DEFORMATIONS OF RESOLUTIONS OF TWO-DIMENSIONAL SINGULARITIES

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§1. *Introduction*

The possible topological structures of normal two-dimensional singularities are essentially known. Given a singularity, one considers its resolution [*11*], [*10*]. Normal singularities are determined by their resolutions so no information is lost. In dimension two, all possible resolutions and point modifications have been described by Mumford [*18*] and Grauert [*8*]. Their criterion, that the intersection matrix of the irreducible components of the exceptional set be negative definite, is purely topological.

This paper is devoted to describing the different singularities which have topologically equivalent resolutions. Examples given in [*8*], [*4*], and [*22*] show that a given topological type may come from exactly one singularity or may come from a complex family of analytically distinct singularities. Here, the general case is treated as follows, via the Kodaira-Spencer [*15*] deformation approach.

Let $p \in V$ be a normal two-dimensional singularity. Let $\pi: M \to V$ be a resolution of V such that the irreducible components A_i, $1 \leqq i \leqq n$, of $A = \pi^{-1}(p)$ are nonsingular and have only normal crossings. Associated to A is a weighted dual graph Γ (e.g., see [*12*] or [*16*]) which, along with the genera of the A_i, fully describes the topology and differentiable structure of A and the topological and differentiable nature of the embedding of A in M. Let $\mathscr{I}_i$ be the ideal sheaf of A_i. Let $m = \Pi \mathscr{I}_i^{r_i}$, $1 \leqq i \leqq n$, where each r_i is a positive integer. Let $A(m)$ be the nonreduced space given by $(A, \mathcal{O}/m\,|\,A)$. Let $\pi': M' \to V' \ni p'$ be a resolution of another normal two-dimensional singularity p'. Suppose that the irreducible components for $A' = (\pi')^{-1}(p')$ are also nonsingular and have only normal crossings. Suppose also that A and A' are topologically the same, i.e., that the dual weighted graph Γ' for A' is the same as Γ and that the corresponding A_i' and A_i have the same genus. Let m' and $A(m')$ be defined as above. Then [*16*, Theorem 6.20, p. 132], depending solely on Γ and the genera, $r_1, \cdots, r_n$ may be chosen so that if $A(m) \approx A'(m')$, then A and A' have biholomorphically equivalent neighborhoods via a map taking A to A'. Thus, rather than deforming A and its embedding in M, it suffices to deform $A(m)$ with $(r_1, \cdots, r_n)$ appropriately chosen. Moreover, by choosing π to be the canonical minimal resolution with nonsingular irreducible components and normal crossings, we can insure that $p \approx p'$ if and only if $A(m) \approx A'(m')$.

Section 2 presents the needed reformulation of the Kodaira-Spencer theory for nonreduced spaces. For use in later proofs it is necessary to consider deformations which fix subspaces. The definitions and proofs in, say, Morrow and Kodaira [*17*] generalize readily from the manifold case to the case of spaces.

Section 3 discusses the deformations of the $A(m)$ above in a slightly more general setting. Every abstractly given $A(m)$ can be realized as a subspace of a 2-manifold (Proposition 3.8) so every abstractly given $A(m)$ corresponds to a singularity. Any deformation of $A(m)$ has the same dual weighted graph (Lemma 3.1). There exists a locally unique, locally complete family of deformations of $A(m)$ which is effectively parametrized at the distinguished point 0 in the parameter space. The parameter space is a manifold of dimension $\dim H^1(A, {}_m\Theta)$, where ${}_m\Theta$ is the tangent sheaf to $A(m)$ introduced by Grauert [*8*, p. 357] (Theorems 3.4, 2.1 and 2.3). There is a neighborhood U of 0 such that the fiber above q is isomorphic to $A(m)$ for only at most countably many q in U (Theorem 3.6). Suppose that A and A' are topologically the same and have the same weights. Then $A(m)$ may be deformed into $A'(m')$ via a finite series of complex analytic deformations (Theorem 3.2). Finally, we give an algorithm (Theorems 3.9 and 3.10) to determine whether or not a singularity with a given weighted graph is taut in the sense of Brieskorn [*4*], Tjurina [*22*], and Wagreich [*23*], i.e., there is exactly one singularity with the given weighted graph and given genera.

We also show that the automorphism group of $A(m)$ is a complex Lie group with $H^0(A, {}_m\Theta)$ as its Lie algebra (Theorem 3.4).

Section 4 consists of discussion, problems, and examples.

§2. *Deformation Theory for Analytic Spaces*

All spaces in this section will be nonreduced unless otherwise specified (e.g., manifolds are reduced).

Let B be a compact analytic space and let C be a closed, possibly empty, subspace of B. B and C will frequently have the same underlying reduced space.

Definition 2.1. A deformation of B, fixing C, consists of the following.

i) There is an analytic space $\mathscr{B}$ and a proper morphism $\omega: \mathscr{B} \to Q$, where Q is a manifold containing a distinguished point 0.

ii) Let $t = (t_1, \cdots, t_k)$ be coordinates near a point $q \in Q$ with q the origin. The fiber B_q over $q \in Q$ is the subspace of $\mathscr{B}$ whose ideal sheaf is generated by $\omega^*(t_1), \cdots, \omega^*(t_k)$. There is an isomorphism $i: B \to B_0$. We shall usually identify B with B_0.

iii) ω is a trivial deformation of C. That is, there is a closed subspace $\mathscr{C}$ of $\mathscr{B}$ and inclusion morphism $\iota: \mathscr{C} \to \mathscr{B}$ such that

a) $\mathscr{C}$ is a product having C and Q as factors, i.e., there is an isomorphism $\psi: \mathscr{C} \to C \times Q$ with $\omega \circ \iota: \mathscr{C} \to Q$ equal, via ψ, to projection onto the second factor, Q.

b) $\iota_0: C_0 \to B_0$ is the inclusion map for C as a subspace of B.

iv) ω is locally trivial in a way which extends the triviality expressed by ψ. That is, for every $b \in \mathscr{B}$, $\omega(b) = q$, there exists a neighborhood $\mathscr{W}$ of b in $\mathscr{B}$, a neighborhood U of q in Q, a neighborhood S of b in B_q and an isomorphism $\phi: \mathscr{W} \to S \times U$ with ω equal to projection onto the second factor, U. With the appropriate restrictions, $\phi \circ \iota = (\iota_q \times id) \circ \psi$.

Suppose that B is locally embedded as a subspace of a polydisc Δ and that n is the ideal sheaf of B. Let $\mathcal{O}$ be the structure sheaf of Δ and let Ω be the sheaf of germs of holomorphic 1-forms on Δ. Recall [7, p. 357] that $\Omega' \subset \Omega$ is the sheaf generated at $x \in \Delta$ by $f_x dg_x + dh_x$, where $g_x \in \mathcal{O}_x$ and $f_x, h_x \in n_x$. ${}_B\Omega = {}_n\Omega = \Omega/\Omega'$ is the sheaf of germs of holomorphic 1-forms on B. Let ${}_B\mathcal{O}$ be the structure sheaf for B.

Definition 2.2. Let B be an analytic space with C a closed subspace of B. Let m be the ideal sheaf for C. Then ${}_{B,C}\theta = \mathscr{H}om_{{}_B\mathcal{O}}({}_B\Omega, m)$ is the sheaf of germs of vector fields on B which vanish on C.

When $C = \varnothing$, ${}_{B,C}\theta$ is the tangent sheaf ${}_n\Theta$ of Grauert. We shall denote ${}_n\Theta = {}_{B,\varnothing}\theta$ by ${}_B\theta$. The sheaf ${}_{n,m}\Theta = \mathscr{H}om_{{}_B\mathcal{O}}({}_C\Omega, m)$ of Grauert is in general different from ${}_{B,C}\theta$.

It will be useful later to represent ${}_{B,C}\theta$ in terms of coordinates on Δ. Suppose $0 \in B \subset \Delta$ and $(z_1, \cdots, z_l)$ are coordinates for Δ, with 0 the origin. Consider $v \in {}_{B,C}\theta_0$. v induces an element $v' \in \mathscr{H}om_{\mathcal{O}}(\Omega, m)_0$. Let $m + n$ be the ideal sheaf of C in Δ. Since Ω is a locally free $\mathcal{O}$-sheaf, v' may be lifted noncanonically to $v'' \in \mathscr{H}om_{\mathcal{O}}(\Omega, m+n)_0 \subset \mathscr{H}om_{\mathcal{O}}(\Omega, \mathcal{O})_0 \approx \mathscr{T}_0$. $\mathscr{T}$ is the tangent sheaf on Δ. Thus v can be represented by an ambient germ of a vector field, $v'' = \sum f_i\, \partial/\partial z_i$ with $f_i \in (m+n)_0$. Any $v'' = \sum f_i\, \partial/\partial z_i$ with $f_i \in (m+n)_0$ will induce an element of ${}_{B,C}\theta_0$ if and only if $v''(dg_0) \in n_0$ for $g_0 \in n_0$.

For a deformation of B, fixing C, as in Definition 2.1, we let Θ be that subsheaf of ${}_{\mathscr{B},\mathscr{C}}\theta$ given by germs of vector fields along the fibers. That is, an element $w \in {}_{\mathscr{B},\mathscr{C}}\theta$ is in Θ if, via the local product structure in Definition 2.1 iv), w vanishes on $dt_1, \cdots, dt_k$. There is a canonical map of $\Theta \mid B_0$ onto ${}_{B,C}\theta$.

Let $\mathscr{T}$ be the tangent sheaf to Q. Let U_r be a neighborhood of 0 in Q and $v \in \Gamma(U_r, \mathscr{T})$. By Definition 2.1 iv), for U_r sufficiently small, we can find a finite cover $\{\mathscr{W}_i\}$ of $\omega^{-1}(U_r)$ with local product maps $\phi_i: \mathscr{W}_i \to S_i \times U_r$. Via ϕ_i and the cartesian product structure, v generates an element $v_i \in \Gamma(\mathscr{W}_{i,\mathscr{B}}\theta)$. v_i may be locally represented by some ambient vector field involving only $t = (t_1, \cdots, t_k)$. Since the deformation of C is trivial, the extensions of v to $\Gamma(\mathscr{W}_{i,\mathscr{C}}\theta)$ and $\Gamma(\mathscr{W}_{j,\mathscr{C}}\theta)$ coincide in $\mathscr{W}_i \cap \mathscr{W}_j$. Let $v_{ij} = v_i - v_j$ be defined on $\mathscr{W}_i \cap \mathscr{W}_j$. Then (v_{ij}) represents a cocycle for ${}_{\mathscr{B},\mathscr{C}}\theta$. However, for $b \in \mathscr{B}$, $v_i(dt_\nu)_b = v(dt_\nu)_{\omega(b)}$. Thus $v_{ij}(dt_\nu)_b = 0$ and $v_{ij} \in \Gamma(\mathscr{W}_i \cap \mathscr{W}_j, \Theta)$. Different choices of ϕ_i will change (v_{ij}) by a coboundary. Thus there is a canonical map $\rho_r: \Gamma(U_r, \mathscr{T}) \to H^1(\omega^{-1}(U_r), \Theta)$. Letting U_r decrease through a fundamental system of neighborhoods of 0 and letting $R^1\omega(\Theta)$ be the first direct image sheaf, we get

$$\rho: \Gamma(0, \mathscr{T}) \to R^1\omega(\Theta)_0. \tag{2.1}$$

The infinitesimal deformation is derived from (2.1) by restricting elements of $\Gamma(0, \mathscr{T})$ to their tangent vectors at 0 and mapping $\Theta \mid B_0$ to ${}_{B,C}\theta$. Letting ${}_QT$ be the tangent bundle to Q, we have

$$\rho_0: {}_QT_0 \to H^1(B, {}_{B,C}\theta). \tag{2.2}$$

Definition 2.3. A deformation of B, fixing C, is effectively parametrized at 0 if (2.2) is injective.

Let $\omega: \mathscr{B} \to Q$ be a deformation of B, fixing C. Suppose that $f: R \to Q$, $f(0) = 0$, is a holomorphic map between manifolds. f induces the following deformation $f^*\omega: \mathscr{P} \to R$. $\mathscr{P} \subset \mathscr{B} \times R$ is defined as follows. $\mathscr{B}$ is locally of the form $S \times U$. In $S \times U \times R$, the ideal sheaf of $\mathscr{P}$ is generated by the condition $f(r) = u$, $r \in R$, $u \in U$. $f^*\omega$ is just projection onto R.

Definition 2.4. A deformation $\omega: \mathscr{B} \to Q$ of B, fixing C, is complete at 0 if, given any deformation $\tau: \mathscr{P} \to R$ of B fixing C, there is a neighborhood R' of 0 in R and a holomorphic map $f: R' \to Q$ such that τ restricted to $\tau^{-1}(R')$ is the deformation $f^*\omega$. ω is complete if it is complete at each $q \in Q$.

Throughout the rest of this paper we will restrict ourselves to the very special B and C that occur in resolutions of two-dimensional singularities. Namely, B will have an underlying reduced space of pure dimension one. Locally, B can be embedded in a two-dimensional polydisc $\Delta = \{(x, y) \mid$

$|x| < \varepsilon, |y| < \varepsilon\}$ with $\mathscr{I}d\, B = (x^a y^b)$ and $\mathscr{I}d\, C = (x^c y^d)$. x and y are, of course, complex variables. Necessarily, $c \leqq a$ and $d \leqq b$. By a change of variables, we can assume that $\varepsilon = 1$.

We adopt the following notation. If S is an analytic space, let $|S|$ denote the underlying reduced space of S. $|S|$ is called the reduction of S.

Theorem 2.1. *Let B and C be as above. Let $\omega: \mathscr{B} \to Q$ be a deformation of B, fixing C. If ρ_0 of (2.2) is surjective, then ω is complete at* 0.

Proof. We essentially mimic the proof of Theorem, p. 56 of [*17*]. Some modifications are needed because B is not reduced. When convenient, we shall omit subscripts and superscripts in local coordinate systems.

$\lambda: \mathscr{P} \to R$ is the given deformation. (x_j, y_j) are local ambient coordinates for B. Let $t = (t_1, \cdots, t_{\beta_0})$ be local coordinates near $0 \in R$ and let $\tau = (\tau_1, \cdots, \tau_{\gamma_0})$ be local coordinates near $0 \in Q$.

For sufficiently small $\varepsilon > 0$, we can cover $\lambda^{-1}(\Delta_\varepsilon)$, $\Delta_\varepsilon = \{t \mid |t_\beta| < \varepsilon\}$ with ambient coordinate patches

$$U_j = \{(x_j, y_j, t) \mid |x_j| < 1, |y_j| < 1, |t_\beta| < \varepsilon\}.$$

Using the same $\{(x_j, y_j)\}$ we can cover a neighborhood of $\omega^{-1}(0)$ by

$$V_j = \{(x_j, y_j, \tau) \mid |x_j| < 1, |y_j| < 1, |\tau_\gamma| < \varepsilon\}.$$

Since the singularities of $|B|$ are isolated, we may assume that for $j \neq k$, $|U_j \cap U_k \cap \mathscr{P}|$ and $|V_j \cap V_k \cap \mathscr{B}|$ are manifolds. On $U_j \cap U_k$, let

$$x_j = f_{jk}^1(x_k, y_k, t),\ y_j = f_{jk}^2(x_k, y_k, t), \qquad t = t$$

be the transition functions. If on $U_j \cap U_k$, $\mathscr{I}d\, \mathscr{B} = (y_j^b)$, then we may assume that y_k divides f^2. Similarly, on $V_j \cap V_k$,

$$x_j = g_{jk}^1(x_k, y_k, \tau),\ y_j = g_{jk}^2(x_k, y_k, \tau), \qquad \tau = \tau$$

and y_k divides g^2. When $t = \tau = 0$, we just get the transition functions for B, so

$$f_{jk}^\alpha(x_k, y_k, 0) = g_{jk}^\alpha(x_k, y_k, 0), \qquad \alpha = 1, 2. \tag{2.3}$$

We need to construct holomorphic functions $\phi_j^1(x_j, y_j, t)$ and $\phi_j^2(x_j, y_j, t)$ and a holomorphic map $\tau = \Phi(t)$ such that

$$\phi_j^\alpha(f_{jk}(x_k, y_k, t), t) = g_{jk}^\alpha(\phi_k(x_k, y_k, t), \Phi(t)), \qquad \alpha = 1, 2 \tag{2.4}$$

and

$$x_j = \phi_j^1(x_j, y_j, 0) \qquad y_j = \phi_j^2(x_j, y_j, 0). \tag{2.5}$$

Equality in (2.4) need only hold modulo $\mathscr{I}d\, \mathscr{B}$ and we require that y_j divide ϕ_j^2.

Let $\mathscr{Q}$ denote the subspace of $\mathscr{P}$ which is a trivial deformation of C. Since $\mathscr{P}$ and $\mathscr{B}$ are trivial deformations of C, f^α and g^α may be chosen independently of t and τ, modulo the ideals of $\mathscr{C}$ and $\mathscr{Q}$ respectively. So y_k^d, the generator of $\mathscr{Id}\,\mathscr{C}$ and $\mathscr{Id}\,\mathscr{Q}$ should divide $f_{jk}^\alpha(x_k, y_k, t) - f_{jk}^\alpha(x_k, y_k, 0)$ and $g_{jk}^\alpha(x_k, y_k, \tau) - g_{jk}^\alpha(x_k, y_k, 0)$.

Later, to insure convergence, we shall need similar covers

$$U_j^0 = \{(x_j, y_j, t) \mid |x_j| < 1 + \nu, |y_j| < 1 + \nu, |t_\beta| < \varepsilon\}$$

for some $\nu > 0$, and similar V_j^0 with $V_j^0 \supset V_j$.

We expand the $f_{jk}^\alpha, g_{jk}^\alpha, \phi_j^\alpha$ and the coordinate functions Φ^γ of Φ into power series in t or τ.

$$(2.6)\quad \begin{aligned} f_{jk}^\alpha(x_k, y_k, t) &= f_{jk|0}^\alpha(x_k, y_k) + f_{jk|1}^\alpha(x_k, y_k)t + \cdots + f_{jk|m}^\alpha(x_k, y_k)t^m + \cdots \\ g_{jk}^\alpha(x_k, y_k, \tau) &= g_{jk|0}^\alpha(x_k, y_k) + g_{jk|1}^\alpha(x_k, y_k)\tau + \cdots + g_{jk|m}^\alpha(x_k, y_k)\tau^m + \cdots \\ \phi_j^1(x_j, y_j, t) &= x_j + \phi_{j|1}^1(x_j, y_j)t + \cdots + \phi_{j|m}^1(x_j, y_j)t^m + \cdots \\ \phi_j^2(x_j, y_j, t) &= y_j + \phi_{j|1}^2(x_j, y_j)t + \cdots \\ \Phi^\gamma(t) &= \Phi_1^\gamma t + \cdots + \Phi_m^\gamma t^m + \cdots \end{aligned}$$

where each $f_{jk|m}^\alpha, g_{jk|m}^\alpha, \phi_{jk|m}^\alpha$ and Φ_m^γ has the appropriate number of components to be coefficients for t^m or τ^m, as needed. Also, y_k divides $f_{jk|m}^2, g_{jk|m}^2$ and $\phi_{k|m}^2$, all m.

