INTEGRAL FORMULAE CONNECTED BY DOLBEAULT'S ISOMORPHISM

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Introduction

Serre [3] describes a natural bilinear form on $H^r(X,\Omega^n) \times H_*^{n-r}(X,\mathcal{O})$. For $p = 1,\cdots,r$ there is a natural bilinear form on

$$H^p(\mathcal{U},\mathcal{Z}_{n,r-p}) \times H_p(\mathcal{U},\mathcal{E}'_{0,n-r+p}/\bar{\partial}\mathcal{E}'_{0,n-r+p-1})$$

(see Section 1 for definitions and notation). The main result (Theorem 1.16) of Section 1 is that these various bilinear forms are all equivalent under Dolbeault isomorphisms acting on $H^r(\mathcal{U},\mathcal{Z}_{n,r-p})$ and adjoint Dolbeault isomorphisms acting on $H_r(\mathcal{U},\mathcal{E}'_{0,n-r+p}/\bar{\partial}\mathcal{E}'_{0,n-r+p-1})$.

In Section 2 we show (Theorem 2.2) that the Cauchy kernel is equivalent to the kernels of Cauchy-Fantappié type under Dolbeault's isomorphisms. In particular, this proves the Cauchy-Fantappié integral formula (Leray [2]), assuming the Bochner-Martinelli integral formula. The next result (Theorem 2.6) is a generalization of Theorem 2.2 where the Cauchy kernel is replaced by the Cauchy-Weil kernel.

Integration over the various strata of an analytic polyhedron determines compactly supported distribution forms of various types. In Proposition 3.6 it is shown that a certain family of such forms (3.4) are all equivalent under adjoint Dolbeault isomorphisms. In particular, integration over the topological boundary of a polydisk and integration over the distinguished boundary of a polydisk are equivalent under adjoint Dolbeault isomorphisms. In Proposition 3.2 analogous results are proved for "smeared" polyhedra. Here the compactly supported forms (3.3) correspond heurestically to integration over strata, but with the advantage that they are smooth.

The results of the first three sections imply that the Cauchy-Weil integral over the distinguished boundary of an analytic polyhedron is equivalent to a generalized Cauchy-Fantappié integral over the topological boundary of the polyhedron. In conclusion we prove these integral formulas (Theorem 4.1) using the above equivalence. These results have applications to the theory of analytic functionals and the residue calculus which will be discussed in another paper.

1. *Dolbeault's Homomorphism and its Adjoint*

Suppose X is a complex manifold countable at infinity. Throughout this paper $\mathscr{E}_{p,q}$ will denote the sheaf of germs of C^∞ forms of type (p,q) on X and $\mathscr{Z}_{p,q}$ will denote the sheaf of germs of $\bar{\partial}$-closed C^∞ forms of type (p,q) on X. We will also denote the sheaf $\mathscr{Z}_{0,0}$ of germs of holomorphic functions on X by $\mathscr{O}$ and the sheaf $\mathscr{Z}_{n,0}$ of germs of holomorphic n-forms on X by Ω^n.

Next we define the usual Čech cohomology groups $H^p(\mathscr{U},\mathscr{F})$. Suppose $\mathscr{U} = \{U_i\}$ is an open covering of X. Let I denote a $(p+1)$-tuple $(i_0,\cdots,i_p)$ where each i_k belongs to the indexing set for $\mathscr{U}$. Abbreviate $U_{i_0} \cap \cdots \cap U_{i_p}$ by U_I and let $|I| = p+1$. Suppose $\mathscr{F}$ is a sheaf of (complex) vector spaces on X. A *p-cochain* f of $\mathscr{U}$ with values in $\mathscr{F}$ is a map which assigns to every I with $|I| = p+1$ a section $f_I \in \Gamma(U_I, \mathscr{F})$ so that f_I is an alternating function of I. Let $C_p(\mathscr{U},\mathscr{F})$ denote the vector space of all p-cochains with values in $\mathscr{F}$. Let $\delta: C^p(\mathscr{U},\mathscr{F}) \to C^{p+1}(\mathscr{U},\mathscr{F})$ denote the usual coboundary map defined by

$$(\delta f)_I = \sum_{k=0}^{p+1} (-1)^k f_{j_0\cdots\hat{j}_k\cdots j_{p+1}}$$

($\wedge$ over a symbol will always indicate deletion of that symbol). Let $Z^p(\mathscr{U},\mathscr{F}) = \{f \in C^p(\mathscr{U},\mathscr{F}) : \delta f = 0\}$ and $H^p(\mathscr{U},\mathscr{F}) = Z^p(\mathscr{U},\mathscr{F})/\delta C^{p-1}(\mathscr{U},\mathscr{F})$.

The standard way to prove that $H^p(\mathscr{U},\mathscr{E}_{n,q}) = 0$ for $p \geqq 1$ is to construct a chain homotopy, as described below (1.1). Suppose that $\mathscr{U} = \{U_i\}$ is an open covering of X.

Lemma 1.1. *There exists a family $\{\phi_i\}$ of functions $\phi_i \in C^\infty(X)$ such that*

i) supp $\phi_i \subset U_i$, *i.e.*, $\phi_i \equiv 0$ *in a neighborhood of* $X - U_i$;

ii) $\{\operatorname{supp}\phi_i\}$ *is locally finite*;

iii) $\sum \phi_i \equiv 1$ *in* X.

Proof. The covering $\mathscr{U}$ has a locally finite refinement $\mathscr{V} = \{V_j\}$. It is sufficient to construct functions ϕ_j satisfying i)–iii) for the covering $\mathscr{V}$. Choose a partition of unity subordinate to $\mathscr{V}$. That is, choose $\psi_k \in C_0^\infty(X)$ with $\operatorname{supp}\psi_k \subset\subset V_{\rho(k)}$ for some index $\rho(k)$ and with $\sum \psi_i \equiv 1$ in X. Now let $\phi_j = \sum_{\rho(k)=j}\psi_k$ and i)–iii) follow.

The above family $\{\varphi_i\}$ can be used to construct a chain homotopy

$$T\colon C^{p-1}(\mathscr{U},\mathscr{E}_{n,q}) \leftarrow C^p(\mathscr{U},\mathscr{E}_{n,q}) \text{ for}$$
$$\delta\colon C^{p-1}(\mathscr{U},\mathscr{E}_{n,q}) \to C^p(\mathscr{U},\mathscr{E}_{n,q}) \text{ as follows.}$$

Given $f^p \in C^p(\mathscr{U},\mathscr{E}_{n,q})$ let

$$(1.1) \qquad (Tf^p)_I = \sum_i \phi_i f^p_{iI}, \text{ for each } |I| = p.$$

Since ϕ_i vanishes on a neighborhood of $X - U_i$, $\phi_i f^p_{iI}$ uniquely determines an element of $\mathscr{E}_{n,q}(U_I)$ which we also denote by $\phi_i f^p_{iI}$. Hence $(Tf^p)_I \in \mathscr{E}_{n,q}(U_I)$, so that T maps $C^p(\mathscr{U},\mathscr{E}_{n,q})$ into $C^{p-1}(\mathscr{U},\mathscr{E}_{n,q})$.

Lemma 1.2. *$T\delta + \delta T\colon C^p(\mathscr{U},\mathscr{E}_{n,q}) \to C^p(\mathscr{U},\mathscr{E}_{n,q})$ is the identity for $p \geqq 1$, and $T\delta + \varepsilon T$ is the identity on $C^0(\mathscr{U},\mathscr{E}_{n,q})$.*

Remark. In order to make the case $p = 0$ meaningful we adopt the following definitions. Let $\varepsilon\colon \mathscr{E}_{n,q}(X) \to C^0(\mathscr{U},\mathscr{E}_{n,q})$ be defined by $(\varepsilon f)_i = f\big|_{U_i}$, and let $T\colon \mathscr{E}_{n,q}(X) \leftarrow C^0(\mathscr{U},\mathscr{E}_{n,q})$ be defined by $Tf = \Sigma\phi_i f_i$.

Proof.

$$(T\delta f^p)_J = \sum_i \phi_i(\delta f^p)_{iJ} = \sum_i \phi_i \left[f^p_J - \sum_{k=0}^{p} (-1)^k f^p_{ij_0\cdots\hat{j}_k\cdots j_p}\right]$$
$$= f^p_J - \sum_i \sum_{k=0}^{p} (-1)^k \phi_i f^p_{ij_0\cdots\hat{j}_k\cdots j_p}.$$
$$(\delta T f^p)_J = \sum_{k=0}^{p} (-1)^k (Tf^p)_{j_0\cdots\hat{j}_k\cdots j_p} = \sum_{k=0}^{p} (-1)^k \sum_i \phi_i f^p_{ij_0\cdots\hat{j}_k\cdots j_p}$$
$$= \sum_{k=0}^{p} \sum_i (-1)^k \phi_i f^p_{ij_0\cdots\hat{j}_k\cdots j_p}.$$

Corollary 1.3. *The sequence*

$$0 \to \mathscr{E}_{n,q}(X) \xrightarrow{\varepsilon} C^0(\mathscr{U},\mathscr{E}_{n,q}) \xrightarrow{\delta} C^1(\mathscr{U},\mathscr{E}_{n,q}) \xrightarrow{\delta} \cdots$$

is exact, and hence $H^p(\mathscr{U},\mathscr{E}_{n,q}) = 0$ for $p \geqq 1$.

Next we construct Dolbeault's homomorphisms:

$$(1.2) \qquad H^r(\mathscr{U},\Omega^n) \to H^{r-1}(\mathscr{U},\mathscr{Z}_{n,1}) \to \cdots \to H^p(\mathscr{U},\mathscr{Z}_{n,r-p}) \to \cdots$$
$$\to H^1(\mathscr{U},\mathscr{Z}_{n,r-1}) \to H^0(\mathscr{U},\mathscr{Z}_{n,r})/\bar{\partial}H^0(\mathscr{U},\mathscr{E}_{n,r+1})$$

which taken together map the Čech cohomology group $H^r(\mathscr{U},\Omega^n)$ into the Dolbeault cohomology group $\mathscr{Z}_{n,r}(X)/\bar{\partial}\mathscr{E}_{n,r-1}(X)$. We will not distinguish between $H^0(\mathscr{U},\mathscr{Z}_{n,q})/\bar{\partial}H^0(\mathscr{U},\mathscr{E}_{n,q-1})$ and $\mathscr{Z}_{n,q}(X)/\bar{\partial}\mathscr{E}_{n,q-1}(X)$.

