SOME NON-ABELIAN PROBLEMS ON COMPACT RIEMANN SURFACES

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A good deal of the classical function theory of compact Riemann surfaces is of an essentially abelian nature, having to do with complex analytic properties of flat complex line bundles. Problems do arise, though, which are really non-abelian in character, having to do with complex analytic properties of flat complex vector bundles. These problems lead in quite interesting but relatively little explored directions. The aim of this lecture is merely to show how the beginnings of the non-abelian theory can be developed in a manner paralleling a familiar development of the abelian theory.

First, to establish a background for the discussion, a few definitions should be recalled. Consider a complex analytic manifold M of complex dimension n, and let $\mathfrak{U} = \{U_\alpha\}$ be a covering of M by open coordinate neighborhoods U_α. A one-cocycle of the covering $\mathfrak{U}$ with coefficients in an arbitrary abstract group G is a collection of elements $\xi_{\alpha\beta} \in G$, indexed by ordered pairs (U_α, U_β) of sets of the covering $\mathfrak{U}$ for which $U_\alpha \cap U_\beta \neq \varnothing$, such that $\xi_{\alpha\alpha} = 1$ and that $\xi_{\alpha\beta}\xi_{\beta\gamma}\xi_{\gamma\alpha} = 1$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$. The set of all such one-cocycles will be denoted by $Z^1(\mathfrak{U}, G)$. Two one-cocycles $(\xi_{\alpha\beta})$ and $(\xi'_{\alpha\beta})$ of $Z^1(\mathfrak{U}, G)$ are called equivalent if there is a collection of elements $\eta_\alpha \in G$, indexed by the sets U_α of the covering $\mathfrak{U}$, such that $\xi'_{\alpha\beta} = \eta_\alpha \xi_{\alpha\beta} \eta_\beta^{-1}$ whenever $U_\alpha \cap U_\beta \neq \varnothing$; it is easy to see that this is an equivalence relation in the usual sense of the term. The set of equivalence classes is called the first cohomology set of the covering $\mathfrak{U}$ with coefficients in the group G, and will be denoted by $H^1(\mathfrak{U}, G)$. The cohomology as thus defined depends on the choice of the covering $\mathfrak{U}$, and not just on the space M; but it can be shown that for well-behaved coverings the cohomologies are in natural one-to-one correspondences. (See for instance the discussion of Leray's theorem in [2].) Hereafter it will be assumed that only such well-behaved coverings are considered, and the common cohomology set will be denoted by $H^1(M, G)$ and called the first cohomology of the space M with coefficients in the group G. If the group G is abelian, it is clear that the set of cocycles form an abelian group; the group operations are compatible with the equivalence relation, so that the cohomology set $H^1(M, G)$ is also an abelian group.

The particular case of this construction of perhaps the greatest interest is that in which the group G is the matrix group of some rank r over the complex numbers, $G = GL(r, \mathbf{C})$. The cohomology classes $\xi \in H^1(M, GL(r, \mathbf{C}))$ are called *flat complex vector bundles of rank* r over the complex manifold M. When $r = 1$ the group $GL(1, \mathbf{C}) = \mathbf{C}^*$ is abelian, and the cohomology classes $\xi \in H^1(M, \mathbf{C}^*)$ are also called *flat complex line bundles* over the complex manifold M.

