NON TRIVIAL CUSP FORMS IN SEVERAL COMPLEX VARIABLES

by H. L. Resnikoff

§1. Poincaré himself showed that the series which today bear his name can represent zero identically [*6*]. This is particularly vexing because the Poincaré series span the vector spaces of cusp forms for certain discrete groups of modular type, and it is of some interest to determine the least weight such that a non trivial cusp form exists for a given discrete group. The problem is no less interesting for automorphic forms in several complex variables, nor is it more tractable.

The purpose of this note is to construct explicit non trivial cusp forms for a wide range of domains and discrete groups in several complex variables by using differential operators. The construction provides an upper bound for the least w such that a non trivial cusp form of weight w exists, but unfortunately this bound is not always the best possible: for the Siegel and Hermitian modular groups it is of the order of the cube of the rank of the domain.

§2. The domains that will be considered are tubes generalizing the classical upper half plane. Let $\mathfrak{A}$ denote a simple compact real Jordan algebra (*simplicity* is an inessential technical hypothesis whose principal purpose is to relieve the burden on the notation), and $Z(\mathfrak{A}) \equiv \mathfrak{A} + i \exp \mathfrak{A}$ the *tube* over $\exp \mathfrak{A}$. Let Γ be a discrete group of analytic automorphisms of $Z(\mathfrak{A})$. If $a \in \mathfrak{A}$, then the mapping $z \to z + a$ is called a *real translation.* We suppose that the subgroup $\Gamma_\infty \subset \Gamma$ of real translations spans $\mathfrak{A}$ in the sense that $\mathfrak{A}/\Gamma_\infty$ is compact, and further that the involution $z \xrightarrow{j} -z^{-1}$ belongs to Γ (this last assumption can be weakened somewhat; cp. [*9*]). Put

$$q = \text{dimension of } \mathfrak{A}/\text{rank } \mathfrak{A},$$

and denote the Jacobian determinant of $\gamma \in \Gamma$ by $J_\gamma(z)$. A holomorphic function $\phi\colon Z(\mathfrak{A}) \to \mathbf{C}$ is a Γ-*automorphic form of weight* w *and multiplier system* v, denoted $\phi \in (\Gamma, w, v)$, if

$$\phi(\gamma z) = v(\gamma) J_\gamma(z)^{-w/2q} \phi(z) \qquad \text{for all } \gamma \in \Gamma, \tag{1}$$

where $v: \Gamma \to \mathbf{C}$ is not the zero map and satisfies the compatibility conditions

$$v(\gamma_1\gamma_2)J_{\gamma_1\gamma_2}(z)^{-w/2q} = v(\gamma_1)v(\gamma_2)J_{\gamma_1}(\gamma_2 z)^{-w/2q}J_{\gamma_2}(z)^{-w/2q}$$

for $\gamma_1, \gamma_2 \in \Gamma$.

It is usual to assume that $|v| = 1$, and to impose a growth condition on ϕ with respect to "large" $\operatorname{Im}(z)$ (measured in terms of the Jordan algebra norm) but this is not necessary for our purposes and in any event is a consequence of the other hypotheses for Γ of the type of Siegel's modular group when rank $\mathfrak{A} > 1$ (cf. [*4*] and [*5*]). Observe that (Γ, w, v) is a complex vector space.

Since $\phi \in (\Gamma, w, v)$ is holomorphic and $\mathfrak{A}/\Gamma_\infty$ is compact, ϕ has a Fourier expansion. We shall make the additional assumption, related to the growth condition mentioned above, that this expansion be restricted to the form

$$\phi(z) = \sum_{\substack{n \geqq 0 \\ n \in \Gamma'_\infty}} \alpha(n) e^{2\pi i \sigma(nz)} . \tag{2}$$

Here $\sigma(x, y) \equiv \sigma(xy)$ denotes the reduced trace of $xy \in \mathfrak{A}(i)$, $n \geqq 0$ means that n lies in the topological closure of $\exp \mathfrak{A}$ with the natural topology inherited from the real vector space substructure of $\mathfrak{A}$ (or, equivalently, that the eigenvalues of n are non-negative), and Γ'_∞ stands for the lattice dual to $\{\gamma(0) : \gamma \in \Gamma_\infty\}$.

