PERSPECTIVE AND PROJECTIVE GEOMETRIES
A COMPARISON

PROJECTIVE Geometry owes its origin to efforts made by mathematicians to find a satisfactory solution for problems in Perspective Drawing,[1] and has been developed as an independent subject far beyond the needs of the practical problem. One of the first to become interested in accurate Perspective Drawing was Leonardo da Vinci, whose name is familiar to all of us for more significant interests. At the present time, students of modern geometry are usually unfamiliar with the theory of Perspective Drawing, and the reverse is even more generally true of architects. The object of this paper is to point out certain fundamental relations between practical Perspective Drawing and Projective Geometry, and to show applications of each to the other.

It is hoped that the reader is familiar with some of the theories and procedures of Projective Geometry and Perspective Drawing, but it is fitting that definitions and theories of both be repeated for clarity of application.

A *Perspective Drawing* of an object is a representation of the same as it would appear to the eye of an observer located in a fixed position with respect to the object. The drawing of perspective is generally projected only on a plane, called the *Picture Plane.* The eye of the observer is called the *Station Point* and is denoted by the letter S. A line joining any point of the object to the station point is called a *Visual Ray.* The perspective of a point is the intersection with the Picture Plane of the visual ray from that point. The foot of the perpendicular from the station point to the picture plane is called the *Center of Vision* and is designated by O. Two visual rays determine a *Visual Plane;*

any two horizontal visual rays determine the *Horizontal Visual Plane*, and the intersection of the *Horizontal Visual Plane* with the picture plane is called the *Common Horizon*, *Floor Horizon* or *Horizon*.

In Projective Geometry projection is the *process* of drawing lines or rays from a point to points or lines, and the *section* of these rays or lines corresponds to the perspective drawn by cutting visual rays to an object by the picture plane in perspective. "The process of projection and section is fundamental in Projective Geometry."[2] Certainly it is no less

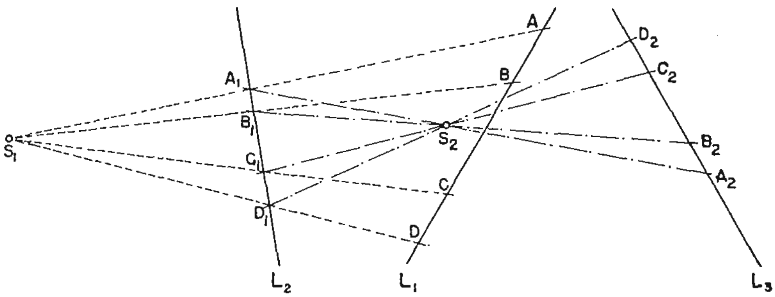

FIGURE 1

important in Perspective Drawing, since *projection* and *visual cone* are but different terms describing the same process. The Visual Cone consists of all the visual rays through the Station point, and corresponds to a "bundle" of lines in Projective Geometry.

In Projective Geometry we speak of transformations. Such a transformation is shown in Figure 1, where the points A, B, C, D are transformed to A_1, B_1, C_1, D_1. Such a transformation is called a perspectivity, since it is accomplished through a single point S_1. A_1, B_1, C_1, D_1 are then projected to A_2, B_2, C_2, D_2 through S_2 and form a second perspectivity.

The transformation of A, B, C, D, into A_2, B_2, C_2, D_2 is a "projectivity" or the sum of more than one perspectivity. If we project any point of a line A to any point of a line B, we obtain a one-to-one correspondence. Such a correspondence may obviously require a double point such as c-c_1 in Figure 2. As seen in Figure 2, it is not in the least disturbing to realize that the point j_1 on line A corresponding to j is infinitely far away and in fact is known as an "ideal point," as is the point i on the line B corresponding to i_1 (on line A).

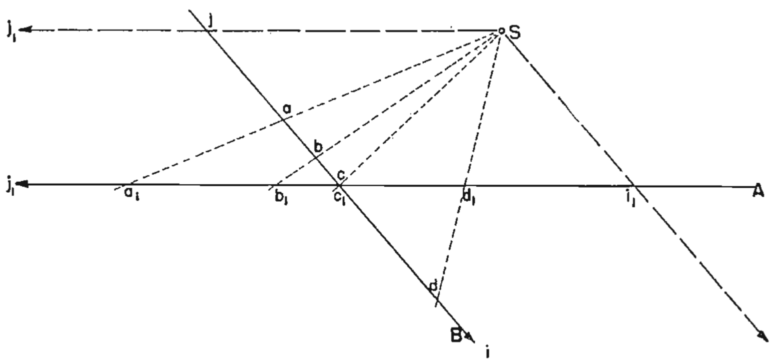

FIGURE 2

We still have a one-to-one correspondence, for not only do we recognize the existence of ideal points but likewise that of ideal lines and planes.

Of course, a perspective drawing is a "Perspectivity" of an object on the picture plane, and in *general* it is a one-to-one perspective transformation. In Projective Geometry all transformations are one-to-one, but in perspective transformations coincidences do sometimes occur.

In order to show the exact duplication of the concept of ideal points in Perspective Drawing, it is necessary at this stage to introduce additional definitions and ideas of the latter field.

The most elementary methods of Perspective Drawing are fundamentally two in number. In the first method a transparent sheet is held between the object and the eye and the points in which the visual rays to points of the object intersect the sheet are marked, being careful all the while to keep the eye in the same position relative to the object. The drawing which is obtained on the sheet by this process is a

