

GENERATING SOME RINGS OF MODULAR FORMS
IN SEVERAL COMPLEX VARIABLES
BY ONE ELEMENT*

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§1. *Introduction*

Suppose that Z is a domain in \mathbb{C}^n and Γ is a properly discontinuous subgroup of the full group $\text{Bih } Z$ of biholomorphic automorphisms of Z . Denote the complex vector space of holomorphic Γ -automorphic forms of weight w and multiplier system v by (Γ, w, v) ; if $v \equiv 1$, write (Γ, w) . Under certain conditions the graded ring $\bigoplus_w (\Gamma, w, v)$ is an integral domain whose quotient field contains the field $\mathfrak{M}(Z/\Gamma)$ of functions meromorphic on the complex space Z/Γ as the subfield of meromorphic Γ -automorphic forms of weight zero, the transcendence degree of $\mathfrak{M}(Z/\Gamma)$ over \mathbb{C} is not greater than $\dim_{\mathbb{C}} Z$, and the transcendence degree of $\bigoplus (\Gamma, w, v)$ is not greater than $(1 + \dim_{\mathbb{C}} Z)$. For example, if Z denotes the upper half plane of classical function theory and Γ is the classical modular group, then $\mathfrak{M}(Z/\Gamma)$ is a field of rational functions of one variable and $\bigoplus (\Gamma, w)$ is generated by the algebraically independent modular forms g_4 and g_6 of weight 4 and 6 respectively. However, these forms are *differentially dependent* in the sense that there is a non trivial polynomial with complex coefficients in two functions f_1 and f_2 and their derivatives which vanishes for the specialization $(f_1, f_2) \rightarrow (g_4, g_6)$. There is such a polynomial in which f_2 appears linearly and none of its derivatives appear. This shows that it is possible to generate $\bigoplus (\Gamma, w)$, and $\mathfrak{M}(Z/\Gamma)$, by using *one* Γ -automorphic form and certain differential operators. Indeed, there are Γ -independent non linear differential operators D^2 and D^3 , of order 2 and 3 respectively, such that g_4 and $D^3 g_4 / D^2 g_4$ generate $\bigoplus (\Gamma, w)$; it can also be shown that there are constants c_1 and c_2 such that $g_4 = c_1 D^2 \Delta / \Delta^2$ and $g_6 = c_2 D^3 \Delta / \Delta^3$, where Δ denotes a cusp form of weight 12 (cf. [6]).

The purpose of this paper is to show that certain graded rings of automorphic forms in several complex variables can be generated by one form

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and differential operators that are independent of the discontinuous group. Whenever this is possible, all information about the group that is contained in $\oplus(\Gamma, w, v)$ or in $\mathfrak{M}(Z/\Gamma)$ is already contained in some one form.

In §2, the differential operators of interest are defined and their properties studied in the context of tube domains arising from simple compact real Jordan algebras.¹ They are applied to study the modular forms of even weight for Siegel's modular group of genus two in §3; our most precise result is Theorem 1 in that section. The normalized cusp form of weight 10 for Siegel's group of genus two is the square of a cusp form of weight 5 associated with a non trivial multiplier system. In §4 we give a differential equation relating it to the cusp form of weight 12 associated with the trivial multiplier. Differential equations for the generators of the ring of forms of even weight for a certain congruence subgroup are also given in §4, but we have not troubled to determine the explicit values of the constants involved. The hermitian modular forms for the gaussian integers are studied in §5; it is shown that by using differential operators, and up to the non vanishing of certain computable constants, the graded ring of all symmetric forms of even weight can be generated by the cusp form of weight 8. Throughout the paper c_1, c_2, \dots denote complex constants.

There are numerous "conjectures" suggested by our results. Unfortunately, there is as yet too little detailed information available to permit us to formalize these in a truly meaningful way. The following remarks may nevertheless be of some use to the reader.

In each of the examples studied here, and for the classical modular group, the graded ring of forms of even weight can be generated by the cusp form of least weight. These rings are not always the full ring of forms: for Siegel's modular group of genus two, there is a cusp form of weight 35 which must be adjoined to obtain the ring of all forms associated with the trivial multiplier. However, in all of these examples, the field of modular functions is determined by the forms of even weight, and hence by one form. Perhaps our results have more to do with function fields than rings of forms.

