LECTURES ON FUNCTION THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

by P. R. Garabedian

The three different topics here discussed are alike in that they demonstrate how partial differential equations may be solved by methods from the theory of functions of a complex variable.

1. Remarks about Hadamard's variational formula.

The Hadamard variational formula

$$\delta G(w,\zeta) = -\frac{1}{2\pi}\int_{\partial D} \frac{\partial G(z,w)}{\partial \nu}\frac{\partial G(z,\zeta)}{\partial \nu}\,\delta\nu\,ds$$

gives the first variation of the Green's function G of Laplace's equation for a plane region D when the boundary ∂D is subjected to an infinitesimal shift $\delta\nu$ along its inner normal. The formula has many interesting applications, but unfortunately most of them involve boundary curves ∂D so lacking in differentiability as to cast doubt on the validity of the results. In this lecture we intend to describe a few of the uses of Hadamard's formula and then to indicate a way of deriving it that is at once simple and direct, yet has sufficient rigor and generality to meet the needs of the applications. For the proof we shall appeal to Dirichlet's principle and to Thomson's principle in order to establish upper and lower bounds on δG that agree to the first order in $\delta\nu$.

Consider the class of normalized schlicht functions

$$w = f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

defined in the circle $|z| < R$. The classical one-quarter theorem asserts that any such function maps the circle $|z| < R$ conformally onto a region D of the w-plane that contains the circle $|w| < R/4$. In terms of the Green's function

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$$G(w,0) = \log\frac{1}{|w|} + \gamma + O(|w|)$$

of D, the theorem merely states that

$$\gamma \leqq \log R$$

whenever the curve ∂D includes a point on the circumference $|w| = R/4$. This follows from the relation

$$G(f(z),0) = \log \frac{R}{|z|}$$

between the mapping function and the Green's function, which also leads us to call the number $R = e^{\gamma}$ the inner mapping radius of D.

We wish to deduce the one-quarter theorem from Hadamard's variational formula

$$\delta\gamma = -\frac{1}{2\pi}\int_{\partial D}\left[\frac{\partial G(w,0)}{\partial v}\right]^2 \delta v\, ds$$

for the constant γ. Let us suppose that the point $w = R/4$ lies outside D. What has to be shown is that under this hypothesis γ becomes a maximum when the boundary ∂D of D reduces to an infinite slit $R/4 \leqq w < \infty$ along the real axis. The corresponding extremal mapping of the circle $|z| < R$ is defined by the Koebe function

$$w = f(z) = \frac{z}{(1 + z/R)^2}.$$

According to Hadamard's formula γ is an increasing functional of the region D. Therefore in maximizing γ we may confine our attention to regions whose boundaries ∂D consist of curvilinear slits extending from the point $w = R/4$ to infinity. Let E stand for a circle lying inside D and enclosing the origin. We introduce a continuous deformation of D by allowing the circle E to expand and push back the slit ∂D without moving the end point $w = R/4$. If we can establish that γ increases under any such deformation, then the one-quarter theorem will follow.

Let C denote the arc along which the boundary of E meets the slit ∂D at some intermediate stage of the deformation just described, and let G^* denote the harmonic function obtained by inverting the Green's function G with respect to E. According to the maximum principle we have $G^* \leqq G$ in E, which shows in turn that the normal derivative $\partial G/\partial v$ is at least as large along the inside edge of C as it is along the outside edge. Substitution of this estimate into Hadamard's formula yields the desired monotonicity of γ in its dependence on our deformation of ∂D. Thus our proof of the one-quarter theorem is complete, but observe that it requires us to apply the Hadamard formula to curves with corners.

We now come to the problem of deriving Hadamard's variational formula in a satisfactory way. What we have in mind is to give a proof based on expressing the Green's function

$$G(w,\zeta) = \log \frac{1}{|w-\zeta|} + \gamma(w,\zeta)$$

as a linear combination of Dirichlet integrals

$$\| u \|^2 = \iint_D (u_x^2 + u_y^2)\,dx\,dy$$

of appropriate harmonic functions u. More precisely, we exploit the identity

$$2\pi \sum_{j,k=1}^{n} \gamma(\zeta_j,\zeta_k)\lambda_j\lambda_k = - \left\| \sum_{j=1}^{n} \lambda_j \gamma(w,\zeta_j) \right\|^2 - \sum_{j,k=1}^{n} \lambda_j\lambda_k \int_{\partial D} (\log|w-\zeta_j|)\left(\frac{\partial}{\partial \nu} \log|w-\zeta_k|\right) ds,$$

where $\lambda_1, \cdots, \lambda_n$ are arbitrary real parameters. This representation enables us to estimate the first variation δG of the Green's function in terms of upper and lower bounds on Dirichlet's integral that follow from the Dirichlet and Thomson principles.