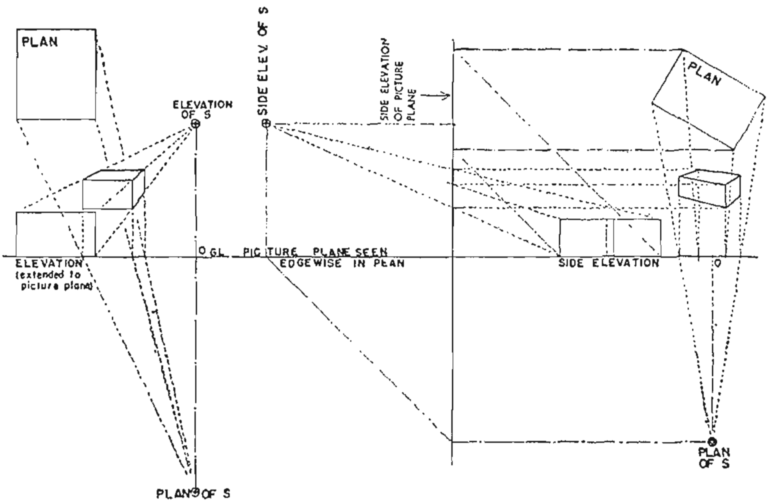

FIGURE 3

perspective, although it is rather crude. The second method is the *Method of Direct Projection* and is shown in Figure 3. It is obtained by drawing a plan and elevation of the station point, the picture plane, and the object to be drawn in perspective. By projecting from the station point in the plan and elevation to the same point of the object in the plan and elevation respectively, the perspective of that point is then found on the elevation of the visual ray by drawing a perpendicular from the point of intersection of the plan of the visual ray and the *Ground Line* (picture plane seen edgewise in plan). By continuing this process for all corner points, and joining the points so obtained by lines corresponding to the edges, the complete perspective is drawn. In the second half of Figure 3, the side elevation and plan are used instead of the elevation and plan—otherwise the construction is similar from beginning to end to that of the first part of Figure 3. These two methods are obviously either very inaccurate or, even in ordinary problems, very laborious, or both. They are impractical, though a modification of the former, employing an *imagined* picture plane at arm's reach, is sometimes used for sketching natural objects.

The only connection between these methods and Projective Geometry is their fundamental ideas of projection and section.

The question arises, "How can accurate perspectives be drawn?" The answer is very easy: by the use of *Vanishing Points,* and these are one of the most valuable possessions of the perspective draftsman. A *Vanishing Point* is that point in the picture plane toward which all the lines of a set of parallel lines, when drawn in perspective, appear to converge. Defining vanishing points projectively: a *Vanishing Point* is the image of the ideal point of a set of parallel lines, or the section by the picture plane of the visual ray to that ideal point. These definitions suggest a convenient method of determining vanishing points in our actual drawings. If a line is drawn through S, parallel to one or to a set of parallel lines, that line will pass through the ideal point of the line or set of parallel lines. The section of this visual ray by the picture plane will be the desired vanishing point. If the set of parallel lines is parallel to the picture plane, the visual ray to their ideal point will not cut the picture plane, and the ideal point itself will be the vanishing point of that set. Therefore that set will be actually parallel in perspective drawing.

The introduction of vanishing points leads to three kinds of perspectives. In any building there are in general three sets of parallel lines, the horizontal set in the front and rear faces, the horizontal set in the end faces, and the vertical set. If the picture plane is parallel to one of the faces of the building, it is parallel to two of these fundamental sets of parallel lines, two of the vanishing points will be their own ideal points, and the third set of parallel lines will have its vanishing point in the picture plane. This first type of perspective is shown in the first part of Figure 3 and is called a *Parallel* or *One-Point Perspective.* If the picture plane is parallel to only one of the fundamental sets of parallel lines (normally the vertical lines), then only that set will have its ideal point as its vanishing point and there will be two vanishing points appearing in the picture plane. This type of perspective is shown in Figure 4 and is called *Angular* or *Two-Point Perspective.* If the picture plane is parallel to none of the fundamental sets of parallel lines, then there will be three vanishing points appearing in the picture plane and this type of perspective is called *Three-Point Perspective.* In two-point perspective the vanishing points of the two sets of parallel lines which are perpendicular to each other are called *Conjugate Vanishing Points,* and each vanishing point is conjugate to the other. In three-point perspective the vanishing points of the three sets of mutually perpendicular lines are called *Tri-Conjugate Vanishing Points* and each is conjugate with respect to the other two.

Any line of an object which lies in the picture plane is its own perspective and may be drawn to scale in the picture plane. From that line other distances in the perspective may be determined by laying off the desired scaled distance on this line, and carrying it to its proper position on the perspective by means explained later. Such a line is called a *Measure Line.* Often there is no line of an object lying in the picture plane, but a measure line may be obtained by drawing the perspective of any line parallel to the picture

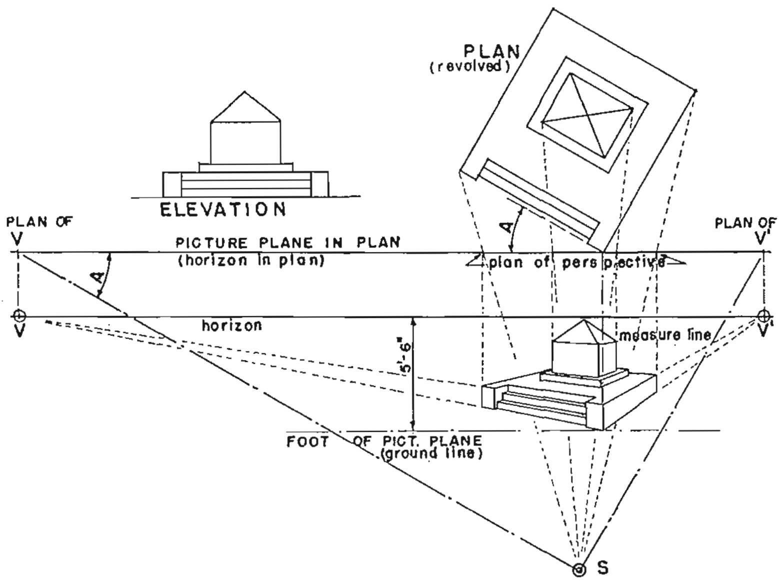

FIGURE 4

plane. The perspective of that line will be parallel to the line itself, and from the ratio of the length of the perspective of the line to the length of that line in the object, a scale may be set up to be used for laying out distances on the lines drawn in perspective.

The first method which introduces vanishing points is the *Mixed Method,* which is illustrated in Figure 4. The revolved plan and the elevation of the monument are shown, and the plan of the station point and of the picture plane are also shown. The height of the eye, for level views in two-point perspectives, is taken to be about five and one-half feet, and from this fact it is possible to locate the main corner, c, five and one-half feet below the horizon. The procedure of the drawing of the two-point perspective is shown in Figure 4. This method, like the method of direct projection, also employs direct projection in plan for the determining of widths in the perspective; but vertical distances are obtained by laying off heights directly on the measure line, which is usually the front corner of the building, resting against the picture plane. The use of vanishing points in this figure eliminates the necessity of drawing the elevation of the station point, the object and the ground line. The *Ground Line* is defined as being the intersection of the picture plane with the ground. This method is less tedious than the former two methods, and is in addition more accurate. But it is extremely difficult to construct a three-point perspective by this method.