Our techniques cannot provide access to forms of odd weight within the confines of the set of forms of integral weight because with this limitation the action of each of the relevant differential operators known to us produces a form of even weight. Also, for related reasons, the differential operators simplify multiplier systems, so these techniques will provide the most information in the case of the trivial multiplier; cp. eq. (3) in §2.

In §4 and §5 there are several constants which, hopefully, are not zero.

They can all be effectively computed with some effort, but it does not at present seem to be of particular interest to know them explicitly. The calculations required are similar to those performed for Theorem 1; their form makes it appear unlikely that the pertinent constants are zero. However, should certain of them vanish, it would imply the existence of differential equations of order less than $(\dim_{\mathbb{C}} Z + \text{rank } Z)$, which we think would be a much more interesting phenomenon (cp. [9]). There is the possibility that a general argument may settle this question without calculation.

If a graded ring can be generated by one form and differential operators, then any relations become differential equations satisfied by the generator; a similar statement holds for the corresponding function field. The reader will think immediately of the Weierstrass \wp -function of an elliptic curve. It would be interesting to know how such differential equations are distinguished from those satisfied by arbitrary forms in the graded ring.

§2. *Differential Operators*

The method of this paper is based on the connection between differential operators that map forms to forms and Jordan algebras. Suppose Z is biholomorphically equivalent to an irreducible bounded symmetric domain of tube type. Then there is a simple compact real Jordan algebra \mathfrak{A} such that Z is equivalent to $Z(\mathfrak{A}) = \mathfrak{A} + i \exp \mathfrak{A}$. Let $\sigma(a)$ denote the reduced trace, $|a|$ the reduced norm, of $a \in \mathfrak{A}$. These functions extend to the complexification of \mathfrak{A} . If f is a differentiable function defined on an open subset of $Z(\mathfrak{A})$, define ∇f , the gradient of f with respect to σ , by

$$\frac{\partial f}{\partial z}(a) = \sigma(\nabla_z f, a).$$

For every simple compact real Jordan algebra of rank r there is a real algebra \mathcal{A} with an involution $*$: $u \rightarrow u^*$ such that \mathcal{A} contains \mathbf{R} as a sub-algebra and every $a \in \mathfrak{A}$ can be represented by an $r \times r$ matrix (a_{ij}) , $a_{ij} \in \mathcal{A}$, which is symmetric with respect to the involution; if $\text{rank } \mathfrak{A} = 2$, there are additional minor technical restrictions. In this representation, $\sigma(z) = \sum_k z_{kk}$, $|z| = \det z$ (the non commutative determinant), and

$$\nabla_z = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & & & \\ & & \frac{1}{2} \frac{\partial}{\partial z_{kl}} & \\ & \frac{1}{2} \frac{\partial}{\partial z_{k^*l}} & & \\ & & & \frac{\partial}{\partial z_{rr}} \end{pmatrix}$$

where $\partial/\partial z_{kl}$ denotes the usual gradient on \mathcal{A} for $k \neq l$. Define

$$\partial_z = |\nabla_z|.$$

Set $q = \dim \mathfrak{A}/\text{rank } \mathfrak{A}$. Then, as Selberg has shown, for every $\Gamma \subset \text{Bih } Z(\mathfrak{A})$, multiplier system v , and positive integer n ,

$$(1) \quad \partial^n: (\Gamma, q-n, v) \rightarrow (\Gamma, q+n, v).$$

Definition 1. Suppose $U \subset Z(\mathfrak{A})$ is an open set. For any $w \in \mathbb{C} - \{0\}$, $n \in \mathbb{Z}^+ - \{q\}$, and $f \in \mathcal{C}^{nr}(U)$, set

$$(2) \quad D^n f = f^{nr+(q-n)/w} \partial^n f^{(n-q)/w}.$$

$D^n f$ is a polynomial in f and its derivatives of order at most nr . It follows from (1), and from a special but elementary consideration in case $w = 0$, that

$$(3) \quad w^{nr} \cdot D^n: (\Gamma, w, v) \rightarrow (\Gamma, n(rw+2), v^{nr}).$$

Suppose $\Gamma \subset \text{Bih } Z(\mathfrak{A})$; put $\Gamma_\infty = \{\gamma \in \Gamma: \gamma z = z + a \text{ with } a \in \mathfrak{A}\}$. We assume that $\mathfrak{A}/\Gamma_\infty$ is compact and consider only those holomorphic Γ -automorphic forms that have a Fourier expansion of the form

$$(4) \quad \sum_{\substack{m \geq 0 \\ m \in M}} a(m) e^{2\pi i \sigma(mz)},$$

where M is a lattice in \mathfrak{A} ; usually we will have $M = \Gamma'_\infty$, where Γ'_∞ is the lattice dual to $\Gamma_\infty(0) = \{\gamma(0): \gamma \in \Gamma_\infty\}$. The partial ordering is defined by $b \geq a \Leftrightarrow b - a \in \overline{\exp \mathfrak{A}}$.