For the purposes of this lecture it will be enough to cite a special example of our method. Let D be a doubly-connected domain bounded by two curves C_1 and C_2, and let u be the harmonic function such that $u=1$ on C_1 and $u=0$ on C_2. Consider the Dirichlet integral

$$P = \| u \|^2 = - \int_{C_1} \frac{\partial u}{\partial \nu}\, ds$$

of u, which may be interpreted as the capacity of a condenser with the cross section D. According to Dirichlet's principle we have

$$P = \min_{\psi} \| \psi \|^2$$

over all continuous functions ψ with the same boundary values as u. On the other hand, Thomson's principle implies that

$$P = \max_{\phi} \frac{1}{\|\phi\|^2},$$

where ϕ ranges over the class of multiple-valued functions in D with the period 1 and with a single-valued gradient. The extremal function v for the latter variational problem is the harmonic conjugate of u divided by P, which means that

$$v_z = -\frac{i}{P} u_z.$$

Our aim is to calculate the first variation δP of the capacity P due to a perturbation of D defined by a variation

$$z^* = z + \varepsilon F(z, \bar{z})$$

of the independent variables x and y. Let us introduce the transformed functions

$$u^*(x^*, y^*) = u(x, y), \ v^*(x^*, y^*) = v(x, y).$$

A direct computation of the Dirichlet integral of u^* shows that

$$\delta \|u\|^2 = -8 \operatorname{Re}\left\{\varepsilon \iint_D u_z^2 F_{\bar{z}}\, dxdy\right\},$$

and the same result is valid for v. Applying Dirichlet's principle to estimate the capacity of the varied region D^*, we conclude that

$$\delta P \leqq \delta \|u\|^2 = -8 \operatorname{Re}\left\{\varepsilon \iint_D u_z^2 F_{\bar{z}}\, dxdy\right\}.$$

Similarly, Thomson's principle yields the lower bound

$$\delta P \geqq -\frac{\delta \|v\|^2}{\|v\|^4} = 8P^2 \operatorname{Re}\left\{\varepsilon \iint_D v_z^2 F_{\bar{z}}\, dxdy\right\}$$

on the first variation of the capacity.

Because $u_z^2 = -P^2 v_z^2$, the upper and lower bounds on δP that we have established actually coincide. Integrating by parts, we deduce from them the final variational formula

$$\delta P = 4 \operatorname{Re}\left\{\varepsilon i \int_{\partial D} u_z^2 F\, dz\right\}.$$

Since

$$\operatorname{Im}\left\{\varepsilon F\frac{d\bar{z}}{ds}\right\} = \delta v,$$

this result is equivalent to the Hadamard formula

$$\delta P = \int\limits_{\partial D}\left[\frac{\partial u}{\delta v}\right]^2 \delta v\, ds.$$

Thus we have developed a rigorous proof of Hadamard's variational formula that is seen to be applicable whenever the integrals involved have a meaning. Finally, it should be mentioned that Kazdan [5] has worked out the above method in a form that leads to a new derivation of Loewner's ordinary differential equation for schlicht functions.

We close by describing briefly how our variational formula for the Dirichlet integral may be used to solve the Plateau problem. The parametric form of the problem asks for a triple of functions $\boldsymbol{r} = (u, v, w)$ that map the boundary of the unit circle D monotonically onto a given curve Γ in space and minimize the integral

$$\|\boldsymbol{r}\|^2 = 4\iint\limits_D (u_z u_{\bar{z}} + v_z v_{\bar{z}} + w_z w_{\bar{z}})\, dx dy.$$

It is not very hard to show that a solution of this formulation of Plateau's problem exists (cf. [2]), and Dirichlet's principle implies that the individual components u, v, w of the extremal vector $\boldsymbol{r}$ are harmonic. What remains to be established is that $\boldsymbol{r}$ defines a conformal mapping of D onto a surface in space which is in consequence a minimal surface through the given curve Γ.