The method which far surpasses all other methods as well in accuracy, simplicity, elasticity, consistency, and speed is called the *45°-Line Method.* By this method one-point, two-point, and three-point perspectives are drawn in exactly the same manner, the only difference being in the laying out of what is known as the *Perspective Diagram.*

Figure 5 shows the various vanishing points which may be used in drawing a two-point perspective, although not all of these would ever be used in preparing any one perspective drawing.

It should be here emphasized again, that the vanishing point of any line is determined by finding the intersection

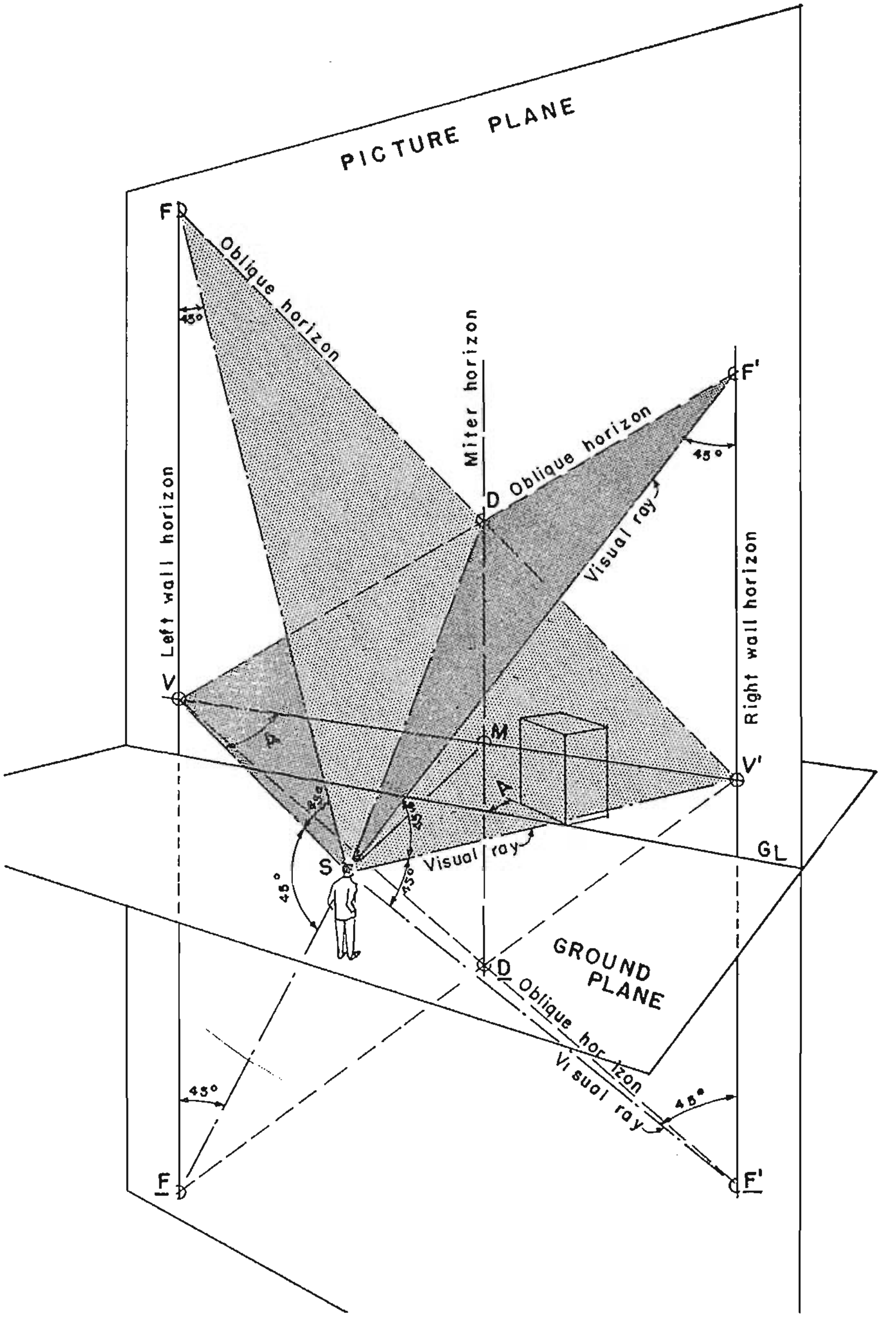

FIGURE 5

NOTE: Barred (underscored) letters on this and subsequent drawings will be referred to in the text by italicized *D*, *F*, etc.

with the picture plane of the visual ray drawn parallel thereto, and is the perspective of the ideal point of the line.

A complete two-point perspective diagram is shown in Figure 6, and its relation to Figure 5 is apparent.