If $P(t) = \Sigma P_n t^n$ and $Q(t) = \Sigma Q_n t^n$ are two power series, $P(t) \equiv_m Q(t)$ means that $Q_n = P_n$ for $n \leqq m$. Let $\phi_j^{\alpha|m}(x_j, y_j, t) = z_j^\alpha + \cdots + \phi_{j|m}^\alpha(x_j, y_j)t^m$, $z^1 = x$, $z^2 = y$, and $\Phi^{\gamma|m} = \Phi_1^\gamma t + \cdots + \Phi_m^\gamma t^m$.

We must solve, omitting the superscript α,

$$(2.7)_m \qquad \phi_j^m(f_{jk}(x_k, y_k, t), t) \equiv_m g_{jk}(\phi_k^m(x_k, y_k, t), \Phi^m(t)).$$

First consider $m = 1$ in $(2.7)_m$.

$$(2.7)_1\quad \begin{aligned} & f_{jk|0}(x_k, y_k) + f_{jk|1}(x_k, y_k)t + \phi_{j|1}(f_{jk|0}^1(x_k, y_k), f_{jk|0}^2(x_k, y_k))t \\ &\equiv_1 g_{jk}(x_k + \phi_{k|1}^1(x_k, y_k)t,\ y_k + \phi_{k|1}^2(x_k, y_k)t, \Phi_1 t) \\ &\equiv_1 g_{jk|0}(x_k, y_k) + \frac{\partial g_{jk|0}}{\partial x_k}(x_k, y_k)\phi_{k|1}^1(x_k, y_k)t \\ &\quad + \frac{\partial g_{jk|0}}{\partial y_k}(x_k, y_k)\phi_{k|1}^2(x_k, y_k)t + g_{jk|1}(x_k, y_k)\Phi_1 t, \end{aligned}$$

but we must verify that the partial derivatives do not make the induced functions on the nonreduced subspaces noncanonical. $\partial g_{jk|0}/\partial x_k$ is not changed if g_{jk} is modified by an element of $\mathscr{Id}\,\mathscr{B}$. $\partial g_{jk|0}/\partial y_k$ is changed by a function divisible by y_k^{b-1}. However, y_k divides $\phi_{k|1}^2(x_k, y_k)$ so the product is unchanged.

$f_{jk|0}$ and $g_{jk|0}$ are equal by (2.3). In $(2.7)_1$ we thus need

$$(2.8) \qquad \begin{aligned} f_{jk|1}(x_k, y_k) = {} & \frac{\partial g_{jk|0}}{\partial x_k}(x_k, y_k)\phi_{k|1}^1(x_k, y_k) + \frac{\partial g_{jk|0}}{\partial y_k}(x_k, y_k)\phi_{k|1}^2(x_k, y_k) \\ & - \phi_{j|1}(x_j, y_j) + g_{jk|1}(x_k, y_k)\Phi_1. \end{aligned}$$

We are operating, just in different coordinate systems, in the sheaf ${}_{B,C}\theta$. $\sum f_{jk|1}^{\alpha}\,\partial/\partial z_j^{\alpha}$ and $\sum g_{jk|1}^{\alpha}\,\partial/\partial z_j^{\alpha}$, $\alpha = 1, 2$, lie in ${}_{B,C}\theta$ since y^d divides $f_{jk|1}$ and $g_{jk|1}$. $f_{jk|1}^{\alpha}$ and $g_{jk|1}^{\alpha}$ are vector valued with as many components as are needed to provide coefficients for t and τ. Each component of $f_{jk|1}$ then determines a cohomology class in $H^1(B, {}_{B,C}\theta)$. ρ_0 of (2.2) is surjective so, by restricting to a submanifold, we may in fact assume that ρ_0 is bijective. The $g_{jk|1}$ determine the cohomology classes of the image of ρ_0. Hence there is a uniquely determined Φ_1, depending on $f_{jk|1}$, so that the cohomology classes on both sides of (2.8) coincide. We then choose ϕ_k and ϕ_j to give equality in (2.8). We shall do the size estimates later.

Suppose that $\phi_j^{\alpha|m}$ and $\Phi^{\gamma|m}$ have been determined so that $(2.7)_m$ holds. That is,

$$(2.9) \qquad \phi_j^m(f_{jk}(x_k, y_k, t), t) - g_{jk}(\phi_k^m(x_k, y_k, t), \Phi^m(t)) \equiv_{m+1} \Gamma_{jk} t^{m+1}.$$

$(2.7)_{m+1}$ may be written

$$\begin{aligned} & \phi_j^m(f_{jk}(x_k, y_k, t), t) + \phi_{j|m+1}(f_{jk}(x_k, y_k, t))t^{m+1} \\ & \equiv_{m+1} g_{jk}(\phi_k^m(x_k, y_k, t) + \phi_{k|m+1}(x_k, y_k)t^{m+1}, \\ & \qquad \Phi^m(t) + \Phi_{m+1}t^{m+1}). \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \phi_j^m(f_{jk}(x_k, y_k, t), t) + \phi_{j|m+1}(f_{jk|0}(x_k, y_k))t^{m+1} \\ & \equiv_{m+1} g_{jk}(\phi_k^m(x_k, y_k, t), \Phi^m(t)) \\ & \qquad + \frac{\partial g_{jk|0}}{\partial x_k}(x_k, y_k)\phi_{k|m+1}^1(x_k, y_k)t^{m+1} \\ & \qquad + \frac{\partial g_{jk|0}}{\partial y_k}(x_k, y_k)\phi_{k|m+1}^2(x_k, y_k)t^{m+1} \\ & \qquad + g_{jk|1}(x_k, y_k)\Phi_{m+1}t^{m+1}, \end{aligned}$$

or

$$\Gamma_{jk}(x_k, y_k) = \frac{\partial g_{jk|0}}{\partial x_k}(x_k, y_k)\phi^1_{k|m+1}(x_k, y_k)$$

$$+ \frac{\partial g_{jk|0}}{\partial y_k}(x_k, y_k)\phi^2_{k|m+1}(x_k, y_k) - \phi_{j|m+1}(x_j, y_j)$$

$$+ g_{jk|1}(x_k, y_k)\Phi_{m+1}.$$

Γ_{jk} is a cocycle by the computation of Lemma 3.4, p. 48, of Morrow and Kodaira. As with $m=1$, the choice of Φ_{m+1} depends on Γ_{jk}. We then choose $\phi_{k|m+1}$ and $\phi_{j|m+1}$.

This gives us the formal power series and it is now necessary to insure convergence. We shall use the following norm on sections of ${}_B\theta$ and thus also on sections of ${}_{B,C}\theta \subset {}_B\theta$. Near a nonsingular point, let $\mathscr{I}d\, B = (y^b)$. Represent a section w of ${}_B\theta$ over some open set U as

$$w = [h_0(x) + yh_1(x) + \cdots + y^{b-1}h_{b-1}(x)]\,\frac{\partial}{\partial x}$$

$$+ [yv_1(x) + \cdots + y^{b-1}v_{b-1}(x)]\,\frac{\partial}{\partial y}.$$

$$(2.10) \qquad \|w\|_U = \max_{i+j\leqq b-1}\ \sup_{x\in U\cap|B|}\ (|h_i^{(j)}(x)|, |v_i^{(j)}(x)|).$$

We shall omit the subscript U when it is clear which open set we are considering. This norm is changed to an equivalent norm under a change of coordinates. Near singular points, which are normal crossings with $\mathscr{I}d\, B = (x^a y^b)$, restrict w to a section of the tangent sheaf for each of the two components, $\{x^a = 0\}$ and $\{y^b = 0\}$. Then take the maximum of the two norms. δ, the coboundary operator for covers, is continuous in the topology defined by $\| \ \|$.

Suppose that $\psi(z,t) = \Sigma\psi_m t^m$, $0 \leqq m < \infty$ is a power series with $t = (t_1, \cdots, t_k)$ and ψ_m vector valued with each component ψ_m^β of ψ_m an element of $\Gamma(U, {}_{B,C}\theta)$. Suppose that $a(T) = \Sigma a_m T^m, 0 \leqq m < \infty$ with a_m real and non-negative. Then, following Kodaira, we write $\psi(z,t) \ll a(T)$ and say that $a(T)$ dominates $\psi(z,t)$ if $\Sigma_\beta \|\psi_m^\beta\| \leqq a_m$ for all m. We have a similar notion for $\Phi(t) = \Sigma\Phi_m t^m$. Let $A(T) = (b_0/16c_0)\,\Sigma(c_0T)^m/m^2, 1 \leqq m < \infty$ with b_0 and c_0 constants to be determined later. For convergence, it suffices to prove that

$$(2.11)_m \qquad \phi_j^{\alpha|m}(x_j, y_j, t) - z_j^\alpha \ll A(T),\ z_j^1 = x_j,\ z_j^2 = y_j$$

$$\Phi_j^m(t) \ll A(T).$$

To satisfy $(2.11)_1$, since all the norms will turn out to be finite, it just suffices for b_0 to be large enough. We now proceed by induction on m, assuming $(2.11)_m$.

Recall that $V_j \subset V_j^0$. If $g^{\beta}_{jk|p}$ is a component of $g_{jk|p}$, then $\|g^{\beta}_{jk|p}\| = \| g^{\beta,1}_{jk|p}(x_k, y_k)\,\partial/\partial x_j + g^{\beta,2}_{jk|p}(x_k, y_k)\,\partial/\partial y_j \|$ and $\| g_{jk|p} \| = \sum_{\beta} \| g^{\beta}_{jk|p} \|$.

Since g_{jk} can be assumed to be holomorphic for $(x_k, y_k, \tau) \in V_j^0 \cap V_k^0$, $\sum g_{jk|p}(x_k + \eta_k, y_k + \xi_k)\tau^p$ and its derivatives with respect to x_k converge for $(x_k, y_k) \in V_j \cap V_k \cap |B|$ and τ, η_k and ξ_k sufficiently small. Expand $g_{jk|p}(x_k + \eta_k, y_k + \xi_k)$ into a power series in η_k and ξ_k. In (2.12) below, domination means the following: Compare coefficients for each $\eta_k^{\mu}\xi_k^{\nu}$. If $c_{\mu\nu}(x_k, y_k)\tau^p$ is the coefficient of $\eta_k^{\mu}\xi_k^{\nu}$ on the left side of (2.12) and $C_{\mu\nu}T^p$ is the coefficient of $\eta_k^{\mu}\xi_k^{\nu}$ on the right side of (2.12), then

$$\| y_k^{\nu} c_{\mu\nu}(x_k, y_k) \|_{V_j \cap V_k} \leqq C_{\mu\nu}.$$

$$g_{jk|p}(x_k + \eta_k, y_k + \xi_k)\tau^p \ll C \sum_{r=0}^{\infty} \kappa^{r+p}(\eta_k + \xi_k)^r T^p. \tag{2.12}$$

(2.12) will hold for appropriate C and κ because only a finite number of derivatives are used in computing norms on the left side.

For $\psi(t) = \sum \psi_m t^m$, let $[\psi(t)]_{m+1} = \psi_{m+1}t^{m+1}$. We first estimate $[g_{jk}(\phi_k^m(x_k, y_k, t), \Phi^m(t))]_{m+1} = \sum_p [g_{jk|p}(\phi_k^m(x_k, y_k, t))\,(\Phi^m(t))^p]_{m+1}$, by (2.6).

Temporarily, let $\psi_k(x_k, y_k, t) = \phi_k^m(x_k, y_k, t) - z_k$. For $p = 0$, since the components of $\psi_k(x_k, y_k, t)$ are polynomials of degree at most m in t,

$$\begin{aligned}[g_{jk|0}(z_k + \psi_k(x_k, y_k, t)]_{m+1} = [g_{jk|0}(z_k + \psi_k(x_k, y_k, t) \\ - z_k - \frac{\partial g_{jk|0}}{\partial x_k}(x_k, y_k)\psi_k^1(x_k, y_k, t) \\ - \frac{\partial g_{jk|0}}{\partial y_k}(x_k, y_k)\psi_k^2(x_k, y_k, t)]_{m+1}.\end{aligned} \tag{2.13}$$

From (2.12)

$$\begin{aligned}g_{jk|0}(x_k + \eta_k, y_k + \xi_k) - z_k - \frac{\partial g_{jk|0}}{\partial x_k}(x_k, y_k)\eta_k \\ - \frac{\partial g_{jk|0}}{\partial y_k}(x_k, y_k)\,\xi_k \ll C \sum_{r=2}^{\infty} \kappa^{r+p}(\eta_k + \xi_k)^r.\end{aligned} \tag{2.14}$$

In (2.14), let $\eta_k = \psi_k^1(x_k, y_k, t)$ and $\xi_k = \psi_k^2(x_k, y_k, t)$. Apply $(2.11)_m$. y_k divides $\psi_k^2(x_k, y_k, t)$ so that the norm preceding (2.12) with its factor of y_k^{ν} is appropriate. When multiplying two functions F and G and using our $\|\ \ \|$ norm on sections of ${}_{B,C}\theta$, $\|FG\| \leqq D\|F\|\,\|G\|$, where D comes from taking derivatives and is independent of F and G. (2.14) and (2.13) thus yield, using Corollary, Morrow and Kodaira, p. 50,

$$[g_{jk|0}(z_k + \psi_k(x_k, y_k, t))]_{m+1} \ll C \sum_{r=2}^{\infty} \kappa^r D^r [2A(T)]^r$$

$$\ll A(T)C \sum_{r=2} (2D\kappa)^r (b_0/c_0)^{r-1}$$

$$= A(T)C \frac{(2D\kappa)^2 b_0}{c_0} \frac{1}{1-(2D\kappa b_0/c_0)}.$$

Choose c_0 so large that $(2D\kappa b_0/c_0) < \frac{1}{2}$. Then

$$[g_{jk|0}(\phi_k^m(x_k, y_k, t))]_{m+1} \ll (2C(2D\kappa)^2 b_0/c_0)A(T).$$

For $p > 0$, more directly from (2.12)

$$g_{jk|p}(z_k + \psi_k(x_k, y_k, t)) \ll C \sum_{r=0}^{\infty} \kappa^{r+p} D^r [2A(T)]^r.$$

For $p = 1$, the summations below start at $r = 1$ since $\Phi^m(t)$ is of degree m. For $p \geqq 2$,

$$[g_{jk|p}(z_k + \psi_k(x_k, y_k, t)) \cdot (\Phi^m(t))^p]_{m+1}$$

$$\ll C \sum_{r=0}^{\infty} \kappa^{r+p} (2D)^r [A(T)]^{r+p}$$

$$\ll C \sum_{r=0}^{\infty} \kappa^{r+p} (2D)^r (b_0/c_0)^{r+p-1} A(T)$$

$$= A(T) C \kappa^p (b_0/c_0)^{p-1} \frac{1}{1-(2D\kappa b_0/c_0)}$$

$$\ll (2C\kappa^p (b_0/c_0)^{p-1}) A(T).$$

Hence for $m \geqq 1$, by a straightforward summation over p,

$$(2.15) \qquad [g_{jk}(\phi_k^m(x_k, y_k, t), \Phi^m(t))]_{m+1} \ll (16CD^2\kappa^2 b_0/c_0)\, A(T).$$

We now wish to estimate the size of $\Gamma_{jk}(x_k, y_k)t^{m+1}$, which from (2.9) satisfies

$$\Gamma_{jk}(x_k, y_k)t^{m+1} = [\phi_j^m(f_{jk}(x_k, y_k, t), t)]_{m+1}$$
$$- [g_{jk}(\phi_k^m(x_k, y_k, t), \Phi^m(t))]_{m+1}.$$

Since Γ is a cocycle and the norm $\| \quad \|$ is equivalent in all coordinate systems, as in Morrow and Kodaira, we can choose

$$U_i^* = \{(x_i, y_i, t) | \ |x_i| < 1-\gamma, |y_i| < 1-\gamma, |t_\beta| < \varepsilon\}$$

with γ sufficiently small so that in estimating $\|\Gamma_{jk}\|_{U_j \cap U_k}$, all j, k, it suffices to make estimates only at points x_j in $|U_j^* \cap U_k \cap B|$.

Since f_{jk} can be assumed to be holomorphic for $(x_k, y_k, t) \in U_j^0 \cap U_k^0$ there are suitable constants b_1 and c_1 so that

$$\begin{aligned} \sigma_{jk}(x_k, y_k, t) &= f_{jk}(x_k, y_k, t) - z_j \ll A_1(T) \\ &= \frac{b_1}{16c_1} \sum_{m=1}^{\infty} \frac{(c_1 T)^m}{m^2}. \end{aligned} \tag{2.16}$$

Our induction hypothesis is

$$\begin{aligned} \phi_j^m(x_j, y_j, t) - z_j &= \sum_{\mu=1}^{m} \phi_{j|\mu}(x_j, y_j) t^\mu \\ \ll A(T) &= \sum_{\mu=1}^{\infty} a_\mu T^\mu, \end{aligned}$$

$(x_j, y_j, t) \in U_j$. The $\phi_{j|\mu}$ are initially defined only on $U_j \cap B$ rather than the ambient space $U_j \cap \{t = 0\}$. Let $\phi_{j|\mu}$ also denote any holomorphic extension of $\phi_{j|\mu}$ to the ambient space. Our estimates below will be independent of the choice of extension. Since $\phi_{j|\mu}$ is holomorphic

$$\begin{aligned} &\phi_{j|\mu}(x_j + \eta, y_j + \xi) - \phi_{j|\mu}(x_j, y_j) \\ &\quad = \sum c_{\nu_1 \nu_2}(x_j, y_j) \eta^{\nu_1} \xi^{\nu_2} \end{aligned} \tag{2.17}$$

with $\nu_1 + \nu_2 \geqq 1$, $(x_j, y_j) \in U_j^*$, $|\eta|, |\xi| < \beta$.

$$c_{\nu_1 \nu_2}(x_j, y_j) = \left(\frac{1}{2\pi i}\right)^2 \iint_{\substack{|\eta| = \beta \\ |\xi| = \beta}} \frac{\phi_{j|\mu}(x_j + \eta, y_j + \xi)}{\eta^{\nu_1 + 1} \xi^{\nu_2 + 1}} d\eta d\xi. \tag{2.18}$$

We are interested in $\| y_j^{\nu_2} c_{\nu_1 \nu_2}(x_j, y_j) \|_{U_j^*}$ if $x_j = y_j = 0$ is not a singular point of $|B|$. Since $c_{\nu_1 \nu_2}$ comes from (2.17), these norms are independent of the extension of $\phi_{j|\mu}$ to the ambient space. The case of $x_j = y_j = 0$ being a singular point reduces to the nonsingular case after $\phi_{j|\mu}$ is restricted to each of the two components of B. $\| y_j^{\nu_2} c_{\nu_1 \nu_2}(x_j, y_j) \|$ may be estimated by estimating $\partial^{p+q}/\partial x^p \partial y^q \, c_{\nu_1 \nu_2}(x_j, 0)$ for appropriate p and q. In (2.18), we can differentiate under the integral sign to get formulae for these derivatives. $\phi_{j|\mu}(x_j + \eta, y_j + \xi)$ can be represented as a polynomial of degree $b - 1$ in $(y_j + \xi)$ and $\| \phi_{j|\mu} \|$ gives the needed estimates of the coefficients and their derivatives. With E a constant depending on b,

$$\| y_j^{\nu_2} c_{\nu_1 \nu_2}(x_j, y_j) \|_{U_j^*} \leqq \frac{E a_\mu}{\beta^{\nu_1 + \nu_2}}. \tag{2.19}$$

Using (2.19) in (2.17),

$$\phi_{j|\mu}(x_j+\eta, y_j+\xi)t^\mu - \phi_{j|\mu}(x_j, y_j)t^\mu$$
$$\ll Ea_\mu \ \Sigma \frac{\eta^{\nu_1}\xi^{\nu_2}}{\beta^{\nu_1+\nu_2}} T^\mu, \qquad \nu_1+\nu_2 \geqq 1.$$

Summing over μ,

$$\phi_j^m(x_j+\eta, y_j+\xi, t) - \phi_j^m(x_j, y_j, t) \tag{2.20}$$
$$\ll E\,\Sigma a_\mu T^\mu \Sigma \frac{\eta^{\nu_1}\xi^{\nu_2}}{\beta^{\nu_1+\nu_2}}, \qquad \mu \geqq 1,\ \nu_1+\nu_2 \geqq 1.$$

Since $\phi_j^m(x_j, y_j, t)$ has only terms of degree at most m in t,

$$[\phi_j^m(f_{jk}(x_k, y_k, t), t)]_{m+1} = [\phi_j^m(z_j+\sigma_{jk}(x_k, y_k, t), t)]_{m+1} - [\phi_j^m(x_j, y_j, t)]_{m+1}. \tag{2.21}$$

Thus, from (2.20) and (2.16),

$$[\phi_j^m(f_{jk}(x_k, y_k, t), t)]_{m+1} \ll EA(T) \sum_{\nu_1+\nu_2\geqq 1} \frac{D^{1+\nu_1+\nu_2}[A_1(T)]^{\nu_1+\nu_2}}{\beta^{\nu_1+\nu_2}}$$
$$= EDA(T)\left\{\left[\sum_{\nu=0}^{\infty}\left(\frac{DA_1(T)}{\beta}\right)^\nu\right]^2 - 1\right\}.$$

Calculating as in Morrow and Kodaira, we use (2.21) and (2.15) to find a constant K_2 such that

$$\Gamma_{jk}(x_k, y_k)t^{m+1} \ll K_2(b_0/c_0)A(T),\ (x_k, y_k, 0) \in U_j^* \cap U_k.$$

Since Γ_{jk} is a cocycle, there is, again as in Morrow and Kodaira, a constant K_1 such that

$$\Gamma_{jk}(x_k, y_k)t^{m+1} \ll K_1K_2(b_0/c_0)A(T),\ (x_k, y_k, 0) \in U_j \cap U_k.$$

Φ_{m+1}, $\phi_{k|m+1}$, and $\phi_{j|m+1}$ depended on Γ_{jk}. Thus, to complete the proof of Theorem 2.1, we only need to bound $|\Phi_{m+1}|$, $\|\phi_{k|m+1}\|$, and $\|\phi_{j|m+1}\|$ in terms of $\|\Gamma_{jk}\|$, for then we just follow the calculations of Morrow and Kodaira. This bound is the analogue of Lemma 3.7 of [*17*, p. 54] on B. It follows from the open mapping theorem and the following lemma.