Suppose $f^p \in C^p(\mathscr{U}, \mathscr{Z}_{n,q})$. *Let*

$$(1.3) \qquad (Df^p)_I = \bar{\partial}(Tf^p)_I = \sum_i \bar{\partial}\phi_i \wedge f^p_{iI} \text{ for each } |I| = p.$$

Note that $(Df^p)_I \in \mathscr{Z}_{n,q+1}(U_I)$ *since* $(Tf^p)_I \in \mathscr{E}_{n,q}(U_I)$.

$$(1.4) \qquad \begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
\cdots \arrow[r, "\delta"] & C^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q}) \arrow[rr, "\delta"] \arrow[d, "i"] & & C^p(\mathscr{U}, \mathscr{Z}_{n,q}) \arrow[r, "\delta"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "\delta"] & C^{p-1}(\mathscr{U}, \mathscr{E}_{n,q}) \arrow[rr, "\delta"] \arrow[d, "\bar{\partial}"] & \arrow[l, phantom, ""] & C^p(\mathscr{U}, \mathscr{E}_{n,q}) \arrow[r, "\delta"] \arrow[d, "\bar{\partial}"] \arrow[ll, bend left, "T"] & \cdots \\
\cdots \arrow[r, "\delta"] & C^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q+1}) \arrow[rr, "\delta"] & & C^p(\mathscr{U}, \mathscr{Z}_{n,q+1}) \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

The map D goes from the upper right to the lower left.

Proposition 1.4. *The operations D and δ anticommute, and hence D induces a map from $H^p(\mathscr{U}, \mathscr{Z}_{n,q})$ into $H^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q+1})$.*

Proof. Suppose $f \in C^p(\mathscr{U}, \mathscr{Z}_{n,q})$. Then

$$(\delta D + D\delta)f = (\delta\bar{\partial}T + \bar{\partial}T\delta)f = \bar{\partial}(\delta T + T\delta)f = \bar{\partial}f = 0,$$

where the third equality follows from Lemma 1.2.

We will refer to the map that D induces as Dolbeault's homomorphism. This homomorphism can be (equivalently) described as follows (a diagram chase in (1.4) establishes the equivalence).

Definition 1.5. The class $[f^{p-1}] \in H^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q+1})$ is the image of the class $[f^p] \in H^p(\mathscr{U}, \mathscr{Z}_{n,q})$ under *Dolbeault's homomorphism* if there exists $g^{p-1} \in C^{p-1}(\mathscr{U}, \mathscr{E}_{n,q})$ such that

$$(1.5) \qquad \delta g^{p-1} = f^p \text{ and } \bar{\partial} g^{p-1}_I = f^{p-1}_I \text{ for all } |I| = p.$$

If $p = 1$, then $[f^0] \in \mathscr{Z}_{n,q+1}(X)/\bar{\partial}\mathscr{E}_{n,q}(X)$ replaces the statement

$$[f^{p-1}] \in H^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q+1}).$$

From this definition it is obvious that Dolbeault's homomorphism is independent of the particular map D (i.e., of the choice of $\{\phi_i\}$).

The following concept (cf. (1.2)) will prove useful in formulating the main result of this section.

Definition 1.6. A sequence $\{f^p\}$, with $f^p \in Z^p(\mathscr{U}, \mathscr{Z}_{n,r-p})$, $p = 0, \cdots, r$; such that $[f^{p-1}]$ is the image of $[f^p] \in H^p(\mathscr{U}, \mathscr{Z}_{n,r-p})$ under Dolbeault's homomorphism for $p = 1, \cdots, r$ will be called a *sequence of Dolbeault representatives* of the cohomology class $[f^r] \in H^r(\mathscr{U}, \Omega^n)$.

Now we wish to dualize Dolbeault's homomorphism. Let $\mathscr{E}'_{p,q}(X)$ denote the space of forms of type (p,q) whose coefficients are compactly supported distributions; or equivalently, $\mathscr{E}'_{n,n}(X)$ is the dual space of the Fréchet space $C^\infty(X)$ and, more generally, $\mathscr{E}'_{p,q}(X)$ is the dual space of the Fréchet space $\mathscr{E}_{n-p,n-q}(X)$ under the pairing $\langle \phi, u \rangle = (\phi \wedge u)$ (1), where $\phi \in \mathscr{E}_{n-p,n-q}(X)$, $u \in \mathscr{E}'_{p,q}(X)$, and hence $\phi \wedge u \in \mathscr{E}'_{n,n}(X)$ (see Serre [3]).

Proposition 1.7. *Suppose U is a Stein manifold. Then the dual space of the Fréchet space $\mathscr{Z}_{n,q}(U)$ is isomorphic to the quotient space $\mathscr{E}'_{0,n-q}(U)/\bar{\partial}\mathscr{E}'_{0,n-q-1}(U)$.*

For the proof see Serre [3]. This proposition provides some motivation for the following definitions. We will assume that $\mathscr{U}$ is a Stein covering.

A *p-chain $\bar{u}$ with values in $\mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}$* is a map which assigns to every I with $|I| = p+1$ an element $\bar{u}_I \in \mathscr{E}'_{0,q}(U_I)/\bar{\partial}\mathscr{E}'_{0,q-1}(U_I)$ so that $\bar{u}_I$ is an alternating function of I, and so that $\bar{u}_I = 0$ except for a finite number of multi-indices I. Let $C_p(\mathscr{U}, \mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1})$ denote the vector space of all p-chains with values in $\mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}$. Here u_I denotes an element of $\mathscr{E}'_{0,q}(U_I)$ and $\bar{u}_I$ denotes the equivalence class in $\mathscr{E}'_{0,q}(U_I)/\bar{\partial}\mathscr{E}'_{0,q-1}(U_I)$ determined by u_I. A boundary map

$$\delta^*: C_p(\mathscr{U}, \mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}) \leftarrow C_{p+1}(\mathscr{U}, \mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1})$$

can be defined by

$$(\delta^*\bar{u})_I = \sum_i \bar{u}_{iI}. \tag{1.6}$$

Here iI denotes the multi-index $(i, i_0, \cdots, i_p)$. Obviously $\delta^*\delta^* = 0$. Let

$$Z_p(\mathscr{U}, \mathscr{E}_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}) = \{\bar{u} \in C_p(\mathscr{U}, \mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}) : \delta^*\bar{u} = 0\},$$

and let

$$H_p(\mathscr{U}, \mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}) = Z_p(\mathscr{U}, \mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1})/\delta^* C_{p+1}(\mathscr{U}, \mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}).$$

Replacing $\mathscr{E}'_{0,q}/\bar{\partial}\mathscr{E}'_{0,q-1}$ by $\mathscr{E}'_{0,q}$, $\bar{u}$ by u, and $\bar{u}_I$ by u_I in the above definitions we obtain the vector space $C_p(\mathscr{U}, \mathscr{E}'_{0,q})$ of p-chains u with values in $\mathscr{E}'_{0,q}$ and the space $H_p(\mathscr{U}, \mathscr{E}'_{0,q}) = Z_p(\mathscr{U}, \mathscr{E}'_{0,q})/\delta^* C_{p+1}(\mathscr{U}, \mathscr{E}'_{0,q})$. Using the map T^* defined below (1.8) and the definition (1.6) of δ^* we could now prove that $\delta^* T + T^* \delta^*$ is the identity on $C_p(\mathscr{U}, \mathscr{E}'_{0,q})$ by a direct calculation. Instead we deduce this fact (Proposition 1.10) from Lemma 1.2 and the fact that δ^* and T^* are the adjoints of δ and T respectively.

Now assume that $\mathscr{U}$ is a countable covering of X. Then $C^p(\mathscr{U}, \mathscr{E}_{n,q})$ with the relative topology induced from the Fréchet space $\prod_{|I|=p+1} \mathscr{E}_{n,q}(U_I)$ is obviously a Fréchet space. If $f^p \in C^p(\mathscr{U}, \mathscr{E}_{n,q})$ and $u^p \in C_p(\mathscr{U}, \mathscr{E}'_{0,n-q})$ let

$$(1.7) \qquad \langle f^p, u^p \rangle = \frac{1}{(p+1)!} \sum_{|I|=p+1} \langle f_I^p, u_I^p \rangle = \sum_{|I|=p+1}{}' \langle f_I^p, u_I^p \rangle$$

where Σ' denotes summation over strictly increasing multi-indices. Now, $C_p(\mathscr{U}, \mathscr{E}'_{0,n-q})$ is easily seen to be isomorphic to the dual space of the Fréchet space $C^p(\mathscr{U}, \mathscr{E}_{n,q})$ under the pairing $\langle f^p, u^p \rangle$. Since $C^p(\mathscr{U}, \mathscr{Z}_{n,q})$ is a closed subspace of $C^p(\mathscr{U}, \mathscr{E}_{n,q})$ it is a Fréchet space. Note that if u^p and v^p determine the same element in $C_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1})$ (i.e. $\bar{u}^p = \bar{v}^p$) then $\langle f^p, u^p \rangle = \langle f^p, v^p \rangle$. Therefore (1.7) defines a bilinear pairing $\langle f^p, \bar{u}^p \rangle$ between $C^p(\mathscr{U}, \mathscr{Z}_{n,q})$ and $C_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1})$. It follows easily from Proposition 1.7 that this pairing is non degenerate (i.e., that $C_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1})$ is isomorphic to the dual of $C^p(\mathscr{U}, \mathscr{Z}_{n,q})$) if each U_i is Stein.

