LOCAL HOLOMORPHIC CONVEXITY OF A TWO-MANIFOLD IN C^2*

by Michael Freeman

1. *Introduction*

A compact set M in C^n is *holomorphically convex* [3, Prop. 2.4] if the natural injection of it into the maximal ideal space [2, p. 2] of $A(M)$ is a homeomorphism. $A(M)$ is the Banach algebra obtained as the uniform closure on M of the continuous functions on M which admit holomorphic extensions to some neighborhood of M. Polynomially convex [2, p. 66] sets are holomorphically convex, as are sets M which are convex with respect to the holomorphic functions on a fixed domain of holomorphy U containing M. So are the rationally convex sets defined below.

Aside from simple sufficient conditions such as ordinary geometric convexity or containment in R^n (as the "real part" of C^n), very few criteria for determining holomorphic convexity are known. Yet this property is extremely important as a condition for the solution of many kinds of function-theoretic approximation problems [2, Ch. III].

This paper discusses the problem of finding differential conditions for holomorphic convexity. The idea is to find conditions on the tangent space, curvature, etc., of a real submanifold of C^n near one of its points p to discern which small compact subsets of it near p are holomorphically convex. This has been done for one-dimensional manifolds by Stolzenberg [8] and others who show that a compact subset of a smooth simple arc in C^n is polynomially convex, and in fact that any continuous function on such a set can be uniformly approximated by polynomials.

The discussion here is restricted to two-manifolds in C^2, although some of the results obtained have ready generalizations to a two-manifold in C^n. Something is already known in this situation, mainly through the efforts of E. Bishop [1] (whose results also apply to many cases of higher dimension). By attaching the boundaries of small analytic discs in C^2 to a two-manifold in a neighborhood of an "elliptic" point, Bishop showed that no neigh-

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borhood of such a point is holomorphically convex. On the other hand, evidence is presented here which indicates that sufficiently small compact neighborhoods of a "hyperbolic" point on a two-manifold in $\boldsymbol{C}^2$ are holomorphically convex.

The terms "elliptic" and "hyperbolic" describe two cases which arise after the following normalization pointed out by Bishop [*1*, p. 4]. Near any point on a two-manifold in $\boldsymbol{C}^2$ it may be represented as the graph of a smooth complex-valued function of two real variables. After a translation which takes this point to the origin in $\boldsymbol{C}^2$, the manifold is coincident near zero with the graph of

$$f(z) = az + b\bar{z} + Q(z) + o(|z|^2), \tag{1.1}$$

where Q is a quadratic form in z and $\bar{z}$ with complex coefficients. Now the holomorphic convexity of sets in the manifold near 0 is invariant under biholomorphic coordinate changes leaving 0 fixed. Denoting the usual coordinates of $\boldsymbol{C}^2$ by (z, w), the particular map $(z, w) \to (z, w - az)$ reduces the representation (1.1) to one in which $a = 0$, so that this restriction may henceforth be assumed. If $b \neq 0$, then it can be shown [*7*] that there is a disc Δ centered at 0 in $\boldsymbol{C}$ and a domain of holomorphy U containing $\{(z, f(z)) : z \in \Delta\}$ such that the latter set is convex with respect to the functions holomorphic on U. The problem thus remains only for the case $b = 0$, in which

$$f = Q + R \tag{1.2}$$

where $R(z) = o(|z|^2)$. By an elementary biholomorphic coordinate change [*1*] which does not affect the conditions $f(0) = a = b = 0$, Bishop showed how to convert (1.2) into a representation of the same form in which Q is a *real*-valued quadratic form. This condition may thus be added to (1.2).

Assume now that zero is not an eigenvalue of Q. The given point in M is *elliptic* if both eigenvalues of Q have the same sign and *hyperbolic* if they have different signs. The conclusions of Theorems 1.1 and 1.3 and certain other examples suggest that small discs in M centered at a hyperbolic point are holomorphically convex.

If f happens to be real, this result is a consequence of a deep theorem due to Mergelyan.

Theorem 1.1 [*4*]. *If f is a real valued continuous function on a closed disc Δ such that for each number β the level set*

$$L_\beta = \{z \in \Delta : f(z) = \beta\} \tag{1.3}$$

has no interior points and does not separate the plane, then every continuous complex valued function on Δ is a uniform limit of polynomial combinations of f and the identity function $z \to z$.

