

GENERALIZATIONS OF FATOU'S THEOREM TO SYMMETRIC SPACES

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1. Introduction

In 1963 Furstenberg [4] introduced the notion of boundary of a symmetric space, and proved some basic facts about the associated Poisson integral. In his *Mathematical Reviews* article on [4], Helgason posed the problem of finding geometric convergence theorems of Fatou's type in this context. Such a theorem, valid for any symmetric space, was proved by Helgason and the author five years later [7]. At about the same time there were other investigations dealing only with special types of symmetric spaces but getting sharper results about them [10], [12], [22], [23]. Both types of results were further improved in [21], [13], [11]; however, the subject is still far from being finished. The purpose of the present paper is, first, to give a survey of the results and problems in this field, and second, to make some new remarks about the proofs, indicating some simplifications and extensions.

There are generalizations of Fatou's theorem in other directions, namely to harmonic functions on domains in \mathbf{R}^n [19], and to holomorphic functions on (not necessarily symmetric) domains in \mathbf{C}^n [1], [14]. These results will not be considered here.

2. The Classical Results

The models for all further generalization are the results of Marcinkiewicz and Zygmund about the polydisc [15], [25], which can also be found in the last chapter of [26]. Here one considers the functions on $D = \{(z_1, \dots, z_n) \mid |z_j| < 1 (1 \leq j \leq n)\}$ harmonic (in the ordinary euclidean sense) in each variable separately; these are exactly the functions that are harmonic in the sense of the theory of symmetric spaces. Every such function, if it is bounded or if it satisfies certain weaker hypotheses, is the "Poisson integral" of a function on the distinguished boundary $B = \{(z_1, \dots, z_n) \mid |z_j| = 1 (1 \leq j \leq n)\}$; the Poisson kernel involved is

the product of the ordinary Poisson kernels for the discs $\{z_j \mid |z_j| < 1\}$. All results are based on the fact that the functions one studies are Poisson integrals; harmonicity is not used directly.

For $0 < \alpha < \pi$ and $|v| = 1$ let $\gamma_\alpha(v)$ be a small triangle inside the unit disc with a vertex at v and the angle at v equal to α . For $u = (u_1, \dots, u_n) \in B$, let $A_\alpha(u) = \gamma_\alpha(u_1) \times \dots \times \gamma_\alpha(u_n)$ and, for $M > 1$, let $R_{\alpha, M}(u)$ be the subset of $A_\alpha(u)$ described by the inequalities

$$1 - |z_j| \leq M(1 - |z_k|) \quad (1 \leq j, k \leq n).$$

We call $A_\alpha(u)$ and $R_{\alpha, M}(u)$ *unrestricted* resp. *restricted non tangential domains at u* .

Given a function F on D , we say that F converges to the number r *non tangentially and unrestrictedly at u* if for all $0 < \alpha < \pi$,

$$\lim_{\substack{z \rightarrow u \\ z \in A_\alpha(u)}} F(z) = r.$$

F converges to r *non tangentially and restrictedly* if for all $0 < \alpha < \pi$ and $M > 1$,

$$\lim_{\substack{z \rightarrow u \\ z \in R_{\alpha, M}(u)}} F(z) = r.$$

If f is a function on B and F a function on D , it is now obvious what we mean by " F converges to f *non tangentially and (un)restrictedly a.e.*"

The principal results are the following:

(i) Suppose that F is the Poisson integral of f .

- (a) If $f \in L^p$ ($1 < p \leq \infty$), then F converges to f non tangentially and unrestrictedly a.e.
- (b) If $f \in L^1$ or if f is a measure, restricted convergence a.e. still holds (in the second case to the Radon-Nikodym derivative of f).
- (c) There are examples of $f \in L^1$ such that unrestricted convergence a.e. fails.

(ii) The boundary of D in C^n can be regarded in a natural way as the union of a finite number of continuous families of lower dimensional polydiscs ("boundary components"; B is just the family of the zero-dimensional boundary components). If, as above, F is the Poisson integral of f , then F has boundary values on every boundary component; these boundary values are themselves (lower dimensional) Poisson integrals of appropriate restrictions of f .

(iii) For small $h > 0$, let $A_\alpha^h(u)$ be the subset of $A_\alpha(u)$ determined by the inequalities $1 - |z_j| \leq h$ ($1 \leq j \leq n$) ("truncated unrestricted non tangential domain").