Proposition 1.8. *The coboundary map $\delta: C^{p-1}(\mathscr{U}, \mathscr{E}_{n,q}) \to C^p(\mathscr{U}, \mathscr{E}_{n,q})$ is a continuous linear map with adjoint*

$$\delta^*: C_{p-1}(\mathscr{U}, \mathscr{E}'_{0,n-q}) \leftarrow C_p(\mathscr{U}, \mathscr{E}'_{0,n-q}).$$

Also the coboundary map $\delta: C^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q}) \to C^p(\mathscr{U}, \mathscr{Z}_{n,q})$ is a continuous linear map with adjoint

$$\delta^*: C_{p-1}(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1}) \leftarrow C_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1}).$$

Proof. Suppose $f^{p-1} \in C^{p-1}(\mathscr{U}, \mathscr{E}_{n,q})$ and $u^p \in C^p(\mathscr{U}, \mathscr{E}'_{0,n-q})$.

$$\langle \delta f^{p-1}, u^p \rangle = \frac{1}{(p+1)!} \sum_{|J|=p+1} \sum_{k=0}^{p} (-1)^k \langle f^{p-1}_{j_0 \ldots \hat{j}_k \ldots j_p}, u_J^p \rangle$$

$$= \frac{1}{p!} \sum_{|J|=p} \sum_i \langle f_I^{p-1}, u_{iI}^p \rangle = \langle f^{p-1}, \delta^* u^p \rangle,$$

where the middle equality follows by letting $I = (j_0, \cdots, \hat{j}_k, \cdots, j_p)$ and $i = j_k$.

A family $\{\phi_i\}$ of functions satisfying the conditions in Lemma 1.1 can be used to construct a chain homotopy

$$T^*: C_{p-1}(\mathscr{U},\mathscr{E}'_{0,q}) \to C_p(\mathscr{U},\mathscr{E}'_{0,q})$$

for

$$\delta^*: C_{p-1}(\mathscr{U},\mathscr{E}'_{0,q}) \leftarrow C_p(\mathscr{U},\mathscr{E}'_{0,q}).$$

Given $u^{p-1} \in C_{p-1}(\mathscr{U},\mathscr{E}'_{0,q})$, let

$$(1.8) \qquad (T^*u^{p-1})_J = \sum_{k=0}^{p} (-1)^k \phi_{j_k} u^{p-1}_{j_1 \dots \hat{j}_k \dots j_p} \text{ for each } |J| = p+1.$$

Since ϕ_{j_k} vanishes in a neighborhood of $X - U_{j_k}$ and $u^{p-1}_{j_0 \cdots \hat{j}_k \dots j_p}$ is compactly supported in $U_{j_0 \dots \hat{j}_k \dots j_p}$, their product is compactly supported in U_J. Hence $(T^*u^{p-1})_J \in \mathscr{E}'_{0,q}(U_J)$ for each $|J| = p+1$, so that T^* defines a map from $C_{p-1}(\mathscr{U},\mathscr{E}'_{0,q})$ into $C_p(\mathscr{U},\mathscr{E}'_{0,q})$.

Proposition 1.9. *The map* $T: C^{p-1}(\mathscr{U},\mathscr{E}_{n,q}) \leftarrow C^p(\mathscr{U},\mathscr{E}_{n,q})$ *defined by* (1.1) *is a continuous linear map with adjoint* $T^*: C_{p-1}(\mathscr{U},\mathscr{E}'_{0,n-q}) \to C_p(\mathscr{U},\mathscr{E}'_{0,n-q})$ *defined by* (1.8).

Proof. Suppose $f^p \in C^p(\mathscr{U},\mathscr{E}_{n,q})$ and $u^{p-1} \in C_{p-1}(\mathscr{U},\mathscr{E}'_{0,n-q})$. Then

$\langle Tf^p, u^{p-1}\rangle = \dfrac{1}{p!} \sum_{|I|=p} \sum_i \langle \phi_i f^p_{iI}, u^{p-1}_I\rangle$ is equal to

$$\frac{1}{(p+1)!} \sum_{|J|=p+1} \sum_{k=0}^{p} (-1)^k \langle f^p_J, \phi_{j_k} u^{p-1}_{j_0 \dots \hat{j}_k \dots j_p}\rangle = \langle f^p, T^*u^{p-1}\rangle,$$

by the change of variables $I = (j_0, \cdots, \hat{j}_k, \cdots, j_p)$ and $i = j_k$ (as in the proof of Proposition 1.8).

Proposition 1.10. $\delta^*T^* + T^*\delta^*: C_p(\mathscr{U},\mathscr{E}'_{0,q}) \to C_p(\mathscr{U},\mathscr{E}'_{0,q})$ *is the identity* $(p \geqq 1)$, *and* $\delta^*T^* + T^*\varepsilon^*$ *is the identity on* $C_0(\mathscr{U},\mathscr{E}'_{0,q})$. *Therefore, the sequence*

$$0 \leftarrow \mathscr{E}'_{0,q}(X) \xleftarrow{\varepsilon^*} C_0(\mathscr{U},\mathscr{E}'_{0,q}) \xleftarrow{\delta^*} C_1(\mathscr{U},\mathscr{E}_{0,q}) \leftarrow \cdots$$

is exact.

Remark. For $p = 0$, let $\varepsilon^*: C_0(\mathscr{U},\mathscr{E}'_{0,q}) \to \mathscr{E}'_{0,q}(X)$ be defined by $\varepsilon^* u = \sum u_i$ and let $T^*: C_0(\mathscr{U},\mathscr{E}'_{0,q}) \leftarrow \mathscr{E}'_{0,q}(X)$ be defined by $(T^*v)_i = \phi_i v$. Define δ^* to be zero on $C_0(\mathscr{U},\mathscr{E}'_{0,q})$ so that $H_0(\mathscr{U},\mathscr{E}'_{0,q})$ is isomorphic to $\mathscr{E}'_{0,q}(X)$.

Proposition 1.11. *The map* $\bar{\partial}: \mathscr{E}_{n,q}(U) \to \mathscr{E}_{n,q+1}(U)$ *is continuous with adjoint* $(-1)^{n+q+1}\bar{\partial}: \mathscr{E}'_{0,n-q}(U) \to \mathscr{E}'_{0,n-q-1}(U)$.

See Serre [3] for the proof.

Next we construct adjoint Dolbeault homomorphisms

$$
\begin{aligned}
(1.9)\qquad & H_r(\mathcal{U},\mathcal{E}'_{0,n}/\bar\partial\mathcal{E}'_{0,n-1}) \leftarrow H_{r-1}(\mathcal{U},\mathcal{E}'_{0,n-1}/\bar\partial\mathcal{E}'_{0,n-2}) \leftarrow \cdots \\
& \cdots \leftarrow H_p(\mathcal{U},\mathcal{E}'_{0,n-r+p}/\bar\partial\mathcal{E}'_{0,n-r+p-1}) \leftarrow \cdots \\
& \cdots \leftarrow H_1(\mathcal{U},\mathcal{E}'_{0,n-r+1}/\bar\partial\mathcal{E}'_{0,n-r}) \leftarrow \{u\in\mathcal{E}'_{0,n-r}(X)\colon \bar\partial u = 0\}/\bar\partial\mathcal{E}'_{0,n-r-1}(X),
\end{aligned}
$$

which taken together map the compactly supported Dolbeault group $\{u\in\mathcal{E}'_{0,n-r}(X)\colon\ \bar\partial u = 0\}/\bar\partial\mathcal{E}'_{0,n-r-1}(X)$ into $H_r(\mathcal{U},\mathcal{E}'_{0,n}/\bar\partial\mathcal{E}'_{0,n-1})$.

Suppose $\bar u^{p-1}\in C_{p-1}(\mathcal{U},\mathcal{E}'_{0,n-q-1}/\bar\partial\mathcal{E}'_{0,n-q-2})$. Let

$$
(1.10)\qquad (D^*\bar u^{p-1})_J = (-1)^{n+q+1} i^*\left[\sum_{k=0}^{p}(-1)^k\phi_{j_k}\bar\partial u^{p-1}_{j_0\ldots\hat{j}_k\ldots j_p}\right],
$$

for each $|J| = p+1$.

Proposition 1.12. *The map* $D\colon C^p(\mathcal{U},\mathcal{Z}_{n,q})\to C^{p-1}(\mathcal{U},\mathcal{Z}_{n,q+1})$ *is continuous and its adjoint is*

$$
D^*\colon C_p(\mathcal{U},\mathcal{E}'_{0,n-q}/\bar\partial\mathcal{E}'_{0,n-q-1}) \leftarrow C_{p-1}(\mathcal{U},\mathcal{E}'_{0,n-q-1}/\bar\partial\mathcal{E}'_{0,n-q-2}).
$$

Proof. Since $D = \bar\partial T i$, the adjoint of D equals $(-1)^{n+q+1}i^*T^*\bar\partial$ by Proposition 1.9 and Proposition 1.11.

