THE ENVELOPE OF HOLOMORPHY OF A TWO-MANIFOLD IN C^2*

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1. *Introduction*

There has been a large amount of study devoted to the problem of analytic continuation. If K is a subset of C^n, $n > 1$, then one wishes to know if there is a larger set K', containing K, such that all holomorphic functions on K can be extended to holomorphic functions on K'. It has been known for some time that the envelope of holomorphy of a domain in C^n is a Stein manifold spread over C^n. One can then consider the case where K is a lower dimensional set in C^n, $n > 1$. For example, Hartogs proved that every function holomorphic in a neighborhood of the boundary of the unit ball in C^n can be extended to a holomorphic function in the interior of the ball. In [12] it was shown that if M is a real $(n+1)$-dimensional differentiable submanifold embedded in C^n, one obtains local extendibility over a manifold of real dimension $n+2$, provided the so-called Levi form does not vanish. Greenfield [5] has proven a similar result for an $(n+k)$-dimensional submanifold of C^n with $1 \leqq k \leqq n-1$.

Denote by $\mathcal{O}(=\mathcal{O}_{C^n})$ the structure sheaf of germs of holomorphic functions on C^n. If K is a subset of C^n, let $\mathcal{O}(K)$ be the algebra of sections of $\mathcal{O}$ over K (germs of holomorphic functions defined near K). If $K \subsetneq K'$ (where K and K' are connected sets), we say that K *is extendible to* K' if the natural restriction map

$$r : \mathcal{O}(K') \to \mathcal{O}(K)$$

is onto.

If K is a compact subset of C^n, then $\mathcal{O}(K)$ is, in a natural way, the inductive limit of Fréchet algebras of the form $\mathcal{O}(U)$ where U is an open set containing K. As such it has a natural locally convex inductive limit topology and becomes a topological algebra. The spectrum (or maximal ideal space) of the topological algebra $\mathcal{O}(K)$ is defined to be the *envelope of holo-*

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morphy of the compact set K, and is denoted by $E(K)$ (see [6] where this definition is seen to be equivalent to an earlier one given in [12]).

The main result of this paper is the following theorem which will be proved in Section 4.

Theorem 1.1. *There is an open dense set U of embeddings of the real two-sphere in C^2 such that each embedded two-sphere from U has the property that its envelope of holomorphy contains a real three-dimensional $\mathscr{C}^n$ manifold $\tilde{M}$ $(1 \leqq n < \infty)$.*

(*Outline of Proof*). Using an iteration technique devised by Bishop [1], we establish a $(k-1)$-parameter family of analytic discs in a neighborhood of an *elliptic point* (to be defined in Section 2) in a real k-dimensional differentiable submanifold M^k of C^k. By computing a certain Jacobian and using a theorem on simultaneous analytic continuation from [11], we are able to say that every function holomorphic on M^k can be extended to a holomorphic function on a real $(k+1)$-dimensional $\mathscr{C}^n$ manifold $(1 \leqq n < \infty)$.

Using Thom Transversality Theory (see Levine [8]), we prove that there exists an open dense set of embeddings of any compact two-manifold in C^2 so that each such embedded manifold has the following properties:

(i) There are at most finitely many exceptional points, and

(ii) Each exceptional point is of the elliptic or hyperbolic type.

(The concepts of exceptional points and hyperbolic points, as well as elliptic points, will be defined in Section 2. They are first and second order conditions on the submanifold at a point).

Bishop [1] proves that for such a two-manifold the number of elliptic points minus the number of hyperbolic points equals the Euler number of the two-manifold. In particular, each two-sphere, embedded by an element in the open dense set, has at least two elliptic points. Using the analytic continuation result for $M^k \subset C^k$ we can easily complete the proof.