Definition 2. $f \in (\Gamma, w, v)$ is a *cusp form* if $a(m) \neq 0$ implies $m > 0$.

Definition 3. If f has an expansion of the form (4), then

$$\text{ord } f \equiv \min_{a(m) \neq 0} \sigma(m).$$

Evidently $\text{ord}(fg) = \text{ord } f + \text{ord } g$ and $\text{ord}(f+g) \geq \min\{\text{ord } f, \text{ord } g\}$. If f is a cusp form, then $\text{ord } f > 0$.

An expression for $D^n f$ more convenient for the applications that we have in mind than eq. (2) is provided by the following

Lemma 1. Let L be an arbitrary homogeneous linear differential operator of order m acting on functions defined on an open subset U of \mathbb{C}^n . If $f \in \mathcal{C}^m(U)$ and $s \in \mathbb{C}$, then

$$(5) \quad L(f^s) = \sum_{k=1}^m \prod_{\substack{j=0 \\ j \neq k}}^m \binom{s-j}{k-j} f^{s-k} L(f^k).$$

Proof. Since a homogeneous linear differential operator of order 1 is a derivation, there are polynomials $E_i(f)$ in the derivatives of f of order i with coefficients which are linear combinations of the coefficient functions of L such that

$$(6) \quad L(f^s) = \sum_{k=1}^m \frac{s!}{(s-k)!} s^{s-k} E_{m+1-k}(f)$$

where $s! = \Gamma(s+1)$. Inductively set $s = 1, 2, \dots, m$ to find expressions for $E_i(f)$: $E_m(f) = L(f)$, $2!E_{m-1}(f) = L(f^2) - 2fL(f)$, etc. Therefore $L(f^s)$ can be written as

$$(7) \quad L(f^s) = \sum_{k=1}^m p_k f^{s-k} L(f^k)$$

with $p_k \in \mathbf{R}[s]$ of degree $\leq m$. Observe that if $1 \leq j \leq m$, then (7) implies that $L(f^j) = p_j L(f^j)$; that is, only the $k = j$ term survives, and $p_j(j) = 1$. Thus, if $k \neq j$, then $(s-j)$ divides p_k for $1 \leq j, k \leq m$. Also, directly from (6), or by noting that $L(f^0) = L(1) = 0$, it is clear that s divides p_k for $1 \leq k \leq m$. Therefore there are constants c_k such that

$$p_k(s) = \frac{c_k}{s-k} \prod_{j=0}^m (s-j).$$

Determine the constants by recalling that $p_k(k) = 1$, so that

$$c_k = 1 / \prod_{\substack{j=0 \\ j \neq k}}^m (k-j).$$

This completes the proof.

Proposition 1. *If $w \in \mathbf{C} - \{0\}$ and $n \in \mathbf{Z}^+ - \{q\}$, then*

$$(8) \quad D^n f = \sum_{k=1}^{nr} \prod_{\substack{j=0 \\ j \neq k}}^{nr} \left(\frac{\binom{q-n-j}{w}}{k-j} \right) f^{nr-k} \partial^n f^k.$$

Proof. ∂^n is linear homogeneous of order nr , so the proposition follows immediately from Definition 1 and Lemma 1.

Proposition 2. *If f is an automorphic form with a Fourier expansion of the form (4), then:*

- (a) $D^n f$ is a cusp form;
 (b) If $\text{ord} f > 0$, then $\text{ord} D^n f \geq nr \text{ord} f$.

Proof. It is easily verified that $\partial_z \exp \sigma(az) = |a| \exp \sigma(az)$. Moreover, if $m \geq 0$ but $m \neq 0$, then $|m| = 0$. Therefore, the Fourier coefficient of $\frac{\partial^n f^k}{\exp \mathfrak{A}}$ corresponding to the lattice point m is not zero only if $m > 0$. Since $\exp \mathfrak{A}$ is a strictly convex cone, the same is true of $f^{nr-k} \partial^n f^k$, and therefore also of the linear combination of such terms in (8). Hence $D^n f$ is a cusp form.