Let $S \subset B$ be a measurable set, and suppose that for each $u \in S$ there is given an α and an h . If F is a harmonic function on D which is bounded in $\bigcup_{u \in S} A_\alpha^h(u)$, then F has unrestricted non tangential boundary values a.e. on S . (In one variable this is due to Privalov; the general case was settled by a new method by Calderón [2].)

3. The General Case: Definitions

Let $X = G/K$ be a symmetric space of non compact type; more exactly, let G be connected semisimple with finite center, such that every simple factor is non compact, and let K be a maximal compact subgroup. Any homogeneous space U of G on which K acts transitively is called a *boundary* of X (although, as will appear below, "distinguished boundary" might perhaps be a better name). Every boundary carries a unique normalized K -invariant measure μ . The Radon-Nikodym derivative

$$P(g, u) = \frac{d\mu(g^{-1}u)}{d\mu(u)}$$

can be shown to exist, and is called the Poisson kernel of X with respect to U ; it clearly depends only on the left K -coset of g , so it can be regarded as a function on $X \times U$. It is now obvious what one means by the Poisson integral of a function given on U ; it is immediate from the Gauss-Godement mean value theorem that every Poisson integral is harmonic (i.e., is annihilated by all G -invariant differential operators without constant term on X).

After changing slightly the definition of a boundary so as to exclude certain trivial redundancies, it can be shown [4] that there exists a unique maximal boundary, which can be regarded as a fiber space over any other boundary, and has the property that every bounded harmonic function on X is a Poisson integral with respect to it. It is always only the maximal one that has the latter property, and this fact would seem to make the other boundaries rather uninteresting. However, in the case of bounded symmetric domains (in their standard realization in C^n) the Bergman-Šilov boundary is in general one of these smaller boundaries; this makes it worthwhile to study all the boundaries together.

To get a more precise description we introduce some notation. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G , let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} , \mathfrak{h} a Cartan subalgebra containing \mathfrak{a} . Com-

plexifying \mathfrak{g} we consider its root structure with respect to \mathfrak{h} . Taking a basis first of \mathfrak{a} , then of its orthogonal complement in \mathfrak{h} , we introduce a lexicographic ordering. The restriction of a root to \mathfrak{a} , if it is non zero, is called a *restricted root*. We denote by F the set of distinct non zero restrictions of the simple roots to \mathfrak{a} .

Let E be a subset of F , $\mathfrak{a}(E)$ the subspace of \mathfrak{a} annihilated by E . Denote the semisimple part of the centralizer of $\mathfrak{a}(E)$ by \mathfrak{g}^E , and the centralizer of $\mathfrak{a}(E)$ in \mathfrak{k} by $\mathfrak{m}(E)$. Let $\mathfrak{k}^E = \mathfrak{k} \cap \mathfrak{g}^E$; then $\mathfrak{m}(E)$ is the direct sum of \mathfrak{k}^E and a subspace of \mathfrak{h} . Denote the sum of all those positive restricted roots which do not vanish on $\mathfrak{a}(E)$ by $2\rho_E$; the sum of the root spaces of these (resp. of their negatives) we denote by $\mathfrak{n}(E)$ (resp. $\overline{\mathfrak{n}}(E)$). The sum of $\mathfrak{n}(E)$ and the centralizer of $\mathfrak{a}(E)$ we denote by $\mathfrak{b}(E)$.

The analytic subgroups of G corresponding to these subalgebras will be denoted by the corresponding Latin capital letters, with two exceptions: $M(E)$ will be the centralizer of $\mathfrak{a}(E)$ in K , $B(E)$ the normalizer of $\mathfrak{n}(E)$ in G . (They have the Lie algebras $\mathfrak{m}(E)$, $\mathfrak{b}(E)$, but they may not be connected.)

It is known [4], [18], [16], that the boundaries of X are exactly the spaces $G/B(E)$; as K -spaces they are isomorphic with $K/M(E)$. So, if l is the rank of X , there are 2^l boundaries, one of which (when $E = F$) degenerates to a point. The case $E = \emptyset$ gives the maximal boundary. In this case we write \mathfrak{a} , \mathfrak{n} , \mathfrak{b} , A, \dots for $\mathfrak{a}(E)$, $\mathfrak{n}(E)$, $\mathfrak{b}(E)$, $A(E), \dots$; the notation still remains consistent.