Lemma 2.2. Let $H_b^1(N(\mathfrak{A}), {}_{B,C}\theta)$ be the first cohomology group for the cover $\mathfrak{A} = \{U_i \cap B\}$ in the sheaf ${}_{B,C}\theta$ using cochains with finite norm. Then the natural map ι: $H_b^1(N(\mathfrak{A}), {}_{B,C}\theta) \to H^1(B, {}_{B,C}\theta)$ is an isomorphism. In particular, $H_b^1(N(\mathfrak{A}), {}_{B,C}\theta)$ is finite dimensional and δ: $C_b^0 \to C_b^1$ has closed range.

Proof. Since $\mathfrak{A}$ is a Leray cover, $H^1(N(\mathfrak{A}), {}_{B,C}\theta) \approx H^1(B, {}_{B,C}\theta)$. ι is onto, since we may restrict representatives of the cohomology classes from $\{U_i^0 \cap B\}$. We just need to show that ι is injective. But if (w_{ij}) is a bounded cocycle with $(w_{ij}) = \delta(v_i)$, then v_i is in fact bounded for the following reason. Let $p \in \partial|U_i \cap B|$. Then $p \in |U_j \cap B|$, some $j \neq i$. $w_{ij} = v_i - v_j$. v_j is bounded near p since it is defined in a neighborhood of p. w_{ij} is bounded near p by hypothesis. Hence v_i is bounded near p. Since $|B|$ is compact, $|U_i \cap B|$ has compact closure and thus v_i is bounded.

The next theorem is the expected existence and uniqueness theorem.

Theorem 2.3. *Let B and C be as in Theorem 2.1. There exists $\omega: \mathscr{B} \to Q$, a deformation of B, fixing C, such that ρ_0 of (2.2) is bijective. Any two such ω are isomorphic near 0, i.e., if $\omega': \mathscr{B}' \to Q'$ has ρ_0' bijective, then there are neighborhoods U and U' of 0 and 0′ respectively with isomorphisms $f: U \to U'$ and $g: \omega^{-1}(U) \to (\omega')^{-1}(U')$ such that $f \circ \omega = \omega' \circ g$.*

Proof. Let us first prove existence. The singular points $\{p_i\}$, $1 \leqq i \leqq m$, of $|B|$ are isolated. Let $\{U_i\}$, $1 \leqq i \leqq m$, be neighborhoods of the p_i in B which are the intersection of B with ambient polydiscs, B locally having $x_i^{a_i} y_i^{b_i}$ as generator of its ideal. We can take $\{U_i\}$ so that $U_i \cap U_j = \emptyset$, $i \neq j$. Let V_i be defined similarly to U_i with $V_i \subset\subset U_i$. Let $U_0 = B - \bigcup_{i=1}^m V_i$. Then $|U_0|$ is Stein [9, Theorem IX.B. 10, p. 270] and $\mathfrak{A} = \{U_i\}$, $0 \leqq i \leqq m$, is a Leray cover for any coherent sheaf over B [7, Satz 3, p. 17]. $\mathfrak{A}$ has the useful property that there are no 2-simplices. The cocycle condition on 1-cochains is then vacuous. $H^1(B, {}_{B,C}\theta)$ is finite dimensional. Let $\theta_1, \cdots, \theta_n$ be cocycles in $Z^1(N(\mathfrak{A}), {}_{B,C}\theta)$ whose images in $H^1(B, {}_{B,C}\theta)$ form a basis. Let $\mathfrak{A}' = \{U_i'\}$, $0 \leqq i \leqq m$, be defined similarly to $\mathfrak{A}$ with $\bar{U}_i' \subset\subset U_i$. The parameter space Q is $\{t = (t_1, \cdots, t_n) \mid |t_i| < \varepsilon\}$ with ε to be later chosen sufficiently small. We wish to integrate along $\theta = t_1\theta_1 + \cdots + t_n\theta_n$ for a time 1 (cf. [19]). ε is to be sufficiently small so that under this integration $|U_0' \cap U_i'|$ remains within $|U_0 \cap U_i|$. The integration will not necessarily be defined for points in $|U_0 \cap U_i| - |U_0' \cap U_i'|$. We now verify that integration along θ can be done in a canonical manner. Since the integration will be canonical it will suffice to do things locally. Near $b \in B$, represent B locally in a polydisc Δ of dimension r and let ω be a vector field on Δ which induces θ. That is, for arbitrary $g_x \in {}_\Delta\mathcal{O}_x$, $\omega(dg_x) \equiv \theta(dg_x)$, mod $\mathscr{I}d B_x$.

Suppose that $\omega = \sum \omega_i(z)\, \partial/\partial z_i$, $1 \leqq i \leqq r$. Integrating along ω means solving the simultaneous ordinary differential equations

$$\frac{dz_i}{d\tau} = \omega_i(z), \qquad 1 \leqq i \leqq r,$$

with initial conditions $z(0) = z_0$. We denote the solution by $z_0(\tau)$. Given any relatively compact subset $\Delta' \subset\subset \Delta$, there is a τ_0 such that for $\tau \leqq \tau_0$, the map $\Omega_\tau : z_0 \to z_0(\tau)$ is a biholomorphic map of Δ' to another subset Δ'' of Δ. We want Ω_τ to induce an isomorphism from $\Delta' \cap B$ to $\Delta'' \cap B$. We first need to show that $\Omega_\tau^* : \mathscr{I}d B \to \mathscr{I}d B$. So suppose $f(z) \in \mathscr{I}d B$.

$$g(z) = \omega(df) = \sum \omega_i(z) \frac{\partial f}{\partial z_i} \in \mathscr{I}d B.$$

Let $z = z_0(\tau)$ and regard both z_0 and τ as variables.

$$\begin{aligned} g(z_0(\tau)) &= \sum \omega_i(z_0(\tau)) \frac{\partial f}{\partial z_i}(z_0(\tau)) \\ &= \sum \frac{\partial f}{\partial z_i}(z_0(\tau)) \frac{dz_{0,i}(\tau)}{d\tau} = \frac{\partial}{\partial \tau} f(z_0(\tau)). \end{aligned}$$

Expand $f(z_0(\tau))$ in a power series in τ about $\tau = 0$.

$$f(z_0(\tau)) = \sum \frac{1}{k!} \frac{\partial^k f}{\partial \tau^k}(z_0(0))\tau^k = \sum \frac{1}{k!} \omega^k(df)(z_0)\tau^k. \tag{2.22}$$

By induction, $\partial^k f/\partial \tau^k \in \mathscr{I}d B$ for all k. Choosing a τ_0 within the radius of convergence of (2.22), we get $f(z) \in \mathscr{I}d B$, $\tau \leqq \tau_0$. To get a common τ_0 for all $f \in \mathscr{I}d B$, $z_0 \in \Delta'$, observe that $\mathscr{I}d B$ is coherent on Δ, and Δ' has compact closure in Δ. Thus a finite number of f will generate $\mathscr{I}d B$. It suffices to take τ_0 within the radius of convergence of these f.

To check that the induced map on B is canonical, we must check what happens when a vector field λ with $\lambda(dh_x) \in \mathscr{I}d B_x$, all $h_x \in {}_\Delta\mathcal{O}_x$, is added to ω. As before, $(\omega + \lambda)\ (dh) = \partial h(z_0(\tau))/\partial\tau = \theta(dh) + \lambda(dh)$. Since $\lambda(dh) \in \mathscr{I}d B$, the power series expansion in (2.22) is just changed by an element of $\mathscr{I}d B$.

The inverse map to Ω_τ is given by integrating $-\omega$, so the induced map on B is an isomorphism. Also, Ω_τ is the identity on C if $\theta \in \Gamma(\Delta', {}_{B,C}\theta)$.

Returning to our proof, we construct $\mathscr{B}$ via coordinate patches. Start with $\{U_i' \times Q\}$, $0 \leqq i \leqq m$. We must give the g_{i0}, the transition morphisms. Let $(x_0, y_0) = z_0$ be local coordinates in U_0'. For each $t \in Q$, integration along $t_1\theta_1 + \cdots + t_n\theta_n$ for time 1 gives a morphism $(x_0, y_0) \to (\Phi_t^1(z_0), \Phi_t^2(z_0))$. As t varies, $\Phi_t^\alpha(z_0)$, $\alpha = 1,2$, gives a function on $\{(x_0, y_0)\} \times Q$. g_{i0} is defined by $x_i = \Phi_t^1(z_0)$, $y_i = \Phi_t^2(z_0)$, $t = t$. There are no compatibility conditions to verify since no three patches intersect. It is necessary, however, to slightly modify the $\{U_i' \times Q\}$, $1 \leqq i \leqq m$, so that the map of g_{i0} on the underlying topological spaces is well-defined. g_{i0} maps $|U_0' \cap U_i'| \times Q$

$\subset |U_0'| \times Q$ to an open subset S_i of $|U_i| \times Q$. S_i lies "near" $|U_0' \cap U_i'| \times Q \subset |U_i'| \times Q$ for Q small. Let U_i'' be defined as was U_i' and U_i, with $|U_i''| \subset |U_i'|$ and $\partial |U_i''| \times Q \subset S_i$. Replace $\{U_i' \times Q\}, 1 \leqq i \leqq m$, by $\{(U_i'' \times Q) \cup S_i\}$, $1 \leqq i \leqq m$. Retain $U_0' \times Q$ in our covei for $\mathscr{B}$. $g_{i0} : |U_0' \cap U_i'| \times Q \to S_i$ is a well-defined change of coordinates. From the nature of our construction, ρ_0 of (2.2) is an isomorphism.

Now for uniqueness. Given ω and ω', by Theorem 2.1, there is a holomorphic $f: U \to U'$ and a g such that $f \circ \omega = \omega' \circ g$. We only need to demonstrate that f and g are isomorphisms. But if f_* represents the induced map on tangent spaces at the origin, $\rho_0 = \rho_0' \circ f_*$. Hence f_* is an isomorphism and f has a holomorphic inverse near the origin. g also induces an isomorphism on the tangent spaces. Hence g can be induced locally by an ambient isomorphism and thus has a local inverse. $g_0 : |B_0| \to |B_0'|$ is a homeomorphism and $|B_0|$ is compact. Hence there is some neighborhood of B_0 in $\mathscr{B}$ where g is an isomorphism. This neighborhood contains $\omega^{-1}(U)$ for U sufficiently small.

§3. *Non-Local Theory*

As in Section 2, we shall only be considering analytic spaces B with subspaces C such that B can be expressed locally as $\{x^a y^b = 0\}$ and $C = \{x^c y^d = 0\}$. Let $|B_i|$ be an irreducible component of $|B|$. Let B_i be that subspace of B given locally by $\{y^b = 0\}$ having $|B_i|$ as its reduction. Such a B_i will be called a component of B. If B_i is not reduced, then it has an associated "topological" invariant c_i, the Chern class of the normal bundle of the embedding of $|B_i|$ in any 2-dimensional ambient space containing B_i. c_i may be intrinsically defined as follows: Let $\mathscr{I}_i$ be the ideal sheaf $|B_i|$. $\mathscr{I}_i/\mathscr{I}_i^2$ is an invertible sheaf over $|B_i|$. c_i equals the Chern class of the dual line bundle for $\mathscr{I}_i/\mathscr{I}_i^2$. We shall sometimes write c_i as $|B_i| \cdot |B_i|$, the self-intersection number.

Lemma 3.1. Suppose that B_i is not reduced. Then c_i is invariant under connected deformations of B.

Proof. The deformation of B restricts to a deformation of $(|B_i|, {}_B\mathscr{O}/\mathscr{I}_i^2)$. This in turn induces a deformation of both $|B_i|$ and the dual line bundle for $\mathscr{I}_i/\mathscr{I}_i^2$. The coordinate transition functions depend holomorphically on the parameter space coordinates t. In computing $c_i(t)$ from the transition functions as in, say, [9, p. 249] we may use the local triviality of the deformation to see that the cocycles representing $c_i(t)$ depend holomorphically on t. Since the cocycles are integer valued they are constant, i.e., independent of t. Any deformation of $|B_i|$ as a differentiable manifold is trivial. Thus the image of $c_i(t)$ in $H^2(|B_i|_0\,; Z)$ is well-defined and depends continuously on t. Hence, $c_i(t)$ is independent of t.

Definition 3.1. Let B and B' be compact one-dimensional analytic spaces which can be expressed locally as $\{x^a y^b = 0\}$ and as $\{x^{a'} y^{b'} = 0\}$ respectively. B and B' are diffeomorphic as nonreduced spaces if there is a diffeomorphism $\phi\colon |B| \to |B'|$ such that ϕ preserves the c_i, if defined, and the (a,b). That is, $|B_i| \cdot |B_i| = \phi(|B_i|) \cdot \phi(|B_i|)$ and if locally $B = \{x^a y^b = 0\}$ with $|B_i| = \{y = 0\}$ and locally $B' = \{x^{a'} y^{b'} = 0\}$ with $|B_i'| = \phi(|B_i|) = \{y = 0\}$, then $b = b'$. If $B_i = |B_i|$, there is no c_i to preserve.

Theorem 3.2. *Let (B,C) and (B',C') be pairs of analytic spaces. C and C' are closed subspaces of B and B' respectively. Let $\psi\colon C \to C'$ be an isomorphism such that $|\psi|\colon |C| \to |C'|$ can be extended to a diffeomorphism $\phi\colon |B| \to |B'|$ such that ϕ is a diffeomorphism of B and B' as nonreduced spaces, as in Definition 3.1. Then there is a finite sequence of pairs of analytic spaces $(B_0,C), (B_1,C), \cdots, (B_{n-1},C), (B_n,C)$ such that there is a deformation of B_i into B_{i+1}, fixing C, $0 \leqq i \leqq n-1$, and such that (B_0,C) is isomorphic to (B,C) and (B_n,C) is isomorphic to (B,C). C and C' have been identified via ψ.*

Proof. Fixing C will cause no real difficulties. Mention of C will be omitted during this proof except for those few steps where a little special care is required. Otherwise, it will be assumed that $C = \varnothing$. The reader may find it helpful to look at the proof of Theorem 3.4 below as he reads this proof.

Let $|B_1|, \cdots, |B_n|$ be the irreducible components of $|B|$. Let $T_1, \cdots, T_n$ be the corresponding Teichmüller spaces. (If $C \neq \varnothing$ and $|B_i| \subset C$, replace T_i by some point in T_i which corresponds to $|B_i|$.) Each T_i is a connected parameter space with projection map $\pi_i\colon \mathscr{V}_i \to T_i$ [5]. The fibers of π_i comprise all Riemann surfaces having the genus g_i of B_i. π_i is complete at each of the points in T_i. We now put the $\mathscr{V}_i$ together to give all possible analytic spaces diffeomorphic to $|B|$. The deformation space will be complete at all points. We first "mark" the singular points appearing in each $|B_i|$. Let m be the number of singular points in $|B_i|$. Start with $m = 1$. Let $\mathscr{W}_1 = \{(v_1, v_2) \in \mathscr{V}_i \times \mathscr{V}_i \mid \pi_i(v_1) = \pi_i(v_2)\}$. Let ω_1 be the projection onto the second factor in the cartesian product. $\omega_1\colon \mathscr{W}_1 \to \mathscr{V}_i$ is a deformation space with Riemann surfaces as fibers. Let $\Delta\colon \mathscr{V}_i \to \mathscr{W}_1$ be the diagonal map. Each fiber F in $\mathscr{W}_1$ now has a marked point, namely $\Delta(\omega_1(F))$. Each Riemann surface of genus g_i with each of its points as the marked point appears as a fiber in $\mathscr{W}_1$. The deformation of $|B_i|$, fixing the marked point, is complete at each point. For $m = 2$, let $\mathscr{W}_2 = \{(v, w) \in \mathscr{V}_1 \times \mathscr{W}_1 \mid \pi_i(v) = \pi_i \circ \omega_1(w)\}$. Let $\omega_2 \colon \mathscr{W}_2 \to \mathscr{W}_1$ be projection onto the second factor. Again, the fibers are Riemann surfaces. Two points are marked as follows. $\Delta(w) = (\omega_1(w), w)$ marks one point and $\Delta_1(w) = (\omega_1(w), \Delta \circ \omega_1(w))$ marks another. To insure that the marked points are distinct, we just let $\mathscr{W}_1^* = \{w \mid w \neq \Delta \circ \omega_1(w)\}$ be the parameter space, instead of $\mathscr{W}_1$. $\mathscr{W}_1^*$ is connected since we have just removed a proper subvariety from the connected manifold $\mathscr{W}_1$. Continuing this construction, we eventually reach m marked points for arbitrary m. Let $\omega_i = \mathscr{W}_i \to P_i$ be this deformation for $|B_i|$. Let $\Delta_1, \cdots, \Delta_{m_i} = P_i \to \mathscr{W}_i$ be the m_i marking maps.