If $\text{ord} f > 0$, then $\text{ord}(f^{nr-k} \partial^n f^k) = \text{ord} f^{nr-k} + \text{ord} \partial^n f^k = (nr-k) \text{ord} f + k \text{ord} f = nr \text{ord} f$, which was to be shown.

Proposition 3. Suppose that M is a lattice in \mathfrak{A} and $f \neq 0$ is a cusp form of weight different from 0 with a Fourier expansion of the form $\sum_{m>0, m \in M} a(m) \varepsilon(mz)$, with $\varepsilon(mz) = e^{2\pi i \sigma(mz)}$. Then $\forall n \in \mathbf{Z}^+ - \{q\}$, $D^n f \neq 0$.

Proof. $H(\alpha) = \{u \in \mathfrak{A} : \sigma(u) = \alpha \in \mathbf{R}\}$ is a hyperplane in \mathfrak{A} perpendicular to the line joining 0 to the unit element of \mathfrak{A} . Put $C = H(\text{ord} f) \cap \{m \in M : a(m) \neq 0\}$. C is a non empty subset of $\exp \mathfrak{A}$. Let m_0 be an extreme point of the convex hull of C . Then, by (8), $a(m_0)$ contributes to the Fourier coefficient of $D^n f$ at the lattice point $(nr)m_0$. The convexity of the Cauchy product of series and the extremal property of m_0 in C imply that only the lattice point m_0 contributes to the Fourier coefficient of $D^n f$ at $(nr)m_0$. Hence, we can write $f = a(m_0) \varepsilon(m_0 z) + f_1$, with f_1 not contributing to the coefficient of $D^n f$ at $(nr)m_0$. Writing $a(m|g)$ for the Fourier coefficient of g at m , we have

$$\begin{aligned} a((nr)m_0 | D^n f) &= a((nr)m_0 | D^n(a(m_0) \varepsilon(m_0 z))) \\ &= a((nr)m_0 | a(m_0)^{nr} \varepsilon\left(\left(nr + \frac{q-n}{w}\right)m_0 z\right) \partial^n \varepsilon\left(\left(\frac{n-q}{w}\right)m_0 z\right)) \end{aligned}$$

by (2),

$$= a((nr)m_0 | a(m_0)^{nr} \left(2\pi i \frac{n-q}{w}\right)^{nr} \varepsilon((nr)m_0 z))$$

since $\partial_z \varepsilon(mz) = (2\pi i)^r |m| \varepsilon(mz)$,

$$= \left(2\pi i \frac{n-q}{w}\right)^{nr} a(m_0)^{nr}.$$

The proof is complete.

Two elementary properties of the D^n operators will prove useful in the sequel.

Proposition 4. *If $f \in (\Gamma, w, v)$, then for $s \in \mathbb{C}$,*

$$D^n(f^s) = f^{(s-1)nr} D^n f.$$

Proof. f^s satisfies the functional equation of an element of (Γ, sw, v^s) so

$$\begin{aligned} D^n(f^s) &= (f^s)^{nr+(n-q)/sw} \partial^n (f^s)^{(n-q)/sw} \\ &= f^{(s-1)nr} D^n f. \end{aligned}$$

Also observe that for $\lambda \in \mathbb{C}$,

$$(9) \quad D^n(\lambda f) = \lambda^{nr} D^n f.$$

§3. Siegel Forms of Genus Two

In this section we restrict ourselves to the Siegel upper half plane of degree 2, so $\mathfrak{H} = \mathfrak{H}(2, \mathbb{R})$, and suppose that Γ is Siegel's modular group. Define the Eisenstein series of weight w by

$$(10) \quad \phi_w(z) = \sum_{\{c,d\}} \det(cz + d)^{-w},$$

where $\{c, d\}$ runs over the classes of coprime symmetric matrices; if $w > 2$ is an even integer, the series converges absolutely to an element of (Γ, w) .