This and standard facts about the maximal ideal space of finitely generated Banach algebras implies that the graph

$$M = \{(z,f(z)) \colon z \in \Delta\}$$

is polynomially convex in C^2. Of course, no mention has been made of smoothness, but it is not difficult to see that if 0 is a hyperbolic point of a smooth real function f then f satisfies the hypothesis of Theorem 1.1 for small discs Δ. This follows from the proof of Theorem 1.3 or from Morse's Lemma [5, Lemma 2.2].

Very little is known if f is permitted to take complex values. It is shown here as a consequence of Theorem 1.3 that if 0 is a hyperbolic point of a smooth function f of the form (1.2) taking complex values, but in such a way that rank $f \leqq 1$ near 0, then the graph of f is locally rationally convex. The rank condition means that the ordinary Jacobian determinant of f, as a map of the underlying real space R^2, vanishes on a neighborhood of 0. The force of it is that f still has a thin image in C. This fact is important in the proof of Theorem 1.3, but there seems to be no reason to believe that it is essential to the conclusion.

Among the several equivalent definitions of rational convexity [2, p. 69], the one which is most convenient here is

Definition 1.2. A compact set M in C^n is *rationally convex* if for each point p not in M, there exists a rational function r holomorphic in a neighborhood of M but with a singularity at p.

Any compact set in C is rationally convex, as is any polynomially convex set in C^n. The graph of a function $f \colon \Delta \to C$ is rationally convex in C^2 if every continuous complex-valued function on Δ is a uniform limit of rational combinations of z and f [2]. That is, when

$$[z,f]_R = C(\Delta),$$

where $[z,f]_R$ is the uniform closure on Δ of the functions $z \to r(z,f(z))$, with r rational and holomorphic in a neighborhood of the graph of f, and where $C(\Delta)$ is the full algebra of continuous functions. This assertion follows from the same arguments [2, pp. 68–69] that establish *polynomial* convexity of the graph when the closure $[z,f]$ of polynomials in z and f is $C(\Delta)$.

Theorem 1.3. *If* 0 *is a hyperbolic point for a smooth function* f *of the form* (1.2) *and for which* $\operatorname{rank} f \leqq 1$ *in a neighborhood of* 0, *then*

$$[z,f]_R = C(\Delta)$$

for all sufficiently small discs Δ *centered at* 0. *In particular*, $\{(z,f(z)) \colon z \in \Delta\}$ *is rationally convex.*

A large class of examples verifying this result is obtained from the functions $f = g \circ Q$, where Q is a real quadratic form with two non zero eigenvalues of opposite sign, and $g \colon \boldsymbol{R} \to \boldsymbol{C}$ is a smooth function with $g(0) = 0$ and $g'(0)$ real and non zero.

It should be noted that the rank condition of Theorem 1.3 is not preserved by Bishop's coordinate change. For example, if $f(z) = \bar{z}^2$ then f is an open map, but Bishop's normalization converts it to the function $f(z) = z^2 + \bar{z}^2$, which has only real values. Moreover, an example is constructed in Section 3 for which it is unlikely that there is a coordinate change converting it to one represented by a function satisfying Theorem 1.3.

2. *Proof of Theorem* 1.3

Two proofs will be offered; the first because of its economy and the second because it avoids the use of the deep Theorem 1.1 and also because it shows somewhat more promise for generalization.

Proof 1. It will be shown first that if Δ is a disc centered at 0 on which $\operatorname{rank} f \leqq 1$ then $[z,f]_R$ contains $\operatorname{Re} f$, and second that Δ may be shrunk to attain $[z, \operatorname{Re} f] = C(\Delta)$.

Since f has rank $\leqq 1$ on Δ, Sard's Theorem [6] shows that $f(\Delta)$ has measure zero in $\boldsymbol{C}$. The classical approximation theorem of Hartogs and Rosenthal [2, p. 47] implies that $\operatorname{Re} f$ can be uniformly approximated on Δ by rational functions of f with poles off $f(\Delta)$. That is, $[z,f]_R$ contains $\operatorname{Re} f$.

Now Δ can be chosen small enough so that for each complex number β, the level set L_β defined by (1.3) does not disconnect $\boldsymbol{C}$ and has no interior points. Once this is demonstrated, it follows immediately from Theorem 1.1 that $[z, \operatorname{Re} f] = C(\Delta)$, which will complete the proof.