The following therefore applies generally to both Figures

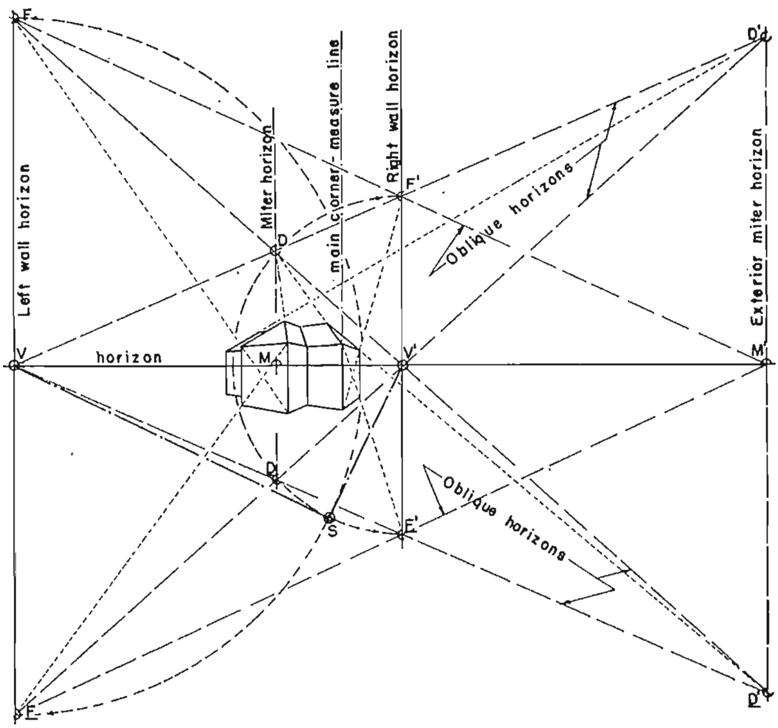

FIGURE 6

5 and 6. The F's shown are the vanishing points of lines inclined at 45° to the horizontal plane and on or parallel to the wall planes of the building in question. Any F is the section of the corresponding parallel visual ray by the picture plane, and obviously lies above and below the horizon and at

a distance therefrom equal to VS or V′S, *i.e.*, the length of the corresponding horizontal visual ray from S to the vanishing point on the picture plane. This definition bears out the construction for locating the F's as shown in Figure 6. The observant reader will notice that the 45° roof on the front of the building contains V- and F′-lines and hence any line in this roof plane vanishes in the oblique horizon VF′. Similarly, any line in the 45°-sloping roof at the ends of the building vanishes in the line V′F. Therefore the hip or valley lines which are the intersections of these two planes vanish at D. In like manner, the other hip and valley lines of intersecting combinations of 45°-sloping planes vanish at *D*, D′ or *D′*. The vertical line joining D and *D*, called the *Miter Horizon*, is obviously the vanishing line of the *Miter Plane*, *i.e.*, the plane bisecting the dihedral angle at the front corner of the building. Horizontal lines in this plane or lines parallel thereto are 45°-lines on or parallel to the floor planes, and must vanish both in the common horizon and in the miter horizon, and hence at their intersection M. D′, M′, and *D′* are identically related points for the planes perpendicular to the miter plane at the front corner and bisecting the dihedral angles at the outer vertical corners of the building, which may be called the *Exterior Miter Planes*, and their vanishing line, obviously D′*D′*, may be called the *Exterior Miter Horizon.*

In the complete diagram (Figure 7) it is clear that the various lines of a cube vanish at one of the twelve vanishing points:

(1) Horizontal lines (12) on or parallel to the left face of the cube vanish in V.

(2) 45°-lines (32) running upward and backward to the left, and lying on the left face or parallel thereto vanish at F.

(3) 45°-lines (14) as in (2) except running *downward* and backward to the left vanish at *F*.

(4) Horizontal lines (15) on or parallel to the right face of the cube vanish at V′.

(5) 45°-lines (35) on or parallel to the right face of the cube and running upward and backward to the right vanish at F′.

(6) 45°-lines (16) as in (5) except running *downward* and backward to the right vanish at F′.

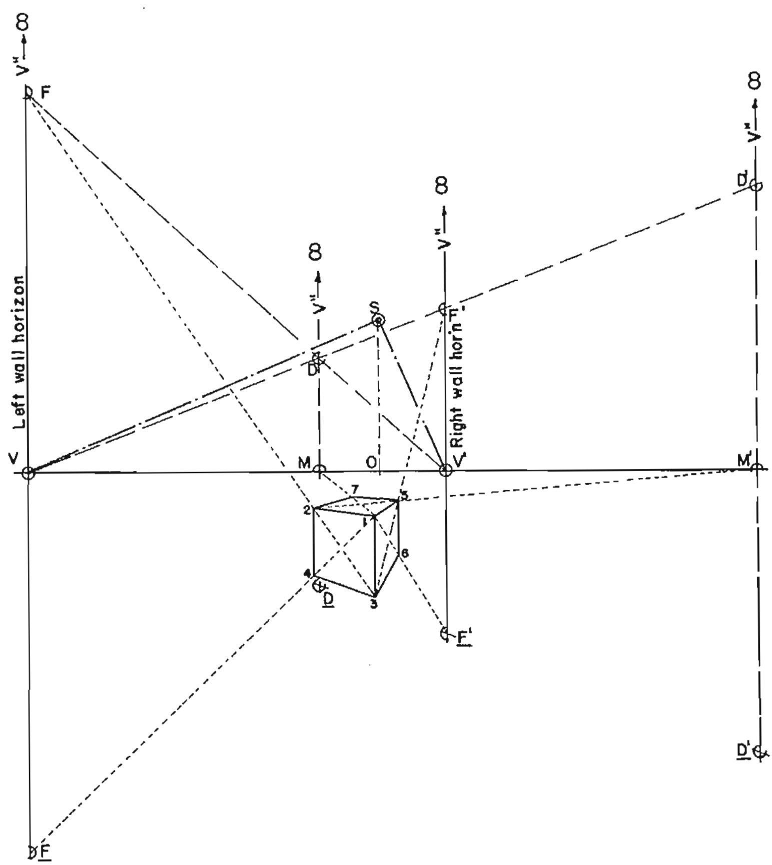

Figure 7

(7) Horizontal lines (17) making 45° angles with the right and left face of the cube and running from front to rear vanish at M.

(8) The diagonal of the cube (37) from the lower front corner to the upper rear corner vanishes at D.

(9) The diagonal of the cube (43) from the upper front corner to the lower rear corner vanishes at D.

(10) Horizontal lines (25) at 45° to the faces of the cube and running from left to right vanish at M′.

(11) The diagonal of the cube (26) running from lower left corner to upper right corner vanishes at D′.

(12) The diagonal of the cube running from lower upper left corner to lower right corner vanishes at D′.

The reader will observe that

(a) the lines of paragraphs (1), (2), and (3) above lie on the left face of the cube or on planes parallel thereto and hence all such planes vanish in the line *F*VF, and this line is called the Vanishing Horizon of this set of vertical planes.

(b) The lines of paragraphs (4), (5) and (6) above lie on the right face of the cube or on planes parallel thereto and such planes vanish in *F*′V′F′.

(c) The lines of (7), (8), and (9) lie on a vertical plane at 45° to the faces of the cube running from front to rear corner and all planes parallel thereto vanish in *D* M D.

