ON HOLOMORPHIC CROSS-SECTIONS IN TEICHMÜLLER SPACES, II*

by Clifford J. Earle

§1. Let X be a closed oriented smooth (= class C^∞) surface of genus $g > 1$. Its Teichmüller space $T(X)$ is a contractible complex manifold of dimension $n = 3g - 3$ whose points correspond to certain equivalence classes of complex structures on X. $T(X)$ can be embedded as a bounded open set in $\boldsymbol{C}^n$ [*2*]; the image is a domain of holomorphy [*3*].

Eells and the author have recently constructed a certain principal fibre bundle over $T(X)$ [*5*]. Its total space $M(X)$ has a natural complex structure which makes the projection

$$\pi : M(X) \to T(X) \tag{1}$$

holomorphic. There are holomorphic local cross-sections ([*1*], [*5*]). In addition, the bundle (1) is topologically trivial because $T(X)$ is contractible. Since $T(X)$ is a Stein manifold it is natural to look for a (global) holomorphic cross-section. Oddly enough, the search is fruitless. We shall prove

Theorem 1. *The fibre bundle* (1) *does not have a holomorphic cross-section.*

A somewhat stronger theorem was proved in [*4*], but the connection between that theorem and Theorem 1 was left rather vague. Here we provide more details. For general accounts of Teichmüller theory see [*1*], [*6*], and [*9*].

§2. First we describe the fibre bundle (1). The total space $M(X)$ is the set of smooth almost complex (= complex) structures on X which induce its given orientation. $M(X)$ is given the C^∞ topology. Let $\mathrm{Diff}_0(X)$ be the topological group of diffeomorphisms of X which are homotopic to the identity, again with the C^∞ topology. The natural action

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$$M(X) \times \mathrm{Diff}_0(X) \to M(X) \tag{2}$$

is defined by letting $\mu \cdot f$ be the pullback of the complex structure μ by the diffeomorphism f. The Teichmüller space $T(X)$ is defined to be the orbit space $M(X)/\mathrm{Diff}_0(X)$, with quotient topology.

Theorem 2. *The quotient map* (1) *produced by the action* (2) *is a trivial fibre bundle. The total space* $M(X)$, *base space* $T(X)$, *and structure group* $\mathrm{Diff}_0(X)$ *of that bundle are all contractible.*

Suitable versions of Theorem 2 can be formulated for any compact surface; the group $\mathrm{Diff}_0(X)$ always has a compact Lie group as strong deformation retract. For details see the author's joint papers with Eells [5] and Schatz [7].

§3. Consider now a cross-section

$$s: T(X) \to M(X)$$

of (1), and choose a complex structure μ_0 in the image of s. According to the uniformization theorem, there is a covering of X by the open unit disc $\Delta = \{z \in \boldsymbol{C};\ |z| < 1\}$ which is holomorphic with respect to μ_0. The cover group Γ is a discrete group of holomorphic automorphisms of Δ. The complex structures on X lift to the Γ-invariant complex structures on Δ. These are in bijective correspondence with the C^∞ functions $\mu: \Delta \to \boldsymbol{C}$ which satisfy the conditions

$$|\mu(z)| < 1 \qquad \text{for all } z \in \Delta \tag{3}$$

$$(\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu \qquad \text{for all } \gamma \in \Gamma. \tag{4}$$

(Under that correspondence, a function $f: \Delta \to \boldsymbol{C}$ is holomorphic with respect to μ if and only if $f_{\bar{z}} = \mu f_z$. Thus $\mu = 0$ corresponds to the standard complex structure on Δ.)

Let $A(\Gamma)$ be the complex Fréchet space of C^∞ functions on Δ which satisfy (4), and $M(\Gamma)$ the convex set of functions in $A(\Gamma)$ which satisfy (3). Since X is compact, Γ has a compact fundamental domain D, and condition (3) is equivalent to

$$\sup\{|\mu(z)|;\ z \in D\} < 1.$$

Thus $M(\Gamma)$ is open in $A(\Gamma)$ (with respect to the C^∞ topology), and the identification of $M(X)$ with $M(\Gamma)$ makes $M(X)$ a complex Fréchet manifold. (More precisely, that identification is biholomorphic with respect to the natural complex structure on $M(X)$, which is independent of the choice of μ_0 and Γ.)

From now on we identify $M(X)$ with $M(\Gamma)$. That means we regard the projection (1) as defined on $M(\Gamma)$ and the cross-section s as a map into $M(\Gamma)$. By our construction of Γ, $0 \in M(\Gamma)$ belongs to the image of s.

