I

FUNCTIONS OF COMPOSITION*

FIRST LECTURE

INTRODUCTION — COMPOSITION, PERMUTABILITY, INTEGRAL POWERS OF COMPOSITION, THE CLOSED-CYCLE GROUP — OBJECT OF THE LECTURES — FRACTIONAL POWERS OF COMPOSITION, INCOMMENSURABLE POWERS, FRACTIONAL AND INCOMMENSURABLE ORDERS OF FUNCTIONS OF COMPOSITION OF A GROUP

1. *Introduction*

1. We call to mind the solutions of the simplest integral equations. In this connection two functions $f(x)$ and $F(x, y)$, which are limited and continuous, are supposed to be given, and we wish to determine $\phi(x)$ so as to satisfy an equation

(1) $$f(x) = \phi(x) + \int_0^x \phi(\xi)\, F(\xi, x)\, d\xi.$$

The solution is given by the formula

(2) $$\phi(x) = f(x) + \int_0^x f(\xi)\, S(\xi, x)\, d\xi,$$

where

(3) $$S(x, y) = -F(x, y) + F_2(x, y) - F_3(x, y) + \dots$$

with

(4) $$F_2(x, y) = \int_x^y F(x, \xi)\, F(\xi, y)\, d\xi$$

(5) $$F_3(x, y) = \int_x^y F_2(x, \xi)\, F(\xi, y)\, d\xi.$$

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* Three lectures delivered at the Rice Institute in the autumn of 1919 by Senator Vito Volterra, Professor of Mathematical Physics and Celestial Mechanics, and Dean of the Faculty of Sciences of the University of Rome.

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The series (3) is uniformly convergent, and defines a function $S(x, y)$ which may be regarded as the first example of a *function of composition* of $F(x, y)$. It is obtained by operations to be performed on $F(x, y)$ and changes when that function changes. And it can therefore be regarded as entering into the class spoken of as *functions depending on other functions*, or *functions of curves*. The equations (4), (5), ... give us moreover the first examples of the operation of *composition of functions*, and of *powers of composition.*

Formula (5) may also be written in the form

$$(5') \qquad F_3(x, y) = \int_x^y F(x, \xi)\, F_2(\xi, y)\, d\xi,$$

and by a comparison of the formulae (5) and (5′) the following equation

$$\int_x^y F_2(x, \xi)\, F(\xi, y)\, d\xi = \int_x^y F(x, \xi)\, F_2(\xi, y)\, d\xi$$

is obtained. Thus we have a first example of *permutable functions.*

Upon these elementary considerations is founded the theory of composition of functions and the permutability of functions. We pass then to give the corresponding general definitions and fundamental properties.

2. *Composition — Permutability — Integral Powers of Composition — Group of the Closed Cycle*

2. The composition of two integrable functions $f(x, y)$, $\phi(x, y)$ is the operation

$$\int_x^y f(x, \xi)\, \phi(\xi, y)\, d\xi.$$

It is understood that these functions remain in the field of real variables, and it will be assumed that $y > x$. If the result of the operation is $\psi(x, y)$, the relation will be written in the form

$$\psi = \overset{*}{f}\,\overset{*}{\phi}.$$

If f and ϕ are equal, we may write

$$\overset{*}{f^2} = \overset{*}{f}\,\overset{*}{f}$$

and also

$$\overset{*}{f^3} = \overset{*}{f^2}\,\overset{*}{f} = \overset{*}{f}\,\overset{*}{f^2}.$$

$$\cdots\cdots\cdots\cdots$$

In general, when m and n are integers

$$\overset{*}{f^{m+n}} = \overset{*}{f^m}\,\overset{*}{f^n},$$

$\overset{*}{f^m}$ being spoken of as the *integral power of composition of degree m.* If a, b, c ... are constants, the quantities

$$af,\ b\phi,\ c\psi,\ \ldots$$

are the products of constants into the functions f, ϕ, ψ, ... and the equation

$$(a\overset{*}{f})\,(b\overset{*}{\phi})\,(c\overset{*}{\psi})\ldots = a\,b\,c\ldots\overset{*}{f}\,\overset{*}{\phi}\,\overset{*}{\psi}\ldots$$

is satisfied.

3. The operation of composition is *associative,* and if the functions happen to be permutable, also *commutative:* it is always *distributive.**

4. Given the series

$$a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

which is supposed to be convergent for $|z| < R$, the series

$$a_1\overset{*}{f} + a_2\overset{*}{f^2} + a_3\overset{*}{f^3} + \ldots$$

is uniformly convergent whatever may be the modulus of the function f, this function being limited; the function defined by the series is permutable with f. The theorem may be extended to power series in more than one variable.†

* V. Volterra, "Leçons sur les fonctions de lignes," Paris, Gauthier-Villars (1913) Chap. IX, §§ 1–5.

† Ibid., Chap. IX, § 10.