We can locally put the $\mathscr{W}_i$ together as follows. Each ω_i is locally trivial. Hence, we can find neighborhoods $U_{i,j}$ in $\mathscr{W}_i$ of the distinguished points in some fiber F_i such that $U_{i,j} = \{(w, p_i) \mid \ |w| < \varepsilon, |p_i| < \varepsilon\}$ with $w = 0$ giving distinguished points on all fibers, $|p_i| < \varepsilon$. We can also cover the rest of F_i by neighborhoods $S_{i,k}$ where ω_i is similarly trivial, but where there are no distinguished points, and again $|p_i| < \varepsilon$. We put the $\mathscr{W}_i$ together near the F_i as follows. Let $P = \times \{\, |p_i| < \varepsilon\}$, the cartesian product. Let $p = (p_1, \cdots, p_n)$. Instead of $S_{i,k} = \{(w, p_i) \mid \ |w| < \varepsilon,\ |p_i| < \varepsilon$, we have $R_{i,k} = \{(w, p) \mid \ |w| < \varepsilon,\ |p| < \varepsilon\}$. If $U_{i',j'} = \{(w', p_i') \mid \ |w'| < \varepsilon, |p_i'| < \varepsilon\}$ has its distinguished points to be identified with those of $U_{i,j}$, we form $U'_{i,j} = \{(w, w', p) \mid ww' = 0,\ |w| < \varepsilon,\ |w'| < \varepsilon,\ |p| < \varepsilon\}$. The changes of coordinates among the $\{R_{i,k}\}$ and $\{U'_{i,j}\}$, all i, are extensions of those for the $\mathscr{W}_i$, all i. Since each P_i is connected, given two points in $\times P_i$, we can connect them by a path, and using compactness of the path, deform along this path via a finite sequence of analytic deformations. This finishes the proof of Theorem 3.2 when B is reduced.

We can realize the above sequence of deformations as a sequence of deformations of nonreduced spaces as follows. Replace one $U'_{i,j}$ on each $|B_i|$ by $U^0_{i,j} = \{(w, w', p) \mid ww' = 0,\ |w| < 2\varepsilon/3,\ |w'| < \varepsilon,\ |p| < \varepsilon\}$ and $U^1_{i,j} = \{(w, p) \mid \varepsilon/3 < |w| < \varepsilon,\ |p| < \varepsilon\}$. We may assume that $U^0_{i,j}$ does not meet any other coordinate patch on $|B_i|$ except for $U^1_{i,j}$. Look at $\{R_{i,k} \times \mathbf{C}\}$, $\{U'_{i,j} \times \mathbf{C}\}$, $\{U^0_{i,j} \times \mathbf{C}\}$, $\{U^1_{i,j} \times \mathbf{C}\}$ with y a variable for $\mathbf{C}$. If C is to be fixed and some of its components are not reduced, use the change of variables which define C to define the change of coordinates involving y. Otherwise, if $c_i = |B_i| \cdot |B_i|$, let the change of variables from $U^0_{i,j} \times \mathbf{C}$ to $U^1_{i,j} \times \mathbf{C}$ be given by $p = p$, $w = w$, $y_0 = w^{c_i} y_1$. For the other patches, let $y = y$ be the change of variables for y. This creates line bundles over the fibers. $\{y_1 = 1\} = \{y_0 = w^{c_i}\}$ extends to a section of this line bundle over $|B_i|$ which has c_i zeros, counting multiplicities. Hence the line bundle has Chern class c_i above $|B_i|$. The constructed deformations of the (reduced) total space of the line bundle induce a deformation of the nonreduced spaces carried on the 0-section of the line bundle. Thus, given two diffeomorphic pairs (B, C) and (B', C), we have constructed a finite sequence of holomorphic deformations, fixing C, deforming (B_1, C) into (B_2, C), (B_3, C) into $(B_4, C), \cdots, (B_{2v-1}, C)$ into $(B_{2v}, C), \cdots, (B_{2\lambda-1}, C)$ into $(B_{2\lambda}, C)$, such that $(|B|, |C|)$ is isomorphic to $(|B_1|, |C|)$, $(|B'|, |C|)$ is isomorphic to $(|B_{2\lambda}|, |C|)$, $(|B_{2v}|, |C|)$ is isomorphic to $(|B_{2v+1}|, |C|)$, $1 \leqq v \leqq \lambda - 1$, and such that (B, C), (B', C) and all the (B_{2v-1}, C), (B_{2v}, C) are diffeomorphic to each other. This reduces Theorem 3.2 to the case where ϕ is an analytic, rather than a differentiable isomorphism. Because C is not necessarily reduced, ϕ may be an extension of $|\psi|$ without ϕ having an extension which induces ψ. Now we give the needed special argument to reduce Theorem 3.2 to the case where ϕ does have an extension which induces ψ, i.e., to the case where $|B| = |C|$.

Let $|B_1|, \cdots, |B_t|$ be the irreducible components of $|B|$ which are not subsets of $|C|$. We shall write down a connected deformation of B, fixing C, which has as fibers all possible analytic spaces S of the differentiable type of $C \cup |B_1| \cup \cdots \cup |B_t|$ and having $C, |B_1|, \cdots, |B_t|$ as subspaces. Namely, to specify an S it suffices to specify the map ι_i of $C \cap |B_i|$ as a nonreduced subspace of C to $C \cap |B_i|$ as a nonreduced subspace of $|B_i|$, $1 \leqq i \leqq t$. Let (x, y) be local coordinates for C, $\mathscr{I}\!d\, C = y^d$, $(0,0) \in |C| \cap |B_i|$. Let (y_i) be local coordinates for $|B_i|$, $0 \in |C| \cap |B_i|$. Then ι_i is given precisely by maps of the type $y_i = a_1 y + a_2 y^2 + \cdots + a_{d-1} y^{d-1}$ with $a_1 \neq 0$. The set of $(d-1)$-tuples $(a_1, a_2, \cdots, a_{d-1})$, $a_1 \neq 0$, serves as the connected parameter space for the deformation.

Thus, finally, we may assume that $|B| = |C|$. It now becomes possible to apply the results of Grauert [*8*, p. 357] on extending isomorphisms of spaces. Let m and m' be the ideal sheaves for the analytic spaces $B(m)$ and $B(m')$ respectively. Let n and n' be the ideal sheaves for $B(n)$ and $B(n')$ respectively, with $B(n)$ and $B(n')$ diffeomorphic, aside from the c_i, as nonreduced spaces. Suppose that $B(m)$ and $B(m')$ are subspaces of $B(n)$ and $B(n')$ respectively.

Lemma 3.3 (Grauert). Given ϕ: $B(m) \to B(m')$, an isomorphism, there is a well-defined obstruction $\mathrm{cls}[\xi] \in H^1(|B|, \mathscr{A}ut(n:m))$ to extending ϕ to ψ: $B(n) \to B(n')$. $\mathrm{cls}[\xi]$ is the distinguished element $*$ in $H^1(|B|, \mathscr{A}ut(n: m))$ if and only if ψ exists. Moreover, if $\tilde{\phi}$: $B(m) \to B(\tilde{m})$ is a similar isomorphism, $B(\tilde{m}) \subset B(\tilde{n})$, then there is an isomorphism Ψ: $B(\tilde{n}) \to B(n')$ which extends $\phi \circ \tilde{\phi}^{-1}$ if and only if $\mathrm{cls}[\xi] = cls[\tilde{\xi}]$.

Lemma 3.3 is slightly stronger than [8, Satz 3, p. 358], but its proof is the same.

It follows from Lemma 3.3 that in order to construct all possible spaces $B(n)$ with a given subspace $B(m)$, it suffices to construct $B(n')$ and ϕ for each element of $H^1(|B|, \mathcal{A}ut(n: m))$. We do this in a step-wise fashion. Let $\mathcal{I}_i = \mathcal{I}d|B_i|$. Let $m = \Pi \mathcal{I}_i^{r_i}$, $r_i \geqq 1$ and $n = \Pi \mathcal{I}_i^{s_i}$ with $s_i = r_i$, all $i \neq j$, and $s_j = r_j + 1$ for one value j. We may assume that $m \subset \mathcal{I}dC$. We start with $r_j = 1$. c_j is given.

From [8, p. 359], (3.1) below is exact.

$$1 \to \mathcal{A}ut(n, m) \xrightarrow{i} \mathcal{A}ut(n: m) \xrightarrow{\rho} \mathcal{A}n(n, m) \to 1 . \tag{3.1}$$

Recall that $\mathcal{A}n(n, m) \approx \mathcal{O}_s^*$, where $\mathcal{O}_s^*$ is that subsheaf of $\mathcal{O}^*$ on $|B_j|$ whose sections near $B_i \cap B_j$ are given by $\exp(\mathcal{I}_i^{r_i})$. $H^1(|B|, \mathcal{O}_s^*)$ is an abelian group whose elements determine the normal bundle of the embedding of $|B_j(n')|$, taking into account the other B_i. $\rho(\mathrm{cls}[\xi])$ is the difference of the normal bundles for $|B_j(n')|$ and $|B_j(n)|$.

$$0 \to Z' \xrightarrow{2\pi i} \mathcal{O}_s \xrightarrow{\exp} \mathcal{O}_s^* \to 1$$

is an exact sequence of sheaves of abelian groups. $\mathcal{O}_s$ is the ideal sheaf of functions on $|B_j|$ which vanish to order r_i at $|B_i| \cap |B_j|$. Z' is that subsheaf of the constant sheaf of integers which has a 0 stalk at the $|B_i| \cap |B_j|$. $H^2(|B_j|, Z') \approx H^2(|B_j|, Z)$. $\delta\rho(\mathrm{cls}[\xi]) \in H^2(|B_j|, Z)$ is the difference of c_j and c_j', which we are assuming is 0. Hence in order to achieve all possible images of $\rho(\mathrm{cls}[\xi])$, it suffices to look at the image of $H^1(|B_j|, \mathcal{O}_s)$, the Picard variety. As in the proof of Theorem 2.3, since $|B_j|$ is one-dimensional we can cover B_j by a Leray cover $\mathfrak{U} = \{U_0, U_1\}$. For use in the proof of Theorem 3.4 below it will be useful to assume that U_0 is a small disc neighborhood of a point in $|B_j| \cap |B_i|$, $i \neq j$. Then $|U_1| = |B_j| - |\bar{V}_0|$, where $V_0 \subset \bar{V}_0 \subset U_0$ is a smaller disc neighborhood. Let $w_1, \cdots, w_\lambda$ be cocycles in $C^1(N(\mathfrak{U}), \mathcal{O}_s)$ whose cohomology classes are a basis in $H^1(|B_j|, \mathcal{O}_s)$. Let (x, y) be coordinates for U_0 with $(y) = \mathcal{I}d|B_j|$. Let (x_1, y_1) be ambient coordinates on U_1 near U_0. Thus $(y_1) = \mathcal{I}d|B_j|$ and $y = y_1$, $x = x_1$ are the transition functions for B. With $\mathbf{C}^\lambda = (t_1, \cdots, t_\lambda)$ as parameter space, deform B by making $x = x_1$, $y = \exp(t_1 w_1 + \cdots + t_\lambda w_\lambda) y_1$ the transition functions. This deformation, of course, changes the normal bundle but retains the c_j. However, we do achieve all possible bundles with the given Chern class c_j.

Let Θ_j be the tangent sheaf for $|B_j|$. In (3.1), by [8, p. 358], $\mathcal{A}ut(n, m) \approx \Theta_j \otimes m/n$, an invertible sheaf. We modify our current cover $\mathfrak{U}$ of B_j to create additional 1-simplices which, like $U_0 \cap U_1$, are annuli about the point in $|B_j| \cap |B_i|$. Let $U_{-1} \subset\subset V_0$ be a disk neighborhood of the point in $|B_j| \cap |B_i|$. Let $V_{-1} \subset\subset U_{-1}$. Let $\mathfrak{A}^1 = \{U_{-1}, U_0', U_1\}$ with $U_0' = U_0 - \bar{V}_{-1}$. Then $U_0' \cap U_1 = U_0 \cap U_1$, so our previous deformation may be defined using $\mathfrak{A}^1$. $\{U_{-1}, U_0' \cup U_1\}$ is a Leray cover for B_j. Essentially, we want to choose a basis of of $H^1(|B_j|, \mathcal{A}ut(n,m))$ represented by cocycles supported on $U_{-1} \cap U_0'$ and deform as above for $H^1(|B_j|, \mathcal{O}_s)$. But m/n affects $\mathcal{A}ut(n,m)$ and the m/n changes as we vary using $H^1(|B_j|, \mathcal{O}_s)$. For the proof of Theorem 3.4 below, it will be necessary to insure that the deformation using $H^1(|B_j|, \mathcal{A}ut(n,m))$ is complete vis-à-vis $H^1(|B_j|, \mathcal{A}ut(n,m))$ for all m/n. We proceed as follows. Let κ be the canonical sheaf on $|B_j|$. Let $\mathcal{S} = \kappa \otimes [\Theta_j \otimes m/n]^*$. By Serre duality, [20], $H^1(|B_j|, \mathcal{A}ut(n,m)) \approx [\Gamma(|B_j|, \mathcal{S})]^*$. For the cover $\{U_{-1}, U_0' \cup U_1\}$, the dual pairing is defined by multiplying (tensoring) the section of $\mathcal{S}$ with a 1-cocycle and then integrating the resulting 1-form about a homology basis. A basis for $\Gamma(|B_j|, \mathcal{S})$ can always be chosen so that different elements of the basis have different order zeros at the point in $|B_j| \cap |B_i|$. c, the Chern class of the line bundle corresponding to $\mathcal{S}$, depends only on topological information. c gives an upper bound on the possible order of the zero of a section of $\mathcal{S}$ at $|B_j| \cap |B_i|$. Recalling that r_i is the exponent of $\mathcal{I}_i$ in m and n, we see that the 1-cocycles on $U_{-1} \cap U_0'$ given by $x^{r_i-1}\,\partial/\partial x \otimes y, x^{r_i-2}\,\partial/\partial x \otimes y, \cdots, x^{r_i-1-c}\,\partial/\partial x \otimes y$, will yield a matrix of maximal rank when evaluated against a basis of $\Gamma(|B_j|, \mathcal{S})$. Hence these cocycles project onto a basis of $H^1(|B_j|, \Theta_j \otimes m/n)$. To realize the deformations in these "directions," let (x, y) be coordinates in U_0' and (x_1, y_1) be coordinates in U_{-1}. Create a deformation with $\mathbf{C}^{c+1} = (s_1, \cdots, s_{c+1})$ as parameter space via transition rules

$$\left(x_1 = x + \sum_{k=1}^{c+1} s_k x^{r_j - k} y, \quad y_1 = y\right).$$

This mapping is biholomorphic on the ambient space for sufficiently small values of y and thus on the nonreduced subspace carried on $|B_j|$.

Observe that the construction of the previous paragraph could have been used to construct cocycles for $H^1(|B_j|, \mathcal{O}_s)$ which were determined solely by topological information and which map onto (rather than bijectively to) a basis of $H^1(|B_j|, \mathcal{O}_s)$. The deformation with $\mathbf{C}^\lambda$ as parameter space for $H^1(|B_j|, \mathcal{O}_s)$ and the deformation with $\mathbf{C}^{c+1}$ as parameter space for $H^1(|B_j|, \Theta_j \otimes m/n)$ may be combined to construct a deformation with $\mathbf{C}^\lambda \times \mathbf{C}^{c+1}$ as parameter space. Suppose a B' is given which is diffeomorphic to B and $B'(m') \approx B(m)$. Then $B'(n')$ must appear as a fiber in this deformation. Indeed, apply Lemma 3.3, but first look at $\rho(\text{cls}[\xi]) \in H^1(|B_j|, \mathscr{A}n(n,m))$. By construction, there is a $(t_1, \cdots, t_\lambda) \in \mathbf{C}^\lambda$ such that any fiber F above a point in $(t_1, \cdots, t_\lambda) \times \mathbf{C}^{c+1}$ has an obstruction $\text{cls}[\xi']$ such that $\rho(\text{cls}[\xi]) = \rho(\text{cls}[\xi'])$. The obstruction $\text{cls}[\zeta]$ to extending the initial isomorphism between $F(m)$ and $B'(m)$ then has $\rho(\text{cls}[\zeta]) = 0$. From (3.1),

$$H^1(|B_j|, \mathscr{A}ut(n,m)) \xrightarrow{\iota} H^1(|B_j|, \mathscr{A}ut(n:m)) \xrightarrow{\rho} H^1(|B_j|, \mathscr{A}n(n,m))$$

is an exact sequence of sets. Then $\text{cls}[\zeta]$ is in the image of ι. By construction, we may choose $(s_1, \cdots, s_{c+1})$ such that the obstruction to extending the initial isomorphism between $F(m)$ and $G(m)$, where G is the fiber for $(t_1, \cdots, t_\lambda, s_1, \cdots, s_{c+1})$, is $\text{cls}[\zeta]$. Then $G(n)$ and $B'(n')$ are isomorphic.

To go from $r_j = 1$ to $r_j = 2$ and so on, we just mimic the above construction, deforming again on small annuli lying within $U_{-1} \cap U'_0$. In all, this gives a deformation with some $\mathbf{C}^\nu$ as a parameter space such that the fibers include all spaces which are diffeomorphic to B and have the same underlying reduced space. $\mathbf{C}^\nu$ is connected, so this concludes the proof of Theorem 3.2.

The following proof of Theorem 3.4 includes another, much more complicated, proof of Theorem 2.3.

Theorem 3.4. *Let $\pi: \mathscr{B} \to Q$ be a deformation of B, fixing C. If ρ_0 is surjective, then ρ_q is surjective for all q sufficiently near to* 0.

Proof. We first verify the theorem for the deformation obtained by combining, locally, the deformations used in the proof of Theorem 3.3. Let us review the proof of Theorem 3.3 in terms of the image of ρ in $H^1(B, {}_{B,C}\theta)$. Temporarily, let $\theta(B, C)$ denote ${}_{B,C}\theta$.

Recall that $|B_1|, \cdots, |B_i|, \cdots, |B_t|$ are the irreducible components of $|B|$ which are not subsets of $|C|$.

$$(3.2) \qquad 0 \to \theta(B, |B_1| \cup \cdots \cup |B_t| \cup C) \to \theta(B, C) \to \theta(|B_1| \cup \cdots \cup |B_t| \cup C, C) \to 0$$

is an exact sheaf sequence. The proof of Theorem 3.3 up to Lemma 3.3 is really concerned with showing that the image of ρ_0 in $H^1(|B|, |B_1| \cup \cdots \cup |B_t| \cup C, C)$ is everything.

$$(3.3) \quad \theta(|B_1| \cup \cdots \cup |B_i| \cup \cdots \cup |B_t|, C) \approx \bigoplus_i \theta(|B_i|, |B_i| \cap \{|B_1| \cup \cdots \cup |\hat{B}_i| \cup \cdots \cup |B_t| \cup C\}),$$

where $|\hat{B}_i|$ means that $|B_i|$ is omitted from the union and we have generalized our notation by considering vector fields which vanish on the zero-dimensional space $|B_i| \cap \{|B_1| \cup \cdots \cup |\hat{B}_i| \cup \cdots \cup |B_t| \cup C\}$. (3.3) is readily verified by writing down the sheaves in local coordinates, as in (2.10).

