}_z = \text{Ext}_z^1(Z_s, \Omega(Z_s))$ since (Z_s, z) is an isolated reduced hypersurface singularity. Q.E.D.

If we denote the Kleinian singularities as usual by A_n , $n \geq 1$, D_n , $n \geq 4$, E_6 , E_7 , and E_8 , we have

$$\dim_{\mathbf{C}} \text{Ext}^1(A_n, \Omega(A_n)) = n, \quad n \geq 1$$

$$\dim_{\mathbf{C}} \text{Ext}^1(D_n, \Omega(D_n)) = n, \quad n \geq 4$$

$$\dim_{\mathbf{C}} \text{Ext}^1(E_n, \Omega(E_n)) = n, \quad n = 6, 7, 8.$$

(These numbers are the numbers of moduli, i.e., the dimension of the — non-singular — base space of the versal family of the corresponding singularity). So our Lemma says for instance that A_n can not be deformed into A_{n+1} , etc.

Unfortunately, not all impossible cases are excluded by that Lemma. Since Kleinian singularities are hypersurface singularities (whose equations can be found in Klein's textbook [12]) it is easy to write down the versal families (cf. [21], [20]):

$$A_n: z_1^2 + z_2^2 + z_3^{n+1} + t_1 z_3^{n-1} + \dots + t_{n-1} z_3 + t_n = 0, \quad n \geq 1$$

$$D_n: z_1^2 + z_2^2 z_3 + z_3^{n-1} + t_1 z_3^{n-2} + \dots + t_{n-1} + t_n z_2 = 0, \quad n \geq 4$$

$$E_6: z_1^2 + z_2^3 + z_3^4 + t_1 z_3^2 + t_2 z_3 + t_3 + z_2(t_4 z_3^2 + t_5 z_3 + t_6) = 0$$

$$E_7: z_1^2 + z_2^3 + z_2 z_3^3 + z_2(t_1 z_3 + t_2) + t_3 z_3^4 + \dots + t_6 z_3 + t_7 = 0$$

$$E_8: z_1^2 + z_2^3 + z_3^5 + z_2(t_1 z_3^3 + \dots + t_4) + t_5 z_3^3 + \dots + t_8 = 0.$$

With some pain one computes all deformations of A_n and D_n and obtains the following diagram (arrows indicating all actually occurring deformations of a given singularity):

$$\begin{array}{cccccccc} \dots & \rightarrow & D_{n+1} & \rightarrow & D_n & \rightarrow & \dots & \rightarrow & D_5 & \rightarrow & D_4 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \dots & \rightarrow & A_n & \rightarrow & A_{n-1} & \rightarrow & \dots & \rightarrow & A_4 & \rightarrow & A_3 & \rightarrow & A_2 & \rightarrow & A_1 & \rightarrow & A_0, \end{array}$$

where A_0 is a symbol for the "regular" singularity.

In order to understand this diagram it seems to be necessary not only to assume the existence of a resolution for the versal family of a Kleinian singularity but also to consider a concrete description of such a resolution like Brieskorn's beautiful construction [6].

We want to discuss finally a second example. Let Σ_m be the Hirzebruch manifold of degree $m \geq 0$. Σ_m is a projective algebraic manifold constructed as follows:

Let \mathbf{C} have coordinate z and \mathbf{P} have inhomogeneous coordinate ζ . Then $\Sigma_m = \mathbf{C} \times \mathbf{P} \cup \mathbf{C} \times \mathbf{P}$, where (z_1, ζ_1) and $(z_2, \zeta_2) \in \mathbf{C} \times \mathbf{P}$ are identified if and only if

$$\zeta_1 = z_2^m \zeta_2, \quad z_1 = \frac{1}{z_2}.$$

For m even the manifolds Σ_m are the only known different complex structures on $\mathbf{P} \times \mathbf{P}$. Let Θ denote the sheaf of germs of holomorphic vector fields on Σ_m , then

$$\dim_{\mathbf{C}} H^1(\Sigma_m, \Theta) = \begin{cases} 0, & m = 0 \\ m - 1, & m > 0. \end{cases}$$

Hence, by the stability theorem of Fröhlicher and Nijenhuis, $\Sigma_0 = \mathbf{P} \times \mathbf{P}$ with canonical product structure and Σ_1 are rigid. If $m \geq 2$ there exists a complete and effectively parametrized family $\tilde{\pi}: \tilde{Z} \rightarrow S$ with $S = \mathbf{C}^{m-1}$ and $\tilde{Z}_0 = Z_m$ (complete and effectively parametrized means that the canonical Kodaira-Spencer mapping ρ of the tangent space of \mathbf{C}^{m-1} at 0 into $H^1(\Sigma_m, \Theta)$ is an isomorphism). Σ_m can be deformed precisely into manifolds Σ_{m-2k} , $k = 0, 1, \dots$, $m - 2k \geq 0$, and $\tilde{Z}_s \not\cong \tilde{Z}_0$ for all $s \neq 0$.