Let $\rho > 0$ be small but fixed, and let

$$z^* = \left(z + \frac{\varepsilon}{z - z_0}\right)\Big/\left(1 + \frac{\bar{\varepsilon} z^2}{1 - \bar{z}_0 z}\right)$$

for $|z - z_0| \geqq \rho$, while

$$z^* = \left(z + \varepsilon\frac{\bar{z} - \bar{z}_0}{\rho^2}\right)\Big/\left(1 + \frac{\bar{\varepsilon} z^2}{1 - \bar{z}_0 z}\right)$$

for $|z - z_0| \leqq \rho$. These relations specify a variation of the independent variables x and y that transforms the unit circle onto itself and is admissible for the Plateau problem. Therefore we have

$$\delta\|\boldsymbol{r}\|^2 = -8\operatorname{Re}\left\{\frac{\varepsilon}{\rho^2}\iint\limits_{|z-z_0|<\rho}(u_z^2 + v_z^2 + w_z^2)\, dx dy\right\} = 0.$$

From the mean value theorem for harmonic functions we conclude that

$$u_z^2 + v_z^2 + w_z^2 = 0,$$

which is the desired conformality condition on the mapping $\mathbf{r}$. Observe that, in the special case where Γ is a plane curve, our procedure furnishes a proof of the Riemann mapping theorem.

2. Symmetric hyperbolic systems in the complex domain.

The type of a partial differential equation is not well defined in the complex domain. We shall see, for example, that it is possible to transform an analytic system of arbitrary type into a symmetric hyperbolic system by means of complex substitutions. The purpose of this lecture is to show how such devices can be of service in solving the Cauchy problem.

The complex substitution $z = it$ converts any analytic solution u of the wave equation

$$u_{xx} + u_{yy} - u_{tt} = 0$$

into a function satisfying Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

If u is harmonic throughout a three-dimensional region D, it has an integral representation

$$u = \frac{1}{4\pi} \iint_{\partial D} \left[u \frac{\partial}{\partial \nu} \frac{1}{r} - \frac{1}{r} \frac{\partial u}{\partial \nu} \right] d\sigma$$

in terms of its Cauchy data over the boundary surface ∂D. We wish to transform this formula into an expression for the solution of Cauchy's problem for the wave equation.

When $z = it$ is pure imaginary, the locus of points (ξ, η, ζ) where

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - it)^2}$$

vanishes consists of a circle

$$(\xi - x)^2 + (\eta - y)^2 = t^2, \qquad \zeta = 0$$

of radius t. For the sake of simplicity let us suppose that $u = 0$ on the plane $\zeta = 0$. By drawing the surface of integration ∂D in our formula for u down around the disc E enclosed by the circle $r = 0$, we obtain the expression

$$u(x, y, t) = \frac{1}{2\pi} \iint_E \frac{f(\xi, \eta)\, d\xi\, d\eta}{\sqrt{t^2 - (\xi - x)^2 - (\eta - y)^2}}$$

for the solution of Cauchy's problem with initial data

$$u(x,y,0) = 0, \quad u_t(x,y,0) = f(x,y)$$

assigned at $t = 0$.

In the above analysis observe that the change in sign of r when we pass from one side of E to the other just compensates for the change of orientation in the normal ν to reduce the surface integral over ∂D to a real integral over E. On the other hand, for the wave equation in three space variables the fundamental solution $1/r$ must be replaced by a single-valued function $1/r^2$. Hence in that case the only contribution remaining in our integral becomes a residue over the perimeter of E. This is seen to be the substance of Huygens' principle.

We now turn our attention to a quasi-linear analytic system

$$u_t = A u_x$$

of arbitrary type. Let us consider the analytic extension of an analytic solution u to complex values $z = x + iy$ of the independent variable x. In the complex domain u has to satisfy the Cauchy-Riemann equations

$$u_{\bar{z}} = 0.$$

Multiplying by the adjoint matrix $\bar{A}'$ and adding to the original system of quasi-linear partial differential equations for u, we obtain

$$u_t = Au_z + \bar{A}'u_{\bar{z}}.$$

This is equivalent to the symmetric hyperbolic system

$$u_t = \frac{A + \bar{A}'}{2} u_x + \frac{A - \bar{A}'}{2i} u_y.$$

Thus any analytic system can be transformed into a symmetric hyperbolic system in a larger number of independent variables.

The symmetric hyperbolic system for u furnishes a tool for proving the Cauchy-Kowalewski theorem without reference to power series. We are thus led to a new explanation of the solvability of Cauchy's problem for analytic equations of arbitrary type. From the opposite standpoint, our symmetric hyperbolic system in the complex domain enables us to establish the existence of global solutions of the original system when it is linear and symmetric hyperbolic to begin with. This is achieved by deriving an energy inequality for the square integral of u in the complex domain which leads by function theoretic methods to direct estimates of u and its derivatives in the real domain.