So far the complex analytic structure of M has played no role; to bring it into the game, the preceding construction can be extended as follows. Suppose that G is not just an abstract group, but is a complex Lie group. Then a one-cocycle of the covering $\mathfrak{U}$ with coefficients in the sheaf of germs of analytic mappings into G is a collection of complex analytic mappings $\xi_{\alpha\beta}: U_\alpha \cap U_\beta \to G$, such that $\xi_{\alpha\alpha}(p) = 1$ for all $p \in U_\alpha$ and that $\xi_{\alpha\beta}(p)\xi_{\beta\gamma}(p)\xi_{\gamma\alpha}(p) = 1$ for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$. The set of all these one-cocycles will be denoted by $Z^1(\mathfrak{U}, \mathcal{O}(G))$. Two one-cocycles $(\xi_{\alpha\beta})$ and $(\xi'_{\alpha\beta})$ are called equivalent if there are complex analytic mappings $\eta_\alpha: U_\alpha \to G$ such that $\xi'_{\alpha\beta}(p) = \eta_\alpha(p)\xi_{\alpha\beta}(p)\eta_\beta(p)^{-1}$ for all $p \in U_\alpha \cap U_\beta$. The set of equivalence classes is called the first cohomology set of the covering $\mathfrak{U}$ with coefficients in the sheaf of germs of complex analytic mappings into G, and will be denoted by $H^1(\mathfrak{U}, \mathcal{O}(G))$; again, all these cohomology sets are in natural one-to-one correspondence for well-behaved coverings, and the common cohomology set will be denoted by $H^1(M, \mathcal{O}(G))$. For the matrix group $G = GL(r, \mathbf{C})$, the cohomology classes $\xi \in H^1(M, \mathcal{O}(GL(r, \mathbf{C})))$ are called *complex analytic vector bundles of rank* r over the complex manifold M. When $r = 1$ the abbreviation $\mathcal{O}^* = \mathcal{O}(\mathbf{C}^*)$ will be used, and the cohomology classes $\xi \in H^1(M, \mathcal{O}^*)$ are called *complex analytic line bundles* over the complex manifold M. (For a discussion of the relationship between this definition and the usual geometric definitions of vector bundles, see for instance [6].)

Note that for any complex Lie group G, a cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, G)$ can be viewed as a cocycle in $Z^1(\mathfrak{U}, \mathcal{O}(G))$, interpreting the constants $\xi_{\alpha\beta}$ as constant-valued analytic mappings; equivalent cocycles in $Z^1(\mathfrak{U}, G)$ are clearly also equivalent when viewed as cocycles in $Z^1(\mathfrak{U}, \mathcal{O}(G))$, hence there results a well-defined mapping

$$\mu: H^1(M, G) \to H^1(M, \mathcal{O}(G)).$$

The interplay between flat and complex analytic vector bundles is expressed in the mapping μ for the case of the matrix group $G = GL(r, \mathbf{C})$. The image of μ is then the set of complex analytic vector bundles which have *flat representatives*, that is to say, which can be described by constant cocycles; and for any flat vector bundle $\xi \in H^1(M, GL(r, \mathbf{C}))$, the set $\mu^{-1}(\mu(\xi))$ consists

of all the flat vector bundles which are *analytically equivalent* to the given bundle ξ.

To digress for a moment, here is an example of a problem, the study of which naturally leads to questions of the relationships between flat and complex analytic vector bundles. It is well known that every compact Riemann surface M (one-dimensional complex analytic manifold) can be represented as the quotient of a subset D of the complex plane by a group Γ of linear fractional transformations. Actually, D can be taken to be either the entire complex plane (when M has genus $g = 1$) or the unit disk (when M has genus $g > 1$); but there are many other such representations which are of importance. (See in this regard the paper $[1]$.) Those subsets of D which contain no points equivalent under the group Γ can be used as coordinate neighborhoods on the surface $M = D/\Gamma$. This provides a coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of the surface M by open sets U_α with local coordinates $z_\alpha : U_\alpha \to \mathbf{C}$ which have the special property that in each intersection $U_\alpha \cap U_\beta \neq \varnothing$ the local coordinate mappings are related by $z_\alpha = T_{\alpha\beta}(z_\beta)$, for some linear fractional transformations $T_{\alpha\beta}$; for a general complex analytic coordinate covering of the surface M, one can only assert that the mappings $T_{\alpha\beta}$ are complex analytic homeomorphisms. Now it is of some interest to determine what are all the possible special complex analytic coordinate coverings of this form for a given Riemann surface M, and one approach to the problem is the following. If $\{U_\alpha, z_\alpha\}$ is one of these special coordinate coverings of M, then to each intersection $U_\alpha \cap U_\beta \neq \varnothing$ there is associated the linear fractional transformation $T_{\alpha\beta}$ such that $z_\alpha = T_{\alpha\beta}(z_\beta)$; clearly $T_{\alpha\alpha} = I$, and $T_{\alpha\beta}T_{\beta\gamma}T_{\gamma\alpha} = I$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$. Of course, replacing the local coordinates z_α by $w_\alpha = S_\alpha(z_\alpha)$ for some linear fractional transformations S_α leads to what must be considered as an equivalent special coordinate covering; and for this equivalent coordinate covering, $w_\alpha = T'_{\alpha\beta}(w_\beta)$ where $T'_{\alpha\beta} = S_\alpha T_{\alpha\beta} S_\beta^{-1}$. Thus to each equivalence class of special coordinate coverings there is naturally associated a cohomology class in $H^1(M, PGL(2, \mathbf{C}))$, where $PGL(2, \mathbf{C})$ is the group of linear fractional transformations, which is a complex Lie group isomorphic to the quotient of the group $GL(2, \mathbf{C})$ by its center $\mathbf{C}^*$. It can be shown that this is a one-to-one correspondence between the set of equivalence classes of these special coordinate coverings of the Riemann surface M and a subset of $H^1(M, PGL(2, \mathbf{C}))$ called the set of indigenous cohomology classes; there remains the problem of describing the indigenous cohomology classes on M. For this latter problem, the natural homomorphism $GL(2, \mathbf{C}) \to PGL(2, \mathbf{C})$ induces a mapping