(d) The lines of (10), (11) and (12) lie on planes at 45° to the faces of the cube passing through the left and right corners thereof, and all planes of this set vanish in the line *D*′ M′ D′.

Here the reader is reminded that in Projective Geometry, the existence of ideal points (points at infinity) and ideal lines (lines at infinity) are fundamental to the subject. In perspective the vanishing line of a plane or a series of parallel planes is determined by drawing the visual plane parallel thereto to its intersection with the Picture Plane and is *in reality the perspective of the infinitely distant line of intersection of the parallel planes.*

The vanishing line of a plane is most easily determined by drawing any two visual rays parallel to the plane and finding their intersections with the picture plane (*i.e.*, their vanishing points). The vanishing line of the plane then lies on the two points and any line on the plane or parallel thereto vanishes in the vanishing line.

Certain correspondences of Projective Geometry and Perspective are appropriately observed at this point:

> The *Common Horizon* is the pencil of points determined by the pencil of all horizontal visual rays. Stated in perspective, the horizon contains the vanishing points of all sets of horizontal sets of lines.
>
> Any *horizon* is the pencil of points determined by the pencil of all visual rays drawn parallel to any given plane, and is, in fact, the intersection with the picture plane of the visual plane parallel thereto.
>
> The pencil of visual rays parallel to the picture plane determines an ideal line or horizon, and all lines parallel to the picture plane vanish in the "ideal horizon," and are parallel in perspective.

The use of proportionate division in perspective provides an interesting variation of the "perspectivity" of Projective Geometry. Normally, in perspective drawing, the problem of laying distances off to scale is solved as previously described by placing one corner of the object on the picture plane so that distances may be laid off at full size on this line. Such distances must be carried back through the perspective to the location when the perspectives of the scaled distances are needed. By the use of proportionate division, however, perspective distances may be determined directly in their perspective locations. Suppose the line A, as in Figure 8, vanishing at V, has been determined to be of the perspective length shown. It is desired to divide this line perspectively into 3 equal (or any number of proprotionate) parts. Three or any number of equal or proportionate distances may be laid off on line B drawn in any direction. The end of A is joined to the last division of B and this joining line is ex-

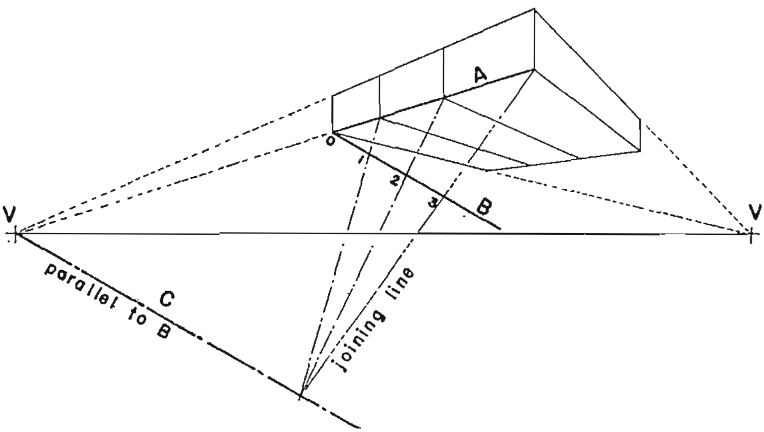

FIGURE 8

tended to a line C through V (the vanishing point of the perspective line to be divided), drawn parallel to B, meeting C in p, called a pivot point; p is also a center of perspectivity in Projective Geometry. p-lines then divide the line A into three *perspectively* equal parts. The truth of this procedure is easily established, for, if B has equal distances equal in perspective, it must be parallel to the picture plane. Hence a plane containing the line A, and B, will intersect the picture plane in a line through V parallel to B and this intersection is the vanishing line of the plane. Since the joining line lies in the plane it vanishes in the line C and lines drawn to p are the perspectives of parallel lines.

In three-point perspective when the picture plane is tilted

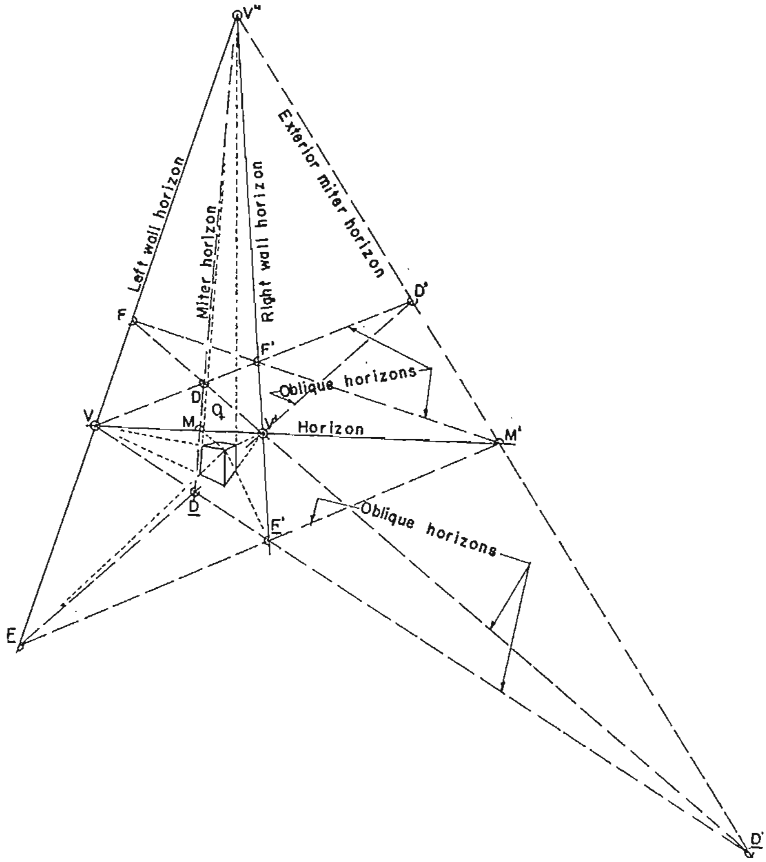

FIGURE 9

away from the perpendicular, the visual ray parallel to the vertical lines of the building is no longer parallel to the picture plane, but intersects it in a nearby vanishing point V″. In Figure 7, the vertical lines of the cube are perpendicular and vanish at the ideal point V″. In Figure 9, the wall and miter horizons are "pulled together" bringing the ideal point in to the nearby location V″, and the complete three-point perspective diagram results. The similarity of Figure 9 to Figures 6 and 7 is obvious. Likewise the horizons of (a) (b) (c) (d), page 13 above, are no longer vertical but pass through V″. That V″ is nearby (Figure 9) or infinitely distant (Figure 7) is entirely immaterial, since the existence of both ideal and real points is recognized.