Since Q is the Hessian form at 0 of $\operatorname{Re} f$, 0 is a non degenerate and therefore isolated critical point of $\operatorname{Re} f$ [5, Cor. 2.3]. Thus Δ may be shrunk so that 0 is the only critical point of $\operatorname{Re} f$ in Δ. For such Δ it is clear that L_β has no interior points. Neither does L_β disconnect the plane, for if B is a non empty bounded connected component of $\boldsymbol{C} - L_\beta$, then $B \subset \Delta$ since the unbounded component of $\boldsymbol{C} - L_\beta$ clearly contains $\boldsymbol{C} - \Delta$. Thus f is defined on $\bar{B}$ and takes the constant value β on ∂B. Now $\operatorname{Re} f$ takes maximum and minimum values on the compact set $\bar{B}$. If both of these are assumed on ∂B, then they are the same and $\operatorname{Re} f = \operatorname{Re} \beta$ is constant on $\bar{B}$. Otherwise $\operatorname{Re} f$ takes an extreme value on B. In either case, B contains a critical point of $\operatorname{Re} f$, which proves that $0 \in B$. But 0 cannot be a relative minimum of $\operatorname{Re} f$ because Q has a negative eigenvalue and $R(z) = o(|z|^2)$. Similarly, 0 cannot be a relative maximum because Q has a positive eigenvalue. This contradiction shows that L_β has no bounded complementary components and completes the proof.

This argument shows that M is really convex with respect to rational functions of the form $p(z)q(w)$, where p is a polynomial and q is rational. The second proof shows this directly as a first step, and then appeals to a technique of Wermer [9] to obtain the conclusion. Since this avoids Mergelyan's Theorem 1.1, it may have some interest as a means for obtaining further results of this type, which may be regarded as differential versions of Theorem 1.1 in which f is permitted to take complex values.

In general, a compact set M is rationally convex exactly when it coincides with its *rational convex hull* $\hat{M}_R$, which is the intersection of all rationally convex sets containing M. The rational hull is clearly rationally convex and the smallest set with this property which contains M. Another description of $\hat{M}_R$ is as the set of all points p in C^n such that each rational function r which is holomorphic near M is also holomorphic near p. For such p, each such function r must also satisfy

$$|r(p)| \leqq \sup_M |r|, \tag{2.1}$$

because if it did not, then $1/(r - r(p))$ would be holomorphic near M but with a singularity at p.

Proof 2. It will be shown that each point (α, β) in $\hat{M}_R$ is in fact contained in M. Let $R(M)$ denote the uniform closure on M of the rational functions holomorphic on M. Then $R(M)$ is a Banach algebra, and because of (2.1) the map $r \to r(\alpha, \beta)$ extends by continuity to an algebra homomorphism λ of $R(M)$ onto C. It is a standard result [2, *pp.* 31–32] that there is a *probability* measure μ on M representing λ in the sense that

$$\lambda(g) = \int g d\mu \tag{2.2}$$

for each g in $R(M)$. It will be proved that (α, β) is in M by examining the support of μ.

The theorems of Sard and Hartogs and Rosenthal show as before that $w \to \bar{w}$ is in $R(M)$, and hence so is $w \to (w-\beta)(\bar{w}-\bar{\beta}) = |w-\beta|^2$. From (2.2) and the fact that λ is multiplicative on $R(M)$,

$$0 = \lambda(w-\beta)\lambda(\bar{w}-\bar{\beta}) = \lambda(|w-\beta|^2) = \int |w-\beta|^2 d\mu.$$

Since both μ and the integrand are non negative, this implies that support $\mu \subset M \cap \{(z,w): w = \beta\}$. Note that the projection $(z,w) \to z$ carries the latter set onto L_β.

Therefore if p is any polynomial in z,

$$(2.3) \qquad |p(\alpha)| = |\lambda(p)| = \left|\int p\,d\mu\right| \leqq \sup\{|p(z)|: z \in L_\beta\}.$$

Now as in Proof 1 it may be assumed that 0 is the only critical point of $\operatorname{Re} f$ in Δ, and hence that $C - L_\beta$ is connected. So if $\alpha \notin L_\beta$, Runge's Theorem yields a polynomial p such that $|p(\alpha)| > \sup\{|p(z)|: z \in L_\beta\}$, in contradiction to (2.3). Thus $\alpha \in L_\beta$; that is, $f(\alpha) = \beta$, or $(\alpha,\beta) \in M$.