Remark: Consider the two-sphere in standard position in $R^3 \subset C^2$ $(R^3 = \{(z_1, z_2) \in C^2 : \operatorname{Im} z_2 = 0\})$. Using a classical argument involving the Cauchy integral formula (cf. Bochner-Martin [2]), we can say that every function holomorphic on the two-sphere can be extended to a function holomorphic in the open ball. In this case the envelope of holomorphy of the two-sphere is the closed unit ball. This is an example of an embedded two-sphere which has the property that its envelope of holomorphy contains a three-manifold, but does not contain a manifold of higher dimension.

A manifold M embedded in C^n is said to be *totally real* if there does not exist a point such that the tangent space to the manifold at this point contains a nonzero complex subspace of C^n.

In [*13*] it was proven that a totally real oriented two-manifold embedded in C^2 must have Euler number zero and thus be a torus. In [*10*] it was shown that a totally real submanifold is not extendible.

Denote by $\mathscr{T}$ the set of embeddings of the torus in C^2 such that the embedded tori are not totally real. In Section 4 we shall prove the following theorem.

Theorem 1.2. *There is an open dense set of embeddings in $\mathscr{T}$ such that each two-torus embedded by an element in this dense set has the property that its envelope of holomorphy contains a real three-dimensional $\mathscr{C}^n$ manifold* $(1 \leqq n < \infty)$.

There is an example of a torus which is embedded in C^2 and is not totally real. Let T^2 be the standard torus in $R^3 \subset C^2$ $(R^3 = \{(z_1, z_2) \in C^2 : \operatorname{Im} z_2 = 0\})$ with its axis of symmetry as the $\operatorname{Re} z_1$ axis. There are exactly four points on the torus T^2 which have tangent spaces parallel to the plane $z_2 = 0$.

If we consider an oriented two-manifold of positive genus, we have the following theorem.

Theorem 1.3. *There is an open set of embeddings of an oriented compact two-manifold of positive genus in C^2 with the following property: each manifold embedded by an element in this open set has an envelope of holomorphy which contains a three-manifold.*

In Section 2 we discuss elliptic and hyperbolic points on a two-manifold in C^2 and state a local theorem on extendibility.

Thom Transversality Theory is the topic of Section 3. We shall give definitions and theorems and apply the theory to a two-manifold in C^2.

Let M^2 be an oriented two-manifold embedded in C^2 such that M^2 has only a finite number of exceptional points, and each exceptional point is of either elliptic or hyperbolic type. In Section 4 we discuss Bishop's proof that the number of elliptic points on M^2 minus the number of hyperbolic points on M^2 equals the Euler number of M^2.

2. *Elliptic and hyperbolic points on a two-manifold in C^2.*

Let M^2 denote a real two-dimensional differentiable manifold embedded in C^2. A point p in M^2 will be called *exceptional* if the tangent space to M^2 at p is a complex-linear subspace of C^2 of complex dimension one.

If $T_x(C^2)$ denotes the real tangent space to C^2 at $x \in C^2$, we have an almost complex tensor $J : T_x(C^2) \to T_x(C^2)$ given by the complex structure on C^2. J is given by multiplication by i.

Define $H_p(M^2) = T_p(M^2) \cap JT_p(M^2)$ where $p \in M^2$ and $T_p(M^2)$ is the real tangent space to M^2 at p. The vector space $H_p(M^2)$ is called the *vector space of holomorphic tangent vectors* to M^2 at p. Then p is an exceptional point in M^2 if the complex dimension of $H_p(M^2)$ is one.

As an example, if S^2 denotes the two-sphere in standard position in $\boldsymbol{R}^3 \subset \boldsymbol{C}^2$, then the exceptional points in S^2 are the north and south poles.

Letting p be an exceptional point of M^2, we choose differentiable coordinates u, v in a neighborhood of p in M^2, vanishing at p, and analytic coordinates z_1, z_2 for a neighborhood of p in $\boldsymbol{C}^2$, vanishing at p. The equations of M^2 in a neighborhood of p can then be written as

$$z_1 = f_1(u,v)$$

$$z_2 = f_2(u,v),$$

where f_1, f_2 are complex valued $\mathscr{C}^\infty$ functions of u and v, defined in a neighborhood of $u = v = 0$. (We denote by $\mathscr{C}^k$ the class of functions continuously differentiable of order k, $1 \leqq k \leqq \infty$.)