(1.11)

$$
\begin{tikzcd}
 & 0 & & 0 & \\
{} & C_{p-1}(\mathcal{U},\mathcal{E}'_{0,n-q}/\bar\partial\mathcal{E}'_{0,n-q-1} \arrow[l, "\delta^*"'] \arrow[u] & & C_p(\mathcal{U},\mathcal{E}'_{0,n-q}/\bar\partial\mathcal{E}'_{0,n-q-1}) \arrow[ll, "\delta^*"'] \arrow[u] & {} \arrow[l, "\delta^*"'] \\
{} & C_{p-1}(\mathcal{U},\mathcal{E}'_{0,n-q}) \arrow[l, "\delta^*"'] \arrow[u, "i^*"'] \arrow[rr, shift right, "T^*"'] & & C_p(\mathcal{U},\mathcal{E}'_{0,n-q}) \arrow[ll, shift right, "\delta^*"'] \arrow[u, "i^*"'] & {} \arrow[l, "\delta^*"'] \\
{} & C_{p-1}(\mathcal{U},\mathcal{E}'_{0,n-q-1}/\bar\partial\mathcal{E}'_{0,n-q-2}) \arrow[l, "\delta^*"'] \arrow[u, "(-1)^{n+q+1}\bar\partial"'] & & C_p(\mathcal{U},\mathcal{E}'_{0,n-q-1}/\bar\partial\mathcal{E}'_{0,n-q-2}) \arrow[ll, "\delta^*"'] \arrow[u, "(-1)^{n+q+1}\bar\partial"'] & {} \arrow[l, "\delta^*"']
\end{tikzcd}
$$

Proposition 1.13. *The operators* D^* *and* δ^* *anticommute. Hence* D^* *induces a map from*

$$
H_{p-1}(\mathcal{U},\mathcal{E}'_{0,n-q-1}/\bar\partial\mathcal{E}'_{0,n-q-2}) \textit{ into } H_p(\mathcal{U},\mathcal{E}'_{0,n-q}/\bar\partial\mathcal{E}'_{0,n-q-1}).
$$

Proof. By Proposition 1.4, $\delta D + D\delta = 0$. Therefore, by Proposition 1.8 and Proposition 1.12, $D^*\delta^* + \delta^* D^* = 0$.

We will refer to the map induced by D^* as the *adjoint Dolbeault homomorphism*. It follows from a diagram chase in (1.11) that this homomorphism can be described (equivalently) as follows.

Definition 1.14. The class $[\bar{v}^p] \in H_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1})$ is the image of the class $[\bar{v}^{p-1}] \in H_{p-1}(\mathscr{U}, \mathscr{E}'_{0,n-q-1}/\bar{\partial}\mathscr{E}'_{0,n-q-2})$ under the *adjoint Dolbeault homomorphism* if there exists $u^p \in C_p(\mathscr{U}, \mathscr{E}'_{0,n-q})$ such that

$$(1.12) \quad (\delta^* u^p)_I = (-1)^{n+q+1}\bar{\partial} v_I^{p-1} \text{ and } u_J^p - v_J^p \in \bar{\partial}\mathscr{E}'_{0,n-q-1}(U_J).$$

If $p = 1$ then $[\bar{v}^0] \in \{w \in \mathscr{E}'_{0,n-q-1}(X) : \bar{\partial} w = 0\}/\bar{\partial}\mathscr{E}'_{0,n-q-2}(X)$ replaces the statement $[\bar{v}^{p-1}] \in H_{p-1}(\mathscr{U}, \mathscr{E}'_{0,n-q-1}/\bar{\partial}\mathscr{E}'_{0,n-q-2})$ where v^0 also denotes $\varepsilon^* v^0 = \sum_i v_i^0 \in \mathscr{E}'_{0,n-q-1}(X)$ (given a $v^0 \in C_0(\mathscr{U}, \mathscr{E}'_{0,n-q-1})$).

Definition 1.15. A sequence $\{\bar{u}^p\}$ with $\bar{u}^p \in Z_p(\mathscr{U}, \mathscr{E}'_{0,n-r+p}/\bar{\partial}\mathscr{E}'_{0,n-r+p-1})$ $p = 0, 1, \cdots, r$ such that $[\bar{u}^p] \in H_p(\mathscr{U}, \mathscr{E}'_{0,n-r+p}/\bar{\partial}\mathscr{E}'_{0,n-r+p-1})$ is the image of $[u^{p-1}]$ under the adjoint Dolbeault homomorphism for $p = 1, 2, \cdots, r$, will be called an *adjoint sequence of Dolbeault representatives* of $[u^r] \in H_r(\mathscr{U}, \mathscr{E}'_{0,n-r}/\bar{\partial}\mathscr{E}'_{0,n-r-1})$. In the special case $p = 0$,

$$[u^0] \in \{w \in \mathscr{E}'_{0,n-r}(X) : \bar{\partial} w = 0\}/\bar{\partial}\mathscr{E}'_{0,n-r-1}(X).$$

Consider the bilinear pairing $\langle f^p, \bar{u}^p \rangle$ between $C^p(\mathscr{U}, \mathscr{Z}_{n,q})$ and $C_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1})$ restricted to $Z^p(\mathscr{U}, \mathscr{Z}_{n,q})$ and

$$Z_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1}).$$

This restricted bilinear pairing vanishes if either $f^p \in \delta C^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q})$ or $\bar{u}^p \in \delta^* C_{p+1}(\mathscr{U}, \mathscr{E}_{0,n-q}/\bar{\partial}\mathscr{E}_{0,n-q-1})$, since $\langle \delta g^{p-1}, \bar{u}^p \rangle = \langle g^{p-1}, \delta^* \bar{u}^p \rangle = 0$ and $\langle f^p, \delta^* \bar{v}^{p+1} \rangle = \langle \delta f^p, \bar{v}^{p+1} \rangle = 0$. Therefore, the bilinear pairing (1.7) induces a bilinear pairing between $H^p(\mathscr{U}, \mathscr{Z}_{n,q})$ and $H_p(\mathscr{U}, \mathscr{E}'_{0,n-q}/\bar{\partial}\mathscr{E}'_{0,n-q-1})$.

Now the main theorem of this section follows easily.

Theorem 1.16. *Suppose $\mathscr{U}$ is a Stein covering of X. If $\{f^p\}$ $(p = 0, \cdots, r)$ is a sequence of Dolbeault representatives of a cohomology class $[f^r] \in H^r(\mathscr{U}, \Omega^n)$ and $\{\bar{u}^p\}$ is an adjoint sequence of Dolbeault representatives of $[\bar{u}_r] \in H_r(\mathscr{U}, \mathscr{E}'_{0,n-r}/\bar{\partial}\mathscr{E}'_{0,n-r-1})$ then $\langle f^p, u^p \rangle$ is the same for all $p = 0, \cdots, r$. In particular,*

$$(1.13) \quad \int_X f^0 \wedge u^0 = \sum_{|I|=r+1}' \int_X f_I^r \wedge u_I^r.$$

Proof. As mentioned above the bilinear pairing $\langle\ ,\ \rangle$ induces a bilinear pairing between $H^p(\mathscr{U}, \mathscr{Z}_{n,r-p})$ and $H_p(\mathscr{U}, \mathscr{E}'_{0,n-r+p}/\bar{\partial}\mathscr{E}'_{0,n-r+p-1})$. Therefore, to prove the theorem it suffices to replace $\{f^p\}$ by any sequence of Dolbeault representatives of $[f^r]$ and replace $\{u^p\}$ by any adjoint sequence of

Dolbeault representatives of $[u']$. Now replace $\{f^p\}$ by $\{D^{r-p}f^r\}$ and $\{u^p\}$ by $\{(D^*)^p u^0\}^r p = 0, \cdots, r$. Then $\langle f^p, u^p\rangle = \langle D^{r-p}f^r, (D^*)^p u^0\rangle = \langle D^{r-p-1}f^r, (D^*)^{p+1}u^0\rangle = \langle f^{p+1}, u^{p+1}\rangle$ by Proposition 1.12.

Remark. If, for any $p = 0, \cdots, r$, $\delta C^{p-1}(\mathscr{U}, \mathscr{Z}_{n,r-p})$ is closed in $Z^p(\mathscr{U}, \mathscr{Z}_{n,r-p})$ and $\delta^* C_{p+1}(\mathscr{U}, \mathscr{E}'_{0,n-r+p}/\bar{\partial}\mathscr{E}'_{0,n-r+p-1})$ is closed in $Z_p(\mathscr{U}, \mathscr{E}'_{0,n-r+p}/\bar{\partial}\mathscr{E}'_{0,n-r+p-1})$ then the same is true for all p, and $H_p(\mathscr{U}, \mathscr{E}'_{0,n-r+p}/\bar{\partial}\mathscr{E}'_{0,n-r+p-1})$ is the dual of the reflexive Fréchet space $H^p(\mathscr{U}, \mathscr{Z}_{n,r-p})$ (i.e., the bilinear pairing $\langle \rho, \sigma\rangle$ is non degenerate). We will not pursue this extension of Serre duality further.

We conclude this section by proving that Dolbeault's homomorphism commutes with "restriction." This fact will enable us to mimick the standard proof of Cauchy's integral formula in one variable in proving integral formula in several variables. Suppose $\mathscr{V}$ is a refinement of $\mathscr{U}$, that is, there exists a map ρ from the index set for $\mathscr{V}$ into the index set for $\mathscr{U}$ such that $V_i \subset U_{\rho(i)}$ for all i. Let $\rho^*: C^p(\mathscr{U}, \mathscr{F}) \to C^p(\mathscr{V}, \mathscr{F})$ be defined by $(\rho^* f)_I = f_{\rho(i_0)\dots\rho(i_p)}$ in V_I. The map ρ^* obviously commutes with δ and hence induces a map $\rho^*: H^p(\mathscr{U}, \mathscr{F}) \to H^p(\mathscr{V}, \mathscr{F})$. This map ρ^* can be shown to be independent of the particular map ρ. We will refer to this map as restriction from $\mathscr{U}$ to $\mathscr{V}$. Note that if $p = 0$ then ρ^* is just restriction of sections of $\mathscr{F}$ from $\bigcup_i U_i$ to $\bigcup_j V_j$.

Proposition 1.17. *Dolbeault's homomorphism commutes with restriction. That is,*

$$\begin{array}{ccc} H^p(\mathscr{U}, \mathscr{Z}_{n,q}) & \longrightarrow & H^{p-1}(\mathscr{U}, \mathscr{Z}_{n,q+1}) \\ \downarrow \rho^* & & \downarrow \rho^* \\ H^p(\mathscr{V}, \mathscr{Z}_{n,q}) & \longrightarrow & H^{p-1}(\mathscr{V}, \mathscr{Z}_{n,q+1}) \end{array}$$

commutes.

Proof. Use Definition 1.5 and the fact that δ commutes with ρ^*.