At this point a boundary is only an abstractly defined space, and it makes no sense to talk about convergence to the boundary on X . For each E it is possible to construct a natural compactification of X (via projective embeddings [18], or equivalently, via embeddings of X into spaces of measures on $G/B(E)$ [4], [16], cf. also [9]) such that the topological boundary of X in it consists of $G/B(E)$ and of certain families of lower dimensional symmetric spaces, called boundary components. We do not pursue this description here; instead, following [7] and [13], we shall define the generalizations of the truncated non tangential domains intrinsically in terms of X . We call them "admissible" instead of non tangential, since in the case of bounded symmetric domains in \mathbf{C}^n they are in general not non tangential [13, §3]. The guiding idea behind this definition is that $G/B(E) = K/M(E)$ can in an obvious way be identified with the set of all conjugates under $\text{ad}(K)$ of the "positive Weyl chamber" $\mathfrak{a}^+(E)$.

For fixed E , choose an element $H \in \mathfrak{a}^+(E)$ (i.e., an element of $\mathfrak{a}(E)$ on which all the positive roots that do not vanish on $\mathfrak{a}(E)$ have a positive value). Let $C \subset X$ be compact, $M(E)$ -invariant, with non empty interior;

let $\tau \in \mathbf{R}$, $k \in K$. We define a *truncated restricted admissible domain* at $\hat{k} \in G/B(E)$ by

$$\mathcal{R}_{H,C}^\tau(k) = \{k(\exp tH) \cdot x \mid t \geq \tau, x \in C\}.$$

We say that a function F on X converges at \hat{k} *admissibly and restrictedly with respect to H* to the number r , if for all $\varepsilon > 0$ and all C there exists τ such that $x \in \mathcal{R}_{H,C}^\tau(k)$ implies $|F(x) - r| < \varepsilon$. If f is a function on $G/B(E)$, it is now clear what we mean by F *converges to f admissibly and restrictedly (with respect to H) a.e.*

As for the truncated *unrestricted* admissible domains, the hermitian symmetric example suggests [13, §3], [21] the definition

$$\{kma \cdot x \mid m \in K^E, a \in A, \log a \geq T, x \in C\}$$

where C is a compact set as above, and $T \in \mathfrak{a}(E)$ ($\log a \geq T$ means that $\log a - T \in \overline{\mathfrak{a}^+}$). However (cf. §4, (i.c)) it seems that the only case where something can be proved about this is $E = \emptyset$. It is possible (and this is new here) to define an *intermediate* type of convergence in terms of the sets

$$\mathcal{A}_C^T(k) = \{ka \cdot x \mid a \in A(E), \log a \geq T, x \in C\}$$

where $C \subset X$ is compact $M(E)$ -invariant with non empty interior and $T \in \mathfrak{a}(E)$ (F converges to r at \hat{k} , if for every $\varepsilon > 0$ and every C there exists T such that...). In the case of the maximal boundary, where $E = \emptyset$, this type of convergence coincides with *unrestricted admissible convergence* as defined in [13] and above. For any E , convergence in the intermediate sense implies convergence with respect to any choice of $H \in \mathfrak{a}^+(E)$.

4. Results and Problems

(i) For given E , let f be a function on $G/B(E)$, and let F be its Poisson integral. The following facts are known about the convergence of F to f .

(a) If $f \in L^\infty$, *restricted admissible convergence a.e.* holds with respect to any choice of $H \in \mathfrak{a}^+(E)$ (shown in [13] by a slight extension of the methods of [7]; cf. also §5 here).

(b) If $f \in L^\infty$ and $E = \emptyset$, *unrestricted admissible convergence a.e.* holds (reduced to a "strong differentiation" theorem in [13]; the latter proved in [11]; cf. also §5).

(c) If $E \neq \emptyset$, there are examples of $f \in L^\infty$ such that *unrestricted convergence a.e.* does not hold [21, §4].