By properly choosing our coordinates these equations may be put in the form

$$z_1 = u + iv = w$$

$$z_2 = g(w),$$

where g is complex valued and vanishes to second order at $u = v = 0$. Then p is exceptional if and only if the determinant of the Jacobian $J = J(z_1, z_2/u, v)$ vanishes at $p = 0$.

Expanding $g(w)$ about $p = 0$ we have

$$g(w) = \alpha w^2 + \gamma w\bar{w} + \beta \bar{w}^2 + \lambda(w),$$

where λ vanishes to third order at $w = 0$. Assuming $\gamma \neq 0$ and using coordinate changes (see [*1*]) we obtain $g(w) = \beta(w^2 + \bar{w}^2) + w\bar{w} + \lambda(w)$, where $\beta \geqq 0$. If we assume $|\beta| \neq \frac{1}{2}|\gamma|$, we find $\beta \neq \frac{1}{2}$.

Definition 2.1. If $\beta < \frac{1}{2}$, then $\beta(w^2 + \bar{w}^2) + w\bar{w} = 2\beta(u^2 - v^2) + u^2 + v^2 = c$, for a positive constant c, is the equation of an ellipse, and p is called a point of *elliptic type*.

Definition 2.2. If $\beta > \frac{1}{2}$, then $\beta(w^2 + \bar{w}^2) + w\bar{w} = c$ is the equation of a hyperbola and p is called a point of *hyperbolic type*.

Definition 2.3. If $\beta = \frac{1}{2}$, we say that p is a point of *parabolic type*.

Note that if $\beta < \frac{1}{2}$, then the point p is a minimum point for the function $\operatorname{Re} g$, while if $\beta > \frac{1}{2}$, the point p is a saddle point for the function $\operatorname{Re} g$.

Since $\left.\frac{\partial g}{\partial u}\right|_{p=0} = \left.\frac{\partial g}{\partial v}\right|_{p=0} = 0$, p is a critical point in Morse Theory (see Milnor [9]). The Hessian matrix is

$$\begin{bmatrix} \frac{\partial^2 g(0)}{\partial u^2} & \frac{\partial^2 g(0)}{\partial u \partial v} \\ \frac{\partial^2 g(0)}{\partial v \partial u} & \frac{\partial^2 g(0)}{\partial v^2} \end{bmatrix} = \begin{bmatrix} 2(2\beta+1) & 0 \\ 0 & 2(1-2\beta) \end{bmatrix}.$$

Thus p is non degenerate if $\beta \neq \frac{1}{2}$ and degenerate if $\beta = \frac{1}{2}$. Hence the elliptic and hyperbolic points are non degenerate critical points, while the parabolic points are degenerate critical points. The signature of the matrix H is 0 if $\beta > \frac{1}{2}$ and 2 if $\beta < \frac{1}{2}$.

We have similar definitions for elliptic and hyperbolic points in a real k-dimensional differentiable manifold M^k embedded in $\boldsymbol{C}^k$.

In [7] the following theorem has been proved.

Theorem 2.1. *Let $M^k \subset \boldsymbol{C}^k$ be a real k-dimensional differentiable submanifold of $\boldsymbol{C}^k$ such that M^k has at least one exceptional point of the elliptic type. Then given any n, $1 \leqq n < \infty$, there exists a real $(k+1)$-dimensional $\mathscr{C}^n$ submanifold $\tilde{M}$ such that M^k is extendible to $M^k \cup \tilde{M}$.*

This theorem was proved by extending the iteration argument devised by Bishop and applying the analytic continuation result for families of analytic discs from [11].