The principal application of our reduction of an arbitrary analytic system to symmetric hyperbolic normal form arises in determining inverse solutions of the detached shock problem. There one wants to find numerical solutions of Cauchy's problem for equations of motion that are of mixed type. The associated symmetric hyperbolic system may be approximated by a second order central difference equation of the form

$$\frac{u_{l,m,n+1}-u_{l,m,n-1}}{\Delta t} = \frac{A_{l,m,n}+\bar{A}'_{l,m,n}}{2}\,\frac{u_{l+1,m,n}-u_{l-1,m,n}}{\Delta x} + \frac{A_{l,m,n}-\bar{A}'_{l,m,n}}{2i}\,\frac{u_{l,m+1,n}-u_{l,m-1,n}}{\Delta y}.$$

Because the matrices of coefficients are Hermitian, this scheme is stable provided that Δt is sufficiently small relative to Δx and Δy. Moreover, the truncation error approaches zero like the square of the largest mesh size when the initial data are handled with enough accuracy.

In the case of plane flow Eva Swenson has programmed the above solution of the detached shock problem for the IBM 7090 computer at New York University. Results for flows with incident Mach numbers very little larger than 1 are being calculated. These examples are of interest because they exhibit a large stand-off distance between the body and the shock wave. The only other method that might be adequate for such a situation is suggested by work of Nagumo and Swigart. It would exploit Cauchy-Kowalewski power series expansions with respect to the variable x alone to develop a system of ordinary differential equations in t for the Taylor series coefficients of the unknowns. Observe that neither this method nor the one described before requires analytic dependence of the data or of the solution on t.

3. General solutions and the method of images applied to the problem of the clamped plate.

The construction of solutions of analytic partial differential equations in the complex domain that we outlined in the previous lecture is related to the question of determining a general solution. We wish to discuss next how the general solution can be used to treat boundary value problems by a function theoretic procedure based on the method of images. The most successful example of this technique is provided by the biharmonic equation.

The equation of any analytic arc C in the z-plane has a representation

$$\bar{z} = g(z)$$

in terms of a function g that is analytic throughout some neighborhood of C. Let u stand for a complex analytic function of z that is real on C. Then we have

$$u(z) = \overline{u(z)} = \bar{u}(\bar{z}) = \bar{u}(g(z)) = \overline{u(\overline{g(z)})}$$

along C. Since the expressions on the extreme left and the extreme right are analytic in z, we conclude that Schwarz's principle of reflection takes the form

$$u(z) = \overline{u(\overline{g(z)})}$$

across C. In particular, the point $\overline{g(z)}$ is seen to be the reflected image of z with respect to the analytic arc C.

Note that for a segment of the real axis g is the function z, whereas for an arc of the unit circle g is the function $1/z$. A more interesting example is furnished by the equilateral hyperbola

$$\bar{z} = \sqrt{2 - z^2}\,.$$

In the latter case the Schwarz principle of reflection yields multiple-valued results at the foci $\pm\sqrt{2}$. More generally, we might call any branch point of the function g a focus of the corresponding analytic arc C. We shall establish that the nonuniform behavior of the reflection process around such a focus stems from the multiple-valuedness of the solution of Cauchy's problem about a characteristic that becomes tangent to the curve where the initial data are assigned.

Consider at first a solution u of Laplace's equation

$$u_{xx} + u_{yy} = 0$$

that satisfies the boundary condition $u(0, y) = 0$. Let us continue u analytically to complex values of the independent variable $y = y_1 + iy_2$, so that it becomes a solution of the wave equation

$$u_{xx} - u_{y_2 y_2} = 0$$

in the (x, y_2)-plane. If u_1, u_2, u_3 and u_4 stand for the values of u at consecutive corners of a square bounded by characteristics of the wave equation, then we have

$$u_1 - u_2 + u_3 - u_4 = 0,$$

which may be viewed as an explicit formula for the solution of a characteristic initial value problem determining u. By choosing the second and fourth corners of the square to lie on the y_2-axis, where u vanishes, we obtain Schwarz's principle of reflection in the form

$$u_1 = -u_3.$$

In this connection observe that both the solution of the characteristic initial value problem and the Schwarz reflection principle exhibit a Huygens principle in as much as only data at isolated points, rather than data along prescribed curves, appear in the final formulas.