Igusa's fundamental Structure Theorem [2] states that the graded ring of modular forms of even weight is generated by $\phi_4, \phi_6, \phi_{10}$, and ϕ_{12} . Moreover, the field of modular functions is the subfield of $C(\phi_4, \phi_6, \phi_{10}, \phi_{12})$ consisting of meromorphic forms of weight zero. The Eisenstein series of weights 10 and 12 can be replaced by cusp forms since $(\phi_4 \phi_6 - \phi_{10})$ and $(3^2 \cdot 7^2 \phi_4^3 + 2 \cdot 5^3 \phi_6^2 - 691 \phi_{12})$ are in the kernel of the Siegel Φ operator. Following Igusa, define normalized cusp forms by

$$\chi_{10} = -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1} (\phi_4 \phi_6 - \phi_{10})$$

and

$$\chi_{12} = 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1} (3^2 \cdot 7^2 \phi_4^3 + 2 \cdot 5^3 \phi_6^2 - 691 \phi_{12});$$

their Fourier expansions are

$$(11) \quad \chi_{10}(z) = \frac{1}{2} \varepsilon(z_1 + z_2) - \frac{1}{4} [\varepsilon(z_1 + z_2 + z_3) + \varepsilon(z_1 + z_2 - z_3)] + \dots$$

and

$$(12) \quad \chi_{12}(z) = \frac{5}{6} \varepsilon(z_1 + z_2) + \frac{1}{12} [\varepsilon(z_1 + z_2 + z_3) + \varepsilon(z_1 + z_2 - z_3)] + \dots$$

Theorem 1.

$$(13) \quad D^1\phi_4 = 2^6 \cdot 3^2 \cdot 5 \cdot 7\pi^2\chi_{10};$$

$$(14) \quad D^1\phi_6 = 2^6 \cdot 3^2 \cdot 7 \cdot 11\pi^2\phi_4\chi_{10};$$

$$(15) \quad D^1\chi_{10} = \frac{3^2}{2^4 \cdot 5^2} \pi^2 \chi_{10}\chi_{12};$$

$$(16) \quad D^1\chi_{12} = -\frac{\pi^2}{2^6 \cdot 3^5} \{2 \cdot 23\phi_6\chi_{10}^2 + 3 \cdot 37\phi_4\chi_{10}\chi_{12}\}.$$

Proof. By (3), $D^1\phi_4 \in (\Gamma, 10)$; Proposition 2 states that it is a cusp form. Hence, Igusa's Structure Theorem implies the existence of a constant c_3 such that $D^1\phi_4 = c_3\chi_{10}$.

Similarly, $D^1\phi_6 \in (\Gamma, 14)$, so the Structure Theorem asserts the existence of c_4 such that $D^1\phi_6 = c_4\phi_4\chi_{10}$.

χ_{10} is a cusp form in $(\Gamma, 22)$. The Fourier expansion (11) shows that $\text{ord } \chi_{10} = 2$, so $\text{ord } D^1\chi_{10} \geq 4$ by Proposition 2. Hence $D^1\chi_{10} = c_5\chi_{10}\chi_{12}$ with $c_5 \neq 0$.

χ_{12} is cusp and $\text{ord } \chi_{12} = 2$ by (12); hence $D^1\chi_{12} \in (\Gamma, 26)$ is of the form $D^1\chi_{12} = c_6\phi_6\chi_{10}^2 + c_7\phi_4\chi_{10}\chi_{12}$, with at least one of c_6, c_7 non zero.

The values of the constants c_3, \dots, c_7 are determined by calculating the first few terms of the Fourier expansions, using the table of Fourier coefficients in [2].

Remark. Elimination of ϕ_6, χ_{10} , and χ_{12} yields a differential equation of order 8 for ϕ_4 (cf. [8]).

Theorem 2. $\bigoplus_{w \in 2\mathbb{Z}}(\Gamma, w)$ and $\mathfrak{M}(Z(\mathfrak{H}(2, \mathbb{R}))/\Gamma)$ can be generated by ϕ_4 .

Proof. Since \mathfrak{M} is in the quotient field of the ring of forms of even weight, it is enough to express ϕ_6, χ_{10} , and χ_{12} in terms of ϕ_4 using D^1 . Eq. (13) yields χ_{10} immediately; division of (15) by χ_{10} then provides an expression for χ_{12} . Finally, ϕ_6 occurs linearly in (16) and all other quantities are expressible in terms of ϕ_4 and D^1 . The proof is complete.

Theorem 3. Using differential operators, $\bigoplus_{w \in 2\mathbb{Z}}(\Gamma, w)$ and $\mathfrak{M}(Z(\mathfrak{H}(2, \mathbb{R}))/\Gamma)$ can be generated by χ_{10} .