2. *Sequences of Dolbeault Representatives*

First, for an arbitrary open covering $\mathscr{U}$ of a complex manifold X countable at infinity, we have the sequence of Dolbeault representatives defined by (1.3).

Proposition 2.1. *Suppose $\{\phi_i\}$ is a family of functions satisfying the conditions in Lemma 1.1. Given $f^r \in Z^r(\mathscr{U}, \Omega^n)$, let*

$$(2.1) \qquad f_J^p = (-1)^{a_p} \sum_{|K|=r-p} \bar{\partial}\phi^K \wedge f_{KJ}^r \quad \text{for } all \ |J| = p+1,$$

where $a_p = \frac{1}{2}(r-p)(r-p-1)$. *Then* $\{f^p\}$ $(p = 0,\cdots,r)$ *is a sequence of Dolbeault representatives of* $[f^r] \in H^r(\mathcal{U}, \Omega^h)$.

Proof. The proposition follows from the fact that $D^{r-p}f^r = f^p$. (Here D is defined by (1.3) and D^k denotes the kth iterate of D.) We prove this fact by induction. Suppose $D^{r-p}f^r = f^p$. Then $(D^{r-p+1}f^r)_I = (Df^p)_I = \sum_i \bar{\partial}\phi_i \wedge f_{iI}^p$ by (1.3). Now by (2.1) this equals $(-1)^{a_p} \sum_i \sum_{|K|=r-p} \bar{\partial}\phi_i \wedge \bar{\partial}\phi^K \wedge f_{KiI}^r$ which equals $(-1)^{a_{p+1}} \sum_{|L|=r-p+1} \bar{\partial}\phi^L \wedge f_{LI}^r$ (let $L = Ki$).

Next we consider certain sequences of Dolbeault representatives which relate to integral formulae in several complex variables. First we examine the Cauchy kernel (Theorem 2.2) and then the more general Cauchy-Weil kernel (Theorem 2.6).

Let K_i $(i=1,\cdots,n)$ denote a compact subset of $\mathbb{C}$ and let $K = K_1 \times \cdots \times K_n$. Let $U_i = \{z \in \mathbb{C}^n : z_i \notin K_i\}$, $i = 1,\cdots,n$. Then each U_i is a domain of holomorphy and $\mathcal{U} = \{U_i\}_{i=1}^n$ is an open covering of $\mathbb{C}^n - K$. For a fixed $z \in K$, the *Cauchy kernel*,

$$k(\xi,z) = (2\pi i)^{-n} \frac{d\xi}{(\xi_1 - z_1)\cdots(\xi_n - z_n)} \in \Omega^n(U_1 \cap \cdots \cap U_n),$$

obviously determines a unique cocycle (also denoted $k(\xi,z)$) in $Z^{n-1}(\mathcal{U},\Omega^n)$. Note $\delta k = 0$ since $C^n(\mathcal{U}\ \Omega^n) = 0$.

Theorem 2.2. *Suppose for a fixed* $z \in K$, *functions* $g_i(\xi,z) \in C^\infty(\mathbb{C}^n - K)$ $i=1,\cdots,n$ *are given such that* $g(\xi,z) = g_1(\xi,z)(\xi_1 - z_1) + \cdots + g_n(\xi,z)(\xi_n - z_n)$ *never vanishes on* $\mathbb{C}^n - K$. *Let* $\phi_i(\xi,z) = g_i(\xi,z)(\xi_i - z_i)/g(\xi,z)$ $i = 1,\cdots,n$. *Then* (2.1) *defines a sequence* $\{k^p(\xi,z)\}$ $(p = n-1,\cdots,0)$ *of Dolbeault representatives of the cohomology class* $[k^{n-1}] \in H^{n-1}(\mathcal{U},\Omega^n)$ *where* $k^{n-1}(\xi,z)$ *is the Cauchy kernel. For* $p = 0$,

$$(2.2) \quad k^0(\xi,z) = (-1)^{\frac{1}{2}n(n-1)}(n-1)!(2\pi i)^{-n}(-1)^{i-1} \frac{\bar{\partial}\phi_1 \wedge \cdots \wedge \widehat{\bar{\partial}\phi_i} \wedge \cdots \wedge \bar{\partial}\phi_n \wedge d\xi}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}$$

which can also be expressed as

$$(2.3) \quad k^0(\xi,z) = (-1)^{\frac{1}{2}n(n-1)}(n-1)![2\pi i g(\xi,z)]^{-n} \times \sum_{j=1}^{n} (-1)^{j-1} g_j(\xi,z) \bar{\partial}g_1 \wedge \cdots \wedge \widehat{\bar{\partial}g_j} \wedge \cdots \wedge \bar{\partial}g_n d\xi.$$

Remarks. If $g_i(\xi,z) = \bar{\xi}_i - \bar{z}_i$ then (2.3) is the Bochner-Martinelli kernel

(2.4)

$$(-1)^{\frac{1}{2}n(n-1)}(n-1)!(2\pi i)^{-n}\frac{\sum_{j=1}^{n}(-1)^{j-1}(\bar{\xi}_j-\bar{z}_j)d\bar{\xi}_1\wedge\cdots\wedge\widehat{d\bar{\xi}_j}\wedge\cdots\wedge d\bar{\xi}_n\wedge d\xi}{|\xi-z|^{2n}}.$$

The general kernel (2.3) is called a kernel of *Cauchy-Fantappié type.*

Note that the above theorem says in particular that a kernel of Cauchy-Fantappié type is $\bar{\partial}$-closed in $\mathbb{C}^n - K$ and that any two such kernels differ by $\bar{\partial}\omega$, where $\omega \in \mathscr{E}_{n,n-2}(\mathbb{C}^n - K)$.

Proof. The proof of all but (2.3) is exactly the same as the proof of Proposition 2.1, since $\sum_{i=1}^{n}\phi_i \equiv 1$ in $\mathbb{C}^n - K$ and $\phi_i k_{iI}^{p+1}$ extends to a form in U_I where $\phi_i(\xi,z)$ is defined to be $g_i(\xi,z)(\xi_i - z_i)/g(\xi,z)$.

Now (2.3) can be proved as follows. Differentiation yields $\bar{\partial}\phi_k = (\xi_k - z_k)g(\xi,z)^{-2}(g\bar{\partial}g_k - g_k\bar{\partial}g)$. Let $\omega_k = g\bar{\partial}g_k - g_k\bar{\partial}g$. Then

$$\bar{\partial}\phi_1\wedge\cdots\wedge\widehat{\bar{\partial}\phi_i}\wedge\cdots\wedge\bar{\partial}\phi_n =$$
$$[(\xi_1-z_1)\cdots(\xi_n-z_n)/(\xi_i-z_i)g(\xi,z)^{2n-2}]\omega_1\wedge\cdots\wedge\widehat{\omega_i}\wedge\cdots\wedge\omega_n.$$

Assume $i \neq 1$. Then since ω_1 occurs in $\omega_1\wedge\cdots\wedge\widehat{\omega_i}\wedge\cdots\wedge\omega_n$ we may replace each ω_k $(k>1)$ by $\omega_k - g_k g_1^{-1}\omega_1 = g(\bar{\partial}g_k - g_k g_1^{-1}\bar{\partial}g_1)$. That is,

$$\omega_1\wedge\cdots\wedge\widehat{\omega_i}\wedge\cdots\wedge\omega_n =$$
$$g^{n-2}(g\bar{\partial}g_1 - g_1\bar{\partial}g)\wedge(\bar{\partial}g_2 - g_2g_1^{-1}\bar{\partial}g_1)\wedge\cdots$$
$$\wedge(\bar{\partial}g_{i-1} - g_{i-1}g_1^{-1}\,\bar{\partial}g_1)\wedge(\bar{\partial}g_{i+1}-g_{i+1}g_1^{-1}\bar{\partial}g_1)\wedge\cdots\wedge(\bar{\partial}g_n - g_ng_1^{-1}\bar{\partial}g_1).$$

Now by a direct calculation this equals

$$g^{n-2}\sum_{j\neq i}(-1)^{i+j}(\xi_i-z_i)g_j\bar{\partial}g_1\wedge\cdots\wedge\widehat{\bar{\partial}g_j}\wedge\cdots\wedge\bar{\partial}g_n$$
$$+\ g^{n-2}\left[g - \sum_{j\neq i}(\xi_j-z_j)g_j\right]\bar{\partial}g_1\wedge\cdots\wedge\widehat{\bar{\partial}g_i}\wedge\cdots\wedge\bar{\partial}g_n.$$

Consequently, $\bar{\partial}\phi_1\wedge\cdots\wedge\widehat{\bar{\partial}\phi_i}\wedge\cdots\wedge\bar{\partial}\phi_n$

$$= (-1)^i(\xi_1 - z_1)\cdots(\xi_n - z_n)g^{-n}\sum_{j=1}^{n}(-1)^j g_j\bar{\partial}g_1\wedge\cdots\wedge\widehat{\bar{\partial}g_j}\wedge\cdots\wedge\bar{\partial}g_n.$$

This completes the proof of (2.3) and hence the theorem.

The construction of the Cauchy-Weil kernel depends on the following lemma.

Lemma 2.3. *Let U denote an arbitrary open set in $\mathbb{C}^n$. Given $f\in\mathcal{O}(U)$,* *there exist functions* $b_i(\xi,z) \in \mathcal{O}(U \times U)$ *such that* $f(\xi) - f(z) = b_1(\xi,z)(\xi_1 - z_1) + \cdots + b_n(\xi,z)(\xi_n - z_n)$ in $U \times U$.

Remark. If $U = \mathbb{C}^n$ then the lemma is trivial (expand $f(\xi)$ in a power series about z).