Let U be an open neighborhood of the origin in $\boldsymbol{R}^2$ and f be an embedding of U into $\boldsymbol{R}^3 \subset \boldsymbol{C}^3$ with $f(0) = 0$ such that $f(0)$ is a saddle point for $f(U)$. Freeman [4] has shown that $f(U)$ is locally polynomially convex. For example, suppose the equations of $f(U)$ are given by

$$z_1 = u + iv = w$$
$$z_1 = u^2 - v^2.$$

Then the point $u = v = 0$ is a saddle point of $f(U)$, and the intersection of a closed ball with $f(U)$ is polynomially convex. Thus hyperbolic points will not in general contribute to the envelope of holomorphy.

Definition 2.4. An exceptional point is called *non degenerate* if it is either elliptic or hyperbolic.

Definition 2.5. By a *non degenerate embedding* we mean an embedding under which a manifold has at most finitely many exceptional points, and all such points are non degenerate.

3. *Thom Transversality Theory*

The following discussion of Thom Transversality Theory is taken from the notes of Levine [*8*].

Let V and M be manifolds of real dimensions n and p respectively. We define the *r-jet from V to M with source x and target y of a* $\mathscr{C}^r$-map $f: V \to M$ as the equivalence class of all $\mathscr{C}^r$-maps from V to M which take x into y, all of whose partial derivatives at x of orders $\leqq r$ are equal to those of f.

We denote the r-jet of f at x by $J^r(f)(x)$. If we let $J^r(n,p)$ denote the space of r-jets of $\mathscr{C}^r$-maps $f: V \to M$ with $f(x) = y$, then $J^r(n,p)$ becomes a euclidean space if we take the values of the partial derivatives at x as the coordinates of a jet. The set of all r-jets from V to M is denoted by $J^r(V,M)$. If we choose local coordinates in V and M, then $J^r(V,M)$ becomes a fibre bundle with fibre $J^r(n,p)$ and group $L^r(n,p)$ (see [*8*]).

If $f: V \to M$ is of class at least $\mathscr{C}^r$, the *r extension of f* is defined by: $J^r(f)$: $V \to J^r(V,M)$, where $x \to J^r(f)(x)$.

Let S be a submanifold of codimension q in M, and let $f: V \to M$ be a differentiable mapping. f is said to be *transversal to the submanifold S at a point* $x \in V$ if either:

(i) $f(x) \notin S$, or

(ii) $f(x) \in S$ and the image under df of the tangent space to V at x and the tangent space to S at $f(x) = y$ span the tangent space to M at y.
If f is transversal to S at every point $x \in V$, we say that f is *transversal to S*. In this case one can prove that $f^{-1}(S)$ is a regular submanifold of V of codimension q in V, or void.

Let $L(V,M,s)$ denote the set of all s-times continuously differentiable maps from V to M. On $L(V,M,s)$ we put the topology of compact convergence of all partial derivatives of orders $\leqq s$.

Assume $s > r \geqq 0$ and let N be an $(s-r)$ differentiable regular submanifold of $J^r(V,M)$ where V and M are at least $\mathscr{C}^s$ differentiable paracompact manifolds of dimensions n and p respectively. Suppose that the codimension of N in $J^r(V,M)$ is q.

Theorem 3.1. (Tranversality Theorem of Thom). *The set of maps in* $L(V,M,s)$ *whose r extensions are transversal to N on V is everywhere dense if* $(s-r) > max(n-q,0)$. *In addition if V is compact, this dense set is open.*

For any pair of positive integers (n,p) a *singularity manifold of order r* is a regular submanifold of $J^r(n,p)$ which is invariant under the group

$L^r(n,p)$. Given a singularity manifold of order r, $S \subset J^r(n,p)$, we can define a submanifold $S(V,M)$ of $J^r(V,M)$, and the codimension of S in $J^r(n,p)$ equals the codimension of $S(V,M)$ in $J^r(V,M)$.

Let $S(f) = (J^r(f)^{-1}(S(V,M)))$, where $f\colon V \to M$ is of class $\mathscr{C}^r$. Assume the codimension of $S(V,M)$ in $J^r(V,M)$ is q.