With these the case of $f \in L^\infty$, i.e., the case of bounded harmonic func-

tions, seems to be rather completely settled. We continue with the results about more general cases:

(d) If X is hermitian symmetric and $G/B(E)$ is the Bergman-Šilov boundary of its standard realization in C^n , then *restricted admissible convergence a.e.* holds for all $f \in L^p$ ($1 \leq p \leq \infty$), and even for all signed measures. (See [22], [23] for $p > 1$, [21] for the general case; the equivalence of the notion of convergence used there with ours is in [13].)

(e) If X has rank 1 (so that there is only one non trivial boundary, and there is no distinction between restricted and unrestricted convergence) then admissible convergence a.e. holds for all $f \in L^p$ ($1 \leq p \leq \infty$) and for all measures. (See [10] for radial convergence, extended to admissible in [13]; new proof in §5 here.)

(f) If X is a product of spaces of rank 1 and E is arbitrary, then unrestricted convergence a.e. holds for $f \in L^p$ ($1 < p \leq \infty$) and restricted convergence for $f \in L^p$ ($1 \leq p \leq \infty$) and measures. (This follows from (e) by the arguments of [15]; cf. also [26, Ch. XVII].)

Nothing is known for general X about the case $f \in L^p$ for $p > 1$.

(ii) For each fixed E , X has a Furstenberg-Satake compactification whose boundary consists of families of "boundary components" (cf. §3). It could be expected that, at least if $f \in L^\infty$ and F is the Poisson integral of f , then F will have boundary values on a.a. components, and these will be (lower dimensional) Poisson integrals of certain restrictions of f . Results of Karpelevič [9] about continuous f and certain partial results of the author indicate that this is indeed the case; the author hopes to return to this question in the future.

(iii) Let $S \subset G/B$ be a measurable set, and suppose that for each $k \in S$ there is given a $T \in \mathfrak{a}$ and a compact $C \subset X$ with non empty interior. If F is harmonic on X and bounded in $\bigcup_{k \in S} \mathcal{A}_C^T(k)$, is it true that F converges unrestrictedly and admissibly at a.a. points of S ?

The answer is *yes* for spaces of rank 1 [12], [13], and this result can also be extended to products of such spaces by using the techniques of Calderón [2]. In the general case the question is open.

5. Remarks About the Proofs

All proofs depend, in the first place, on the following construction, which amounts to transforming the Poisson integral from the disc to the upper halfplane and substituting the natural non tangential domains of the halfplane for those of the disc.

The inclusion of $\bar{N}(E)$ in G induces a map $\iota: \bar{N}(E) \rightarrow G/B(E)$. By the lemma of Bruhat and Harish-Chandra this map is regular, one-to-one, with open dense range. One has $\iota(\bar{n}) = k(\bar{n})$, where $k(\bar{n})$ is defined by the Iwasawa decomposition $G = KAN$, $\bar{n} = k(\bar{n})an$.

For $\bar{n} \in \bar{N}(E)$, $T \in \mathfrak{a}(E)$, and for U, V compact sets with non empty interior in $\bar{N}(E)$, A , respectively, we define

$$\Gamma_{U,V}^T(\bar{n}) = \{\bar{n}a\bar{n}_1ma_1 \cdot o \mid a \in A(E), \log a \geq T, \bar{n}_1 \in U, m \in K^E, a_1 \in V\}$$

where o denotes the identity coset in $X = G/K$. Then the following is true.

In the definition of convergence of intermediate type (end of §3) one can replace the sets $\mathcal{A}_C^T(k(\bar{n}))$ by $\Gamma_{U,V}^T(\bar{n})$.

More exactly: F converges to r at $k(\bar{n})$ in the intermediate sense if and only if for all $\varepsilon > 0$ and all U, V , there exists T such that $x \in \Gamma_{U,V}^T(\bar{n})$ implies $|F(x) - r| < \varepsilon$. (Of course, since a.a. points of the boundary are of the form $\iota(\bar{n}) = k(\bar{n})$ for some \bar{n} , it will always be enough to consider only these.)

The proof of this is an easy extension of the proof of [I3, Prop. 4.2], the key point being that a.e. convergence of intermediate type is a G -invariant notion. There is also a similar result about restricted admissible convergence [I3, Prop. 2.4], but that is not needed in the present simplified treatment.

It is now possible to prove the following theorem, which contains both (i.a) and (i.b) of §4.