Proof. Let $\tilde{U}$ denote the envelope of holomorphy of U with projection $\pi: \tilde{U} \in \mathbb{C}^n$. The functions $f(\xi)$ and ξ_i extend to holomorphic functions on $\tilde{U}$ which we also denote by $f(\xi)$ and ξ_i. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on $\tilde{U} \times \tilde{U}$ and let $\lambda: \mathcal{O}^n \to \mathcal{O}$ denote the sheaf homomorphism defined by $\lambda:(g^1,\cdots,g^n) \to g^1(\xi_1 - z_1) + \cdots + g^n(\xi_n - z_n)$. Let $\mathscr{R}$ denote the kernel of λ and $\mathscr{F}$ the image of λ. Then $0 \to \mathscr{R} \to \mathcal{O}^n \to \mathscr{F} \to 0$ is an exact sequence of coherent analytic sheaves on the Stein manifold $\tilde{U} \times \tilde{U}$. Therefore $H^1(\tilde{U} \times \tilde{U}, \mathscr{R}) = 0$ and hence $\lambda: \mathcal{O}(\tilde{U} \times \tilde{U}) \to \Gamma(\tilde{U} \times \tilde{U}, \mathscr{F})$ is surjective. It remains to show that $f(\xi) - f(z) \in \Gamma(\tilde{U} \times \tilde{U}, \mathscr{F})$. By using power series we have that the germ induced by $f(\xi) - f(z)$ belongs to $\mathscr{F}$ at each point $(\xi_0, z_0) \in U \times U$. This completes the proof if U is a domain of holomorphy. Suppose $x_0 \in \tilde{U}$ and $\pi x_0 = z_0$. Let $g(\xi) = f(\pi^{-1}(\xi))$ for ξ near z. Expanding in power series gives $g(\xi) - g(z) = \sum_{i=1}^n a_i(\xi,z) \cdot (\xi_i - z_i)$ for ξ, z both near z_0. Now $f(y) = g(\pi y)$ for y near x_0. Therefore $f(x) - f(y) = \sum_i a_i(\pi x, \pi y)(\xi_i - z_i)$ for x, y near x_0.

Suppose U is an open set in $\mathbb{C}^n$ and functions $h_i \in \mathcal{O}(U)$ are given. Let K_i denote a compact subset of $\mathbb{C}$ and let $U_i = \{z \in U : h_i(z) \notin K_i\}$. Then $\mathscr{U} = \{U_i\}$ is an open covering of $U - K$ where $K = \{z \in U : h_i(z) \in K_i$ for all $i\}$. Utilizing Lemma 2.3 (for each i) choose $a_{ij} \in \mathcal{O}(U \times U)$ such that $h_i(\xi) - h_i(z) = \sum_{j=1}^n a_{ij}(\xi,z)(\xi_j - z_j)$. Let A denote the matrix whose ith row is $(a_{i1},\cdots,a_{in})$. Given $|I| = n$, let $A_I(\xi,z)$ denote the $n \times n$ matrix obtained from the $i_0, i_1, \cdots, i_{n-1}$ rows of A.

Definition 2.4. For a fixed $z \in K$,

$$k_I(\xi,z) = (2\pi i)^{-n} \frac{\det A_I(\xi,z)d\xi}{\prod_{i \in I}(h_i(\xi) - h_i(z))} \in \Omega^n(U_I), \tag{2.5}$$

with $|I| = n$, determines a cochain $\{k_I(\xi,z)\} \in C^{n-1}(\mathscr{U}, \Omega^n)$ called the *Cauchy-Weil kernel.*

This cochain is in fact a cocycle.

Proposition 2.5. $\delta k = 0$.

Proof. Suppose $|J| = n+1$ is given. Consider the $(n+1) \times (n+1)$ matrix with rows $(a_{j_k 1,\ldots,} a_{j_k n}, h_{j_k}(\xi) - h_{j_k}(z))$ $k = 0, 1, \cdots, n$. Its determinant is zero since each row is orthogonal to the vector $((\xi_1 - z_1), \cdots, (\xi_n - z_n), -1)$. Computing this determinant by expanding about the last column gives $\sum_{k=0}^{n}(-1)^k(h_{j_k}(\xi) - h_{j_k}(z))\det A_{j_0 \ldots \hat{j}_k \cdots j_n}$ which must therefore equal zero. Now dividing this last expression by $\prod_{j \in J}(h_j(\xi) - h_j(z))$ we obtain $\delta k = 0$.

Now we can generalize Theorem 2.2.

Theorem 2.6. *Consider the covering* $\mathscr{U}$ *of* $U - K$ *described in Definition* 2.4. *Suppose for a fixed* $z \in K$ *that functions* $g_i(\xi, z) \in C^\infty(U - K)$ *are given such that* $\{\operatorname{supp} g_i(\xi, z)\}$ *is locally finite and*

$$g(\xi, z) = \Sigma_i g_i(\xi, z) \cdot (h_i(\xi) - h_i(z))$$

never vanishes on $U - K$. *Define* $\varphi_i(\xi, z) = g_i(\xi, z)(h_i(\xi) - h_i(z)) / g(\xi, z)$. *Then* (2.1) *defines a sequence* $\{k^p\}$ $(p = n - 1, \cdots, 0)$ *of Dolbeault representatives of the cohomology class* $[k^{n-1}] \in H^{n-1}(\mathscr{U}, \Omega^n)$ *determined by the Cauchy Weil kernel* $k^{n-1} = \{k_I(\xi, z)\}$. *The special case* $(p = 0)$,

$$(2.6) \quad k^0(\xi, z) = (-1)^{\frac{1}{2}n(n-1)}(n-1)! \sum_{|I| = n-1}{}' \bar{\partial}\phi^I(\xi) \wedge k_{iI}(\xi, z) \text{ in } U_i$$

can also be expressed as

$$(2.7) \quad k^0(\xi, z) = (-1)^{\frac{1}{2}n(n-1)}(n-1)![2\pi i\, g(\xi, z]^{-n}$$
$$\sum_{|J| = n}{}' \sum_{k=0}^{n-1} (-1)^k \det A_J(\xi, z) g_{j_k}(\xi, z) \bar{\partial} g_{j_0} \wedge \cdots \wedge \widehat{\bar{\partial} g_{j_k}} \wedge \cdots \wedge \bar{\partial} g_{j_{n-1}} \wedge d\xi.$$

Proof. The proof of all but (2.7) is exactly the same as the proof of Theorem 2.2. The proof of (2.7) is similar to the proof of its special case (2.3), except more complicated. The proof of (2.7) is omitted.

3. *Adjoint Sequences of Dolbeault Representatives*

The next proposition is an analogue of Proposition 2.1 in the dual situation. As before, we assume that X is countable at infinity, and that $\mathscr{U}$ is a Stein covering of X.

Proposition 3.1. *Suppose* $\{\phi_i\}$ *is a family of functions satisfying the conditions in Lemma* 1.1. *Given* $u^0 \in \mathscr{E}'_{0,n-r}(X)$ *with* $\bar{\partial} u^0 = 0$, *let*

$$(3.1) \quad u_J^p = (-1)^{a_p} p! \sum_{k=0}^{p} (-1)^k \phi_{j_k} \bar{\partial}\phi_{j_0} \wedge \cdots \wedge \widehat{\bar{\partial}\phi_{j_k}} \wedge \cdots \wedge \partial\phi_{j_p} \wedge u^0,$$

$p = 0, 1, \cdots, r$. Then $\{\bar{u}^p\}$ is an adjoint sequence of Dolbeault representatives of $[u^0] \in H_*^{n-r}(X, \mathscr{O})$. Here $a_0 = 0$ and $a_p = (n - r) + \frac{1}{2}p(p-1)$ for $p > 0$.

Proof. We prove more, namely that $u^p = (D^*)^p u^0$. Let v^p denote $(D^*)^p u^0$. Then by (1.9),

$$v_J^p = (-1)^{n-r+p-1} \sum_{k=0}^{p} (-1)^k \phi_{j_k} \bar{\partial} v^{p-1}_{j_2 \cdots \hat{j}_k \cdots j_p}, \tag{3.2}$$

with $v_i^0 = \phi_i u^0$. Now we prove that $v^p = u^p$ by induction. Differentiating (3.1) gives $\bar{\partial} v_I^{p-1} = \bar{\partial} u_I^{p-1} = (-1)^{a_{p-1}} p! \bar{\partial}\phi^I \wedge u^0$. By substituting for $\bar{\partial} v_I^{p-1}$ in the right-hand side of (3.2) we immediately obtain $v_J^p = u_J^p$.

Next we consider the important case $r = n-1$ and construct some special adjoint sequences of Dolbeault representatives.

Proposition 3.2. *Suppose* $\psi_i \in C^\infty(X)$ *with* $\operatorname{supp} \bar{\partial}\psi_i \subset\subset U_i$ *and* $\{\operatorname{supp}(1-\psi_i)\}$ *locally finite. Let* $u^0 = \bar{\partial} \prod_i \psi_i$ *and*

$$u_J^p = (-1)^{\frac{1}{2}p(p+1)} \prod_{j \notin J} \psi_j \bar{\partial} \psi^J, \quad p = 0, \cdots, n-1. \tag{3.3}$$

Then $\{\bar{u}^p\}$ *is an adjoint sequence of Dolbeault representatives of* $[u^0] \in H^1_*(X, \mathcal{O})$.

Proof.

$$(\delta^* u^p)_I = \sum_i u^p_{iI} = (-1)^{\frac{1}{2}p(p+1)} \sum_i \prod_{j \notin iI} \psi_j \bar{\partial} \psi^{iI} = (-1)^p \bar{\partial} u_I^{p-1}.$$

Therefore $[\bar{u}^p]$ is the image of $[\bar{u}^{p-1}]$ under the adjoint Dolbeault homomorphism (see Definition 1.15).

The assumption that $\psi_i \in C^\infty(X)$ can be weakened considerably. In particular, the formulas (3.3) can be made to include integration over the various strata of the topological boundary of the polydisc. This is made precise in the rest of this section. First we formalize the hypothesis of Proposition 3.2 as a definition.

Definition 3.3. A collection $\{\psi_i\}$ satisfying the hypothesis of Proposition 3.2 will be called a *smeared* C^∞*-polyhedron subordinate to* $\mathcal{U}$, and the u_J^p defined by (3.3) will be called *generalized strata*.