Thom's Transversality Theorem gives us the following results (assume $(s-r) > \max(n-q,0)$):

(i) If $q > n$, the set of maps f in $L(V,M,s)$ such that $S(f) = \varnothing$ is dense in $L(V,M,s)$.

(ii) If $q \leqq n$, the set of maps f in $L(V,M,s)$ such that $S(f) = \varnothing$ or $S(f)$ is a submanifold of V of codimension q is dense in $L(V,M,s)$.

We want to apply the transversality theory to the case $V = M^2$ (M^2 is compact, $\mathscr{C}^\infty$, real two-dimensional), $M = \boldsymbol{C}^2$, and $f\colon M^2 \to \boldsymbol{C}^2$ is an embedding.

If p is a point in M^2, $J^1(2,2)$ can be identified with four-dimensional complex euclidean space. If p is an exceptional point of M^2 under a $\mathscr{C}^\infty$ map f, then the complex Jacobian of f has a vanishing determinant at p. Since the singularity S_1 of $J_1(2,2)$, which is defined by the vanishing of the Jacobian determinant, is invariant under coordinate changes on M^2 and $\boldsymbol{C}^2$, it is a singularity manifold of order 1, and we can thus define $S_1(M^2,\boldsymbol{C}^2)$ in $J^1(M^2,\boldsymbol{C}^2)$.

Consider the set $L(M^2,\boldsymbol{C}^2,\infty)$. Since the real codimension of $S_1(M^2,\boldsymbol{C}^2)$ in $J^1(M^2,\boldsymbol{C}^2)$ is 2, we have that the set of maps f in $L(M^2,\boldsymbol{C}^2,\infty)$ such that $S_1(f) = \varnothing$ or $S_1(f)$ is a submanifold of dimension 0 in M^2 is open and dense in $L(M^2,\boldsymbol{C}^2,\infty)$.

Because the embeddings are an open set in $L(M^2,\boldsymbol{C}^2,\infty)$, we find that the set of embeddings f such that $S_1(f) = \varnothing$ or $S_1(f)$ contains a finite number of points is open and dense in the set of all $\mathscr{C}^\infty$ embeddings. If $S_1(f) = \varnothing$ and M^2 is orientable, then M^2 is totally real under the embedding f and is thus a torus, as mentioned earlier.

If p is an exceptional point in an embedded two-manifold in $\boldsymbol{C}^2$, the equations for a neighborhood of p are

$$z_1 = u + iv = w$$
$$z_2 = \alpha w^2 + \beta \bar{w}^2 + \gamma w\bar{w} + \lambda(w).$$

If $\gamma \neq 0$ and $|\beta| \neq \frac{1}{2}|\gamma|$ these equations can be put in the form

$$z_1 = u + iv = w$$
$$z_2 = \beta(w^2 + \bar{w}^2) + w\bar{w} + \lambda(w)$$

where $\beta \geqq 0$ and $\beta \neq \frac{1}{2}$. In this case p is a non degenerate exceptional point.

$J^2(2,2)$ can be identified with ten-dimensional complex euclidean space. Let p be an exceptional point of a manifold M^2 embedded in $\boldsymbol{C}^2$ in such a way that $\gamma = 0$. Since the condition $\gamma = 0$ is not invariant under coordinate changes on M^2, the submanifold S_2 of $J^2(2,2)$ which arises from the fact that p is an exceptional point with $\gamma = 0$ is not a singularity manifold. However, if M^2 has only a finite number of exceptional points under an embedding f into $\boldsymbol{C}^2$, we may apply the following lemma at each of these points.

We have the notation:

$$T = \boldsymbol{R}^p \times J^r(n,p)$$

$$F: \quad \boldsymbol{R}^n \to T: x \to (f(x), J^r(f)(x))$$

$$G: \quad \boldsymbol{R}^n \to T: x \to (f(x), J^r(g)(x)).$$

Lemma 3.1 (Local lemma). *Suppose* $f \in L(\boldsymbol{R}^n, \boldsymbol{R}^p, s)$ *and* $N \subset T$ *is an* $(s-r)$ *differentiable regular submanifold of codimension* q. *If* $(s-r) >$ *max* $(n-q, 0)$, *then for each* $x \in \boldsymbol{R}^n$ *and each* $u \in N \subset T$ *such that* $f(x) = u$ *we can find*:

(1) *A neighborhood* V_u *of* u *in* T.