If $f \in L^\infty(G/B(E))$ and F is its Poisson integral, then F converges to f in the intermediate sense a.e.

When $E = \emptyset$, this is just (i.b). In the case of general E it clearly implies (i.a); it even strengthens it slightly, by making the exceptional set of measure zero independent of the choice of $H \in \mathfrak{a}^+(E)$.

The proof runs as follows: One shows first that a.a. points $\bar{n}_0 \in \bar{N}(E)$ have the property that for every compact $U \subset \bar{N}(E)$ with non empty interior,

$$\lim_{a \rightarrow \infty} \frac{1}{|U^a|} \int_{U^a} |(f \circ \iota)(\bar{n}_0\bar{n}) - (f \circ \iota)(\bar{n}_0)| d\bar{n} = 0$$

where $a \rightarrow \infty$ means $\lambda(\log a) \rightarrow \infty$ for all those positive roots λ that do not vanish on $\mathfrak{a}(E)$. This is a trivial extension of Theorem 4.2 of Knapp and Williamson [I1], which in turn is a non trivial extension of the method of [8] for proving the strong differentiation theorem. One finishes the proof of our theorem by showing that whenever \bar{n}_0 has this property, F converges to $f(\iota(\bar{n}_0))$ at $\iota(\bar{n}_0)$ in the intermediate sense; this is an easy modification of the proof of [I3, Prop. 4.3].

Next we want to indicate a simplified new way of proving (i.d) of §4. For this we need the following result (cf. [13, Prop. 2.5]).

Let $1 \leq p \leq \infty$. If for all f such that $f \circ \iota$ is in $L^p(\bar{N}(E))$ the Poisson integral of f converges admissibly (restrictedly or in the intermediate sense) a.e. to f , then the same is true for all $f \in L^p(G/B(E))$. A similar statement holds for finite signed measures.

Another basic fact needed is an extension of the Hardy-Littlewood maximal theorem to \bar{N} ; this can be found formulated and proved in a form most convenient for our purpose in [21, §2]. Its proof is essentially due to Wiener [24], and was successively extended to more general situations in [17], [2a], [19], [3], [20].

After these one only needs an estimate on the Poisson kernel transformed to \bar{N} , and (i.d) follows by simple classical methods [13, §5]. Such an estimate is given in [13, Lemma 5.2]; its proof relies on some relatively complicated computations of Knapp [10]. Recently, however, Helgason [6] obtained an explicit formula for the Poisson kernel in the case of rank 1; this makes it very easy to get the estimate we need.

In fact, if X has rank 1, there are at most two positive restricted roots, α and 2α ; let their respective multiplicities be $m_\alpha, m_{2\alpha}$. As in [13] we normalize $H \in \mathfrak{a}^+$ by the condition $\alpha(H) = 1$; then we have $2\rho(H) = m_\alpha + 2m_{2\alpha}$, which we denote by m for brevity. Every element $\bar{n} \in \bar{N}$ can be written in the form $\bar{n} = \exp(X + Y)$ with X, Y in the root spaces for $-\alpha, -2\alpha$, respectively. By Helgason's formula

$$e^{-2\rho(H(\bar{n}))} = [(1 + c|X|^2)^2 + 4c|Y|^2]^{-m/2}$$

where $c^{-1} = 4(m_\alpha + 4m_{2\alpha})$. The gauge on \bar{N} [13, §2] now has the expression

$$|\bar{n}| = \text{Max}\{|X|, |Y|^{\frac{1}{2}}\}.$$

(One could also use $(c^2|X|^4 + 4c|Y|^2)^{\frac{1}{2}}$ as gauge, which would make the proof run even more smoothly.) It is now immediately apparent that

$$e^{-2\rho(H(\bar{n}))} \leq \begin{cases} 1 \\ M|\bar{n}|^{-2m} \end{cases}$$

with some constant M . Since $|\bar{n}^{e^{xp-tH}}| = e^t|\bar{n}|$, it follows that

$$e^{-2\rho(H(\bar{n}^{e^{xp-tH}})-tH)} \leq \begin{cases} e^{mt} \\ M \frac{e^{-mt}}{|\bar{n}|^{2m}} \end{cases}.$$

This estimate is of the type of [13, Lemma 5.2]; it is even a little stronger, and can be used in the same way to finish the proof of (i.d).

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