Proof. It is enough to obtain ϕ_4 from χ_{10} . As in the proof of the previous theorem, obtain χ_{12} from $D^1\chi_{10}$. Now consider the non zero form $D^2\chi_{10} \in (\Gamma, 44)$; since $\text{ord } D^2\chi_{10} \geq 8$, there are constants c_8, c_9 such that

$D^2\chi_{10} = c_8\phi_4\chi_{10}^4 + c_9(\chi_{10}\chi_{12})^2$. It can be checked that $c_8 \neq 0$, so the equation can be solved for ϕ_4 , and the theorem is proved.

§4. *Related Results for Siegel Subgroups*

Some peripheral results related to Siegel modular forms of genus two can be easily obtained, so it seems worthwhile to mention them here.

Up to a constant factor, the cusp form χ_{10} is the square of a form $\theta_5 \in (\Gamma, 5, v)$ where v denotes the non trivial multiplier system for Siegel's group of genus two; cf. [3], [5]. An immediate consequence of Theorem 3 is

Theorem 4. $\oplus_{w \in 2\mathbb{Z}}(\Gamma, w)$ and $\mathfrak{M}(Z(\mathfrak{H}(2, \mathbf{R}))/\Gamma)$ can be generated by θ_5 . Furthermore,

Theorem 5. $D^1\theta_5 = -\frac{2^{10} \cdot 3^2}{5^2} \pi^2 \chi_{12}$.

Proof. θ_5 is a cusp form since $\theta_5^2 \propto \chi_{10}$ is, so, by (3) and Proposition 3, $D^1\theta_5$ is a non zero cusp form in $(\Gamma, 12, 1)$, where 1 denotes the trivial multiplier system. Now apply Igusa's Structure Theorem to conclude that there is a non zero constant c_{10} such that $D^1\theta_5 = c_{10}\chi_{12}$. From [3, p. 403], the expansion of $\theta_5 \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ in the neighborhood of $z_3 = 0$ is

$$(17) \quad \theta_5(z) = -2^7 i \varepsilon \left(\frac{z_1 + z_2}{2} \right) (\pi z_3) + \dots;$$

comparison with the corresponding expansion of χ_{10} given in [2] shows that

$$(18) \quad \theta_5^2 = -2^{14} \chi_{10}.$$

Hence, by Proposition 4, and eq. (9),

$$D^1\chi_{10} = 2^{-28} D^1\theta_5^2 = 2^{-28} \theta_5^2 D^1\theta_5 = -2^{-14} \chi_{10} D^1\theta_5,$$

but eq. (15) of Theorem 1 implies

$$D^1\chi_{10} = \frac{3^2}{2^4 5^2} \pi^2 \chi_{10} \chi_{12};$$

comparison completes the proof.

Denote by $\Gamma(T)$ the congruence subgroup of level T of Siegel's modular group of genus two. Define $\hat{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ by

$$\hat{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cup \left\{ \gamma \begin{pmatrix} u' & 0 \\ 0 & u^{-1} \end{pmatrix} : \gamma \in \Gamma \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, u = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}$$

where u' denotes the transpose of u . $\hat{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is a group containing $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ as a subgroup of index 2; see [I] for details. Freitag [I] has determined generators of the graded ring of $\hat{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ -automorphic forms of even weight for the trivial multiplier system. With the subscript denoting the weight, and h indicating cusp forms, they can be selected to be of the form:

$$f_4, f_6, h_8, h_{10}, h_{12}.$$

Theorem 6. a) *There are constants c_{11}, \dots, c_{20} such that*

$$(19) \quad D^1 h_8 = c_{11} h_8 h_{10};$$

$$(20) \quad D^1 h_{10} = c_{12} f_4 h_8 h_{10} + c_{13} h_{10} h_{12};$$

$$(21) \quad D^2 h_8 = c_{14} f_4 h_8^4 + c_{15} h_8^3 h_{12} + c_{16} h_8^2 h_{10}^2;$$

$$(22) \quad D^1 \cdot D^1 h_8 = c_{17} f_4 h_8^3 h_{10} + c_{18} h_8^2 h_{10} h_{12} + c_{19} f_6 h_8^4 + c_{20} h_8 h_{10}^3.$$

$$c_{11} \neq 0, \quad |c_{12}| + |c_{13}| \neq 0, \quad |c_{14}| + |c_{15}| + |c_{16}| \neq 0,$$

$$|c_{17}| + |c_{18}| + |c_{19}| + |c_{20}| \neq 0.$$

b) *If $c_{12}c_{15} - c_{13}c_{14} \neq 0$ and $c_{19} \neq 0$, then $\bigoplus_{w \in 2\mathbf{Z}} \left(\hat{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, w \right)$ is generated by h_8 .*

Proof. The proof is analogous to that of Theorem 1; Proposition 3 is used repeatedly.