(2) *A neighborhood* W_f *of* f *in* $L(\boldsymbol{R}^n, \boldsymbol{R}^p, s)$.

(3) *A compact neighborhood* U_x *of* x *in* $\boldsymbol{R}^n$ *such that*

(a) *for each* $g \in W_f$, $G(U_x) \subset V_u$;

(b) *for each* $h \in W_f$, *there exists a* $g \in W_f$ *arbitrarily close to* h *such that* $G \mid U_x$ *is transversal to* N.

If we set $N = \boldsymbol{C}^2 \times S_2$ and note that the real codimension of S_2 in $J^2(2,2)$ is 4, we find by applying the lemma at each exceptional point, that arbitrarily close to the embedding f is an embedding g which has a finite number of exceptional points with $\gamma \neq 0$ at each such point.

We use the lemma again with the condition $\gamma = 0$ replaced by the condition $|\beta| = \frac{1}{2}|\gamma|$. Thus the set of embeddings under which a manifold has no exceptional points or a finite number of exceptional points with $\gamma \neq 0$ and $|\beta| \neq \frac{1}{2}|\gamma|$ at each such point is dense in the set of all embeddings. Therefore the non degenerate embeddings of M^2 into $\boldsymbol{C}^2$ are an open dense set in the set of all $\mathscr{C}^\infty$ embeddings.

4. *The Gauss Mapping and Intersection Theory*

Let $M \subset \boldsymbol{C}^2$ be a compact oriented two-manifold with a given orientation. Assume M has been embedded in $\boldsymbol{C}^2$ by a non degenerate embedding.

The following theorem is proved by Bishop [1].

Theorem 4.1. *The number of elliptic points minus the number of hyperbolic points equals* $\chi(M)$, *where* $\chi(M)$ *denotes the Euler number of* M.

Proof. If M is a totally real torus, then M has no exceptional points. However, the Euler number of M in this case is 0, and the theorem holds.

Therefore assume that M is not totally real. Let G denote the Grassmann manifold of all oriented two-dimensional real-linear subspaces of C^2. By mapping each point into its oriented tangent plane we obtain the Gauss map $t: M \to G$.

Using Plücker coordinates we may identify G with the product of unit two spheres S_1 and S_2. Denote by H the subset of G consisting of those two-dimensional real-linear subspaces of C^2 which also have a complex structure, and whose orientation is induced by this complex structure. Then p in M is exceptional if and only if $t(p) \in H$ or $-t(p) \in H$, where $-t(p)$ denotes $t(p)$ with orientation reversed. Again using Plücker coordinates we find that $H = (1,0,0) \times S_2$.

We next prove that t (actually an order 2 approximation of t) is transversal to H on M. If $t(p) \in H$, by computing a certain determinant we have $\operatorname{sgn}(p) = +1$ if p is an elliptic point and $\operatorname{sgn}(p) = -1$ if p is a hyperbolic point. If $p_1, \cdots, p_N$ are the points of M such that $t(p_i) \in H$, we define the *intersection number* of $t(M^2)$ and H as $\sum_{i=1}^{N} \operatorname{sgn}(p_i)$. Chern and Spanier [3] have shown that the intersection number is $(\frac{1}{2})\chi(M)$. By reversing the orientation, we find that the intersection number of $-t(M^2)$ and H is $(\frac{1}{2})\chi(M)$. Therefore the number of elliptic points minus the number of hyperbolic points is equal to $\chi(M)$. Q.E.D.

Now we are able to prove Theorems 1.1, 1.2, and 1.3.