If $n \in \mathfrak{A}$ has only positive eigenvalues, write $n > 0$; if $n \geqq 0$ but $n \not> 0$, write $n \sim 0$. Then $\sum_{n \geqq 0} = \sum_{n \sim 0} + \sum_{n > 0}$.

If $\sum_{n \sim 0} \alpha(n) e^{2\pi i \sigma(nz)} \equiv 0$ in equation (2), then ϕ is a *cusp form*. This definition is weaker, i.e., admits a larger class of cusp forms, than that usual in the theory of automorphic forms. But if Γ has only one cusp (which our previous hypotheses insure is "at infinity") then this definition coincides with the usual one; this is true in particular for the Siegel modular groups [*10*].

Our immediate task is the construction of a linear differential operator mapping (Γ, w, v) into (Γ, w', v') for some $\{(w, v), (w', v')\}$ and all Γ. From it we will construct non-linear differential operators mapping (Γ, w, v) into (Γ, w', v') for any (Γ, w, v) and appropriate (w', v'). The procedure generalizes that in our paper [*8*] for the classical case.

Let $|z|$ denote the reduced norm of $z \in \mathfrak{A}(i)$, and ∇_z the gradient operator defined with respect to the symmetric bilinear form σ. As in [*9*], define the linear differential operator $\partial_z \equiv |\nabla_z|$ by introducing the Peirce decomposition $\mathfrak{A} = \sum_{m \leqq n} \mathfrak{A}_{mn}$ with respect to a complete orthonormal set of primitive idempotents $\{c_k\}$ of $\mathfrak{A}$ and a canonical vector space basis

$\{b^l_{mn}\}$ for $\mathfrak{A}_{mn}$, $m < n$ (one will do since the simplicity of $\mathfrak{A}$ insures that the $\mathfrak{A}_{mn}$ are isomorphic for $m < n$). Then $z \in \mathfrak{A}(i)$ can be expressed as

$$z = \sum_{k=1}^{\operatorname{rank}\mathfrak{A}} c_k z_k + \sum_{m<n} \sum_{l=1}^{\dim \mathfrak{A}} b^l_{mn} z^l_{mn},$$

and

$$\nabla_z = \sum_{k=1}^{\operatorname{rank}\mathfrak{A}} c_k \alpha_k \frac{\partial}{\partial z_k} + \sum_{m<n} \sum_{l=1}^{\dim \mathfrak{A}mn} \alpha^l_{mn} b^l_{mn} \frac{\partial}{\partial z^l_{mn}}$$

with constants $\alpha_k, \alpha^l_{mn} \in \mathbf{R}$. The reduced norm $|z|$ is a certain homogeneous polynomial of degree rank $\mathfrak{A}$ in $\{z_k\} \cup \{z^l_{mn}\}$, say $N(z_k, z^l_{mn})$; we define

$$\partial_z \equiv |\nabla_z| \equiv N\left(\alpha_k \frac{\partial}{\partial z_k},\ \alpha^l_{mn} \frac{\partial}{\partial z^l_{mn}}\right).$$

In [9] we proved the elementary formula

$$\partial_z e^{\sigma(az)} = |a| e^{\sigma(az)}, \quad a \in \mathfrak{A}(i), \tag{3}$$

and the Selberg operator identity

$$\partial^n_{-z^{-1}} = |z|^{n+q} \partial^n_z |z|^{n-q}. \tag{4}$$

Hence ∂^n: $(\Gamma, q-n, v) \to (\Gamma, q+n, v)$.