For simplicity assume that X is an open subset of $\mathbb{C}^n$ and that $\mathcal{U}$ is a finite covering of X. Suppose D is an open subset of X with the topological boundary ∂D compact. Suppose that pairwise disjoint oriented (not necessarily closed) C^∞ submanifolds $(\partial D)_I$ of U_I are given for all strictly increasing multi-indices I, and that $\partial D = \bigcup_I (\partial D)_I$. Extend the notation $(\partial D)_I$ to all multi-indices by skew-symmetry.

Definition 3.4. Suppose

1) $(D, \bigcup_i(\partial D)_i)$ is an (not necessarily closed) oriented C^∞-manifold with boundary,

2) $((\partial D)_I, \bigcup_i(\partial D)_{iI})$ is an (not necessarily closed) oriented C^∞-manifold with boundary,

3) Each $(\partial D)_I$ has finite volume and $(\partial D)_I$ for $|I| = n$ is a compact manifold without boundary·

Then D will be called a *C^∞-polyhedron in general position with respect to $\mathscr{U}$.*

Proposition 3.6. *Suppose D is a C^∞-polyhedron in general position with respect to $\mathscr{U}$. Let*

$$\langle f, u_J^p \rangle = \int_{(\partial D)_J} f \text{ for all } f \in \mathscr{E}_{n-1-p}(U_J), \tag{3.4}$$

$p = 0, \cdots, n-1$. *Then $\{\bar{u}^p\}$ is an adjoint sequence of Dolbeault representatives of $[u^0] \in H^1_*(X, \mathscr{O})$, where $\langle f, u^0 \rangle = \int_{\partial D} f$ for all $f \in \mathscr{E}_{n,n-1}(X)$.*

Proof. The proof is formally the same as the proof of Proposition 3.2. Here $\Sigma_i u_{iI}^p = (-1)^p \bar{\partial} u_I{}^{p-1}$ means

$$\left\langle f, \sum_i u_{iI}^p \right\rangle = \sum_i \int_{(\partial D)_{iI}} f = \int_{(\partial D)_I} \bar{\partial} f = \langle \bar{\partial} f, u_I^{p-1} \rangle = \langle f, (-1)^p \bar{\partial} u_I^{p-1} \rangle$$

for all $f \in \mathscr{E}_{n-1-p}(U_I)$. The second equality is true by a general version of Stokes' theorem (see Stolzenberg [4] and Federer [1]) since $\bar{\partial} f = df$.

The general version of Cauchy's integral formula in several complex variables, which we prove in the next section, will involve the following configuration. Suppose $h = (h_1, \cdots, h_N) \in \mathscr{O}(U)^N$ is a proper map of a domain of holomorphy U into $\mathbb{C}^N$, and that $K_i (i = 1, \cdots, N)$ are compact subsets of $\mathbb{C}$. Let $U_i = \{\xi \in U : h_i(\xi) \notin K_i\}$ and let $K = \{\xi \in U : h_i(\xi) \in K_i \text{ for all } i\}$. Then $\mathscr{U} = \{U_i\}$ is a Stein covering of $X = U - K$.

Lemma 3.6. *For each neighborhood V of K,*

1) *there exists a smeared C^∞-polyhedron $\{\psi_i\}$ subordinate to $\mathscr{U}$ such that $\Pi \psi_i$ extends as the constant function 1 across K and the extension is compactly supported in V;*

2) *there exists a C^∞-polyhedron D in general position with respect to $\mathscr{U}$ such that $G = D \cup K$ is an open, relatively compact subset of V.*

Proof of 1). Pick neighborhoods D_i of K_i such that $\bar{G} = \{\xi \in U : h_i(\xi) \in \bar{D}_i$ for all $i\}$ is a compact subset of V. Pick $\psi_i \in C^\infty(U)$ such that $\psi_i \equiv 1$ in a neighborhood of $\{\xi \in U: h_i(\xi) \in K_i\}$ and $\psi_i \equiv 0$ in a neighborhood of $\{\xi \in U: h_i(\xi) \notin D_i\}$. Then $\Pi\ \psi_i \in C_0^\infty(V)$ and is identically one in a neighborhood of K. Also, $\operatorname{supp} \bar\partial\psi_i$ is contained in the strip $\{\xi \in U: h_i(\xi) \in D_i - K_i\}$ which is contained in U_i.

Proof of 2). Pick neighborhoods D_i' of K_i with C^∞ boundaries $\partial D_i'$ such that $\bar G = \{\xi \in U: h_i(\xi) \in D_i$ for all $i\}$ is a compact subset of V. Let

$$\phi_i(z) = \begin{cases} d(z, \partial D_i') & \text{if } z \notin D_i' \\ -d(z, \partial D_i') & \text{if } z \in D_i'. \end{cases}$$

Then $\phi_i^0 h_i$ is a C^∞ map of a neighborhood of $\{\xi \in U: h_i(\xi) \in \partial D_i'\}$ into a neighborhood of $0 \in \boldsymbol{R}$. Sard's theorem can be used to insure that for arbitrarily small $r = (r_1, \cdots, r_N)$, the sets $S_i = \{\xi \in U: \phi_i(h_i(\xi)) = r_i\}$ are C^∞ manifolds and that for each I, the manifolds $S_{i_0}, \cdots, S_{i_p}$ intersect transversely. Let $D = \{\xi \in U: \phi_i^0 h_i(\xi) < r_i$ for all $i\}$ and let $(\partial D)_I = \{\xi \in U: \phi_i^0 h_i(\xi) = r_i$ for all $i \in I$ and $\phi_i^0 h_i(\xi) < r_i$ if $i \notin I\}$. Since $(\partial D)_I$ is the intersection of the bounded domain $\{\xi \in U: \phi_i^0 h_i(\xi) < r_i + \varepsilon$ for $i \in I$ and $\phi_i^0 h_i(\xi) < r_i$ for $i \notin I\}$ with the C^∞ manifold $S_{i_0} \cap \cdots \cap S_{i_p}$, $(\partial D)_I$ has finite volume.

4. *Cauchy's Integral Formula*

As before, suppose $h = (h_1, \cdots, h_N) \in \mathcal{O}(U)^N$ is a proper map of a domain of holomorphy U into $\mathbb{C}^N$, and that K_i $(i = 1, \cdots, N)$ are compact subsets of $\boldsymbol{C}$. Let $U_i = \{\xi \in U: h_i(\xi) \notin K_i\}$ and let $K = \{\xi \in U: h_i(\xi) \in K_i$ for all $i\}$. Then $\mathscr{U} = \{U_i\}$ is a Stein covering of $U - K$.

In the following theorem we assume the following:

1) An $N \times n$ matrix $A(\xi, z)$ is given with entries holomorphic in $U \times U$ and with $A(\xi, z)$ mapping the vector $\xi - z$ into $h(\xi) - h(z)$. Let $A_I(\xi, z)$ denote the matrix whose 1st row is the i_0th row of $A(\xi, z)$, etc.

2) A family $\{\phi_i\}$ of functions satisfying Lemma 1.1 is given.

3) There exists a neighborhood of K, and for each fixed z in this neighborhood, functions $g_i(\xi, z) \in C^\infty(U)$ such that

$$g(\xi, z) = \sum_{i=1}^{N} g_i(\xi, z)(h_i(\xi) - h_i(z)) \neq 0 \text{ for } \xi \neq z.$$

4) A family of smeared C^∞ polyhedra is given with each smeared polyhedron $\{\psi_j\}$ subordinate to $\mathscr{U}$ such that $\Pi\psi_i$ extends as the constant function 1 across K and $\{(\operatorname{supp} \Pi\ \psi_j) \cup K\}$ is a fundamental neighborhood system for K in U.

5) A family of C^∞ polyhedra is given with each polyhedron D in general

position with respect to $\mathscr{U}$ such that $\{D \cup K\}$ is a fundamental neighborhood system for K in U.

Theorem 4.1. i) *For all* $f \in \mathcal{O}(K)$,

$$(4.1)' \qquad f(z) = (-1)^{\frac{1}{2}(n-p-1)(n-p-2)}(2\pi i)^{-n}$$

$$\sum_{|J|=p+1}' \int_{(\partial D)_J} f(\xi) \sum_{|K|=n-p-1} \bar{\partial}\phi^K(\xi) \frac{\det A_{KJ}(\xi,z)d\xi}{\prod_{i \in KJ}(h_i(\xi)-h_i(z))}$$

holds in a neighborhood of K, *where* D *is a* C^∞-*polyhedron chosen so that* $f \in \mathcal{O}(\bar{D} \cup K)$. *Using cutoff functions instead of* C^∞ *polyhedra yields*

ii) *For all* $f \in \mathcal{O}(K)$,

$$(4.1)'' \quad f(z) = (-1)^{\frac{1}{2}(n-p-1)(n-p-2)+\frac{1}{2}p(p+1)}(2\pi i)^{-n}$$

$$\sum_{|J|=p+1}' \int f(\xi) \sum_{|K|=n-p-1} \bar{\partial}\phi^K(\xi) \frac{\det A_{KJ}(\xi,z)d\xi}{\prod_{i \in KJ}(h_i(\xi)-h_i(z))} \prod_{j \notin J} \psi_j \bar{\partial}\psi^J$$

holds in a neighborhood of K, *where the* ψ_j *are chosen so that* $f \in \mathcal{O}(\operatorname{supp} \Pi\psi_j)$.