§5. Hermitian Forms of Degree 2

In this section we consider a hermitian modular group of degree two. The associated Jordan algebra is $\mathfrak{H}(2, C)$, for which $q = (\dim/\text{rank}) = 2$; hence D^2 is not defined in this case.

Let Γ now denote the Hermitian modular group of degree two with respect to the gaussian integers; cf. [I]. If z' denotes the transpose of $z \in Z(\mathfrak{H}(2, C))$, then an $f \in (\Gamma, w)$ is *symmetric* if $f(z) = f(z')$. Freitag [I] has shown that the graded ring of symmetric Γ -automorphic forms of even weight is generated by six forms, of weights 4, 8, 10, 12, 12, and 16; of these, the one of weight 4, denoted ϕ_4 , and one of the two of weight 12, denoted $(\eta_6)^2$, are not cusp forms. The others can be assumed to be cusp

forms, which we will denote by χ_8 , χ_{10} , χ_{12} , and χ_{16} . It is easy to show from the construction given in [I] that

$$\text{ord } \chi_8 = 2; \text{ ord } \chi_{10} = 2; \text{ ord } \chi_{12} = 2; \text{ ord } \chi_{16} = 2.$$

Theorem 7. a) *There are constants c_{21}, \dots, c_{40} such that*

$$(23) \quad D^1 \chi_8 = c_{21} \chi_8 \chi_{10};$$

$$(24) \quad D^1 \chi_{10} = c_{22} \phi_4 \chi_8 \chi_{10} + c_{23} \chi_{10} \chi_{12};$$

$$(25) \quad D^1 \cdot D^1 \chi_8 = c_{24} \phi_4 \chi_8^3 \chi_{10} + c_{25} \chi_8^2 \chi_{10} \chi_{12} + c_{26} \chi_8 \chi_{10}^3;$$

$$(26) \quad D^1 \chi_{12} = c_{27} \chi_8^2 \chi_{10} + c_{28} \phi_4^2 \chi_8 \chi_{10} + c_{29} \phi_4 \chi_{10} \chi_{12} + c_{30} \chi_{10} \chi_{16};$$

$$(27) \quad D^1 \chi_{16} / \chi_{10} = (c_{31} \chi_{12} + c_{32} \phi_4 \chi_8) \eta_6^2 + c_{33} \phi_4^4 \chi_8 + c_{34} \phi_4^3 \chi_{12} + \\ c_{35} \phi_4^2 \chi_{16} + c_{36} \phi_4^2 \chi_8^2 + c_{37} \phi_4 \chi_8 \chi_{12} + c_{38} \chi_8 \chi_{16} + \\ c_{39} \chi_{12}^2 + c_{40} \chi_8^3.$$

$$c_{21} \neq 0; |c_{22}| + |c_{23}| \neq 0; |c_{24}| + |c_{25}| + |c_{26}| \neq 0; |c_{27}| + |c_{28}| + \\ |c_{29}| + |c_{30}| \neq 0; |c_{31}| + \dots + |c_{35}| \neq 0.$$

b) If $c_{22}c_{25} - c_{23}c_{24} \neq 0$, $c_{30} \neq 0$, and $|c_{31}| + |c_{32}| \neq 0$, then the graded ring of symmetric hermitian forms of even weight can be generated by χ_8 .

Remarks. A consequence of the main result in [9] is that every symmetric hermitian modular form satisfies an algebraic differential equation of order at most 6. It is not known if this is in fact the exact order. Now suppose that $c_{22} = 0$ or $c_{23} = 0$. Then it can be shown that there is an equation for χ_8 of order less than 6. Similar remarks hold for various other combinations of vanishing constants here, and also for the generators described in Theorem 6, which suggests that the hypotheses of Theorems 6b and 7b are probably satisfied.

NOTE

1. Jordan terminology is as in H. Braun and M. Koecher, *Jordan-Algebren*, Springer-Verlag, Berlin (1966) and M. Koecher, *Jordan Algebras and their Applications*, Lecture notes, University of Minnesota (1962).

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