Suppose that $w \neq 0$ and $\phi \in (\Gamma, w, v)$; then $\phi^{(q-n)/w}$ satisfies the functional equation (1) with weight $q-n$ and multiplier $v^{(q-n)/w}$ and is holomorphic except on a thin set of points in $Z(\mathfrak{A})$. This will be denoted by writing $\phi^{(q-n)/w} \in (\Gamma, q-n, v^{(q-n)/w})_0$. Then $\partial^n \phi^{(q-n)/w} \in (\Gamma, q+n, v^{(q-n)/w})_0$ and consequently, when $n \neq q$ is a positive integer and $r \equiv \operatorname{rank}\mathfrak{A}$,

$$\mathscr{D}^n \phi \equiv \phi^{((n-q)/w)+nr} \partial^n \phi^{(q-n)/w} \in (\Gamma, n(rw+2), v^{nr})_0. \tag{5}$$

Since ∂^n is a linear differential operator of order nr, multiple application of Leibniz' product differentiation formula shows that $\mathscr{D}^n\phi$ is actually a polynomial in ϕ and its first nr partial derivatives. This proves the basic

Lemma 1. $\mathscr{D}^n: (\Gamma, w, v) \to (\Gamma, n(rw+2), v^{nr})$.

The next result shows that it is unusual for $\mathscr{D}^n$ to be surjective in Lemma 1.

Lemma 2. *If $\phi \in (\Gamma, w, v)$ and w is a rational number, then $\mathscr{D}^n\phi$ is a cusp form.*

Proof. In this case $\phi^{(q-n)/w}$ has a Fourier expansion of the form

$$\phi^{(q-n)/w}(z) = \sum_{k \geq 0} \beta(k) e^{2\pi i \sigma(kz)}$$

where k runs over a certain lattice. Equation (3) shows that

$$\mathscr{D}^n\phi = \left(\sum_{k\geqq 0}\beta(k)e^{2\pi i\sigma(kz)}\right)^p \sum_{k>0}\beta(k)\left|2\pi ik\right|^n e^{2\pi i\sigma(kz)}$$

for some integer p, which simplifies to a sum over positive k, i.e., $\mathscr{D}^n\phi$ is a cusp form.

Thus, the main purpose of this note will be fulfilled if it can be shown that $\mathscr{D}^n\phi \not\equiv 0$.

Suppose that $w \neq 0$, $q \neq n$, and $\phi \in (\Gamma, w, v)$, and put $\psi = \phi^{(q-n)/w}$. Then

$$0 \equiv \mathscr{D}^n\phi \Leftrightarrow 0 \equiv \partial^n\psi \Leftrightarrow 0 \equiv \partial\psi,$$

the last equality following from the uniqueness of the Fourier representation.

A necessary condition that $\partial\psi \equiv 0$ hold is given in Theorem A of [*9*]. In order to state it, some additional notation is necessary. Let $\{c_m\}$ be a complete set of primitive orthogonal idempotents of $\mathfrak{A}$, and let $\mathfrak{A}_k = \mathfrak{A}_1(c_1 + \cdots + c_k)$ be the Peirce 1-algebra with respect to the idempotent $c_1 + \cdots + c_k$. Put $q_k = \dfrac{\text{dimension } \mathfrak{A}_k}{\text{rank } \mathfrak{A}_k}$; in particular, $q \equiv q_r$. Suppose ψ has a Fourier expansion of the form (2) and satisfies the equation (1) for Γ, with weight w^*. Then Theorem A states that $\partial\psi \equiv 0$ implies w^* is a *critical weight*, that is,

$$w^* \in \{q_k - 1 : 1 \leqq k \leqq r\}.$$

For $\psi = \phi^{(q-n)/w}$ and w rational, $w^* = q - n$.

Theorem. *If w is a non-zero rational, $0 \not\equiv \phi \in (\Gamma, w, v)$, and n is a positive integer such that $n \notin \{q_k : 1 \leqq k \leqq r\}$, then $\mathscr{D}^n\phi$ is a non trivial cusp form.*

The proof follows immediately from the preceding remarks by observing that $\{q_r - q_k + 1 : 1 \leqq k \leqq r\} = \{q_k : 1 \leqq k \leqq r\}$.