There is a sheaf $\mathscr{S}$, supported on the set $|B_i| \cap \{|B_1| \cup \cdots \cup |\hat{B}_i| \cup \cdots \cup |B_t| \cup |C|\}$ such that (3.4) is exact. $\mathscr{S}$ is independent of the point q in the parameter space.

$$(3.4) \qquad 0 \to \theta(|B_i|, |B_i| \cap \{|B_1| \cup \cdots \cup |\hat{B}_i| \cup \cdots \cup |B_t| \cup C\}) \\ \to \theta(|B_i|, \varnothing) \to \mathscr{S} \to 0.$$

The Teichmüller space deformation in the proof of Theorem 3.3 insures that the image of ρ_q is onto $H^1(|B_i|_q, \theta(|B_i|_q, \varnothing))$ for all q. The marking process and the separate argument for $C \neq \varnothing$ two paragraphs before Lemma 3.3 insure that, for all sufficiently small q, the image of ρ_q contains $\delta(\Gamma(|B_i|_q, \mathscr{S}))$. Since (3.4) is exact, the image of ρ_q is onto

$$H^1(|B_i|_q, \theta(|B_i|_q, |B_i|_q \cap \{|B_1|_q \cup \cdots \cup |\hat{B}_i|_q \cup \cdots \cup |B_t|_q \cup C_q\})).$$

Thus from (3.3) and (3.2) it suffices to show that the deformation has the image of ρ_q onto $H^1(B_q, \theta(B_q, |B_1|_q \cup \cdots \cup |B_t|_q \cup C_q)$ for all sufficiently small q. This is accomplished after the proof of Lemma 3.3 as follows. Let $|B_j|^2$ be subspace of B with ideal sheaf $\mathscr{I}_j^2$. Then

$$\begin{aligned} 0 &\to \theta(B, |B_1| \cup \cdots \cup |B_j|^2 \cup \cdots \cup |B_t| \cup C) \\ &\to \theta(B, |B_1| \cup \cdots \cup |B_j| \cup \cdots \cup |B_t| \cup C) \\ &\to \theta(|B_1| \cup \cdots \cup |B_j|^2 \cup \cdots \cup |B_t| \cup C, |B_1| \cup \cdots \cup |B_j| \\ &\qquad\qquad \cup \cdots \cup |B_t| \cup C) \\ &\to 0 \end{aligned}$$

is an exact sheaf sequence. Let m and n be the ideal sheaves for $|B_1| \cup \cdots \cup |B_j| \cup \cdots \cup |B_t| \cup C$ and $|B_1| \cup \cdots \cup |B_j|^2 \cup \cdots \cup |B_t| \cup C$ respectively. Then $\mathscr{A}ut(n, m) \approx \Theta_j \otimes m/n$ and $\mathscr{A}n(n, m) \approx \mathscr{O}_s^*$. Also, with the obvious notation

$$0 \to \Theta_j \otimes m/n \to \theta(n, m) \to \mathscr{O}_s \to 0$$

is an exact sheaf sequence. The construction in the proof of Theorem 3.2 used only topological information and gave ρ_q onto $H^1(B_q, (\mathscr{O}_s)_q)$ and $H^1(B_q, (\Theta_j \otimes m/n)_q)$ for all q sufficiently near to 0.

The rest of the proof proceeds similarly. In going from $|B_j|^2$ to $|B_j|^3$,

$\mathscr{A}n(n, m) = 1$. Let $\mathscr{N}_j$ be the sheaf of germs of sections of the normal bundle to $|B_j|$. Then

$$0 \to \Theta_j \otimes m/n \to \theta(n, m) \to \mathscr{N}_j \otimes m/n \to 0$$

is an exact sequence. $\Theta_j \otimes m/n$ and $\mathscr{N}_j \otimes m/n$ are sheaves of germs of sections of line bundles whose Chern classes are independent of q. Hence again it is possible to construct a deformation such that ρ_q is onto $H^1(B_q, (\theta(n, m))_q)$ for all sufficiently small q. Let ω denote this specially constructed deformation such that ρ_q: $T_q \to H^1(B_q, \theta(B_q, C_q))$ is surjective for all sufficiently small q.

Now look at the π of the theorem's hypothesis. Since π is complete at 0 by Theorem 2.1, ω may be induced from π. The induced map on the parameter spaces, which are manifolds, is a submersion since the induced tangent map is surjective at the origin. Since the maps for ρ to the tangent spaces of the fibers commute with the inducing map, ρ_q is surjective for π, all sufficiently small q.

Recall from the proof of Theorem 2.3 that if $\omega \in \Gamma(B, {}_{B,C}\theta)$ then we can canonically integrate along ω to get an automorphism of B which induces the identity on C. Let $\exp \omega$ be that automorphism obtained by integrating for a time $t = 1$. The action on functions is given by (2.22), with $t = 1$. In $\Gamma(B, {}_{B,C}\theta)$, the Lie bracket operation $[\omega, \lambda](df) = \omega(d(\lambda df)) - \lambda(d(\omega(df)))$ is canonically defined.

Lemma 3.5. Let B be a compact analytic space and C a closed subspace. There are arbitrarily small neighborhoods U and V of the origin in $\Gamma(B, {}_{B,C}\theta)$ such that given $\omega, \lambda \in U$, there is $\sigma \in V$ such that $\exp \omega \circ \exp \lambda = \exp \sigma$.

Proof. The proof is essentially a matter of checking convergence of the Campell-Hausdorff formula [*13*, pp. 111–112].

$\Gamma(B, {}_{B,C}\theta)$ is finite dimensional since B is compact and ${}_{B,C}\theta$ is a coherent sheaf. Thus the topology on $\Gamma(B, {}_{B,C}\theta)$ is that of a finite dimensional Euclidean space. Suppose that ω and λ have suitably small norm. Let σ be given by the Campell-Hausdorff formula. By [*13*, p. 112] this series for σ converges for all sufficiently small ω and λ. We must verify that indeed $\exp \omega \circ \exp \lambda = \exp \sigma$. Consider some ambient polydisc Δ in which B is locally embedded. Let Θ be the tangent sheaf to Δ. Since $\Gamma(B, {}_{C,B}\theta)$ is finite dimensional, there is a linear, bounded map ρ from $\Gamma(B, {}_{C.B}\theta)$ to $\Gamma(\Delta, \Theta)$ such that $\rho(\alpha)$ is an ambient vector field which locally induces α. Thus in inducing σ from an ambient vector field, use $\rho(\sigma)$, rather than applying the Campell-Hausdorff formula to $\rho(\omega)$ and $\rho(\lambda)$ (which need not converge).

Theorem 3.6. *Let B and C be as in Theorem 2.1. Then the group* $\mathrm{Aut}(B:C)$ *of automorphisms of B, fixing C, is a Lie group with at most countably many connected components.* $\Gamma(B, {}_{B,C}\theta)$ *is the Lie algebra for* $\mathrm{Aut}(B:C)$ *with* exp *given by integrating along a vector field for time* $t = 1$. *Let* $\mathrm{Def}(B:C)$ *be the parameter space for some deformation of B, fixing C, which has ρ_0 an isomorphism. Then in some sufficiently small neighborhood U of* 0 *in* $\mathrm{Def}(B:C)$, *only at most countably many fibers* (B_q, C), $q \in U$, *are isomorphic to the pair* (B, C).

Proof. When B is reduced, these results are classical or known from Teichmüller space theory [5]. The results also follow easily from classical information when C is not reduced but B differs from C only in that B has some additional reduced components.

Let $\mathscr{Id}B = \Pi \mathscr{I}_i^{r_i}$, $\mathscr{Id}C = \Pi \mathscr{I}_i^{s_i}$, $s_i \leqq r_i$. Fix $(s_1, \cdots, s_n)$. Our proof will be an induction on the multi-index $(r_1 - s_1, \cdots, r_n - s_n)$. As stated in the previous paragraph, we may start the induction with $r_i = \max(1, s_i)$, all i.

The sheaf $\mathscr{Aut}(n:m)$ of Grauert [8] plays the crucial role.

First consider $r_i - s_i = 1$, $r_j = s_j$, $j \neq i$. We are assuming, by induction, that $r_i \geqq 2$. Let $n = \mathscr{Id}B$, $m = \mathscr{Id}C$. $\Gamma(B, \mathscr{Aut}(n:m)) = \mathrm{Aut}(B:C)$. There are two cases. For $s_i \geqq 2$, ${}_{B,C}\theta \approx {}_{n,m}\Theta \approx \mathscr{Aut}(n:m)$ and the second isomorphism is given by exp. Thus $\exp: \Gamma(B, {}_{B,C}\theta) \approx \mathrm{Aut}(B:C)$. By Lemma 3.3, the elements of $H^1(B, \mathscr{Aut}(n:m))$ correspond in a one-to-one manner to the analytically distinct pairs (B', C), with B and B' diffeomorphic. Construct $\mathrm{Def}(B:C)$ via a coordinate cover as in the proof of Theorem 2.3. Because for $s_i \geqq 2$ and $\omega \in {}_{B,C}\theta_p$, $(\exp\omega)^*(f) = f + \omega(df)$, $\mathrm{Def}(B:C)$ may be identified with $H^1(B, \mathscr{Aut}(n:m))$. Thus all of the fibers above $\mathrm{Def}(B:C)$ are distinct.

For $s_i = 1$, $r_i = 2$, a different argument is needed. Recall from (3.1) and its following paragraph the maps $\rho: H^1(B, \mathscr{Aut}(n:m)) \to H^1(B, \mathscr{An}(n,m))$ and $\delta: H^1(B, \mathscr{O}_s^*) \to H^2(B, Z')$. $\mathscr{An}(n, m) \approx \mathscr{O}_s^*$. The elements of $H^1(B, \mathscr{Aut}(n:m))$ whose image under $\delta \circ \rho$ is trivial correspond in a one-to-one manner to the analytically distinct pairs (B', C) with B and B' diffeomorphic. The rest of the proof will be similar to the general induction step. Some simplifications for this proof have been omitted so that, with minor modification, this proof may be used for the general induction step.

Recall (3.1). Consider diagram (3.5) below. The top and bottom rows are exact.

$$(3.5)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(B, {}_{n\,m}\Theta) & \xrightarrow{\iota'} & \Gamma(B, {}_{B,C}\theta) & \xrightarrow{\rho'} & \Gamma(B, \mathcal{O}_s) & \xrightarrow{\delta'} & \\ & & \Big\downarrow \exp_1 & & \Big\downarrow \exp_2 & & \Big\downarrow \exp_3 & & \\ 0 & \longrightarrow & \Gamma(B, \mathcal{A}ut(n,m)) & \xrightarrow{\iota} & \mathrm{Aut}(B:C) & \xrightarrow{\rho} & \Gamma(B, \mathcal{A}n(n,m)) & \xrightarrow{\delta} & \end{array}$$

$$\begin{array}{ccccccccc} \xrightarrow{\delta'} & H^1(B, {}_{n,m}\Theta) & \xrightarrow{\iota'} & & H^1(B, {}_{B,C}\theta) & \xrightarrow{\rho'} & H^1(B, \mathcal{O}_s) & \longrightarrow & 0 \\ & & & & \mathrm{Def}(B:C) & & & & \\ & & & \overset{\iota''}{\nearrow} & \Big\downarrow \pi & \overset{\rho''}{\searrow} & & & \\ \xrightarrow{\delta} & H^1(B, \mathcal{A}ut(n,m)) & \xrightarrow{\iota} & & H^1(B, \mathcal{A}ut(n:m)) & \xrightarrow{\rho} & H^1(B, \mathcal{A}n(n,m)). & & \end{array}$$

π is the map taking $q \in \mathrm{Def}(B: C)$ to that element of $H^1(B, \mathcal{A}ut(n: m))$ which corresponds to the analytic type of the fiber above q. $\mathrm{Def}(B: C)$ may be constructed as in the proof of Theorem 2.3. Using the same cover, we may also integrate along cocycles for ${}_{n\,m}\Theta$ which give a basis of $H^1(B, {}_{n,m}\Theta)$. This constructs a deformation of B, fixing C. ι'', not canonical, is the map induced by the completeness property, Theorem 2.1, of $\mathrm{Def}(B: C)$. In particular, ι'' is only defined in some suitably small neighborhood of the origin. $\iota = \pi \circ \iota''$ and ι', which is canonical, is the tangent map at the origin for ι''.

$\exp_1$ is an isomorphism. $\exp_3$ is an isomorphism for some neighborhoods of the origins. δ' is the tangent map at the origin for δ. Also, ρ' is the tangent map for $\rho'' = \rho \circ \pi$ at the origin.

$\mathrm{Aut}(B: C)$ receives its topology as a subset of $\mathrm{Aut}(B: \varnothing) = \mathrm{Aut}(B)$. $\mathrm{Aut}(B)$ is a topological group with a neighborhood of the identity given by elements which induce elements lying within some neighborhood of the identity in $\mathrm{Aut}(|B|)$ and whose induced map on the structure sheaves takes local coordinates (over $|B|$) to nearby local coordinates (over the image of $|B|$). Topologize $H^1(B, \mathcal{A}ut(n: m))$ by the strongest topology that makes π continuous for all B' diffeomorphic to B. By Theorem 3.4, for a suitably small neighborhood of $0 \in \mathrm{Def}(B: C)$, this corresponds to taking the quotient topology on $\mathrm{Def}(B: C)$ via the equivalence relation of having isomorphic fibers. π is an open mapping near each point where $\mathrm{Def}(B: C)$ is complete. ρ'' is holomorphic so ρ is continuous. All the other maps in (3.5) are also continuous.

ρ', the tangent map for ρ'', is surjective so that ρ'' is locally a submersion. Let $*$ denote the distinguished elements in $H^1(B, \mathcal{A}ut(n, m))$, $H^1(B, \mathcal{A}ut(n: m))$ and $H^1(B, \mathcal{A}n(n, m))$. $(\rho'')^{-1}(*)$ is a submanifold M in $\mathrm{Def}(B: C)$. We want to show that $\pi^{-1}(*)$ is countable. $\pi^{-1}(*) \subset M$, so that it suffices to show that only countably many elements of M project onto $*$. $\mathrm{im}\ \iota'' \subset M$ by exactness and commutativity. Since the tangent map,

ι', for ι'' has maximal rank, $H^1(B,\mathscr{Aut}(n,m))$ locally fibers as a submersion onto M. The dimension of the fibers is the dimension of $\ker \iota' = \operatorname{im}\delta'$. We must show that only countably many of these fibers project onto $*$.

The proof of the following lemma is straightforward and will be omitted.

Lemma 3.7. Let $1 \to \mathscr{F} \xrightarrow{\iota} \mathscr{G} \xrightarrow{\rho} \mathscr{H} \to 1$ be an exact sequence of sheaves of non-abelian groups over a paracompact Hausdorff space X. Let $\mathscr{F}$ be a sheaf of abelian groups. Then $\mathscr{H}$ operates on $\mathscr{F}$ via conjugation in $\mathscr{G}$, i.e., if $h \in \mathscr{H}_x$, $f \in \mathscr{F}_x$, $h(f) = \iota^{-1}(\rho^{-1}(h) \circ \iota(f) \circ [\rho^{-1}(h)]^{-1})$ is well defined. $\delta\colon \Gamma(X,\mathscr{H}) \to H^1(X,\mathscr{F})$ satisfies $\delta(c \circ d) = \delta(c) + c(\delta(d))$. Hence $h\colon b \to h(b) + \delta h$ is a group action of $\Gamma(X,\mathscr{H})$ on $H^1(X,\mathscr{F})$. Let ι_1: $H^1(X,\mathscr{F}) \to H^1(X,\mathscr{G})$. Then $\iota_1(a) = \iota_1(b)$ if and only if there is an element $h \in \Gamma(X,\mathscr{H})$ such that $a = h(b) + \delta h$.

Return now to the proof of Theorem 3.6. The map ι is given by mapping points of $H^1(B,\mathscr{Aut}(n,m))$ into their orbits under the group action of Lemma 3.7. But given any Lie group action on a manifold, each orbit, while not necessarily closed, is the image of a one-to-one immersion of a manifold S. The orbit of $*$ in $H^1(B,\mathscr{Aut}(n,m))$ is the image of $\Gamma(B,\mathscr{An}(n,m))$ under δ, by exactness. Let G be the isotropy subgroup for $* \in H^1(B,\mathscr{Aut}(n,m))$. $G = \delta^{-1}(*)$ by Lemma 3.7. $S = \Gamma(B,\mathscr{An}(n,m))/G$ and δ induces the immersion having the orbit of $*$ as the image of S. Since δ' is the tangent map for δ at the identity element in $\Gamma(B,\mathscr{An}(n,m)$, $\ker \delta' = \mathfrak{G}$, the Lie algebra for G. Also, the dimension of S equals the dimension of the image of δ'. But the dimension of the image of δ' is also the dimension of the fibers of the map ι''. Points within the same fiber lie in the same orbit since the points have the same image in $H^1(B,\mathscr{Aut}(n\colon m))$. Thus the fibers above $\pi^{-1}(*)$ correspond to disjoint open subsets of S. Since $\Gamma(B,\mathscr{An}(n,m))$ has a countable topology, S has a countable topology and only countably many fibers may appear in the orbit of $*$, as was to be shown. We also know that G has at most countably many components and that (3.6) below is exact and commutative.

$$
(3.6)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(B,{}_{n,m}\Theta) & \xrightarrow{\iota'} & \Gamma(B,{}_{B,C}\theta) & \xrightarrow{\rho'} & \mathfrak{G} & \longrightarrow & 0 \\
 & & \downarrow \exp_1 & & \downarrow \exp_2 & & \downarrow \exp_3 & & \\
0 & \longrightarrow & \Gamma(B,\mathscr{Aut}(n,m)) & \xrightarrow{\iota} & \operatorname{Aut}(B\colon C) & \xrightarrow{\rho} & G & \longrightarrow & 1.
\end{array}
$$

We must show that $\exp_2$ is an isomorphism for suitably small neighborhoods of $0 \in \Gamma(B,{}_{B,C}\theta)$ and of $1 \in \operatorname{Aut}(B\colon C)$ and that $\operatorname{Aut}(B\colon C)$ has countably many connected components. We know these facts for $\exp_1$ and $\exp_3$. Let $a \in V$, V to be a suitably small neighborhood of $1 \in \operatorname{Aut}(B\colon C)$. Then $b = \rho(a) = \exp_3(\beta)$ for some β near $0 \in \mathfrak{G}$. $\beta = \rho'(\gamma)$ for some γ near 0 since ρ' is a linear map. Let $c = \exp_2(\gamma)$. $\rho(ac^{-1}) = 1$. $ac^{-1} = \iota(d)$ and d will be near 1. $d = \exp_1(\delta)$. Then $\exp_2(\iota'(\delta)) = ac^{-1} = a[\exp_2(\gamma)]^{-1}$. By Lemma 3.5, there is an α near $0 \in \Gamma(B, {}_{B,C}\theta)$ such that $a = \exp_2(\alpha)$. Thus $\exp_2$ is onto some neighborhood of the identity and $\mathrm{Aut}(B\colon C)$ is a finite-dimensional Lie group. Exactness in the bottom row gives that $\mathrm{Aut}(B\colon C)$ has countably many components.