5. If m is an integer and the relation

$$\psi(x, y) = (y-x)^m f(x, y)$$

is valid, $f(x, y)$ being limited and continuous, and $f(x, x)$ always different from zero, the function $\psi(x, y)$ is said to be of *order* $m+1$.

The resultant of composition of two functions of order m and n respectively is of order $m+n$, and the power of composition of degree m of a function of order n is of order mn.

6. Knowing a function ϕ of order 1 permutable with ψ of order m, it is possible to calculate a function θ of the first order whose mth power of composition is ψ, provided that ψ and ϕ have limited derivatives up to and including the mth order. In this case then we write:

$$\theta = \overset{*}{\psi}^{\frac{1}{m}}.*$$

7. Let $\alpha(x)$ and $\beta(x)$ be two functions, limited and continuous, which do not vanish, and write

$$\frac{dx}{\alpha(x)\,\beta(x)} = dx_1,$$

from which x_1 and x are determined as functions of each other:

$$x_1 = \lambda(x), \; x = \mu(x_1).$$

Form then the function $\alpha(x)\;\beta(y)\;f(x, y)$ and write it as a function of x_1, y_1, that is,

$$f_1(x_1, y_1) = \alpha(x)\;\beta(y)\;f(x, y).$$

If now we write, by means of the change of variable given above,

$$\frac{1}{\alpha(x)} = \alpha_1(x_1), \;\; \frac{1}{\beta(x)} = \beta_1(x_1),$$

we shall have the equations

$$\frac{dx_1}{\alpha_1(x_1)\,\beta_1(x_1)} = dx,$$

$$f(x, y) = \alpha_1(x_1)\;\beta_1(y_1)\;f_1(x_1, y_1).$$

* V. Volterra, loc. cit., Chap. XI, § 8.

If again we apply the same transformation to $\phi(x, y)$, and obtain thereby $\phi_1(x_1, y_1)$; and if we let $\xi = \mu(\xi_1)$, we shall have

$$\alpha(x)\,\beta(y)\int_x^y f(x, \xi)\,\phi(\xi, y)\,d\xi$$
$$= \int_{x_1}^{y_1} \alpha(x)\,\beta(y) f(x, \xi)\,\phi(\xi, y)\,\alpha(\xi)\,\beta(\xi)\,d\xi_1$$
$$= \int_{x_1}^{y_1} f_1(x_1, \xi_1)\,\phi_1(\xi_1, y_1)\,d\xi_1.$$

From this equation we deduce that *the resultant of the composition of two transformed functions is the transform of the resultant of the two functions themselves*, and hence that *a power of composition of a transformed function is the transform of the power of composition of the function itself*, and finally, that *the transformation does not alter the property of permutability*, that is to say, *it transforms a group of permutable functions into a new group of permutable functions.*

8. Given a function $F(x, y)$ of order 1 all the functions which are permutable with it can be found. For this purpose the question may first be reduced to the case in which

$$\text{(1)} \qquad F(x, x) = 1, \quad \left(\frac{\partial F}{\partial x}\right)_{x=y} = \left(\frac{\partial F}{\partial y}\right)_{x=y} = 0.$$

In fact, if $F(x, y)$ does not happen to satisfy these conditions, it may be reduced to one that does by means of a transformation of the type just considered.* We shall say that a function F which satisfies (1) is reduced to *canonical form.* On the assumption that the function F is limited and continuous with its derivatives of the first two orders, the solution of the problem is then given by the formula

$$\text{(2)} \qquad \lambda(y-x) + \int_0^{y-x} \lambda(\xi)\,\Phi(\xi \mid x, y)\,d\xi,$$

* V. Volterra, loc. cit., Chap. XI, §§ 1, 2.

in which λ is an arbitrary function, and Φ can be calculated from F and its derivatives of the first two orders.*

9. Another fundamental property of permutability is expressed in the following theorem: *Two functions permutable with a third are permutable with each other.* We omit the proof of this theorem, referring merely to the paper of Professor Vessiot.†

10. A group of permutable functions is characterized by a function of the first order of which the first and second partial derivatives exist and are finite. Consequently when we consider a group of permutable functions, *we shall always assume that there is known to us a function of the first order which has finite derivatives of the first and second orders and belongs to the group.* This function shall be spoken of as the *fundamental function of the group.* When a fundamental function of the group has the canonical form, we shall speak of the group as a *canonical group.*

11. A remarkable group of permutable functions is the so-called *closed-cycle group*,‡ which is made up of functions of the form

$$f(y-x).$$

Unity belongs to this group, and it is deduced immediately that

$$\overset{*}{1}{}^m = \frac{1}{(m-1)!}(y-x)^{m-1}.$$

3. *Plan of the Lectures*

12. On the basis of these general ideas it is the plan of the following lectures to develop a complete theory of permutable

* V. Volterra, loc. cit., Chap. XI, p. 162.