It is known that Σ_m contains an exceptional subvariety $E_m \cong \mathbf{P}$ of self-intersection number $-m$ for $m \geq 2$. By blowing down E_m one obtains an isolated rational singularity (X_m, x_0) and hence by Theorem 2 a commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tau'} & Z' \\ & \searrow \tilde{\pi} & \swarrow \pi' \\ & & S \end{array}$$

where $Z' \xrightarrow{\pi'} S$ is a deformation of the rational singularity (X_m, x_0) .

At first glance one will expect that all singularities X_{m-2k} will occur in fibers of π' near $0 \in S$. However, the fibers Z'_s , $s \neq 0$, are nonsingular. This can be proved in the following way:

By [5] (X_m, x_0) is a quotient singularity \mathbf{C}^2/G , where $G \subset GL(2, \mathbf{C})$ is generated by

$$\begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m \end{pmatrix}, \quad \zeta_m \text{ a primitive } m\text{th root of unity.}$$

If $\mathbf{C}\langle u, v \rangle$ denotes the \mathbf{C} -algebra of holomorphic functions of \mathbf{C}^2 at 0, \mathcal{O}_{X_m, x_0} is the subalgebra of $\mathbf{C}\langle u, v \rangle$ generated by

$$z_i = u^{m-i}v^i, \quad i = 0, \dots, m,$$

(cf. for instance [8], p. 162). It is then easy to see that the embedding-dimension of (X_m, x_0) is $m+1$ and (X_m, x_0) will be described by $m(m-1)/2$ equations

$$\frac{z_0}{z_1} = \frac{z_1}{z_2} = \dots = \frac{z_{m-1}}{z_m}$$

at $0 \in \mathbf{C}^{m+1}$ with coordinates z_0, \dots, z_m . Although X_m is never a complete intersection for $m > 2$, it turns out that

$$\text{Ext}_{x_0}^2(X_3, \Omega(X_3)) = 0.$$

Hence the singularity X_3 has a versal family $\pi: Z \rightarrow T$ with nonsingular base T (cf. [20]). Since

$$\dim_{\mathbf{C}} \text{Ext}_{x_0}^1(X_3, \Omega(X_3)) = 2,$$

we will have $\dim T = 2$. If t_1, t_2 denote complex coordinates of T about 0, Z can be described as follows:

$$\frac{z_0}{z_1} = \frac{z_1 + t_1}{z_2} = \frac{z_2 + t_2}{z_3},$$

and it is easy to see that all fibers Z_t , $t \neq 0$, are nonsingular.

Since $\pi: Z \rightarrow T$ is versal there exists a map $\sigma: S \rightarrow T$ such that $Z' \cong Z \times_T S$, and fibers Z'_s near 0 can only have singularities of type (X_m, x_0) . But $\tilde{Z}_s \not\cong \tilde{Z}_0$ for $s \neq 0$ and hence $Z'_s = Q(\tilde{Z}_s) \not\cong Q(\tilde{Z}_0) = Z'_0$ such that Z'_s has no singularities at all, if $s \neq 0$.

This consideration shows moreover that $\sigma^{-1}(\sigma(0)) = 0$. Hence σ is discrete at $0 \in S$. Since $\dim S = \dim T = 2$, σ is finite and surjective in a neighborhood of $0 \in S$; this means: the canonical diagram

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\tau'} & Z' \cong Z \times_T S & \rightarrow & Z \\ \tilde{\pi} \downarrow & & \downarrow \pi' & & \downarrow \pi \\ S & \xrightarrow{id} & S & \xrightarrow{\sigma} & T \end{array}$$

is a resolution for the versal family $\pi: Z \rightarrow T$ of the singularity (X_3, x_0) .

For $m \geq 4$ we cannot follow the same pattern of reasoning since in that case X_m is obstructed with

$$\dim_{\mathbf{C}} \text{Ext}_{x_0}^1(X_m, \Omega(X_m)) = 2m - 4$$

(cf. D. Mumford, "A remark on the paper of M. Schlessinger", pp. 113-117 in this volume). Hence the corresponding formula

$$\frac{z_0}{z_1} = \frac{z_1 + t_1}{z_2} = \dots = \frac{z_{m-1} + t_{m-1}}{z_m},$$

given by Tjurina ([20], Example 2), does not describe the versal family of X_m . However, one can prove by direct computation — as was pointed out to me by F. Huikeshoven in Göttingen — that this is precisely the de-

formation of X_m , $m \geq 2$, obtained by blowing down the complete family of the Hirzebruch manifold Σ_m .

We finally remark that our examples are not only rational singularities but moreover are contained in the subclass of quotient singularities and their local deformations also turn out to be in that subclass. This might be a general principle. In higher dimensions it is even true that isolated quotient singularities are rigid (Schlessinger [17]).

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