$$\rho : H^1(M, GL(2, \mathbf{C})) \to H^1(M, PGL(2, \mathbf{C})),$$

which is not onto; but it can be shown that the indigenous cohomology classes all lie in the image of ρ, hence are the images of a class of flat vector bundles of rank 2 on the surface M, which bundles will be called the indigenous flat vector bundles on M. Indeed, the indigenous flat vector bundles can all be taken to have determinant 1, in the obvious sense. Finally, for a Riemann surface of genus $g > 1$, this class of indigenous flat vector bundles can be described as precisely the class of those flat vector bundles which are analytically equivalent to complex analytic vector bundles represented by cocycles of the form

$$
\begin{bmatrix} \phi_{\alpha\beta} & 2\dfrac{d}{dz_\beta}\phi_{\alpha\beta} \\[2mm] 0 & \phi_{\alpha\beta}^{-1} \end{bmatrix} \in Z^1(\mathfrak{U}, \mathcal{O}(GL(2,\mathbf{C}))),
$$

where $(\phi_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ are cocycles representing any complex line bundles ϕ for which $\phi^2 = \kappa$. The canonical line bundle κ is that line bundle defined by the cocycle $(\kappa_{\alpha\beta}) = (dz_\beta/dz_\alpha) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$; there are 2^g such line bundles ϕ, hence there are 2^g complex analytic vector bundles of this form. (For the proofs and further details see [3].)

Returning once more to the main thread of the discussion, recall that the mapping μ arises by viewing elements of the complex Lie group G as constant-valued complex analytic mappings into G. The constant functions can be described quite conveniently for these purposes as the class of complex analytic functions with zero exterior derivative; but there is the complication that the functions considered here have values in the group G, so that some care must be taken in defining the exterior derivative. Suppose that $f: U \to G$ is a complex analytic mapping from an open neighborhood of a point p_0 in M into the group G. The Lie algebra $\mathfrak{g}$ of the group G can be used to provide canonical coordinates in an open neighborhood of the point $f(p_0)$ in G by the mapping

$$
X \in \mathfrak{g} \to f(p_0) \cdot \exp X \in G.
$$

(See [5], for instance.) In terms of these coordinates, the mapping f can be expressed as the complex analytic mapping $F_{p_0}: U \to \mathfrak{g}$ such that

$$
f(p) = f(p_0) \cdot \exp F_{p_0}(p) \qquad \text{for all } p \in U,
$$

provided that U is small enough. The mapping F_{p_0} has values in the complex vector space $\mathfrak{g}$, so that its exterior derivative dF_{p_0} is a well-defined complex analytic $\mathfrak{g}$-valued differential form of type $(1,0)$ in the neighborhood U.