Hence in both Figures 7 and 9 on any of the horizons of the complete diagram there are four points:

F V F V″
D M D V″
F′ V′ F′ V″
D′ M′ D′ V″

Likewise all the other horizons of the diagram contain four points:

F D V′ *D′*
V *D F′ D′*
V M V′ M′
V D F′ D′
F D V′ D′

In Projective Geometry a complete 4-line in a plane is defined as four lines, no three of which are concurrent, unlimited in length with the six vertices and three diagonals which are determined thereby, as in Figure 10. This figure is so constituted that any four points on one of the diagonals indicated are said to be "Harmonic" and this relation is written H (ab, i_3i_2), *i.e.*, a and b are harmonic with respect to i_3 and i_2.

Similarly

$H\,(fc,\ i_1i_3)$

$H\,(ed,\ i_1i_2)$.

In a harmonic set of points, if three are given, the fourth may be determined singularly.

Observe then the perspective cdef of a square in Figure 11. Lines 1 and 2 are drawn through e, and V- and V′-lines are horizontal and perpendicular to each other. Any M-line,

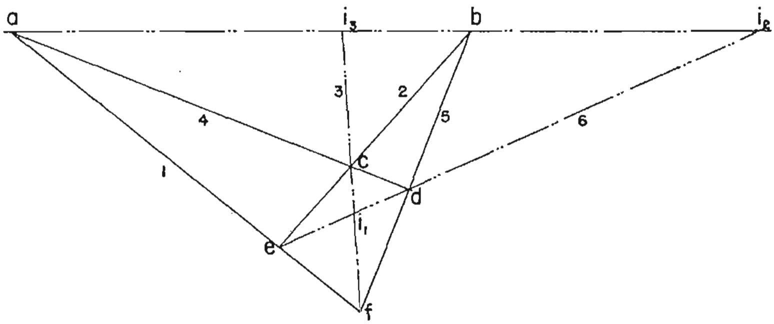

FIGURE 10

3, is drawn cutting the lines 1 and 2 in c and f. This is then the perspective of a horizontal 45° right triangle and hence half of the perspective of a square. Completing the perspective of the square, we draw V- and V′-lines 4 and 5 determining point d. The perspective of the square is then cdef; and the other 45° diagonal, 6, should be an M′-line. From the similarity of Figures 10 and 11, it is clear that V, M, V′, M′ comprise a harmonic set of points, and such a relation would be most useful in drawing lines to distant vanishing points without going to the effort of locating those distant points.

From Projective Geometry again, if four harmonic points are perspective from a point S as in Figure 12, the projecting

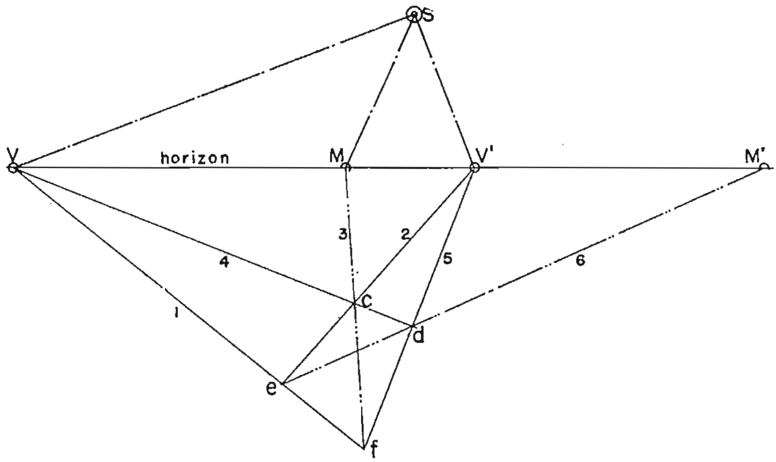

FIGURE 11

lines are harmonic lines, and cut any transversal in a harmonic set of points. A "perspectivity" of harmonic points is a set of harmonic points, and a "projectivity" being the

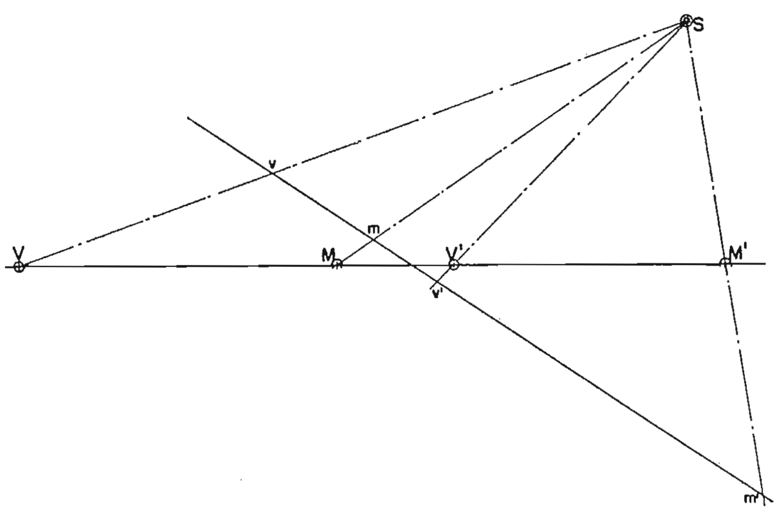

FIGURE 12

resultant of two or more perspectivities is a harmonic set. Therefore, in the complete 2-point diagram of Figure 7, if V″ is the ideal point of the vertical horizons passing through V, M, V′ M′, and these vertical horizons are a harmonious set through the ideal point,

$$\mathrm{H}\,(\mathrm{FV}', DD')$$
$$\mathrm{H}\,(\mathrm{V}F', D\,D')$$
$$\mathrm{H}\,(\mathrm{VF}', \mathrm{D\,D}')$$
$$\mathrm{H}\,(F\,V', \mathrm{D\,D}').$$