Proof of Theorem 1.1. From Section 3 we know that the non degenerate embeddings are an open dense set in the set of embeddings of the two-sphere in C^2. If S^2 is a two-sphere embedded in C^2 by a non degenerate embedding, then S^2 has at least two elliptic points by Theorem 4.1. Let $p_1, p_2, \cdots, p_l$ be the elliptic exceptional points. Using Theorem 2.1, we find that S^2 is extendible to $S^2 \cup \tilde{M}_i$, where $\tilde{M}_i$ is the three-dimensional $\mathscr{C}^n$, $n \geqq 1$, real manifold related to the point p_i, $i = 1,2,\cdots,l$. Choosing the $\tilde{M}_i$ to be disjoint, we set $\hat{M} = \bigcup_{i=1}^{l} \tilde{M}_i$, and thus we have that S^2 is extendible to $S^2 \cup \hat{M}$ and $\hat{M} \subset E(S^2)$. Q.E.D.

The proof of Theorem 1.2 is the same except that we may have only one exceptional point of the elliptic type.

Proof of Theorem 1.3. If M^2 is an oriented compact two-manifold of positive genus, then the non degenerate embeddings are an open dense set in the set of embeddings of M^2 into C^2. If M^2 is embedded by an element in this dense set, by using Theorem 4.1 we find that there may be no exceptional points (if the genus of M^2 is one), or there are at least two hyperbolic points (if the genus of M^2 is greater than one). Thus the algebraic topology does not allow us to conclude the existence of elliptic points in this case. In [*13*] it was shown that there is at least one non degenerate embedding of an oriented two-manifold of positive genus into C^2, such that at least two of the exceptional points are elliptic. Hence, we can only conclude that there is an open (but not necessarily dense) set of embeddings of a two-manifold of positive genus so that each such embedded manifold has the property that its envelope of holomorphy contains a three-manifold. Q.E.D.

5. *Remarks*

1. Let M^k be a real k-dimensional differentiable manifold embedded in C^k, $k > 2$. Using Thom Transversality Theory we find that there exists a dense set of embeddings of M^k into C^k such that there are no exceptional points or the exceptional points are a submanifold of dimension $k-2$. If M^k is compact and orientable, it was shown in [*13*] that M^k is totally real only if $\chi(M^k) = 0$. Otherwise, if M^k is compact and oriented and $\chi(M^k) \neq 0$ there exists an open dense set of embeddings of M^k into C^k such that the exceptional points form a submanifold of M^k of dimension $k-2$. Since $k \geqq 3$, these exceptional points are not isolated, and we cannot use the local lemma as in the two-dimensional case. Also, if we could find that the non degenerate embeddings are an open dense set in the set of embeddings, we have no theorem analogous to that of Chern and Spanier to complete the process.

2. We have given an example of a two-sphere in C^2 with two elliptic points and no hyperbolic points. Does there exist a compact two-manifold which can be embedded in C^2 in such a way that all exceptional points are of the hyperbolic type?

3. Consider a real k-dimensional differentiable manifold embedded in C^n where $k \geqq n+1$, $n > 1$. A point p in M^k will be called *exceptional* if $\dim_C H_p(M^k) = k - n + 1$. Elliptic and hyperbolic points can be defined for this case and a local extension theorem similar to Theorem 2.1 of this paper has been proved (see [*7*]).

4. It was shown in [*14*] that there is a dense open set of embeddings

of a k-manifold in C^n, $k > n$, with a one-higher dimensional envelope of holomorphy. For real codimension 2, all such embeddings have this property. One could conjecture that:

a) All compact submanifolds of C^n, of real dimension $> n$, have an envelope of holomorphy of at least one higher dimension.

b) All compact submanifolds of C^n, of real dimension n, have an envelope of holomorphy of at least one higher dimension, provided that the manifolds are not totally real.

It is possible that the results mentioned in Remark No. 3 will be applicable in proving a) for five-dimensional submanifolds of C^4.

ADDED IN PROOF: S. Greenfield has recently given an affirmative answer to the question in Remark 2.

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