§3. The classification of simple compact real Jordan algebras [*2*] enables us to determine the least positive integer n_0 not a member of $\{q_k : 1 \leqq k \leqq r\}$. The results are collected in the following table (following page):

§4. $Z(\mathfrak{A})$ is Siegel's upper half plane of degree r when rank $\mathfrak{A} = r$ and $\dim \mathfrak{A} = r(r+1)/2$. Suppose that Γ is Siegel's modular group acting on $Z(\mathfrak{A})$, and denote the Eisenstein series of weight w by $g_w(z)$ [*10*]. A well-known result of H. Braun states that the series defining g_w converges to a non trivial holomorphic automorphic form if and only if $w > r + 1$ and

TABLE 1

rank $\mathfrak{A}$	dim $\mathfrak{A}$	$q_k(\mathfrak{A})$	$n_0(\mathfrak{A})$
$r \geqq 1$	$\frac{r(r+1)}{2}$	$\frac{k+1}{2}$	$\frac{2r+5-(-1)^r}{4}$
$r \geqq 1$	r^2	k	$r+1$
$r \geqq 1$	$2r^2 - r$	$2k-1$	2
2	$d > 2$	$q_1 = 1,\ q_2 = \frac{d}{2}$	3 if $d = 4$ 2 if $d \neq 4$
3	27	$q_1 = 1,\ q_2 = 5,\ q_3 = 9$	2

$w \equiv 0 \bmod 2$. Therefore the least w such that g_w is a non trivial modular form is $(r+2) + \frac{1-(-1)^r}{2}$. The theorem and the first line of the table imply

Corollary 1. *Let g_w denote the Eisenstein series of weight w for Siegel's modular group of degree r. Then*

$$\mathscr{D}^{(2r+5-(-1)^r)/4} g_{(2r+5-(-1)^r)/2}$$

is a non trivial cusp form of weight

$$w_0 = \{(2r+5-(-1)^r)(2r^2+(5-(-1)^r)r+4)\}/8 .$$

If $r = 1$, then $w_0 = 12$ and indeed one easily calculates that

$$\mathscr{D}^2 g_4 = +240\pi^2 \Delta$$

where Δ is the normalized cusp form $e^{2\pi i z}\{\prod_{n=1}^{\infty}(1-e^{2\pi i n z})\}^{24}$ for the classical modular group (cf. [7], [8]).

If $r = 2$, then $\mathscr{D}^2 g_4$ is a cusp form of weight 20 for the Siegel modular group of degree 2. Igusa [3] has determined the generators of the ring of modular forms for this case: they are g_4 and g_6, and cusp forms χ_{10} and χ_{12} of weight 10 and 12 respectively. Therefore $\mathscr{D}^2 g_4$ is not the cusp form of least weight, but can be expressed as

$$\mathscr{D}^2 g_4 = \alpha \chi_{10}^2 + \beta g_4^2 \chi_{12}$$

for certain constants α and β.

One can show, however, that $\mathscr{D}^1 g_4$ is proportional to χ_{10}, so the theorem is not the best that can be done using differential operators.

If rank $\mathfrak{A} = r$ and $\dim \mathfrak{A} = r^2$, then $Z(\mathfrak{A})$ is the upper half plane associated with the Hermitian modular forms of Hel Braun [*1*]. Let Γ denote the Hermitian modular group and g_w the Eisenstein series of weight w for Γ. The series defining g_w converges to a non trivial holomorphic Hermitian modular form if $w \equiv 0 \bmod 2$ and $w > 2r$ (cf. [*1*]). Application of the theorem and the second row of the table yields

Corollary 2. *Let g_w denote the Eisenstein series of weight w for the Hermitian modular group. Then $\mathscr{D}^{r+1} g_{2r+2}$ is a non trivial cusp form of weight $2(r+1)(r^2+r+1)$.*

Observe that the weights given in Corollaries 1 and 2 insuring the existence of non trivial cusp forms are $\mathcal{O}(r^3)$. If, however, the algebras corresponding to the last three rows of Table 1 are considered, there are large enough gaps between q_k and q_{k+1} so that n_0 no longer is constrained to grow with rank or dimension of $\mathfrak{A}$, and therefore the weights insuring the existence of non trivial cusp forms grow only as the product of r by the minimal weight for a non trivial form, which is probably $\mathcal{O}(r^2)$. But very little is known about these cases.

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RICE UNIVERSITY