Further, if three points are equally spaced along a line, they are harmonic with the ideal point of that line. Hence

$$\mathrm{H}\,(\mathrm{F\,F}, \mathrm{V\,V}'')$$
$$\mathrm{H}\,(D\,\mathrm{D}, \mathrm{M\,V}'')$$
$$\mathrm{H}\,(F'\,\mathrm{F}', \mathrm{V}'\,\mathrm{V}'')$$
$$\mathrm{H}\,(D'\,\mathrm{D}', \mathrm{M}'\,\mathrm{V}'').$$

All of the harmonic relations of the two-point diagram exist in the three-point perspective diagram, without change, except that V″ is not an ideal point, but is a nearby vanishing point instead, as in Figure 9.[8]

Figure 13 shows a further application of the use of harmonic points for avoiding the use of distant vanishing points in perspective[4] for

$$\mathrm{H}\,(\mathrm{V'M}, \mathrm{HV})$$
$$\mathrm{H}\,(\mathrm{V'D}, \mathrm{JF}).$$

Because of the conception of ideal points and lines in Perspective Geometry, it is possible to classify conics (or geometric curves) on the basis of the relation of a conic to the ideal line, or line at infinity. For example, an ellipse and an ideal line being drawn in a plane, if the line does not intersect the ellipse, the conic is defined as an ellipse. If, on the other hand, the ideal line is tangent to the ellipse, then one point of the ellipse is an ideal point, and the curve is a parabola. Also, if the ideal line intersects the ellipse in two

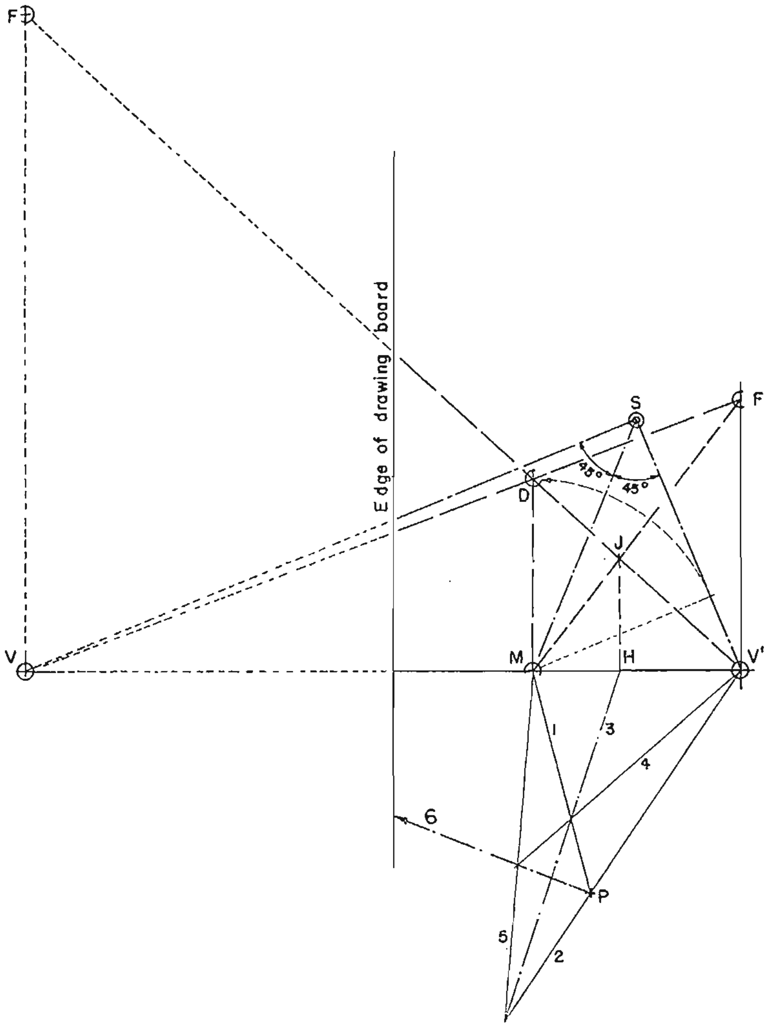

FIGURE 13

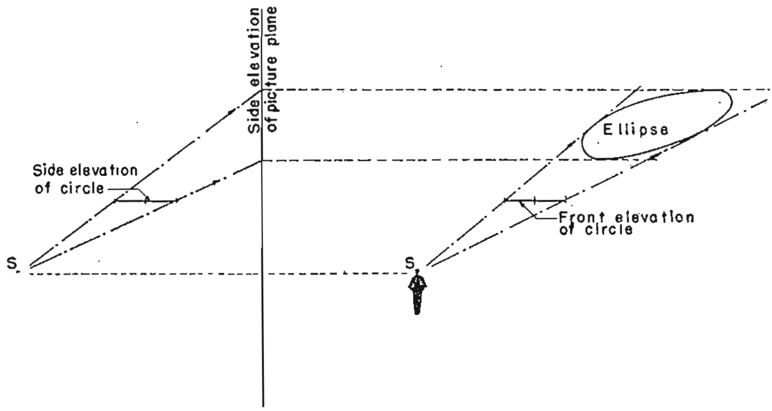

FIGURE 14

points, these two intersections are ideal points and the conic is called a hyperbola.

While the above may seem a bit far-fetched to the student of Euclidean geometry, its direct application to Perspective Drawing is easily understood. Suppose we sketch in three stages the perspective of a circle which is in a horizontal plane:

(1) The circle is between the Picture Plane and the Station Point (Figure 14). The perspective is obviously an ellipse.