In these formulas $(4.1)'$ *and* $(4.1)''$, $p = 0, 1, \cdots, n-1$.

iii) *If* $p = n-1$ *then* $(4.1)'$ *is the Cauchy-Weil formula,*

$$(4.2)' \quad f(z) = (2\pi i)^{-n} \sum_{|J|=n}' \int_{(\partial D)_J} f(\xi) \frac{\det A_J(\xi,z)d\xi}{\prod_{j \in J}(h_j(\xi)-h_j(z))}, \quad \textit{and } (4.1)'' \textit{ is}$$

$$(4.2)'' \quad f(z) = (-1)^{\frac{1}{2}n(n-1)}(2\pi i)^{-n} \sum_{|J|=n}' \int f(\xi) \frac{\det A_J(\xi,z)d\xi}{\prod_{j \in J}(h_j(\xi)-h_j(z))} \prod_{j \notin J} \psi_j \bar{\partial}\psi^J .$$

iv) *If* $p = 0$ *then*

$$(4.3)' \quad f(z) =$$

$$(-1)^{\frac{1}{2}n(n-1)}(2\pi i)^{-n} \sum_{j=1}^{N} \int_{(\partial D)_J} f(\xi) \sum_{|K|=n-1} \bar{\partial}\phi^K(\xi) \frac{\det A_{jK}(\xi,z)d\xi}{\prod_{i \in jK}(h_i(\xi)-h_i(z))}, \text{ or}$$

$$(4.3)'' \quad f(z) =$$

$$(-1)^{\frac{1}{2}n(n-1)}(2\pi i)^{-n} \sum_{j=1}^{N} \int f(\xi) \sum_{|K|=n-1} \bar{\partial}\phi^K(\xi) \frac{\det A_{jK}(\xi,z)d\xi}{\prod_{i \in jK}(h_i(\xi)-h_i(z))} \prod_{i \neq j} \psi_i \bar{\partial}\psi_j,$$

or equivalently,

$$(4.4)' \qquad f(z) = (-1)^{\frac{1}{2}n(n-1)}(n-1)!(2\pi i)^{-n} \int_{\partial D} f(\xi)g(\xi,z)^{-n}$$

$$\sum_{|J|=n}{}' \sum_{k=0}^{n-1} (-1)^k \det A_J(\xi,z) g_{j_k}(\xi,z)\bar{\partial}g_{j_0} \wedge \cdots \wedge \widehat{\bar{\partial}g_{j_k}} \wedge \cdots \wedge \bar{\partial}g_{j_{n-1}} \wedge d\xi.$$

$$(4.4)'' \qquad f(z) = (-1)^{\frac{1}{2}n(n-1)}(n-1)!(2\pi i)^{-n} \int f(\xi)g(\xi,z)^{-n}$$

$$\sum_{|J|=n}{}' \sum_{k=0}^{n-1} (-1)^k \det A_J(\xi,z) g_{j_k}(\xi,z)\bar{\partial}g_{j_0} \wedge \cdots \wedge \widehat{\bar{\partial}g_{j_k}} \wedge \cdots \wedge \bar{\partial}g_{j_{n-1}} \wedge d\xi \prod_{i \neq j} \psi_i \bar{\partial}\psi_j.$$

Remark. The Cauchy integral formula for a polydisc is a special case of (4.2)′ and the Bochner-Martinelli integral formula is a special case of (4.4)′.

Proof. Consider the sequence $\{k^p\}$ $p=0,\cdots,n-1$ of Dolbeault representatives of the Cauchy-Weil kernel (see Definition 2.4 for the definition of k^{n-1} and see (2.6) or (2.7) for the definition of k^0). Consider the adjoint sequence $\{\bar{u}^p\}$ of Dolbeault representatives given in either Proposition 3.5 or Proposition 3.2. The pairings $\langle f^p, u^p \rangle$ make up the right-hand sides of the above formulas. Note that $\bar{\partial}\, \Pi \psi_j$ and $\bar{\partial}\chi_D$ represent the same class in $H^1_*(U-K,\mathcal{O})$, where χ_D denote the characteristic function of D. By Theorem 1.16, this proves that the right-hand sides in all the above formulas are the same. Consequently, it suffices to prove any one of the above formulas. Now, of course, there are many ways to complete the proof of the theorem. In order to use Proposition 1.17 we construct auxiliary coverings $\mathscr{V}$ and $\mathscr{W}$ as follows.

Suppose z is fixed. Let $K_{N+j} = \{z_j\}$ and $h_{N+j}(\xi) = \xi_j - z_j$, and let $V_{N+j} = \{\xi \in U : h_{N+j}(\xi) \notin K_{N+j}\} = \{\xi \in U : \xi_j - z_j \neq 0\}$ for $j = 1,\cdots,n$. Let $\mathscr{V} = \{V_j\}_{j=N+1}^{N+n}$ and $\mathscr{W} = \mathscr{U} \cup \mathscr{V}$. Then $\mathscr{W}$ is a Stein covering of $U-\{z\}$, $\mathscr{V}$ is a Stein covering of $U-\{z\}$, and $\mathscr{U}$ is a Stein covering of $U-K$. Extend the matrix $A(\xi,z)$ by adding the $n \times n$ identity matrix to the bottom of A. Call the extended matrix B. Then

$$h_i(\xi) - h_i(z) = \sum_{j=1}^{n} b_{ij}(\xi_j - z_j) \quad \text{for} \quad i = 1,\cdots,N+n.$$

Now, consider the Cauchy-Weil kernel $E^{n-1} \in Z^{n-1}(\mathscr{W},\Omega^n)$ with

$$E_I^{n-1} = \frac{1}{(2\pi i)^n} \frac{\det B_I(\xi)\, d\xi}{\prod_{i \in I} (h_i(\xi) - h_i(z))}.$$

The covering $\mathscr{V}$ is a refinement of $\mathscr{W}$ with an obvious refinement map ρ_1. Similarly, $\mathscr{U}$ is a refinement of $\mathscr{W}$ with an obvious refinement map ρ_2. The restriction $\rho_1^* E^{n-1}$ is just the Cauchy kernel, which we denote by k^C. That is

$$(\rho_1^* E^{n-1})_I = (k^C)_I = \frac{1}{(2\pi i)^n} \frac{\partial\zeta}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)},$$

where $I = (N+1, \cdots, N+n)$. The restriction $\rho_2^* E^{n-1}$ is just the Cauchy-Weil kernel k^{n-1} (see Definition 2.4). That is

$$(\rho_2^* E^{n-1})_I = (k^{n-1})_I = \frac{1}{(2\pi i)^n} \frac{\det A_I(\zeta, z) d\zeta}{\prod\limits_{i \in I} (h_i(\zeta) - h_i(z))}$$

where $I = (i_0, \cdots i_{n-1})$ with each $1 \leqq i_k \leqq N$.

The following diagram commutes by Proposition 1.17.

$$\begin{tikzcd}
 & & {[k^C] \in H^{n-1}(\mathscr{V}, \Omega^n)} \arrow[dd] & \\
{[E^{n-1}] \in H^{n-1}(\mathscr{W}, \Omega^n)} \arrow[urr, "\rho_1^*"] \arrow[rrr, "\rho_2^*"] \arrow[dd, dash] & & & {[k^{n-1}] \in H^{n-1}(\mathscr{U}, \Omega^n)} \arrow[dd] \\
 & & {[k^{B-M}] \in H^{n-1}(U - \{z\}, \Omega^n)} \arrow[dr] & \\
{[E^0] \in H^{n-1}(U - \{z\}, \Omega^n)} \arrow[urr, "\rho_1^*"] \arrow[rrr, "\rho_2^*"] & & & {[k^0] \in H^{n-1}(U - K, \Omega^n)}
\end{tikzcd}$$

Therefore $[E^0]$, the image of $[E^{n-1}]$ under Dolbeault's homomorphism, restricts to $[k^0]$ the image of $[k^{n-1}]$ under Dolbeault's homomorphism. Also $[E^0]$ restricts to $[k^{B-M}]$ where $k^{B-M}(\xi, z)$ denotes the Bochner-Martinelli. Therefore k^0 and k^{B-M} determine the same class in $H^{n-1}(U - K, \Omega^n)$.

The right-hand sides of $(4.3)''$ and $(4.4)''$ can both be written as $\langle f(\zeta)k^0(\zeta, z), \bar{\partial}\Pi\psi_j(\zeta)\rangle$ by Theorem 2.6. Since $[k^0] = [k^{B-M}]$ in $U - K$ and $\bar{\partial}\Pi\psi_j$ is compactly supported in $U - K$, $\langle f(\zeta)k^0(\zeta, z), \bar{\partial}\Pi\psi_j(\zeta)\rangle = \langle f(\zeta)k^{B-M}(\zeta, z), \bar{\partial}\Pi\psi_j(\zeta)\rangle$. Now, integrating by parts and using tbe fact that $\bar{\partial}k^{B-M}(\zeta, z) = \delta_z(\zeta)$ (the form which evaluates $g \in \mathscr{E}_{0,0}(\mathbb{C}^n)$ at z) this equals $\langle \bar{\partial}(f(\zeta)k^{B-M}(\zeta, z)), \Pi\psi_j(\zeta)\rangle = \langle f(\zeta)\bar{\partial}k^{B-M}(\zeta, z), \Pi\psi_j(\zeta)\rangle = f(z)$. This proves both $(4.3)''$ and $(4.4)''$. That is, $f(z) = \langle f(\zeta)k^0(\zeta, z), \bar{\partial}\Pi\psi_j(\zeta)\rangle$. The proof of the theorem is now complete.

Remark. If $k(\xi) \in \mathscr{E}_{n,n-1}(U - K)$ with $\bar{\partial}k = 0$ in $U - K$ and there exists $\omega \in \mathscr{E}_{n,n-2}(U - K)$ with $k(\xi) = k^{B-M}(\xi, z) + \bar{\partial}\omega(\xi)$, where z is some fixed point in K (i.e., if $[k] = [k^{B-M}]$ in $U - K$), then there exists a hyperfunction extension of k to U with $\bar{\partial}k(\xi) = \delta_z(\xi)$. This can be seen as follows. First extend $\omega(\xi)$to $\bar{\omega}(\xi)$ on U. Then $k(\xi) = k^{B-M}(\xi, z) + \bar{\partial}\bar{\omega}(\xi)$ and hence $\bar{\partial}k(\xi) = \delta_z(\xi)$.

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