Our discussion of the Schwarz principle of reflection shows that it is actually a relationship among four values of u, two of which occur at image points z and $\overline{g(z)}$ in the real domain, and two of which occur at points on the complex extension of the reflecting curve C where the characteristics through z and $\overline{g(\bar{z})}$ intersect. Now let us return to the example of the equilateral hyperbola

$$x^2 - y^2 = 1.$$

In the (x, y_2)-plane it has a realization as the unit circle

$$x^2 + y_2^2 = 1.$$

The pair of characteristics through the focus $x = \sqrt{2}$, $y = 0$ are tangent to this circle. Therefore the correspondence between any point and its reflected image with respect to the equilateral hyperbola becomes double-valued in the neighborhood of the focus. In other words, the reflection process breaks down there for precisely the same reason as does the solution of Cauchy's problem for the wave equation with initial data assigned on the unit circle.

The ideas we have just been exploring have an application to the inverse detached shock problem in the case of supersonic flow past an obstacle with a pointed nose. At the vertex of the body the flow quantities become singular and exhibit the same behavior as does an analytic function at a branch point. Since the shock conditions lead to Cauchy data depending only on the slope of the curve defining the shock wave, our problem is to choose that curve so as to generate a singularity of the desired kind. This is achieved by arranging that the complex characteristics through the vertex of the body become tangent to the analytic extension of the shock wave into the complex domain. Numerical experiments with such a construction are being carried out on the IBM 7090 computer by Eva Swenson.

The method of images in the form in which we have presented it is especially helpful for treating the biharmonic equation

$$\Delta\Delta u = 0.$$

Consider a plane region D bounded by a simple closed analytic curve C whose equation has been expressed in terms of an analytic function g as described above. In D we may represent the general solution of the biharmonic equation in the form

$$u = \operatorname{Re}\left\{\frac{\bar{z}-g(z)}{2}\,\frac{\Psi'(z)-\Phi'(z)}{g'(z)}+\Psi(z)\right\},$$

where Φ and Ψ stand for arbitrary analytic functions of the complex variable z. We are interested in developing a rule for the analytic continuation of u across an arc of C where it satisfies the homogeneous boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0,$$

which occur in the theory of infinitesimally thin clamped plates.

In terms of the complex analytic functions Φ and Ψ our boundary conditions on u become

$$\operatorname{Im}\{\Phi\} = \operatorname{Re}\{\Psi\} = 0.$$

Therefore Schwarz's reflection principle asserts that

$$\Phi(z) = \overline{\Phi(\overline{g(z)})}, \quad \Psi(z) = -\overline{\Psi(\overline{g(z)})}.$$

These formulas provide the desired rule for continuing the biharmonic function u across C. We shall indicate next how they lead to an explicit solution of the biharmonic Dirichlet problem for the class of regions D in which the analytic function g is regular except for poles.

The particular expression we have given for the general solution of the biharmonic equation has the advantageous property that prescribing u and $\partial u/\partial \nu$ along the boundary curve C becomes equivalent to prescribing $\operatorname{Re}\{\Psi\}$ and $\operatorname{Im}\{\Phi\}$. Hence the Dirichlet problem for u reduces to a pair of boundary value problems for the analytic functions Φ and Ψ. The only complication that arises in the reduction is that Φ and Ψ will usually have singularities in D at the poles of g. However, since g is single-valued and regular in D except for its poles, the requirement that u be a regular biharmonic function yields a finite set of conditions on the principal parts of Φ and Ψ that determine them uniquely. Thus it is possible to solve Dirichlet's problem in closed form.

It is of interest to examine more carefully the significance of our hypothesis that g is regular in D except for a finite number of poles. Let $z = z(w)$ be a conformal mapping of the unit circle $|w| < 1$ onto D. Clearly $z + g(z)$

and $z - g(z)$ are meromorphic functions of w in the closure of the unit circle, and their boundary values are real and pure imaginary, respectively. Consequently the Schwarz principle of reflection implies that they are meromorphic in the entire w-plane, which means that they must be rational functions. Thus our hypothesis about the analytical curve C is equivalent to the assumption that the conformal mapping z of the unit circle onto D is accomplished by a rational function of w.

For the class of regions just described, Muskhelishvili [7] has solved the Dirichlet problem for the biharmonic equation many years ago. His results are derived by means of either Fourier series or the Cauchy integral formula, whereas ours emphasize the role of the method of images more pointedly. Despite their apparent differences, all these procedures make essential use of an exact knowledge of the general solution of the biharmonic equation.

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