The exterior derivative of the mapping f at the point p_0 can then be defined by

$$Df(p_0) = dF_{p_0}(p_0);$$

and it is easy to see that Df is also a complex analytic $\mathfrak{g}$-valued differential form of type $(1,0)$ in the neighborhood U. This derivative has the property that for any two complex analytic mappings $f, g: U \to G$,

$$D(fg) = Ad(g^{-1}) \cdot Df + Dg,$$

where Ad denotes the usual action of G on the vector space $\mathfrak{g}$. To see this, write $f(p) = f(p_0) \cdot \exp F(p)$, $g(p) = g(p_0) \cdot \exp G(p)$, and $f(p) \cdot g(p) = f(p_0) \cdot g(p_0) \cdot \exp H(p)$; then

$$\exp H(p) = g(p_0)^{-1} \cdot \exp F(p) \cdot g(p_0) \; \exp G(p)$$

$$= \exp[Ad(g(p_0)^{-1}) \cdot F(p)] \cdot \exp G(p),$$

from which the desired result follows.

Specializing to the case of a one-dimensional complex manifold M (a Riemann surface) for the remainder of the lecture, every complex analytic $\mathfrak{g}$-valued differential form of type $(1,0)$ is locally of the form Df, and the mapping f is constant precisely when $Df = 0$. What has been said so far can then be summarized in the assertion that the following is a twisted exact sequence of sheaves (of not necessarily abelian groups)

$$0 \to G \xrightarrow{i} \mathcal{O}(G) \xrightarrow{D} \mathcal{O}^{(1,0)}(\mathfrak{g}) \to 0,$$

where i is the inclusion of elements of G as germs of constant-valued analytic mappings, and D is the exterior derivative as just defined. (To say that this sequence is twisted means that i is a sheaf homomorphism, while D is only required to satisfy a condition of the sort already noted; D is a sheaf homomorphism twisted by means of the adjoint action of G on $\mathfrak{g}$. To say that this sequence is exact means that for each sheaf, the image of the mapping from the left is the kernel of the mapping to the right; this is well-defined, even for twisted homomorphisms D.) When G is an abelian group, this is an exact sequence in the usual sense.

The use of sheaf terminology is not really essential; it merely suggests passing to an associated cohomology sequence. There are limits to what can be done in this direction, since the group G may not be abelian and the sequence is twisted; but at least the following portion of an exact cohomology sequence always arises, in the familiar and very straightforward manner:

$$0 \longrightarrow G \xrightarrow{\ i\ } \Gamma(M, \mathcal{O}(G)) \xrightarrow{\ D\ } \Gamma(M, \mathcal{O}^{1,0}(\mathfrak{g})) \xrightarrow{\ \delta\ }$$

$$\xrightarrow{\ \delta\ } H^1(M, G) \xrightarrow{\ \mu\ } H^1(M, \mathcal{O}(G)) \xrightarrow{\ \nu\ } \bigcup_{\xi \in H^1(M, \mathcal{O}(G))} H^1(M, \mathcal{O}^{1,0}(Ad\xi)).$$

(See the discussions of the abelian case in [2] or [6], for instance.) It is necessary to be a bit careful about the meaning of this exact cohomology sequence, since the cohomology sets need not have any group structure; but everything becomes quite clear, upon examining the mappings more closely.