§4. Now let $L^\infty(\Gamma)$ be the Banach space of bounded measurable functions on Δ which satisfy (4), with the L^∞ norm. Because Γ has a compact fundamental domain, there is a continuous inclusion map $j: A(\Gamma) \to L^\infty(\Gamma)$. Under $j, M(\Gamma)$ is mapped into the open unit ball $M^\infty(\Gamma)$ of $L^\infty(\Gamma)$. The theory of quasiconformal mappings tells us that the map (1) factors through $M^\infty(\Gamma)$; there is a commutative diagram

$$\begin{array}{ccc} M(\Gamma) & \xrightarrow{\ j\ } & M^\infty(\Gamma) \\ \pi \big\downarrow & & \big\downarrow \pi^\infty \\ T(X) & \xrightarrow{\ 1\ } & T(X) \end{array}$$

whose bottom row is the identity. Moreover, all the above maps are holomorphic. The differential of π^∞ at 0 is known; its kernel is

$$Q(\Gamma)^\perp = \{\mu \in L^\infty(\Gamma): \int_{\Delta/\Gamma} \mu\phi = 0, \quad \text{all } \phi \in Q(\Gamma)\}$$

where $Q(\Gamma)$ is the space of holomorphic functions ϕ on Δ which satisfy

$$(\phi \circ \gamma)(\gamma')^2 = \phi \quad \text{for all } \gamma \in \Gamma$$

(see [1] and [2]). $Q(\Gamma)$ is a complex vector space of dimension $3g-3$; it is the lift to Δ of the space of holomorphic quadratic differentials on Δ/Γ.

§5. Suppose the cross-section $s: T(X) \to M(\Gamma)$ is holomorphic. Then so is the map $h: M^\infty(\Gamma) \to M^\infty(\Gamma)$ given by

$$h = j \circ s \circ \pi^\infty .$$

It is easy to see that $h(0) = 0$, $\pi^\infty \circ h = \pi^\infty$, and $h^2 = h$. Let $h'(0) = P: L^\infty(\Gamma) \to L^\infty(\Gamma)$. Then

$$P^2 = P, \tag{5}$$

$$\text{kernel } P = \text{kernel}(\pi^\infty)'(0) = Q(\Gamma)^\perp, \tag{6}$$

$$\|P\| \leqq 1. \tag{7}$$

(5) and (6) are consequences of the chain rule; (7) follows from the Cauchy derivative estimate or by Schwarz's lemma. To prove Theorem 1 we will show that the above properties are contradictory. P cannot exist. For the proof we need a

Definition. $\mu \in L^\infty(\Gamma)$ is a *Hamilton differential* if $\|\mu\| \leqq \|\mu + \lambda\|$ for all $\lambda \in Q(\Gamma)^\perp$.

Lemma 1. *Every μ in the image of P is a Hamilton differential.*

Proof. If $\mu = P\mu$ and $\lambda \in Q(\Gamma)^\perp$, then

$$\|\mu\| = \|P\mu\| = \|P(\mu + \lambda)\| \leqq \|\mu + \lambda\|.$$

Lemma 2. (Hamilton). *Every Hamilton differential $\mu = \|\mu\| \, |\phi|/\phi$, some $\phi \in Q(\Gamma)$.*

Proof (from [8]). By assumption, $\|\mu\|$ equals the norm of μ as a functional on $Q(\Gamma)$. Since $Q(\Gamma)$ has finite dimension, there is a non zero ϕ in $Q(\Gamma)$ with

$$\int_{\Delta/\Gamma} \mu\phi = \int_{\Delta/\Gamma} \|\mu\| \, |\phi|.$$

Since $\phi \neq 0$ a.e., Lemma 2 follows.

We can conclude immediately that P does not take its values in $j(A(\Gamma))$, for each $\phi \in Q(\Gamma)$ has zeros, and $\mu = |\phi|/\phi$ is not smooth at the zeros of ϕ. Theorem 1 follows at once, but we prefer to show that P cannot even take its values in $L^\infty(\Gamma)$. (Hence the map π^∞ has no holomorphic cross-section.) The point is that $Q(\Gamma)$ and the image of P both have dimension $3g - 3 \geqq 3$. But each function in the image of P has constant modulus, by Lemmas 1 and 2. That contradicts the obvious

Lemma 3. *Let V be a complex vector space of functions of constant modulus. Then* $\dim V \leqq 1$.

Remark. We are grateful to Hugo Rossi for Lemma 3 and for pointing out that our argument proves this

Proposition. Let E be any complex linear subspace of L^1 such that

(a) $1 < \dim E < \infty$;

(b) *if $\phi \in E$, then either $\phi = 0$ a.e. or $\phi \neq 0$ a.e.*

Then there is no linear map $P: L^\infty \to L^\infty$ with $P^2 = P$, $\|P\| \leqq 1$, and kernel $P = E^\perp$. In other words the natural map from L^∞ onto E^* cannot be split by a linear isometry.

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HARVARD UNIVERSITY and CORNELL UNIVERSITY