A theorem of Wermer [9] is applicable, now that $\hat{M}_R = M$, to show that $[z,f]_R = C(\Delta)$. As stated, Wermer's theorem applies to polynomially convex sets M and concludes that $[z,f] = C(\Delta)$ in that case. However, his techniques will establish the rational version of this result needed here, and so complete the proof.

It seems likely that under the hypotheses of Theorem 1.3 M is in fact polynomially convex, and that a method similar to the second proof could be used to show it. If so, Wermer's theorem would give a deeper result on polynomial approximation more closely comparable with Theorem 1.1: It would yield that $[z,f] = C(\Delta)$ if $f = Q + o(|z|^2)$ has rank $\leqq 1$ near 0 and Q is a real quadratic form with non zero eigenvalues of opposite sign.*

3. *A Counterexample*

The techniques employed to prove Theorem 1.3 might be generalized to obtain the same result in the situation where M admits a non constant function which is holomorphic in a neighborhood of 0 in C^2 and which takes real values on M. The idea is that this function would play the same general role in the proofs as that of the coordinate function w. This is an attractive possibility, especially if there are very many manifolds M which admit such a holomorphic function. Unfortunately, there do not seem to be

* (Added in proof.) This result is true as stated (for small Δ). The proof, along the indicated lines, will appear in [10]. An exposition of it is in [11].

very many, and in fact the set of manifolds which do not have this property is dense, in a special sense.

It will be shown how to construct a function f_0 of the form (1.2) for which there exists no non constant function ϕ, holomorphic in a neighborhood of 0 in C^2 and such that

$$z \to \phi(z, f_0(z)) \tag{3.1}$$

is real for all z near 0. A function f_0 with this property can be found with any prescribed order of contact at 0 to a given infinitely differentiable representation of the form (1.2). In particular, there is no prospect for finding a biholomorphic coordinate change to convert an arbitrary representation (1.2) into one of the same form where f takes real values, for one of the coordinate functions of such a transformation would have to take real values on M.

These examples probably permit no non constant holomorphic functions for which (3.1) has rank $\leqq 1$ near 0, but here it is only shown that (3.1) cannot be real.

An example can be found easily if it is not required to satisfy (1.2): $f(z) = z^2\bar{z}$ is a function whose graph permits no non constant holomorphic function ϕ which takes real values on it. This is seen by equating the coefficients of the general term $z^p\bar{z}^q$ in the power series relation

$$\phi(z, z^2\bar{z}) = \bar{\phi}(z, z^2\bar{z}).$$

To find this behavior in a function f of the form (1.2) seems to require more effort. The construction below produces one whose graph M is in fact a determining set for real-analytic functions, meaning that any real-analytic function in a connected neighborhood U of zero which is zero on M vanishes identically on U. Once this is established and ϕ is holomorphic on U and real on M, then $\operatorname{Im}\phi$ vanishes on U, which implies that ϕ is constant.

The idea of the construction is roughly that M should be a determining set if it has infinite order of contact with the graph of Q at 0, but is not contained in any of the obvious real-analytic varieties through the graph of Q. It is convenient to make the construction in several steps.

Lemma 3.1. *Suppose $\{(x_n, y_n)\}$ is a sequence in R^2 such that $x_n \neq 0$ and $y_n \neq 0$ for any n, $x_n \to 0$ and for any integer $p \geqq 0$,*

$$y_n/x_n^p \to 0.$$

If h is real-analytic in a connected neighborhood of 0 in R^2 and $h(x_n, y_n) = 0$ for all n, then $h = 0$.

Thus a sequence in $\boldsymbol{R}^2$ with "infinite order of contact to the x-axis at 0" but which is never in the x-axis is a determining set. Examples are the sequences of the type $\{(x_n, g(x_n))\}$ where

$$g(t) = \exp(-1/t^2), \qquad t \neq 0,$$

and $x_n \to 0$. In particular, Lemma 3.1 shows that the graph of g is a determining set for functions real-analytic near 0 in $\boldsymbol{R}^2$. Later, other graphs constructed from g will be shown to be determining sets for real-analytic functions of three and four variables.