First, consider an element $\xi \in H^1(M, \mathcal{O}(G))$ represented by a cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}(G))$; recall that the cocycle condition is just that $\xi_{\alpha\beta}\xi_{\beta\gamma}\xi_{\alpha\gamma}^{-1} = 1$ in $U_\alpha \cap U_\beta \cap U_\gamma$. Applying the differential operator D, it follows easily that the $\mathfrak{g}$-valued differential forms $D\xi_{\alpha\beta}$ defined in the various intersections $U_\alpha \cap U_\beta$ satisfy $D\xi_{\alpha\gamma} = Ad(\xi_{\beta\gamma}^{-1}) \cdot D\xi_{\alpha\beta} + D\xi_{\beta\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$; and this can be viewed as the condition that these differential forms compose a one-cocycle $(D\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}^{1,0}(Ad\xi))$. The mapping ν in the exact cohomology sequence is the mapping which assigns to the cohomology class ξ that cohomology class in $H^1(M, \mathcal{O}^{1,0}(Ad\xi))$ represented by the cocycle $(D\xi_{\alpha\beta})$; the cohomology class ξ is in the image of μ if and only if $\nu(\xi) = 0$. In the special case that $G = GL(1, \mathbf{C}) = \mathbf{C}$, all the sets $H^1(M, \mathcal{O}^{1,0}(Ad\xi))$ coincide with the set $H^1(M, \mathcal{O}^{1,0})$, which is a group in this case; and the Serre duality theorem provides a natural isomorphism $H^1(M, \mathcal{O}^{1,0}) \cong \mathbf{C}$, when M is a compact Riemann surface. The image $\nu(\xi) \in \mathbf{C}$ is an integer called the Chern class of the line bundle ξ; and ξ has flat representatives precisely when $c(\xi) = 0$. (See [2] for further details.) In the special case that $G = GL(r, \mathbf{C})$ for $r > 1$, the sets $H^1(M, \mathcal{O}^{1,0}(Ad\xi))$ can be distinct; when M is a compact Riemann surface, Serre duality can again be applied, and $H^1(M, \mathcal{O}^{1,0}(Ad\xi))$ can be identified with the dual vector space to the algebra of complex analytic endomorphisms of the complex analytic vector bundle ξ. The analysis is a bit more involved, but there eventually results the theorem of Weil that $\xi \in H^1(M, \mathcal{O}(GL(r, \mathbf{C})))$ has flat representatives if and only if, for each indecomposable component $\xi_i \subset \xi$, the line bundle determinant (ξ_i) has zero Chern class. (The first statement and proof of this theorem were given in [9]; see also the discussion in [4].)

Next, the set of those $\xi \in H^1(M, G)$ such that $\mu(\xi) \in H^1(M, \mathcal{O}(G))$ is the trivial cohomology class is just the set $\mu^{-1}(0) = \delta\Gamma(M, \mathcal{O}^{1,0}(\mathfrak{g}))$, where the coboundary mapping δ is defined as follows. For any section $\lambda \in \Gamma(M, \mathcal{O}^{1,0}(\mathfrak{g}))$ (that is, for any complex analytic $\mathfrak{g}$-valued differential form of type $(1,0)$ defined on the entire Riemann surface M), and for any sufficiently fine open

covering $\mathfrak{U} = \{U_\alpha\}$ of M, there will exist complex analytic mappings $f_\alpha : U_\alpha \to G$ such that $Df_\alpha = \lambda$ in U_α; the functions $\xi_{\alpha\beta} = f_\alpha f_\beta^{-1}$ are easily seen to be constants in the sets $U_\alpha \cap U_\beta$, and the cohomology class $\xi \in H^1(M, G)$ represented by the cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, G)$ is the image $\xi = \delta(\lambda)$. If the group G is abelian, then the cohomology set $H^1(M, G)$ also has an abelian group structure, and the mapping δ is a group homomorphism; it is then clear that for any element $\xi \in H^1(M, G)$, the subset $\mu^{-1}(\mu(\xi)) \subset H^1(M, G)$ is a coset of the subgroup $\mu^{-1}(0) \subset H^1(M, G)$, and hence

$$\mu^{-1}(\mu(\xi)) = \xi \cdot \delta\Gamma(M, \mathcal{O}^{1,0}(\mathfrak{g})) \cong \frac{\Gamma(M, \mathcal{O}^{1,0}(\mathfrak{g}))}{D\Gamma(M, \mathcal{O}(\mathfrak{g}))}.$$

If the group G is not abelian, there is a similar but slightly weaker statement. For any element $\xi \in H^1(M, G)$ there is a mapping