Now for the general induction step. Let $n = \mathscr{I}\!\mathscr{d} B = \Pi \mathscr{I}_i^{r_i}$, $n' = \mathscr{I}\!\mathscr{d} B' = \Pi \mathscr{I}_i^{r'_i}$, $m = \mathscr{I}\!\mathscr{d} C = \Pi \mathscr{I}_i^{s_i}$ with $s_i \leqq r'_i \leqq r_i$ and $r_i = r'_i$ for all but one value, j, of i and $r_j = r'_j + 1$. By induction, we may assume the theorem proved for the pairs (B', C) and (B, B') and that $|B| = |B'|$.

Label the $|B_i|$ by their indices i, $1 \leqq i \leqq n$. Let $\mathrm{Aut}_l(B\colon C)$ be that subset of $\mathrm{Aut}(B\colon C)$ which preserves the labeling. $\mathrm{Aut}_l(B\colon C)$ is an open and closed normal subgroup of $\mathrm{Aut}(B\colon C)$. $\mathrm{Aut}_l(B\colon C)/\mathrm{Aut}(B\colon C)$ can be thought of as a subgroup of the (finite) group of permutations on n letters. Let $S(B\colon C)$ denote the set of analytically distinct pairs (B'', C) with B'' diffeomorphic to B as in Definition 3.1 and Theorem 3.2. Let $S_l(B\colon C)$ denote the set of analytically distinct pairs (B'', C) with B'' labeled and B'' diffeomorphic to B. That is, (B'', C) and (B''', C) are the same element of $S_l(B\colon C)$ if there is an isomorphism $\Phi\colon B'' \to B'''$ such that $\phi = |\Phi|\colon |B''_i| \to |B'''_i|$ and Φ extends the identity map on C. By disregarding the labeling, we get a canonical map $\omega\colon S_l(B\colon C) \to S(B\colon C)$. $\omega^{-1}(a)$ has at most $n!$ elements. ω is one-to-one if $|B| = |C|$.

As before, for $r'_j \geqq 2$, $H^1(B, \mathscr{A}\!ut(n\colon n')) \approx S(B\colon B')$. For $r'_j = 1$, $S(B\colon B')$ is isomorphic to that subset of $H^1(B, \mathscr{A}\!ut(n\colon n'))$ whose elements have trivial image under the $\delta \circ \rho$ of (3.1) and its following paragraph.

In (3.7) below, the upper row is an exact sequence of abelian groups. The lower row is an exact sequence of pointed sets. The given structure and labeling on (B, B', C) determine the distinguished elements. Exactness will be verified below.

(3.7)
$$\begin{array}{ccccccccc}
0 \to & \Gamma(B, {}_{B,B'}\theta) & \xrightarrow{\iota'} & \Gamma(B, {}_{B,C}\theta) & \xrightarrow{\rho'} & \Gamma(B, {}_{B',C}\theta) & \xrightarrow{\delta'} \\
 & \downarrow \exp_1 & & \downarrow \exp_2 & & \downarrow \exp_3 & \overset{\delta''}{\nearrow} \\
0 \to & \mathrm{Aut}(B\colon B') & \xrightarrow{\iota} & \mathrm{Aut}_l(B\colon C) & \xrightarrow{\rho} & \mathrm{Aut}_l(B'\colon C) & \xrightarrow{\delta}
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{\delta'} & H^1(B, {}_{B,B'}\theta) & \xrightarrow{\iota'} & H^1(B, {}_{B,C}\theta) & \xrightarrow{\rho'} & H^1(B, {}_{B',C}\theta) & \to 0 \\
 & \mathrm{Def}(B\colon B') & \xrightarrow{\iota''} & \mathrm{Def}(B\colon C) & \xrightarrow{\rho''} & \mathrm{Def}(B'\colon C) & \\
\overset{\delta''}{\nearrow} & \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 & \\
\xrightarrow{\delta} & S(B\colon B') & \xrightarrow{\iota} & S_l(B\colon C) & \xrightarrow{\rho} & S_l(B'\colon C). &
\end{array}$$

ι and ρ are defined in the obvious way. Exactness at $\mathrm{Aut}(B: B')$, $\mathrm{Aut}_l(B: C)$ and $S_l(B: C)$ is immediate. For $\phi \in \mathrm{Aut}_l(B': C)$, $\delta(\phi)$ is the obstruction of Lemma 3.3 to extending ϕ to an automorphism of B. Thus we have exactness at $\mathrm{Aut}_l(B': C)$. More explicitly, δ may be defined as follows. ϕ extends to local isomorphisms $\phi_i: U_i \to V_i$, where U_i and V_i are open subsets of B. $\phi_j^{-1} \circ \phi_i$ is an automorphism of $U_i \cap U_j$ which is the identity on B'. $\{\phi_j^{-1} \circ \phi_i\}$ is a set of coordinate changes for $\delta(\phi)$. $\{\phi_i: U_i \to V_i\}$ are compatible with the coordinate changes for $\delta(\phi)$ and are the identity on C. Hence $\{\phi_i\}$ establishes an isomorphism between $\iota \circ \delta(\phi)$ and $*$, the given structure on (B, C). Hence $\iota \circ \delta = *$. Let $\psi = (\tilde{B}, B')$ and suppose that $\iota(\psi) = *$. Then there is an isomorphism $\phi: \tilde{B} \to B$ which is the identity on C. Since $B' \subset \tilde{B}$, ϕ induces an isomorphism $\tilde{\phi}: B' \to \phi(B')$. In fact, $\tilde{\phi}(B') = B'$ since ϕ maps each $|B_i'|$ to itself by our definition of $S_l(B: C)$. $\delta(\tilde{\phi}^{-1}) = \psi$ and we have exactness at $S(B: B')$.

π_1, π_2, and π_3 are like the π of (3.5). ι'' and ρ'' are the (noncanonical) induced maps of Theorem 2.1. ι' and ρ' are the tangent maps for ι'' and ρ'' at the distinguished points. δ'' is (noncanonically) defined in some neighborhood of 1 as follows: $a \in \mathrm{Aut}_l(B': C)$. $a = \exp_3(\alpha)$. α may be locally extended to vector fields $\alpha_i \in \Gamma(U_i, {}_{B,C}\theta)$. The map $\alpha \to \alpha_i$ is chosen to be linear. $\exp(-\alpha_j) \circ \exp(\alpha_i)$ provides changes of coordinates which are the identity on B'. Thus $\{\alpha\}$ provides a parameter space for a deformation of B, fixing B'. δ'' is the induced map of Theorem 2.1. δ' is the tangent map for δ'' at 1 (compare with (3.16) below). Also $\pi_1 \circ \delta'' = \delta$. (3.7) is a commutative diagram.

By Theorem 3.4, we may assume that $\mathrm{Def}(B: B')$, $\mathrm{Def}(B: C)$ and $\mathrm{Def}(B': C)$ are complete at each of their points. ω_2: $S_l(B: C) \to S(B: C)$ and $\omega_3 : S_l(B': C) \to S(B': C)$ forget about the labeling. It suffices to prove that $(\omega_2 \circ \pi_2)^{-1}(*)$ is countable. ρ' is onto, so we may assume that ρ'' is a submersion. By the induction hypothesis, $(\omega_3 \circ \pi_3)^{-1}(*)$ is countable. Thus it suffices to prove that for $s \in (\omega_3 \circ \pi_3)^{-1}(*)$, only countably many points in $(\rho'')^{-1}(s)$ are isomorphic to $* = (B, C)$. But if $\rho''(t) = s$ and t is isomorphic to (B, C), we may just change the labeling and regard t as the distinguished point in $S_l(B: C)$ and s as the distinguished point in $S_l(B': C)$. $\mathrm{Def}(B: C)$ and $\mathrm{Def}(B': C)$ are effectively parametrized at t and s respectively since they are complete there and have the smallest possible dimension. Thus we may keep π_2 and π_3, change base points and change the rest of (3.7), and always take $s = 0$. It thus suffices to prove only that countably many points in $(\rho'')^{-1}(0)$ are isomorphic (without labeling) to $(B: C)$. $\omega_2^{-1}(*)$ is finite. Hence, by another change of base points, it suffices to prove only that countably many points in $(\rho'')^{-1}(0)$ are isomorphic (with labeling) to $(B: C)$.

We may take $\mathrm{Def}(B: B')$ to be connected. Then $\rho'' \circ \iota''(\mathrm{Def}(B: B')) \subset \pi_3^{-1}(*)$ is connected. But $\pi_3^{-1}(*)$ is countable by the induction hypothesis. Hence $\rho'' \circ \iota''$ $(\mathrm{Def}(B: B')) = 0$. That is, $\mathrm{Def}(B: B')$ is mapped by ι'' into the fiber $(\rho'')^{-1}(0) = M$ of the submersion ρ''. By exactness of the top row, ι'' is a submersion onto M.

The needed analogue of Lemma 3.7 is given by the following group action of $\mathrm{Aut}_l(B': C)$ on $S(B: B')$. Let $\psi = \mathrm{cls}[\xi] \in S(B: B') \subset H^1(B, \mathcal{Aut}(n: n'))$. Then ψ is the obstruction to extending the identity isomorphism $\xi: B' \to B'$ to an isomorphism between B and B''; $\psi = (B'', B')$. If $a \in \mathrm{Aut}_l(B': C)$, $a(\psi)$ is the obstruction to extending the isomorphism $a \circ \xi$ to an isomorphism between B and B''. Then $\delta a = a(1)$ and $\iota(\psi) = \iota(\psi')$ if and only if there is an $a \in \mathrm{Aut}_l(B': C)$ such that $\psi' = a(\psi)$.

For $r_j' \geqq 2$, π_1 is an isomorphism and the rest of the proof is just like the proof using (3.5).

For $r_j' = 1$, π_1 is not necessarily an isomorphism. $S(B: B')$ in fact need not even be a manifold so special care is needed. We do know that π_1 is an open mapping. $|B|$ has at least two irreducible components because we are at the second step at least in the induction process and $r_j' = 1$. Then in (3.5), $\Gamma(B, \mathcal{A}n(n, n')) \approx \Gamma(B, \mathcal{O}_s^*) = 1$. There is no group action and ι is an injection. $\mathrm{Def}(B: B')$ may be constructed more explicitly as follows. $\mathfrak{U} = \{U_0, U_1\}$ is a Leray cover of B_j with $|U_0|$ a disc and $|U_1| = |B_j| - \bar{V}_0$, $V_0 \subset\subset |U_0|$. Let (x, y) be ambient local coordinates for U_0 with $|B_j| = \{y = 0\}$. Let $g_1(x), \cdots, g_v(x)$ be cocycles in $C^1(N(\mathfrak{U}), \mathcal{O}_s)$ which project onto a basis of $H^1(B, \mathcal{O}_s)$. Let $yf_1(x)\partial/\partial x, \cdots, yf_u(x)\partial/\partial x$ be cocycles in $C^1(N(\mathfrak{U}), {}_{n,n'}\Theta)$ which project onto a basis of $H^1(B, {}_{n,n'}\Theta)$. Let $(yf(x)\partial/\partial x, yg(x)\partial/\partial y)$ denote $(yf_1(x)\partial/\partial x, \cdots, yf_u(x)\partial/\partial x,\ yg_1(x)\partial/\partial y, \cdots, yg_v(x)\partial/\partial y)$. Then $(yf(x)\partial/\partial x, yg(x)\partial/\partial y)$ projects onto a basis of $H^1(B, {}_{B,B'}\theta)$. Let $(t, \tau) = (t_1, \cdots, t_u, \tau_1, \cdots, \tau_v)$ be the parameter space for $\mathrm{Def}(B: B')$. From (2.22), the change of coordinates for the fiber above (t, τ) is given by

$$x \to x + tyf(x)\left[1 + \frac{\tau g(x)}{2!} + \frac{(\tau g(x))^2}{3!} + \cdots\right],\ y \to \exp(\tau g(x))y.$$

Bound $|\tau|$ so that for fixed t, ρ'' is an injection. $\mathrm{Def}(B: B')$ is complete near $(0, 0)$. Suppose that $\mathrm{Def}(B: B')$ is also effectively parametrized at $q = (t, \tau)$. Think of τ as fixed.

$$yf(x)\left[1 + \frac{\tau g(x)}{2!} + \frac{(\tau g(x))^2}{3!} + \cdots\right]\frac{\partial}{\partial x}$$

represents a u-tuple of cohomology classes in $H^1(B_{q,n,n'}\Theta)$. Since Def$(B: B')$ is complete and effectively parametrized at q,

$$yf(x)\left[1+\frac{\tau g(x)}{2!}+\frac{(\tau g(x))^2}{3!}+\cdots\right]\frac{\partial}{\partial x}$$

in fact represents a basis of $H^1(B_{q,n,n'}\Theta) \approx H^1(B_q, \mathscr{Aut}(n, n'))$. But $H^1(B_q, \mathscr{Aut}(n, n'))$, like $H^1(B, \mathscr{Aut}(n, n'))$, is injected into $S(B: B') \subset H^1(B, \mathscr{Aut}(n: n'))$. Thus, if $\pi(q) = \pi(q')$, then $q = q'$. We have thus shown that π is injective and hence a homeomorphism on that subset of Def$(B: B')$ where Def$(B: B')$ is effectively parametrized. Also, since the fibers of ρ'' are closed and $(\rho'')^{-1}(1)$ is embedded in $H^1(B, \mathscr{Aut}(n: n'))$ via π and since π is an open map, each point in $S(B: B')$ is closed.

Now return to (3.7) and the Lie group action of $\mathrm{Aut}_l(B': C)$ on $S(B: B')$. We need to show that only countably many of the fibers of ι'' are mapped by π_1 to the orbit of $*$. Since each point in $S(B: B')$ is closed, $\delta^{-1}(*) = G \subset \mathrm{Aut}_l(B': C)$, the isotropy subgroup of $*$, is closed. $S = \mathrm{Aut}_l(B': C)/G$ is then a manifold which is injected into $S(B: B')$. Let Orb$(*)$ denote the orbit of $*$. On U, a suitably small neighborhood of $1 \in \mathrm{Aut}_l(B': C)$, δ factors through δ'', which has δ' as its tangent map. $\pi_1 \circ \delta''(s) \in$ Orb$(*)$. All points in Orb$(*)$ are isomorphic so Def$(B: B')$ is effectively parametrized at points of $\pi_1^{-1}(\mathrm{Orb}(*))$. As shown above, $\pi_1 : \pi_1^{-1}(\mathrm{Orb}(*)) \to \mathrm{Orb}\,(*)$ is a homeomorphism onto its image. In particular, $\delta'' = \pi_1^{-1} \circ \delta$ is uniquely determined. $\delta''(G) = 0$. Let $\mathfrak{G}$ be the Lie algebra of G. Then $\mathfrak{G} \subset \ker \delta'$. If $t \in \ker \delta'$, $t = \rho'(w)$ by exactness. $\delta \circ \exp_3(t) = \delta \circ \rho \circ \exp_2(w) = *$. Thus $G \supset T$, the one-parameter subgroup generated by t. Then $t \in \mathfrak{G}$. $\mathfrak{G} = \ker \delta'$.

Hence δ'' immerses U/G, an open subset of S, into $\pi_1^{-1}(\mathrm{Orb}(*))$. $\dim S = \dim\ \mathrm{im}\,\delta'$. Changing the basepoint in $S(B': B)$, but keeping Def$(B: B')$, we see that $\pi_1^{-1}(\mathrm{Orb}(*))$ is the image of a one-to-one immersion of an open subset of S. $(\pi_2 \circ \iota'')^{-1}(*) \subset \pi_1^{-1}(\mathrm{Orb}(*))$ by exactness of the bottom row of (3.7). But the fibers of ι'' have the dimension of $\ker \iota' = \mathrm{im}\,\delta'$, which is also the dimension of S. Hence the fibers of ι'' above $\pi_2^{-1}(*)$ correspond to disjoint open subsets of S. Since S has a countable topology, there are only countably many fibers.

The proof that $\mathrm{Aut}_l(B: C)$ has countably many components is the same as for (3.6).

Proposition 3.8. Let B be a compact analytic space which can locally be expressed via $\mathscr{Id}B = (x^a y^b)$. Then there is a two-dimensional manifold M in which B can be embedded as a subspace.

Proof. Let $\mathfrak{U}$ be the cover of B constructed in the proof of Theorem 2.3. $|U_0|$ is a one-dimensional Stein manifold. All obstructions to extending maps in [8, p. 357] vanish over U_0. Thus U_0 can be realized as a subset of $|U_0| \times \mathbf{C}$. The transition functions on B for going from $U_0 \subset |U_0| \times \mathbf{C}$ to U_i are of the form $y_i = yf(x,y)$, $x_i = g(x,y)$ with $f(x,0) \neq 0$ and $\partial g/\partial x(x,0) \neq 0$. Then these coordinate changes on B are locally isomorphisms between some ambient open sets in $|U_0| \times \mathbf{C}$ and $\Delta_i \supset U_i$. Since the change of coordinates is one-to-one on $|B|$, it is one-to-one on some ambient open sets. Then the ambient neighborhoods patch together to form a manifold M. There are no compatibility conditions to check since no three distinct U_i intersect.

Suppose that Γ is a given weighted graph with specified genera for the vertices. We can construct as follows a two-dimensional complex manifold M with a reduced compact analytic subset A such that Γ is the weighted graph for A. Let $A_1, \cdots, A_n$ be Riemann surfaces with the genera of the vertices. Embed each A_i in a line bundle L_i whose Chern class is given by the weight of the vertex A_i in Γ. If x is a local coordinate for A_i and y is a fiber coordinate for L, L is locally $\{(x,y) |\ |x| < 1, |y| < 1\}$ and $A_i = \{y = 0\}$. To achieve the appropriate graph, we plumb the L_i together in the obvious manner, i.e., if L_i is locally (x_i, y_i) and L_j is locally (x_j, y_j), we let $y_j = x_i$, $x_j = y_i$ be the change of coordinates at a point of $A_i \cap A_j$. Locally A is $\{x_i y_i = 0\} = \{x_j y_j = 0\}$. This construction is in general not canonical. Call any such embedding $A \subset M$ an embedding obtained by plumbing from Γ.

Theorem 3.9. *Let $B = (B, \varnothing)$ be as in Theorem 2.1. Let $A \subset M$ be obtained by plumbing from the weighted graph for B in the particular manner to be described below. Suppose that all the weights are negative. Let P be that nonreduced subspace of M such that $|P| = A$ and P is diffeomorphic to B. Then all B' diffeomorphic to B are analytically equivalent to B if and only if $H^1(P, {}_p\theta) = 0$. There is an algorithm for determining whether $H^1(P, {}_p\theta) = 0$.*

Proof. We will exhibit the algorithm in the course of the proof.

For the moment, P may be obtained from any plumbing construction.

If $H^1(P, {}_P\theta) \neq 0$, then Theorem 3.6 shows that there is a B'' diffeomorphic to P (and hence to B) but such that B'' is not analytically equivalent to P.

We must show that $H^1(P, {}_P\theta) = 0$ implies that P and B' are analytically equivalent. If $H^1(P, {}_P\theta) = 0$, then necessarily, (3.2) and (3.3), each A_i has genus 0 and may contain at most three points in $\cup A_j$, $j \neq i$. These properties must also hold for B', so $|B'| \approx |P|$. We can now omit the prime in B'. Let V_i be a small open neighborhood of B_i, the nonreduced component of B carried on $|B_i|$. Then $V_i \cap V_j \cap V_k = \varnothing$, i, j, k distinct. By a straightforward generalization of the Mayer-Vietoris sequence [*1*, p. 236]

$$(3.8) \qquad 0 \to \Gamma(B, {}_B\theta) \xrightarrow{\iota} \bigoplus_i \Gamma(V_i, {}_B\theta) \xrightarrow{\rho} \bigoplus_{i \neq j} \Gamma(V_i \cap V_j, {}_B\theta)$$

$$\xrightarrow{\delta} H^1(B, {}_B\theta) \xrightarrow{\iota} \bigoplus_i H^1(V_i, {}_B\theta)$$

is an exact sequence. ι is induced by the restriction maps and $\rho(\oplus a_i)(V_i \cap V_j) = a_i - a_j$.