Proof. It will be shown first that $a_{io} = 0$ for all $i \geqq 0$ in the expansion

$$h(x, y) = \sum_{i,j} a_{ij} x^i y^j.$$

For $i = 0$ this is clear by continuity of h at 0, since $0 = \lim_{n \to \infty} h(x_n, y_n) = h(0,0) = a_{oo}$. If $p > 0$ and it is true for all $i < p$, then for each n,

$$0 = \frac{1}{x_n^p} h(x_n, y_n) = \frac{1}{x_n^p} \sum_{i<p,\, j \geqq 0} a_{ij} x_n^i y_n^j + \sum_{i \geqq p,\, j \geqq 0} a_{ij} x_n^{i-p} y_n^j.$$

By the induction hypothesis, the first sum on the right is taken only over positive j, so that this sum is divisible by y_n. It therefore becomes y_n/x_n^p times a function analytic in a neighborhood of zero. By hypothesis, the first sum thus tends to zero. The constant term of the second sum on the right is a_{po}. Therefore as n increases the right side tends to a_{po}, so that $a_{po} = 0$.

This proves that any function real-analytic in a connected neighborhood of 0 and which vanishes on $\{(x_n, y_n)\}$ is divisible by y.

Now h can be rewritten as an infinite sum of homogeneous monomials. If h is not identically zero, there is in this expression a non zero monomial of least (total) degree m, and consequently a largest integer $p \leqq m$ such that

$$h(x, y) = y^p k(x, y)$$

for some real-analytic function k and all (x, y) near 0. Thus for all n, $0 = y_n^p k(x_n, y_n)$, and since $y_n \neq 0$, k also vanishes on $\{(x_n, y_n)\}$. Therefore k is divisible by y, which contradicts the choice of p. This proves that h is zero.

This result is used to construct a determining set in $\boldsymbol{R}^3$.

Lemma 3.2. *If h is a real-analytic function on a connected neighborhood of* 0 *in $\boldsymbol{R}^3$ which vanishes on*

$$\{(t, g(t)\sin(1/t),\ g(t)\cos(1/t)) : t > 0\}$$

then $h = 0$.

Proof. It will suffice to prove that h is zero on a small open ball B centered at 0. If H is any plane in $\boldsymbol{R}^3$ through the t-axis, it is obvious that there is a sequence $\{t_n\}$ of positive real numbers such that $t_n \to 0$ and the point

$$p_n = (t_n, g(t_n)\sin(1/t_n),\ g(t_n)\cos(1/t_n))$$

is in $H \cap B$ for each n. Up to an isomorphism of H with $\boldsymbol{R}^2$, $\{p_n\}$ and $h \mid H \cap B$ satisfy Lemma 3.1. Therefore $h \mid H \cap B = 0$. Since H is arbitrary, $h \mid B = 0$.

Now let

$$k(t) = g(t)\cos(1/t) + ig(t)\sin(1/t), \qquad t \neq 0,$$

and F be any real-analytic function in two real variables.

Theorem 3.3. *If ϕ is holomorphic in a connected neighborhood of 0 in $\boldsymbol{C}^2$ and takes real values on*

$$M = \{(z, F(z) + k(|z|)) : z \neq 0\}$$

then ϕ is constant.

If f is any infinitely differentiable function on a neighborhood of 0 in $\boldsymbol{C}$ and f_m is the partial sum of degree m of the Taylor series for f, it follows that $f_0(z) = f_m(z) + k(|z|)$ admits no non constant holomorphic functions taking real values on its graph, and $f(z) - f_0(z) = o(|z|^m)$. In this sense, examples of this behavior are "dense." Note: It is not asserted that an f_0 with this property can be found arbitrarily close to f in the $\boldsymbol{C}^\infty$ topology of some neighborhood of 0. This may be true, but no proof is offered here.

Proof. As noted already, it is enough to show that M is a determining set for real-analytic functions. Since the map

$$(z, w) \to (z, w - F(z))$$

is a real-analytic coordinate change which carries M onto

$$M' = \{(z, k(|z|)) : z \neq 0\},$$

it will suffice that M' is a determining set for real-analytic functions of four real variables.

Suppose h is such a function defined on an open ball B centered at 0. If $h \mid M' = 0$, $z \neq 0$ is any point of $\boldsymbol{C}$, and K_z is the hyperplane

$$K_z = \{tz : t \text{ real}\} \times \boldsymbol{C},$$

then $h|K_z$ is defined on the connected neighborhood $K_z \cap B$ of 0 in K_z, and vanishes on

$$M' \cap K_z = \{(tz, k(|tz|)) : t \neq 0\}.$$

Up to an isomorphism of K_z with $\boldsymbol{R}^3$, $h|K_z \cap B$ and $M' \cap K_z$ satisfy Lemma 3.2, so that $h|K_z = 0$. Since z is arbitrary, $h = 0$.

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