$$\delta_\xi : \Gamma(M, \mathcal{O}^{1,0}(Ad\xi)) \to H^1(M, G)$$

such that $\mu^{-1}(\mu(\xi)) = \delta_\xi \Gamma(M, \mathcal{O}^{1,0}(Ad\xi))$; this modified coboundary mapping δ_ξ is defined as follows. For any sufficiently fine open covering $\mathfrak{U} = \{U_\alpha\}$ of M and for any section $\lambda \in \Gamma(M, \mathcal{O}^{1,0}(Ad\xi))$ (that is, for any collection of complex analytic $\mathfrak{g}$-valued differential forms λ_α in the various sets U_α, such that $\lambda_\alpha = Ad(\xi_{\alpha\beta}) \cdot \lambda_\beta$ in $U_\alpha \cap U_\beta$), there will exist complex analytic mappings $f_\alpha : U_\alpha \to G$ such that $Df_\alpha = \lambda_\alpha$; the functions $\xi'_{\alpha\beta} = f_\alpha \xi_{\alpha\beta} f_\beta^{-1}$ are easily seen to be constants in the sets $U_\alpha \cap U_\beta$, and the cohomology class $\xi' \in H^1(M, G)$ represented by the cocycle $(\xi'_{\alpha\beta}) \in Z^1(\mathfrak{U}, G)$ is the image $\xi' = \delta_\xi(\lambda)$. For the trivial cohomology class $\xi = 0$ represented by the cocycle $\xi_{\alpha\beta} = 1$, the mapping δ_ξ coincides with the standard coboundary operator δ. The groups $\Gamma(M, \mathcal{O}^{1,0}(Ad\xi))$ depend upon the cohomology ξ to such an extent that the images $\delta_\xi \Gamma(M, \mathcal{O}^{1,0}(Ad\xi)) \subset H^1(M, G)$ may be inequivalent in any natural sense, for different cohomology classes ξ. (The proofs of these assertions are all quite trivial consequences of the previously noted properties of the differential operator D; details for the case that $G = GL(r, \mathbf{C})$ are worked out quite explicitly in [4].)

To conclude this discussion, it should perhaps be pointed out that the mapping μ can be used to impose an additional structure on the cohomology set $H^1(M, \mathcal{O}(G))$, generalizing the classical structure of an abelian variety on the cohomology group $H^1(M, \mathcal{O}^*)$. (This sort of structure was first suggested in [9].) The first step is to observe that the set $H^1(M, G)$ can be given some sort of complex analytic structure. To see this, select a finite open covering $\mathfrak{U} = \{U_\alpha\}$ of the compact Riemann surface M, such that $H^1(\mathfrak{U}, G) \cong H^1(M, G)$; let v_0 be the number of open sets in the covering $\mathfrak{U}$, v_1 be the number of

ordered pairs of open sets in $\mathfrak{U}$ with non-vacuous intersection, and v_2 be the number of ordered triples of open sets in $\mathfrak{U}$ with non-vacuous intersection. The set $Z^1(\mathfrak{U}, G)$ is then the complex analytic subvariety of the product manifold $G^{v_1} = \{(\xi_{\alpha\beta})\}$ defined by the cocycle conditions $\xi_{\alpha\beta}\xi_{\beta\gamma}\xi_{\gamma\alpha} = 1$. The complex analytic Lie group $G^{v_0} = \{(\eta_\alpha)\}$ acts as an analytic transformation group on the manifold G^{v_1} under the mapping

$$[(\eta_\alpha), (\xi_{\alpha\beta})] \in G^{v_0} \times G^{v_1} \to (\eta_\alpha \xi_{\alpha\beta} \eta_\beta^{-1}) \in G^{v_1},$$

and the subset $Z^1(\mathfrak{U}, G) \subset G^{v_1}$ is clearly preserved under this group action. The quotient space

$$Z^1(\mathfrak{U}, G)/G^{v_0} = H^1(\mathfrak{U}, G)$$

therefore has the structure of the quotient space of a complex analytic variety under a complex Lie group of transformations acting on that variety.