Let W_i be a V_i for P, i.e., W_i is an open subset of P which is a small neighborhood of the image of $|B_i|$ under the isomorphism $|B| \approx |P|$. We now show that for suitably chosen V_i and W_i, W_i and V_i are isomorphic. Consider first the case $A_i \cdot A_i = -1$. Then ambient neighborhoods of $|B_i|$ in the manifold of Proposition 3.8 and of $|P_i|$ in the manifold of the plumbing construction can be achieved as a result of a quadratic transformation [*8*, pp. 364–365]. Upon blowing down $|B_i|$ and $|P_i|$, the other irreducible components meeting $|B_i|$ and $|P_i|$ become transversely intersecting submanifolds. There are at most three other irreducible components. Any two plane curve singularities arising from three transversely intersecting manifolds are isomorphic. Thus suitable neighborhoods of the blown down singularities are isomorphic via an ambient isomorphism. Hence V_i and W_i can be made isomorphic.

For $A_i \cdot A_i \leqq -2$, we wish to apply the obstruction theory of Grauert [*8*, p. 357] to extend the isomorphisms of the reduced spaces. Let $m = \mathscr{I}_i^s \mathscr{I}d|V_i|$, $n = \mathscr{I}_i^{s+1} \mathscr{I}d|V_i|$. Then $H^1(V_i, \mathscr{A}ut(n:m)) = 0$ if $s \geqq 1$. Also, since $|V_i| \cap |B_j|$, $i \neq j$, is a Stein manifold, the obstructions to extending the isomorphism to $(\mathscr{I}d|V_i|)^{s+1}$ also vanish. Thus it suffices to establish an isomorphism between the spaces $V_i(\mathscr{I}_i \mathscr{I}d|V_i|) = (|V_i|, {}_{V_i}\mathcal{O}/\mathscr{I}_i \mathscr{I}d|V_i|)$ and $W_i(\mathscr{I}_i \mathscr{I}d|W_i|)$. Since we are only interested in V_i and W_i as some neighborhoods of $|B_i|$ and $|P_i|$, $V_i(\mathscr{I}_i \mathscr{I}d|V_i|)$, for our purposes, differs only from $|B_i|^2 = (|B_i|, {}_{B_i}\mathcal{O}/\mathscr{I}_i^2)$ in specifying the intersection points of $|B_i| \cap |B_j|$, $i \neq j$, and the tangent direction in which $|B_j|$ meets $|B_i|$. In $W_i(\mathscr{I}_i \mathscr{I}d|W_i|)$, the $A_j = |P_j|$ are fibers and the local defining equation for a singular point is $xy = 0$, $(y) = \mathscr{I}d A_i$, $(x) = \mathscr{I}d A_j$. Since A_i is the Riemann sphere and $A_i \cdot A_i \leqq 0$, the values of a section s of $T \otimes N^*$, where T is the tangent bundle of A_i and N is the normal bundle of the embedding of A_i, can be specified at any three points. In particular, we can specify values at each point of $\{A_i \cap A_j\}$. $\mathscr{A}ut(\mathscr{I}_i^2, \mathscr{I}_i)$ is isomorphic to the sheaf of germs of sections of $T \otimes N^*$. If (x, y) are local ambient coordinates for A_i, then the automorphism specified by s is of the form $x \to x + ys(x)$, $y \to y$. Thus by a suitable choice of s, the fiber $\{x = 0\} = A_j$ may be mapped to a submanifold with any tangent direction different from that of $A_i = \{y = 0\}$. Thus W_i and V_i can be chosen to be isomorphic.

Let us now show that $H^1(W_i, {}_P\theta) = 0$. Let ${}_P\Omega$ be the sheaf of germs of holomorphic 1-forms and let ${}_P\mathcal{O}$ be the structure sheaf for P. ${}_P\theta = \mathcal{H}om({}_P\Omega, {}_P\mathcal{O})$. Let $\theta_\nu = \mathcal{H}om({}_P\Omega, \mathcal{I}_i^\nu)$, $\nu \geqq 0$. ${}_P\theta/\theta_1$ may be identified with the sheaf $\mathcal{T}_s$ of germs of sections of the tangent bundle to $|P_i| = A_i$ which vanish at the points of $\{A_i \cap A_j\}$, $i \neq j$. Hence $H^1(W_i, {}_P\theta/\theta_1) = 0$ and it suffices to show that $H^1(W_i, \theta_1) = 0$. Let $\mathcal{N}$ be the sheaf of germs of sections of N, the normal bundle of the embedding of A_i. There is a canonical exact sequence, $\nu \geqq 1$,

$$0 \to \mathcal{T}_s \otimes \mathcal{I}_i^\nu/\mathcal{I}_i^{\nu+1} \to \theta_\nu/\theta_{\nu+1} \to \mathcal{N} \otimes \mathcal{I}_i^\nu/\mathcal{I}_i^{\nu+1} \to 0. \tag{3.9}$$

From Chern class considerations over A_i, $H^1(W_i, \mathcal{T}_s \otimes \mathcal{I}_i^\nu/\mathcal{I}_i^{\nu+1}) = H^1(W_i, \mathcal{N} \otimes \mathcal{I}_i^\nu/\mathcal{I}_i^{\nu+1}) = 0$. Thus it suffices to prove that $H^1(W_i, \theta_\nu) = 0$ for some sufficiently large ν. For ν sufficiently large, θ_ν is supported on $\cup(W_i \cap A_j), j \neq i$, whose underlying space is Stein. Hence [7, Satz 3, p. 17], $H^1(W_i, \theta_\nu) = 0$. Hence $H^1(W_i, {}_P\theta) = 0$.

Thus in (3.8), the term $\oplus_i H^1(V_i, {}_B\theta)$ vanishes. In (3.8), let the V_i decrease through a fundamental sequence of neighborhoods of the B_i and take the direct limit. Taking direct limits preserves exactness, so

$$H^1(B, {}_B\theta) \approx \bigoplus_{i \neq j} \Gamma(B_i \cap B_j, {}_B\theta)/\rho\,(\bigoplus_i \Gamma(B_i, {}_B\theta)). \tag{3.10}$$

Temporarily choose local coordinates at a point of $|B_i| \cap |B_j|$ so that $\mathcal{I}dB = (x^a y^b)$, $|B_i| = \{y = 0\}$. Let $\mathcal{O}(x, y)$ be the ring of convergent power series in the two variables x and y. Modulo $x^a y^b$, $\Gamma(B_i \cap B_j, {}_B\theta)$ may be represented near $(0, 0)$ as $x\mathcal{O}(x, y)\,\partial/\partial x \oplus y\mathcal{O}(x, y)\,\partial/\partial y$. $\Gamma(B_i, {}_B\theta)$ includes all vector fields v of the following form. $v \in y^b x\,\mathcal{O}(x, y)\,\partial/\partial x \oplus y^b\mathcal{O}(x, y)\,\partial/\partial y$ for (x, y) near $(0, 0) \in |B_i| \cap |B_j|$ and $v = 0$ near other points in $|B_i|$. Thus in computing $H^1(B, {}_B\theta)$ from (3.10) it suffices to consider only those elements in $\oplus_{i \neq j} \Gamma(B_i \cap B_j, {}_B\theta)$ of the form

$$\Sigma \alpha_{st} x^s y^t \frac{\partial}{\partial x} + \Sigma \beta_{uv} x^u y^v \frac{\partial}{\partial y}, \tag{3.11}$$

$$1 \leqq s \leqq a - 1, 0 \leqq t \leqq b - 1, 0 \leqq u \leqq a - 1, 1 \leqq v \leqq b - 1;$$

$$\alpha_{s,t}, \beta_{u,v} \in \mathbf{C}.$$

The elements of (3.11) form a finite dimensional vector space.

For the particular P we are about to construct, it is a simple matter to list the elements of $\Gamma(P_{i}, {}_{P}\theta)$ whose images in (3.11) are non-zero. Namely, think of $|P_i|$ as $\mathbf{C} \cup \{\infty\}$. Put the (at most) three points of $\{|P_i| \cap |P_j|\}$, $j \neq i$, successively at 0, ∞, and 1. If $|P_i| \cdot |P_i| = -\nu$, the change of coordinates on W_i, with (x, y) coordinates near $0 \in |P_i|$ and (x_1, y_1) coordinates near $\infty \in |P_i|$ is chosen to be $x = 1/x_1$, $y = x_1^{\nu} y_1$. The plumbing construction coordinates near 1 are to be $(x-1)$ and y. There are three cases, corresponding to whether $\{|P_i| \cap |P_j|\}$ has one, two, or three points in it. (The case of no points corresponds to one vertex in the graph and all obstructions to extending maps vanish.) With the only point of $\{|P_i| \cap |P_j|\}$ at 0, the relevant basis of sections of $\Gamma(P_{i}, {}_{P}\theta)$ for non-zero image in (3.11) is the appropriate finite subset of the following list. The list is derived using the θ_{ν} from (3.9).

$$
\begin{aligned}
(3.12)\qquad & x\frac{\partial}{\partial x},\ x^2\frac{\partial}{\partial x} - \nu xy\frac{\partial}{\partial y} \\
& y\frac{\partial}{\partial y} \\
& yx\frac{\partial}{\partial x}, \cdots, yx^{\nu+1}\frac{\partial}{\partial x}, yx^{\nu+2}\frac{\partial}{\partial x} - \nu y^2 x^{\nu+1}\frac{\partial}{\partial y} \\
& y^2\frac{\partial}{\partial y}, y^2 x\frac{\partial}{\partial y}, \cdots, y^2 x^{\nu}\frac{\partial}{\partial y} \\
& y^2 x\frac{\partial}{\partial x}, \cdots, y^2 x^{2\nu+1}\frac{\partial}{\partial x}, y^2 x^{2\nu+2}\frac{\partial}{\partial x} - \nu y^3 x^{2\nu+1}\frac{\partial}{\partial y} \\
& \cdots
\end{aligned}
$$

For two points in $\{|P_i| \cap |P_j|\}$, we have the following sublist of (3.12). We also give the vector fields in (x_1, y_1) coordinates. The last element of the odd numbered rows is lost because of the vanishing condition at $x_1 = 0$.

$$
\begin{aligned}
(3.13)\qquad & x\frac{\partial}{\partial x} = -x_1\frac{\partial}{\partial x_1} + \nu y_1\frac{\partial}{\partial y_1} \\
& y\frac{\partial}{\partial y} = y_1\frac{\partial}{\partial y_1} \\
& yx\frac{\partial}{\partial x} = -y_1 x_1^{\nu+1}\frac{\partial}{\partial x_1} + \nu y_1^2 x_1^{\nu}\frac{\partial}{\partial y_1}, \cdots \\
& \qquad \cdots, yx^{\nu+1}\frac{\partial}{\partial x} = -y_1 x_1\frac{\partial}{\partial x_1} + \nu y_1^2\frac{\partial}{\partial y_1} \\
& \cdots
\end{aligned}
$$

Finally, for three points, the odd numbered rows again have one fewer element than in (3.13). In particular, there is no first row.

$$(3.14)\qquad \begin{aligned} y\frac{\partial}{\partial y} &= y_1\frac{\partial}{\partial y_1} \\ yx(x-1)\frac{\partial}{\partial x} &= -y_1x_1^{\nu}(1-x_1)\frac{\partial}{\partial x_1} + \nu y_1^2x_1^{\nu-1}(1-x_1)\frac{\partial}{\partial y_1}, \cdots, \\ yx^{\nu}(x-1)\frac{\partial}{\partial x} &= -y_1x_1(1-x_1)\frac{\partial}{\partial y_1} + \nu y_1^2(1-x_1)\frac{\partial}{\partial y_1} \\ &\cdots \end{aligned}$$

$H^1(P, {}_P\theta)$ being 0 is equivalent to the condition that the elements of (3.12)–(3.14) map onto a spanning set for the direct sum over all singular points of $|P|$ of the elements of (3.11). This is the desired algorithm. It is a finite process because the elements in (3.12)–(3.14) having factors y^b may be disregarded.

Let $\mathscr{I}_B = \Pi\mathscr{I}_i^{r_i}$. Let $s = (s_1, \cdots, s_n)$, $1 \leqq s_i \leqq r_i$ and $B(s) = (|B|, {}_B\mathcal{O}/\Pi\mathscr{I}_i^{s_i})$. Since ${}_{P(s)}\theta$ is a quotient of ${}_P\theta$, if $H^1(P, {}_P\theta) = 0$, then $H^1(P(s), {}_{P(s)}\theta) = 0$ for all $s \leqq r = (r_1, \cdots, r_n)$. To prove that any B diffeomorphic to P is analytically equivalent to P, we will proceed by induction on the s_i. We induct on the formally stronger statement that $s_i(P(s)) = *$. We start our induction at $s = (2, \cdots, 2)$. To prove that if $B(s)$, $s = (2, \cdots, 2)$, is diffeomorphic to $P(s)$, then $B(s)$ is analytically equivalent to $P(s)$ via an isomorphism which preserves the labeling, it suffices by Theorem 2.1 and Theorem 3.2 to prove that $H^1(B(s), {}_{B(s)}\theta) = 0$ for all $B(s)$. We know that $H^1(P(s), {}_{P(s)}\theta) = 0$. We use (3.10) to calculate $H^1(B(s), {}_{B(s)}\theta)$ and see how the images of elements of (3.12)-(3.14) differ in (3.11) for $B(s)$ and $P(s)$. For $B(s)$, $V_i \approx W_i$ in (3.8) but the transition rule for $V_i \cap V_j$ need not be given by the plumbing transformation. However $|B| = \{x_iy_i = 0\} = \{x_jy_j = 0\}$ because $V_i \approx W_i$ and $V_j \approx W_j$. Thus the change of coordinates for $B(s)$ is of the form

$$(3.15)\qquad x_j = y_i(a_0 + \cdots),\ y_j = x_i(b_0 + \cdots),\ a_0 \neq 0,\ b_0 \neq 0.$$

For $P(s)$, the change of coordinates is $x_j = y_i$, $y_j = x_i$. The relevant terms in (3.12)–(3.14) for $B(s)$ are those with coefficients x, y, or xy. Look at the linear terms. The change of coordinates in (3.15) may add higher order terms to the linear terms, but the effect on the linear terms (interchanging x and y) is exactly the same as the effect of the change of coordinates for $P(s)$. Therefore (3.12)–(3.14) produce the same linear terms in (3.11) for $B(s)$ and $P(s)$. In $P(s)$, the xy terms in (3.11) can only come from the first row in (3.12) or the third row in (3.12)–(3.13) or the second grouping in (3.14).

$H^1(P(s), {}_{P(s)}\theta) = 0$ so the linear terms in (3.12)–(3.14) map onto the linear terms in (3.11) and the xy-terms in (3.11) must be the image of the following terms. An $A_i = |P_i| = |B_i|$ of the form (3.12) contributes both $xy\ \partial/\partial y$ (from $x^2\,\partial/\partial x - vxy\,\partial/\partial y$) and $xy\,\partial/\partial x$. Modulo the ideal (x^2, y^2), the $B(s)$ and $P(s)$ change of coordinates just differ in effect on $xy\,\partial/\partial y$ and $xy\,\partial/\partial x$ by multiplication by a non-zero constant. Thus they do not affect the span of the image. An A_i of the form (3.13) contributes $xy\,\partial/\partial x$, which equals 0 in (3.11) in the (x_1, y_1) system, and $x_1y_1\ \partial/\partial x_1$. Again, the span of these contributions is the same for $B(s)$ and $P(s)$. For $v \geqq 3$, (3.14) similarly contributes $xy\,\partial/\partial x$, $(x-1)y\ \partial/(\partial(x-1))$, $x_1y_1\ \partial/\partial x_1$, and each of these vector fields may be chosen to vanish in the other coordinate systems, modulo (x^2, y^2). $v = 1$ and $v = 2$ must be treated separately. For $v = 1$, the one relevant section in $\Gamma(A_i, {}_{B(s)}\theta)$ has the following images in the $(x, y), (x-1, y)$, and (x_1, y_1) systems, modulo second order terms besides xy:

$$-xy\frac{\partial}{\partial x} \equiv (x-1)y\ \frac{\partial}{\partial(x-1)} \equiv -y_1x_1\ \frac{\partial}{\partial x_1}.$$

For $H^1(P(s), {}_{P(s)}\theta)$ to be zero, two more terms involving $xy\partial/\partial x$, $(x-1)y\,\partial/\partial(x-1)$, or $x_1y_1\partial/\partial x_1$ are needed and these can only come from an A_j of the (3.12) type. The $P(s)$ and $B(s)$ change of coordinates just differ in effect by multiplication by a non-zero constant. Thus if $H^1(P(s), {}_{P(s)}\theta) = 0$, then also for $B(s)$ the $xy\,\partial/\partial x$ terms in (3.11) for the three points in $\{A_i \cap A_j\}$, $A_i \cdot A_i = -1$, $j \neq i$, are in the image of terms from (3.12)—(3.14). Finally, if $v = -A_i \cdot A_i = -2$, there are two sections

$$-xy\frac{\partial}{\partial x} \equiv (x-1)y\ \frac{\partial}{\partial(x-1)} \equiv 0 \text{ and}$$

$$0 \equiv (x-1)y\ \frac{\partial}{\partial(x-1)} \equiv -x_1y_1\ \frac{\partial}{\partial x_1}.$$

As for $v = 1$, for $H^1(P(s), {}_{P(s)}\theta)$ to be zero, one more $xy\,\partial/\partial x$ term is needed and this can only come from an A_j of the (3.12) type. Again the $xy\partial/\partial x$ terms in (3.11) for the three points in $\{A_i \cap A_j\}$, $j \neq i$, are in the span of terms from (3.12)–(3.14) for $B(s)$ as well as for $P(s)$.

The linear terms for $B(s)$ are, as noted, changed from those for $P(s)$ only by the addition of quadratic terms. Hence if $H^1(P(s), {}_{P(s)}\theta) = 0$, then also $H^1(B(s), {}_{B(s)}\theta) = 0$. Furthermore, $s_l(P(s))$ has only one element.

This particular P achieved by plumbing has an important special property that is independent of s. In computing $H^1(P(s), {}_{P(s)}\theta)$, the linear terms act independently of the higher order terms. Any linear terms in the kernel of ρ in (3.10) and (3.8) give rise to an ambient vector field on M. Integration along this field gives an automorphism of $P(s)$ for all s, not just $s = (2, \cdots, 2)$. Locally, the automorphism is $(x, y) \to (a_0 x, b_0 y)$, for appropriate non-zero a_0 and b_0. These automorphisms form a subgroup $L = L(s) \subset \mathrm{Aut}_l(P(s))$. Let $\mathscr{L}(s)$ be the Lie algebra of $L(s)$. Let $\mathscr{M}_l(s) = \mathscr{M}(s)$ be the Lie subalgebra of $\Gamma(P(s), {}_{P(s)}\theta)$ given by the higher order (i.e., non-linear) terms of (3.12)–(3.14) which are in the kernel of ρ. $\mathscr{M}(s)$ is the Lie algebra for the closed normal subgroup $M_l(s) \subset \mathrm{Aut}_l(P(s))$ whose induced map on $|P_j|$ is either the identity or a parabolic map with $|P_j| \cap |P_k|$, some necessarily unique $k \neq j$, as its only fixed point. As vector spaces, $\Gamma(P(s), {}_{P(s)}\theta) \approx \mathscr{L}(s) \oplus \mathscr{M}(s)$.

