# A REMARK ON THE PAPER OF M. SCHLESSINGER

*by David Mumford*

In the conference itself, I spoke on a theorem asserting the existence of "semi-stable" reductions for analytic families of varieties over a disc, smooth outside the origin. This talk turned out to be difficult to transcribe into a paper of moderate size and instead will be incorporated into the notes of a seminar which I am running together with G. Kempf, B. Saint-Donat, and Tai, which we will publish in the Springer Lecture Notes.

Here I would like to add a footnote to Schlessinger's calculations of versal deformations.[1] He studied the situation: $V$ = complex $n+1$-dimensional vector space; $\mathbf{P}(V)$ = $n$-dimensional projective space of 1-dimensional subspaces of $V$; $Y \subset \mathbf{P}(V)$ a smooth $r$-dimensional variety, $r \geqq 1$; $C \subset V$ the cone over $Y$.

Let $L = \mathcal{O}_Y(1)$. Assume:

$$H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(k)) \rightarrow H^0(Y, L^k) \text{ is surjective, } k \geqq 1$$

(We may also assume by replacing $\mathbf{P}(V)$ by a linear space that it is an isomorphism for $k = 1$). Then he proved:

a) There is a natural injection of functors:

$$\bar{H} = \begin{Bmatrix}\text{Deformations}\\ \text{of } Y \text{ in } \mathbf{P}(V)\end{Bmatrix} \Big/ \begin{matrix}\text{projective}\\ \text{automorphisms}\end{matrix} \rightarrow \begin{Bmatrix}\text{Deformations}\\ \text{of } C\end{Bmatrix}$$

b) $T_C^1$ has a natural graded structure

$$T_C^1 = \bigoplus_{k=-\infty}^{+\infty} (T_C^1)_k$$

such that $(T_C^1)_0 \cong$ image of Zariski tangent space to $\bar{H}$,

c) If $(T_C^1)_k = (0)$ for $k \neq 0$, then $\bar{H}$ is *isomorphic* to the functor of deformations of $C$, i.e., all deformations of $C$ remain conical.

d) If $r \geqq 2$ and $L$ is sufficiently ample on $Y$, then the condition in (c) is satisfied.

What I would like to show here is:

d′) If $r = 1$, $L$ is sufficiently ample on $Y$ and $Y$ has genus $\geqq 2$ and is not hyperelliptic, then again the condition in (c) is satisfied.

This gives:

**Corollary.** There exist normal singularities of surfaces with no non-singular deformation!

To prove (d′), we let $U = C - (0)$ and use the exact sequences:

$$\begin{array}{ccccc} \Gamma(V, \theta_V) & \overset{\alpha}{\to} & \Gamma(C, N_C) & \to & T_C^1 \to 0 \\ \| & & \| & & \\ \Gamma(V-(0), \theta_V) & & \Gamma(U, N_C). & & \end{array}$$

Now $\mathbf{C}^*$ acts in a natural way on both $\theta_V$ and $N_C$, and if $\pi: V-(0) \to \mathbf{P}(V)$ is the projection, then both $\pi_*\theta_V$ and $\pi_*N_C$ decompose into direct sums of their eigenspaces for the various characters of $\mathbf{C}^*$. Moreover, the $\mathbf{C}^*$ invariant sections are:

$$(\pi_*\,\theta_V)^{\mathbf{C}^*} \cong \mathcal{O}_{\mathbf{P}(V)}(1) \otimes_{\mathbf{C}} V$$

$$(\pi_* N_C)^{\mathbf{C}^*} \cong N_Y$$

and $\alpha$ induces the map $\alpha' = \gamma \circ \beta$

$$\alpha' : \mathcal{O}_{\mathbf{P}(V)}(1) \otimes_{\mathbf{C}} V \overset{\beta}{\to} \theta_{\mathbf{P}(V)} \overset{\gamma}{\to} N_Y$$

($\beta$ = standard map).

Thus we get:

$$\begin{array}{ccc} \Gamma(V-(0), \theta_V) & \xrightarrow{\alpha} & \Gamma(U, N_C) \\ \| & & \| \\ \bigoplus_{\nu=-\infty}^{+\infty} \Gamma(\mathbf{P}(V), \mathcal{O}(\nu+1) \otimes_{\mathbf{C}} V) & \longrightarrow & \bigoplus_{\nu=-\infty}^{+\infty} \Gamma(Y, N_Y(\nu)). \end{array}$$

So if

$$(T_C^1)_\nu = \operatorname{coker}[\Gamma(\mathbf{P}^n, \mathcal{O}(\nu+1) \otimes_{\mathbf{C}} V) \xrightarrow{\alpha'_\nu} \Gamma(Y, N_Y(\nu))],$$

then $T_C^1 = \oplus_{\nu=-\infty}^{+\infty} (T_C^1)_\nu$. We must compute these groups.