The variety $Z^1(\mathfrak{U}, G)$ generally has singularities, and the group action can be rather nasty at some points; so the quotient structure must be analyzed a bit more carefully. For this purpose, introduce the complex analytic mapping $f: G^{v_1} \to G^{v_2}$ defined by $(f_{\alpha\beta\gamma}(\xi)) = (\xi_{\alpha\beta}\xi_{\beta\gamma}\xi_{\alpha\gamma}^{-1}) \in G^{v_2}$ for $\xi = (\xi_{\alpha\beta}) \in G^{v_1}$. (To simplify the discussion slightly, the conditions $\xi_{\alpha\alpha} = 1$ are being ignored here; this is not serious, since it merely introduces extraneous components, which can in turn be ignored.) The analytic subvariety $Z^1(\mathfrak{U}, G) \subset G^{v_1}$ is just the set $Z^1(\mathfrak{U}, G) = \{\xi \in G^{v_1} | f(\xi) = 1\}$, where $1 \in G^{v_2}$ is the identity element of the group G^{v_2}, and the regular part of this analytic variety is the subset $Z_0^1(\mathfrak{U}, G) \subset Z^1(\mathfrak{U}, G)$ consisting of those points for which the differential of the mapping f has maximal rank. To calculate the differential of f it is convenient again to use the canonical coordinates provided by the Lie algebra $\mathfrak{g}$ of the group G. Selecting a fixed point $\xi = (\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, G)$, introduce coordinates $X = (X_{\alpha\beta}) \in \mathfrak{g}^{v_1}$ in a neighborhood of ξ by the mapping $X_{\alpha\beta} \to \xi_{\alpha\beta} \cdot \exp X_{\alpha\beta}$; and introduce coordinates $Y = (Y_{\alpha\beta\gamma}) \in \mathfrak{g}^{v_2}$ in a neighborhood of $f(\xi)$ by the mapping $Y_{\alpha\beta\gamma} \to f_{\alpha\beta\gamma}(\xi) \cdot \exp Y_{\alpha\beta\gamma}$. In terms of these coordinates, f is represented by the mapping $F: \mathfrak{g}^{v_1} \to \mathfrak{g}^{v_2}$ given by

$$f(\xi) \cdot \exp F(X) = f(\xi \cdot \exp X),$$

or more explicitly by

$$\exp[Ad(\xi_{\alpha\gamma}^{-1}) \cdot F_{\alpha\beta\gamma}(X)] = \exp[Ad(\xi_{\beta\gamma}^{-1}) \cdot X_{\alpha\beta}] \cdot \exp X_{\beta\gamma} \cdot \exp[-X_{\alpha\gamma}].$$

The differential of f at the point ξ can then be viewed as the linear mapping $df_\xi: \mathfrak{g}^{v_1} \to \mathfrak{g}^{v_2}$ given by

$$df_\xi(X)_{\alpha\beta\gamma} = Ad(\xi_{\alpha\gamma}) \cdot [Ad(\xi_{\beta\gamma}^{-1}) \cdot X_{\alpha\beta} + X_{\beta\gamma} - X_{\alpha\gamma}].$$

Note that the kernel of the linear mapping df_ξ is the subset of $\mathfrak{g}^{\nu_1}$ consisting of those elements $(X_{\alpha\beta}) \in \mathfrak{g}^{\nu_1}$ such that $X_{\alpha\gamma} = Ad(\xi_{\beta\gamma}^{-1}) \cdot X_{\alpha\beta} + X_{\beta\gamma}$; that is to say, the kernel of df_ξ is just the space of cocycles $Z^1(\mathfrak{U}, Ad(\xi))$. On the one hand, it follows from this that the regular part of the variety $Z^1(\mathfrak{U}, G)$ can be characterized as

$$Z_0^1(\mathfrak{U}, G) = \{\xi \in Z^1(\mathfrak{U}, G) \,|\, \dim Z^1(\mathfrak{U}, Ad(\xi)) \text{ is maximal}\};$$

on the other hand, the tangent space to the manifold $Z_0^1(\mathfrak{U}, G)$ at a point $\xi \in Z_0^1(\mathfrak{U}, G)$ can be identified with the space $Z^1(\mathfrak{U}, Ad(\xi))$. It is evident that the subset $Z_0^1(\mathfrak{U}, G) \subset Z^1(\mathfrak{U}, G)$ is preserved under the action of the group G^{ν_0}, so that attention can be restricted to the set $H_0^1(\mathfrak{U}, G) = Z_0^1(\mathfrak{U}, G)/G^{\nu_0}$, which is the quotient space of a complex analytic manifold under a complex Lie group of transformations. The orbit of a point $\xi \in Z_0^1(\mathfrak{U}, G)$ under this group action is the image of the group G^{ν_0} under the mapping

$$\eta = (\eta_\alpha) \in G^{\nu_0} \to (\eta_\alpha \xi_{\alpha\beta} \eta_\beta^{-1}) \in G^{\nu_1}.$$