Now for the general induction argument starting at $s = (2, \cdots, 2)$. First look at one A_i such that $\{A_i \cap A_j\}$, $j \neq i$, has only one point in it. Let $\mathscr{J} = \mathscr{I}_1 \cdots \mathscr{I}_n = \mathscr{I}d A$. $\mathscr{A}ut(\mathscr{I}_i^{k+1}\mathscr{J}^2 : \mathscr{I}_i^k \mathscr{J}^2) = 0$, all $k \geqq 0$. This means that any diffeomorphic spaces with ideals of the form $\mathscr{I}_i^k \mathscr{J}^2$, $k \geqq 0$, are analytically equivalent. Next look at another $A_{i'}$ such that $\{A_{i'} \cap A_j\}$, $j \neq i'$, has only one point. If $A_i \cap A_{i'} \neq \varnothing$, then $A = A_i \cup A_{i'}$ and the following special argument suffices to finish the proof of the theorem. For $A_i \cdot A_i \leqq -2$, alternately extend isomorphisms to $\mathscr{I}_i^k \mathscr{I}_{i'}^{k+1}$ and to $\mathscr{I}_i^{k+1} \mathscr{I}_{i'}^{k+1}$, $k = 2, 3, \cdots$. For $A_i \cdot A_i = -1$, blow down A_i. $A_{i'}$ becomes a curve A' with $A' \cdot A' \leqq 0$. Again all obstructions to extending isomorphisms vanish. By [*16*, Lemma 6.11, p. 113] we can perform a quadratic transformation to restore A_i and obtain the desired analytic equivalence between diffeomorphic spaces. We can thus assume that $A_i \cap A_{i'} = \varnothing$. Then $\mathscr{A}ut(\mathscr{I}_{i'}^{u+1} \mathscr{I}_i^k \mathscr{J}^2 : \mathscr{I}_{i'}^u \mathscr{I}_i^k \mathscr{J}^2) = 0$, all $u \geqq 0$, $k \geqq 0$. Thus any diffeomorphic spaces with ideals of the form $\mathscr{I}_{i'}^u \mathscr{I}_i^k \mathscr{J}^2$, $u \geqq 0$, $k \geqq 0$ are analytically equivalent. Continuing similarly, we see that $B(s')$ and $P(s')$ are analytically equivalent for any $\{s_i'\}$, so long as $\{A_i \cap A_j\}$, $j \neq i$, has only one point.

Now let $s = (s_1, \cdots, s_n)$ and $s' = (s_1', \cdots, s_n')$ satisfy $s_j = s_j'$, $j \neq i$, and $s_i' + 1 = s_i$. We may assume by the previous paragraph that $\{A_i \cap A_j\}$, $j \neq i$, has at least two points. We assume by induction that $S_l(P(s')) = *$. We must prove that $S_l(P(s)) = *$. Look at the bottom row in (3.7); B and B' in (3.7) are replaced by $P(s)$ and $P(s')$, $C = \varnothing$. By induction, $S_l(P(s')) = * = P(s')$. By exactness, it suffices to show that δ is surjective. We will in fact show that $\delta \circ \exp_3(\mathscr{M}(s')) = S(P(s): P(s'))$.

$$S(P(s): P(s')) \approx H^1(P(s), \mathscr{A}ut(n: n')) \approx H^1(P(s), {}_{P(s), P(s')}\theta)$$

since $s_i \geqq 2$.

The map $\delta \circ \exp_3$ may be given as follows. $\Gamma(P(s'), {}_{P(s')}\theta)$ is the kernel of the map ρ from (3.12)–(3.14) to (3.11). $\omega \in \Gamma(P(s'), {}_{P(s')}\theta)$ corresponding to $\{w_j \in \Gamma(W_j, {}_{P(s')}\theta)\}$ from (3.12)–(3.14) is then locally represented as follows. Suppose that (x, y) are local coordinates, as in (3.11), near a point in $|P_i| \cap |P_j|$. $\mathscr{I}d\, P_i(s) = (y^{s_i})$. $\mathscr{I}d\, P_i(s') = (y^{s_i - 1})$. $\mathscr{I}d\, P_j(s) = \mathscr{I}d\, P_j(s') = (x^{s_j})$. ω_i is a linear combination of terms from (3.12)–(3.14). ω_j is a linear combination of terms from (3.12)–(3.14) with the roles of x and y reversed. $(\omega_j\} \in \ker \rho$ means that the terms in ω_i and ω_j with coefficients $x^u y^v$, $u < s_j$, $v < s_i - 1$ coincide. Let $\Delta_{ij}(s)$ be a small polydisc neighborhood of $(0,0)$ in $P_i(s) \cup P_j(s)$. Let $R_i(s)$ be the open subset of $P_i(s)$ carried on the regular points of $|P|$. A term in ω_i with coefficient $x^u y^v$ with $u \geqq s_j$ vanishes on $P_j(s')$. ω includes this term on $R_i(s')$ and $\Delta_{ij}(s')$ but not on $R_j(s')$. $\delta'(\omega)$ will not include this term. A term in ω_j with $v \geqq s_i - 1$ is included on $P_j(s')$ and $\Delta_{ij}(s')$ but not on $P_i(s')$. $\delta'(\omega)$ will include this term when $v = s_i - 1$. We thus have local extensions for ω to sections of ${}_{P(s)}\theta$. Denote these extensions by $\omega_k \in \Gamma(R_k(s), {}_{P(s)}\theta)$ and $\omega_{ij} \in \Gamma(\Delta_{ij}(s), {}_{P(s)}\theta)$. At least for ω near $0 \in H^1(P(s'), {}_{P(s')}\theta)$, on $R_i(s) \cap \Delta_{ij}(s)$,

$$(3.16) \qquad \delta \circ \exp_3(\omega) = \exp(-\omega_{ij}) \circ \exp(\omega_i)$$

$$= \exp(\omega_i - \omega_{ij} + \tfrac{1}{2}[-\omega_{ij}, \omega_i] + (\text{higher order brackets}))$$

by the Campell-Hausdorff formula [*13*, pp. 111-112].

Now look at $H^1(P(s), {}_{P(s), P(s')}\theta)$ and the top row of (3.7). In terms of the Leray cover $\mathfrak{A} = \{R_j(s), \Delta_{ij}(s)\}$, on $R_i(s) \cap \Delta_{ij}(s)$, $\delta'(\omega) = \mathrm{cls}[\omega_i - \omega_{ij}]$. Let $\mathscr{T}$ be the tangent sheaf to $|P_i|$ and let $\mathscr{N}$ be the sheaf of germs of sections of the normal bundle of the embedding of $|P_i|$. Let $n = \mathscr{I}d\, P(s)$ and $n' = \mathscr{I}d\, P(s')$. Recall the canonical exact sequence

$$0 \to \mathscr{T} \otimes n'/n \to {}_{P(s), P(s')}\theta \to \mathscr{N} \otimes n'/n \to 0.$$

Thus

$$(3.17) \qquad H^1(P(s), \mathscr{T} \otimes n'/n) \xrightarrow{\iota} H^1(P(s), {}_{P(s), P(s')}\theta)$$

$$\xrightarrow{\mu} H^1(P(s), \mathscr{N} \otimes n'/n) \to 0$$

is an exact sequence. Now examine (3.16) more closely for $\omega \in \mathscr{M}(s')$ and for our cover $\mathfrak{A}$. ω_{ij} differs from ω_i on $R_i(s) \cap \Delta_{ij}(s)$ only by an expression of the form

$$\Sigma\sigma_u y^{s_i-1}x^u\frac{\partial}{\partial x}+\tau_v y^{s_i-1}x^v\frac{\partial}{\partial y},$$

$$1\leqq u\leqq s_j-1,\ 0\leqq v\leqq s_j-1;\ \sigma_u,\ \tau_v\in\mathbf{C}.$$

Recall that $\{A_i\cap A_j\}$, $j\neq i$, has at least two points. Thus ω_i comes from terms from (3.13) or (3.14). ω_i has no terms of the form $\alpha x\,\partial/\partial x+\beta y\,\partial/\partial y$, $\alpha,\beta\in\mathbf{C}$, since $\omega\in\mathscr{M}(s)$. ω_i has no terms of the form $\alpha yx^u\,\partial/\partial y$, $u\geqq 1$, since no terms of this form appear in (3.13)–(3.14). Thus $[-\omega_{ij},\omega_i]$ defines a cohomology class $\text{cls}[-\omega_{ij},\omega_i]\in\ker\rho$, ρ from (3.17). The higher order brackets of (3.16) all involve $[-\omega_{ij},\omega_i]$. Since $[-\omega_{ij},\omega_i]$ is of the form $y^{s_i-1}f(x)\,\partial/\partial x$, with f a holomorphic function of x on $|R_i(s)\cap\Delta_{ij}(s)|$, all the higher brackets vanish. Finally, $\text{cls}[-\omega_{ij},\omega_i]=\text{cls}[\omega_i-\omega_{ij},\omega_i]$ $=[\rho\circ\delta'(\omega),\omega]$ in the following natural sense. $\delta'(\omega)=\text{cls}[\omega_i-\omega_{ij}]$. If ω_i and ω_i' are both extensions of ω to $\Gamma(R_i(s),{}_{P(s)}\theta)$ and λ is any cocycle in $C^1(N(\mathfrak{A}),{}_{P(s),P(s')}\theta)$ such that $\rho(\text{cls}[\lambda])=\rho\circ\delta'(\omega)$, then $[\lambda,\omega_i']$ and $[-\omega_{ij},\omega_i]$ are cohomologous in $C^1(N(\mathfrak{A}),{}_{P(s),P(s')}\theta)$. Namely, for fixed λ, $[\lambda,\omega_i']$ depends only on the $yx^u\,\partial/\partial x$ terms of ω_i' and these are the same for both ω_i and ω_i'. For fixed ω_i', changing $(\omega_i-\omega_{ij})$ by elements of $\Gamma(\Delta_{ij}(s),{}_{P(s),P(s')}\theta)$ or $\Gamma(R_i(s),{}_{P(s),P(s')}\theta)$ changes $[\omega_i-\omega_{ij},\omega_i']$ by elements of $\Gamma(\Delta_{ij}(s),{}_{P(s),P(s')}\theta)$ or $\Gamma(R_i(s),{}_{P(s),P(s')}\theta)$ respectively. Thus $\text{cls}[\lambda,\omega_i']$ depends only on $\text{cls}[\lambda]$. Finally, changing $\text{cls}[\lambda]$ by an element in $\ker\rho=\text{im}\,\iota$ of (3.17) corresponds to modifying λ by a cochain of the form $y_i^{s_i-1}f(x)\,\partial/\partial x$, with f a holomorphic function of x on $|R_i(s)\cap\Delta_{ij}(s)|$. As before, this does not change $[\lambda,\omega_i']$. Thus, $\text{cls}[-\omega_{ij},\omega_i]=[\rho\circ\delta'(\omega),\omega]$ as stated.

Under the isomorphism $S(P(s)\colon P(s'))\approx H^1(P(s),{}_{P(s),P(s')}\theta)$, for $\omega\in\mathscr{M}(s')$, (3.16) thus becomes

(3.18) $$\delta\circ\exp_3(\omega)=\delta'(\omega)+\tfrac{1}{2}[\rho\circ\delta'(\omega),\omega],$$

$$[\rho\circ\delta'(\omega),\omega]\in\ker\rho.$$

$H^1(P,{}_P\theta)=0$ by the theorem's hypothesis. There is a natural surjective map ${}_P\theta\to{}_{P(s)}\theta$ with coherent kernel over P. Thus also $H^1(P(s),{}_{P(s)}\theta)=0$. Hence δ' of (3.7) is surjective. Recall that $\mathscr{L}(s')\subset\ker\delta'$. Hence δ': $\mathscr{M}(s')\to H^1(P(s),{}_{P(s'),P(s)}\theta)$ is surjective. All the terms in an element of $\mathscr{M}(s')$ have coefficients of total degree at least two. Hence the Lie bracket operation in $\mathscr{M}(s')$ strictly increases the total degree. For fixed s', the possible degrees occurring in (3.12)–(3.14) are bounded. Hence, any sufficiently high number of successive Lie bracket operations results in 0 and so $\mathscr{M}(s')$ is nilpotent. Then $\exp_3:\mathscr{M}(s')\to M_l(s')$ maps onto the connected component $M_0(s')$ of the identity in $M_l(s')$. In (3.7), $\delta\colon \mathrm{Aut}_l(P(s')) \to S(P(s)\colon P(s'))$ is given by $\delta(g) = g(*)$, the image of $*$ under the action of g. δ', the tangent map for δ, maps $\mathscr{M}(s')$ onto $H^1(P(s), {}_{P(s'),P(s)}\theta)$. Hence the orbit of $*$ under the action of $M_0(s')$ is an open subset of $S(P(s)\colon P(s'))$. Then $\delta \circ \exp_3$ has an open image.

In (3.17), $\rho \circ \delta \circ \exp_3 = \rho \circ \delta'$ by (3.18). δ' is surjective so $\rho \circ \delta' = \rho \circ \delta \circ \exp_3$ is surjective. Fix a class $a \in H^1(P(s), \mathscr{N} \otimes n'/n)$. $\rho \circ \delta \circ \exp_3 = \rho \circ \delta'$ is a linear map so that $(\rho \circ \delta \circ \exp_3)^{-1}(a)$ is a non-empty affine subspace of $\mathscr{M}(s')$. $\rho^{-1}(a)$ is an affine subspace of $H^1(P(s), {}_{P(s),P(s')}\theta)$. $\delta \circ \exp_3\colon (\rho \circ \delta \circ \exp_3)^{-1}(a) \to \rho^{-1}(a)$ is an affine map by (3.18) and has an open image since $\delta \circ \exp_3(\mathscr{M}(s))$ is open. Thus $\delta \circ \exp_3\colon (\rho \circ \delta \circ \exp_3)^{-1}(a) \to \rho^{-1}(a)$ is surjective. Thus $\delta \circ \exp_3$ and necessarily δ is surjective.

Recall the following definition [*4*], [*22*], [*23*].

Definition 3.2. Let p be a normal two-dimensional singularity. Then p is taut if all normal two-dimensional singularities having the same minimal weighted dual graph as has p are isomorphic.

The following theorem provides an algorithm for determining whether or not a weighted dual graph Γ is the dual graph of a taut singularity.

Theorem 3.10. *Let Γ be a dual weighted graph which comes from a negative definite intersection matrix and represents a minimal resolution among resolutions such that the irreducible components A_i, $1 \leqq i \leqq n$, of the exceptional set are non-singular and have only normal crossings. Let $r = (r_1, \cdots, r_n)$ be chosen sufficiently large so that if $B(r)$ is analytically equivalent to $B'(r)$, then $B(r)$ and $B'(r)$ determine isomorphic normal singularities. Let $A \subset M$ be the plumbing construction of Theorem* 3.9. *Then Γ is the dual graph of a taut singularity if and only if $H^1(P(r), {}_{P(r)}\theta) = 0$.*

Proof. Minimal resolutions of the type of this theorem are unique [*14*], [*3*, Lemma, p. 81], [*9*, Theorem 5.12, p. 91]. r may be chosen by [*9*, Theorem 6.20, p. 132]. The theorem is now an immediate consequence of Theorem 3.9 and Proposition 3.8.

§4. *Discussion, Problems, and Examples*

Let $A \subset M$ be the exceptional set in the resolution of a singularity p. Then Γ, the weighted dual graph, and the genera of the A_i determine both the topological and differentiable type of the embedding of A in M (where we always assume non-singular embeddings of the A_i and normal crossings). In dealing with nonreduced spaces it seems more natural to use the differentiable category. The definition of diffeomorphism in Definition 3.1 is equivalent to requiring that B and B' be diffeomorphic under the natural definitions of being nonreduced differentiable spaces. Reduced spaces have been studied by Spallek [*21*] and Ephraim [*6*]. Section 3 can be entirely rephrased in terms of putting complex analytic structures on nonreduced spaces with differentiable structures. Theorem 3.9 gives a necessary and sufficient condition for a differentiable structure of a certain type to carry a unique analytic structure. One would like to relate this to the singularity p itself. It is not known on just how much of the structure of p Γ depends. Certainly Γ is determined by the analytic structure of p. Γ is probably also determined by the differentiable structure on p. However, this would still be much too strong a structure because Ephraim has shown [*6*] that in many cases there are at most two analytic structures on a given differentiable structure. Γ, on the other hand, is not determined by the topological type of p, although counterexamples seem to be quite scarce. The only examples known to the author come from homeomorphic lens spaces. [*4*] has a fuller discussion.

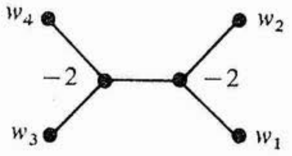
The relationship between $\mathrm{Def}(A(r))$, which preserves Γ, and the deformation of p itself, which can change Γ, (as studied by Schlessinger, Tjurina, Grauert, and Elhik) is not fully understood.

Perhaps "countable" in Theorem 3.6 can be replaced by "finite."

In all graphs to follow, each vertex has genus 0.

Tautness does not imply rationality (see [*2*] for rationality). The simplest examples are singularities with dual weighted graphs like (4.1).

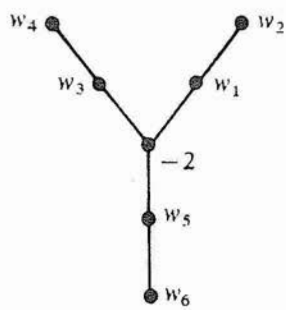
(4.1)



with the w_i large and negative; say, $w_i \leqq -10$. The corresponding singularities are taut but not rational.

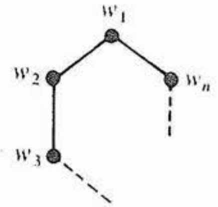
Structure jumping is quite a common phenomenon. In fact, look at (3.10). If the singularity obtained by plumbing has automorphisms arising from the linear terms in (3.12)–(3.14), then a different change of coordinates which destroys these automorphisms will usually lower the dimension of $H^1(B, {}_B\theta)$. For the weighted graph (4.2) with, say, $w_i \leqq -10$, there are exactly two distinct singularities. One is obtained by plumbing and $H^1(P, {}_P\theta) = \mathbf{C}$. The other singularity has $H^1(B, {}_B\theta) = 0$. $S(B) = \{B, P\}$ has $\varnothing$, B, and $\{B, P\}$ as its open subsets.

(4.2)



(4.3) is the dual graph of a cusp singularity, as defined by Hirzebruch. (4.3) consists of a polygon with weights at the n vertices. The singularity of (4.3) is taut.

(4.3)



Graphs of the form (4.3) are the only graphs which are not trees which can possibly be graphs for taut singularities. In particular, using [*18*], if the first Betti number of a deleted good neighborhood of p is greater than 1, then p is not taut. For a first Betti number equal to 1, [*23*] contains a characterization of the possible fundamental groups for taut singularities. If the graph Γ is a tree and the associated singularity is taut, then Γ can have at most two vertices of degree 3. More complete results will appear in "Taut two-dimensional Singularities," to be published in the *Mathematische Annalen*.

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