The idea is to determine $N_Y$ explicitly on $Y$ without actually using the embedding of $Y$ defined by $L$. Consider in fact $N_Y^*(1)$ via the dual of $\alpha'$ as a subbundle of $\mathcal{O}_Y \otimes_{\mathbf{C}} V^*$

$$N_Y^*(1) \subset \theta_{\mathbf{P}(V)}^*(1)\big|_Y \subset \mathcal{O}_Y \otimes_{\mathbf{C}} V^*$$

hence for every $x \in Y$:

$$[N_Y^*(1) \otimes \mathcal{O}_x/m_x] \subset [\overset{*}{\theta}_{P(V)}(1) \otimes \mathcal{O}_x/m_x] \subset V^*.$$

It is easy to see that under these inclusions, if $x' \in C$ lies over $x$:

$$\theta_{P(V)}^*(1) \otimes \mathcal{O}_x/m_x = \left\{\begin{matrix}\text{space of linear forms } l \text{ on } V \\ \text{such that } l(x') = 0\end{matrix}\right\}$$

$$N_Y^*(1) \otimes \mathcal{O}_x/m_x = \left\{\begin{matrix}\text{space of linear forms } l \text{ on } V \\ \text{such that } l(x') = 0 \text{ and} \\ l = 0 \text{ is tangent to } Y \text{ at } x\end{matrix}\right\}$$

But now by assumption:

$$V^* \cong \Gamma(\mathbf{P}(V), \mathcal{O}_{P(V)}(1)) \xrightarrow{\approx} \Gamma(Y, L)$$

and under this isomorphism, the linear forms $l$ such that $l=0$ and is tangent to $Y$ at $x$ go over to the sections of $L$ vanishing at $x$ to 2nd order, i.e. $\Gamma(Y, m_x^2 \cdot L)$. Now consider

$$\begin{array}{c} \Delta \subset Y \times Y \text{ with } p_1^*L(-2\Delta) \\ \downarrow p_2 \\ Y \quad \text{with } p_{2,*}[p_1^*L(-2\Delta)]. \end{array}$$

Then it is easily seen that $p_{2,*}[p_1^*L(-2\Delta)]$ is a locally free sheaf on $Y$ and that

$$\begin{aligned} p_{2,*}[p_1^*L(-2\Delta)] \otimes \mathcal{O}_x/m_x &\cong \Gamma(Y \otimes \{y\}, p_1^*L(-2\Delta) \otimes_{\mathcal{O}_Y} \mathcal{O}_x/m_x) \\ &\cong \Gamma(Y, m_x^2 \cdot L). \end{aligned}$$

Thus the two sub-bundles:

a) $p_{2,*}[p_1^*L(-2\Delta)] \subset p_{2,*}[p_1^*L] = \Gamma(Y, L) \otimes_{\mathbf{C}} \mathcal{O}_Y$

b) $N_Y^*(1) \subset V^* \otimes_{\mathbf{C}} \mathcal{O}_Y \cong \Gamma(Y, L) \otimes_{\mathbf{C}} \mathcal{O}_Y$

are equal. Now assume $r = 1$, so that $Y$ is a curve and $\mathcal{O}(-2\Delta)$ is an invertible sheaf on $Y \times Y$. Then by Serre duality for the morphism $p_2$, we can identify $N_Y(-1)$ as a quotient of $V \otimes_{\mathbf{C}} \mathcal{O}_Y$ or $\Gamma(Y, L)^* \otimes_{\mathbf{C}} \mathcal{O}_Y$:

$$\begin{array}{ccccc} V \otimes_{\mathbf{C}} \mathcal{O}_Y & \xrightarrow{\alpha'(-1)} & N_Y(-1) & \to 0 \\ \wr\| & & \wr\| & \\ \Gamma(Y, L)^* \otimes_{\mathbf{C}} \mathcal{O}_Y & \longrightarrow & \mathrm{Hom}(p_{2,*}[p_1^*L(-2\Delta)], \mathcal{O}_Y) & \to 0 \\ \wr\| & & \wr\| & \\ R^1 p_{2,*}(\mathrm{Hom}(p_1^*L, \Omega_{Y \times Y/Y})) & \to & R^1 p_{2,*}(\mathrm{Hom}(p_1^*L(-2\Delta), \Omega_{Y \times Y/Y})) & \to 0 \\ \wr\| & & \wr\| & \\ R^1 p_{2,*}(p_1^*(\Omega_Y \otimes L^{-1})) & \longrightarrow & R^1 p_{2,*}(p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)) & \to 0. \end{array}$$

We want to show that $(T_C^1)_\nu = (0)$ if $\nu \neq 0$, i.e.,

$$\Gamma(Y, R^1 p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})] \otimes L^\nu) \to \Gamma(Y, R^1 p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)] \otimes L^\nu)$$

is surjective if $\nu \neq 1$. If $\deg L > 2g$, then $p_{2,*}$ of the two sheaves in square brackets is zero, hence by the Leray spectral sequence for $p_2$, the above map is the same as:

$$\begin{aligned} &H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu) \\ &\quad \to H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu \otimes \mathcal{O}(2\Delta)). \end{aligned}$$

We treat the surjectivity in three cases:

*Case I.* $\nu \geqq 2$: Consider the sheaf cokernel

$$p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu \to p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu \otimes \mathcal{O}(2\Delta) \to K_\nu \to 0.$$