In addition to the canonical coordinates already used, introduce coordinates in a neighborhood of the identity in the group G^{ν_0} by the mapping $\eta_\alpha = \exp Z_\alpha$ where $Z = (Z_\alpha) \in \mathfrak{g}^{\nu_0}$; in these terms, the orbit of the point ξ is described locally as the image of the mapping $H: \mathfrak{g}^{\nu_0} \to \mathfrak{g}^{\nu_1}$ given by

$$\exp H_{\alpha\beta}(Z) = \exp[Ad(\xi_{\alpha\beta}^{-1}) \cdot Z_\alpha] \cdot \exp[-Z_\beta].$$

It follows that the tangent space to the orbit at ξ is the image of the differential $dH_\xi: \mathfrak{g}^{\nu_0} \to \mathfrak{g}^{\nu_1}$, where

$$dH_\xi(Z)_{\alpha\beta} = Ad(\xi_{\alpha\beta}^{-1}) \cdot Z_\alpha - Z_\beta,$$

and this image is precisely the subspace $B^1(\mathfrak{U}, Ad(\xi)) \subset Z^1(\mathfrak{U}, Ad(\xi))$ of coboundaries. Thus, whenever the quotient space $H_0^1(\mathfrak{U}, G)$ has a complex manifold structure, its tangent space at any point $\xi \in H_0^1(\mathfrak{U}, G)$ can be identified in a natural manner with the cohomology group $H^1(M, Ad(\xi))$. (Details for the case of complex analytic vector bundles are worked out in [4]. It is shown that $Z_0^1(\mathfrak{U}, G)$ corresponds precisely to the irreducible flat vector bundles, and that the quotient space $H_0^1(\mathfrak{U}, G)$ does have the natural structure of a complex analytic manifold.)

It is easy to see that the subset

$$\delta_\xi^{-1} H_0^1(M, G) = \Gamma_0(M, \mathcal{O}^{1,0}(Ad\xi)) \subset \Gamma(M, \mathcal{O}^{1,0}(Ad\xi))$$

is the complement of a complex analytic subvariety of $\Gamma(M, \mathcal{O}^{1,0}(Ad\xi))$, and that the restriction of δ_ξ is a complex analytic mapping

$$\delta_\xi^0 \colon \Gamma_0(M, \mathcal{O}^{1,0}(Ad\xi)) \to H_0^1(M, G)$$

between these two complex manifolds. (Again, details for the vector bundle case can be found in [4].) The image of δ_ξ^0 is an analytic subvariety of $H_0^1(M, G)$, if the mapping δ_ξ^0 is proper; it is probably true that the image is always a complex submanifold. Thus the manifold $H_0^1(M, G)$ is in some sense analytically fibred over the set $H_0^1(M, \mathcal{O}(G))$, the fibres being the images of the various mappings δ_ξ. This casts a new and quite suggestive light on the mapping μ, and introduces a sort of complex analytic structure on the cohomology set $H_0^1(M, \mathcal{O}(G))$. (This should be compared with the complex structure considered in [7] and [8].) In the particular case that $G = \mathbf{C}^*$, the cohomology set $H^1(M, G)$ is an abelian complex Lie group, and the mapping μ is the natural fibration associated to factoring by a subgroup; the quotient space is itself a complex Lie group, the Picard variety attached to the Riemann surface M.

REFERENCES

[1] BERS, L., Simultaneous uniformization, Bull. Amer. Math. Soc. **66** (1960), 94–97.

[2] GUNNING, R. C., Lectures on Riemann Surfaces, Princeton (1966).

[3] ———, Special coordinate coverings of Riemann surfaces, Math. Ann. **170** (1967), 67–86.

[4] ———, Lectures on Vector Bundles over Riemann Surfaces, Princeton (1967).

[5] HELGASON, S., Differential Geometry and Symmetric Spaces, New York (1962).

[6] HIRZEBRUCH, F., New Topological Methods in Algebraic Geometry, Berlin (1966).

[7] NARASIMHAN, M. S. and C. S. SESHADRI, Holomorphic vector bundles on a compact Riemann surface, Math. Ann. **155** (1964), 69–80.

[8] SESHADRI, C. S., Space of unitary vector bundles on a compact Riemann surface, Ann. of Math. **85** (1967), 303–336.

[9] WEIL, A., Généralisation des fonctions abéliennes, J. Math. Pures Appl. **17** (1938), 47–87.