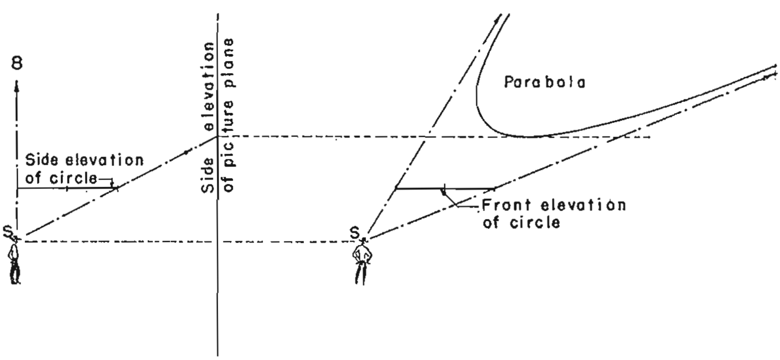

FIGURE 15

(2) The circle is so located that one point thereof is the same distance from the Picture Plane as the Station Point (Figure 15). Then one visual ray is parallel to the picture plane and the perspective of that point is infinitely distant (an ideal point), and we have a parabola.

(3) The circle is so placed that part of it is on both sides of the station point with relation to the picture plane. Then

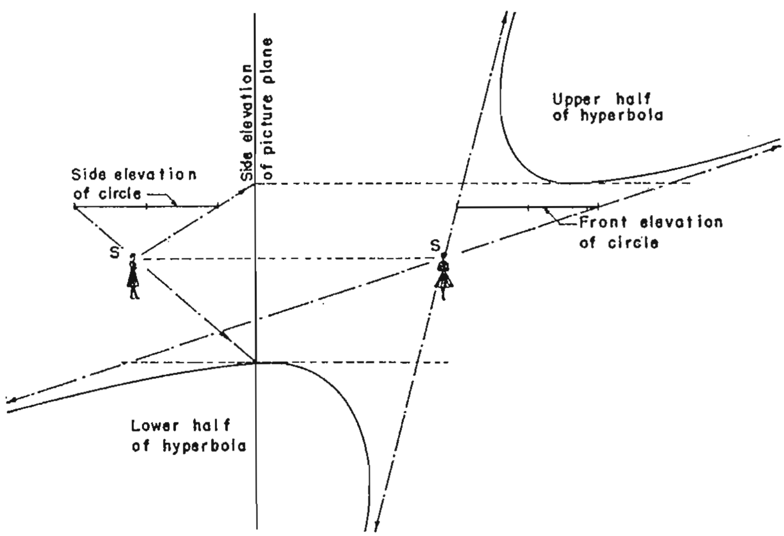

FIGURE 16

two visual rays are parallel to the picture plane and the perspective of two points thereof are infinitely distant, and are ideal points (Figure 16). The perspective is a hyperbola.

The fact that we can not actually see these curves does not prevent their accurate construction by perspective methods.

The rare student of both fields will find it interesting to develop the studies of the perspectives of circles more completely, noting the ease through which poles, polars, asymptotes, and conjugate axes may be found by perspective.

The perspective draftsman often finds it desirable to use additional vanishing points of horizontal lines other than those to V and V′. Such a pair could immediately be utilized

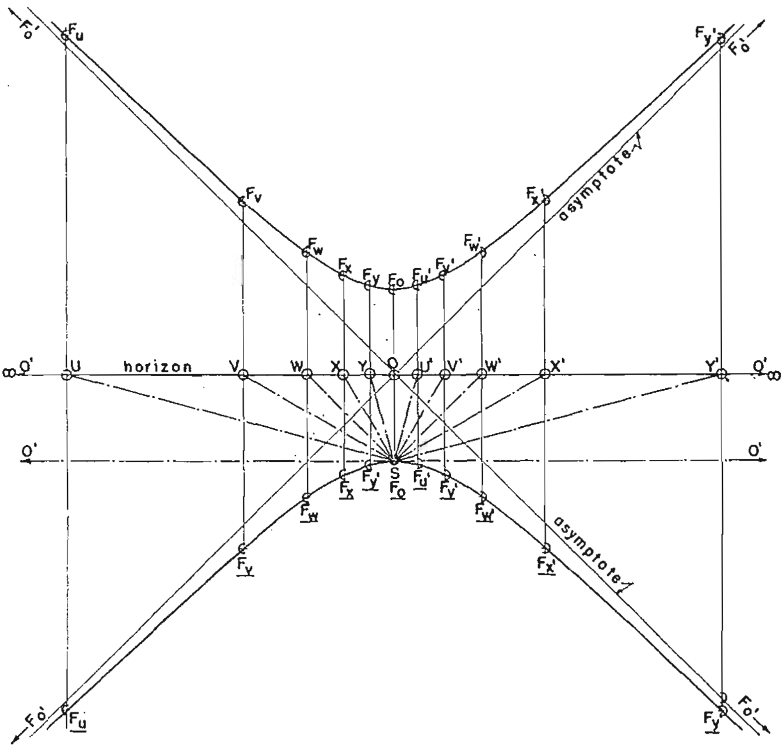

FIGURE 17

in O, and O′, the latter being the ideal point of the horizon. All lines vanishing at O′ are horizontal in perspective. Such a procedure is particularly useful for drawing the perspective of large circles where many radii are required for accuracy. Such a diagram is called a multiple diagram and is shown in part in Figure 17.

It is an interesting fact in the general laying out of the two-point perspective diagram, that the F's, and *F*'s lie on the branches of an equilateral hyperbola whose conjugate axis is the horizon, whose tranverse axis is twice the station point distance and whose asymptotes are 45°-lines through O, the center of vision (Figure 17).

In the foregoing pages and figures, it has been attempted to indicate some of the points of contact between Projective Geometry and practical Perspective Drawing. The uses of the complete quadrilateral have been emphasized a good deal, but only insofar as the practical need of emphasis in that direction has warranted. The link of the perspectives of circles also was developed only so far as the need of the practical problem made necessary. It is undoubtedly a fact that a great amount of the theory of Projective Geometry might be developed quite fully by an acceptance not only of the practical side of Perspective Drawings, but also of the purely mathematical theory of the subject. However, such a task steps far beyond the limits of this paper, and must remain as a problem for future thought and study.

JAMES C. MOREHEAD, JR.

NOTES

1. J. W. Young, *Projective Geometry* (Chicago, 1930), Chap. I, p. 1.
2. *Ibid.*, p. 2.
3. For a more elaborate and complete treatment of harmonic parts in perspective, see J. C. Morehead, *Perspective Drawing*, rev. ed. (Houston, 1952).
4. *Ibid.*, p. 68.