It is a sheaf of $\mathcal{O}_{2\Delta}$-modules so it lies in an exact sequence between $\mathcal{O}_\Delta \cong \mathcal{O}_Y$-modules

$$\begin{array}{ccccc} 0 \to (\mathcal{O}(\Delta) \otimes \mathcal{O}_\Delta) \otimes L^{\nu-1} \otimes \Omega_Y & \to & K_\nu & \to & (\mathcal{O}(2\Delta) \otimes \mathcal{O}_\Delta) \otimes L^{\nu-1} \otimes \Omega_Y \to 0 \\ \wr\Vert & & & & \wr\Vert \\ L^{\nu-1} & & & & L^{\nu-1} \otimes (\Omega_Y)^{-1}. \end{array}$$

So if $\deg L > 4g-4$, $H^1(K_\nu) = (0)$ when $\nu \geqq 2$.

*Case II.* $\nu = 0$: Consider the Leray spectral sequence for $p_1$. Since we have assumed $Y$ is not hyperelliptic

a) $p_{1,*}\mathcal{O}_{Y \times Y}(2\Delta) \cong p_{1,*}\mathcal{O}_{Y \times Y}$

and

b) $R^1 p_{1,*}\mathcal{O}_{Y \times Y}(2\Delta)$ is a locally free sheaf $\mathcal{E}$ of rank $g-2$. Now we have:

$$\begin{array}{ccccccc} 0 \to H^1(Y, \Omega_Y \otimes L^{-1}) & \to & H^1(Y \times Y, p_1^*\Omega_Y \otimes L^{-1}) & \to & H^0(Y, \Omega_Y \otimes L^{-1} \otimes R^1 p_{1,*}\mathcal{O}_{Y \times Y}) & \to & 0 \\ \downarrow \wr \text{ by (a)} & & \downarrow & & \downarrow & & \\ 0 \to H^1(Y, \Omega_Y \otimes L^{-1} \otimes p_{1,*}\mathcal{O}(2\Delta)) & \to & H^1(Y \times Y, p_1^*\Omega_Y \otimes L^{-1}(2\Delta)) & \to & H^0(Y, \Omega_Y \otimes L^{-1} \otimes \mathcal{E}) & \to & 0 \end{array}$$

Note that $\mathcal{E}$ does not depend on $L$. So by (b) there is an integer $n_0$ depending only on $Y$ such that if $\deg L > n_0$, then $(\Omega_Y \otimes \mathcal{E}) \otimes L^{-1}$ has no sections.

*Case III*: $\nu \leqq -1$: Surjectivity in this case always follows from surjectivity when $\nu = 0$. In fact, if we know that

$$V \to \Gamma(Y, N_Y \otimes L^{-1}) \to 0$$

is surjective, I claim $\Gamma(Y, N_Y \otimes L^{-\nu}) = (0)$, $\nu \geqq 2$. If not, $N_Y \otimes L^{-2}$ has a non-zero section $s$. Then for all $t \in \Gamma(Y, L) \cong V^*$, $t \otimes s$ is a non-zero section of $N_Y \otimes L^{-1}$. Thus we must get *all* sections of $N_Y \otimes L^{-1}$ in this way. But this means that all these sections are proportional, hence do not generate $N_Y \otimes L^{-1}$. But since

$$V \otimes \mathcal{O}_Y \rightarrow N_Y \otimes L^{-1}$$

is surjective and $V \otimes \mathcal{O}_Y$ is generated by its sections, so is $N_Y \otimes L^{-1}$. This is a contradiction, so $s = 0$.

This completes the proof of (d′). Finally two remarks:

(A) If you look at the case $Y = \mathbf{P}^1$, $L = \mathcal{O}_{\mathbf{P}^1}(k)$, then $C$ = cone over the rational curve of degree $n$ in $\mathbf{P}^n$ and the sequences we have used enable us to compute $T_C^1$ easily. In fact it turns out that if $k \geqq 3$,

$$\begin{aligned} (T_C^1)_l &= (0), \quad \text{if } l \neq -1 \\ \dim (T_C^1)_{-1} &= 2k-4. \end{aligned}$$

It seems most reasonable to conjecture that the versal deformation space of this $C$ is a non-singular $k-1$-dimensional space but with a 0-dimensional embedded component at the origin if $k \geqq 4$.[2]

(B) What happens in the hyperelliptic case? If, for instance, $\pi: Y \rightarrow \mathbf{P}^1$ is the double covering and $L = \pi^* \mathcal{O}_{\mathbf{P}^1}(k)$, then $C$ is itself a double covering of the rational cone considered in (A) which is known to have non-singular deformations. Do these lift to deformations of this $C$?

## NOTES

1. M. Schlessinger, "On Rigid Singularities," in this volume, pp. 147–162.

2. H. Pinkham has recently proved that this is true if $k \geqq 5$, but if $k = 4$, the versal deformation space has two components, a smooth 3-dimensional one and a smooth 1-dimensional one crossing normally at the origin! (Cf. "Deformations of cones with negative grading," *J. of Algebra*, to appear.)

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