† Vessiot, "Sur les fonctions permutables et les groupes continus de transformations fonctionelles lineaires," Comptes Rendus, 1912, p. 682.

‡ V. Volterra, loc. cit., Chap. VII.

functions and their properties, analogous to the usual algebra and analysis.

In the first place, we observe that the operation of composition of permutable functions is analogous to multiplication, in common with which it possesses the commutative, distributive and associative properties. The algebra of permutable functions has already been studied by Professor Evans.

Now if we follow the historic development of the usual analytic theories, we see first unfolded the theory of *integral powers*, then *fractional* and *negative* powers. Afterwards comes the theory of *logarithms*, which barely precedes the *infinitesimal calculus.* In fact the very definition of logarithm as given by Napier involves implicitly the idea of *derivative.* And finally comes the *general theory of functions*, which crowns the whole structure. We observe that at first the name *function* was applied to powers, and then gradually extended its significance to cover the modern interpretation.

We shall follow the same road in the theory of functions of composition, and since we have already discussed the integral powers, we shall proceed first to treat the *fractional*, then the *negative powers of composition* and then the *logarithms of composition.* This leads us to the *differential and integral calculus of composition*, of which we shall give the foundations and the elementary applications to the logarithms of composition. And we shall develop in its principal lines the *theory of functions of composition.*

In this way it will appear clearly that the logical process which serves as a guide in our path is the one that reproduces the evolution of ordinary analysis in its development from the finite to the infinite.

4. *Fractional Powers of Composition — Incommensurable Powers — Fractional and Incommensurable Orders of Functions of a Group.*

13. If ϕ is of the first order and we propose to ourselves the problem of finding a function f which will satisfy the equation

$$\overset{*}{f^n} = \phi, \tag{1}$$

we cannot find a solution in terms of a function which remains finite. The problem however can be solved by means of a function which becomes infinite but remains integrable.

To be convinced at once of this possibility it is sufficient to call to mind the first result which was known about integral equations, namely the solution of the integral equation of Abel:

$$f(x) = \int_0^x \phi(\xi) \frac{1}{\sqrt{x-\xi}} d\xi,$$

which is

$$\phi(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x f(\xi) \frac{1}{\sqrt{x-\xi}} d\xi,$$

since

$$\int_x^y \frac{1}{\sqrt{y-\xi}} \frac{1}{\sqrt{\xi-x}} d\xi = \pi.$$

If we write down the function

$$F(x, y) = \frac{1}{\sqrt{y-x}},$$

we shall evidently have

$$\overset{*}{F^2} = \pi$$

and thus we see that the square of the function $1/\sqrt{y-x}$, which becomes infinite for $x = y$, is nevertheless a constant.

We now proceed to show that if $\psi_1(x, y)$ is a function of the first order, and if

$$\theta(x, y) = \frac{\psi_1(x, y)}{(y-x)^{\frac{n-1}{n}}},$$

then $\overset{*}{\theta}{}^n$ is of the first order.

In fact, we shall have

$$\overset{*}{\theta}{}^2 = \frac{\psi_2(x, y)}{(y-x)^{\frac{n-2}{n}}},$$

where

$$\psi_2(x, y) = \int_0^1 \frac{\psi_1(x, x+(y-x)\eta)\, \psi_2(x+(y+x)\eta, y)}{\eta^{\frac{n-1}{n}}\, (1-\eta)^{\frac{n-1}{n}}}\, d\eta,$$

and consequently ψ_2 remains finite like ψ_1 and is continuous, and of the first order. Similarly it is evident that

$$\overset{*}{\theta}{}^3 = \frac{\psi_3(x, y)}{(y-x)^{\frac{n-3}{n}}},$$

where $\psi\ (x, y)$ is a function of the first order, and so on; whence it follows that the function

$$\overset{*}{\theta}{}^n = \psi_n(x, y)$$

is a function of the first order.

It is evident that if ψ_1 possesses finite and continuous derivatives up to a certain order, the same is true for $\psi_2, \psi_3, \ldots \psi_n$.

14. Let us assume that the function θ is permutable with ϕ, and that ϕ and ψ possess finite and continuous first derivatives. In this case it is possible to calculate in a simple manner the function f which satisfies (1).

In the first place we may point out that ψ_n must be permutable with ϕ, for we have the equation

$$\overset{*}{\theta}{}^n \overset{*}{\phi} = \overset{*}{\theta}{}^{n-1} \overset{*}{\phi} \overset{*}{\theta} = \overset{*}{\theta}{}^{n-2} \overset{*}{\phi} \overset{*}{\theta}{}^2 = \ldots = \overset{*}{\phi} \overset{*}{\theta}{}^n.$$

It follows that

$$\phi(x, x) = C\, \psi_n(x, x),$$

C being a constant,* and hence, since ϕ and ψ_n possess finite and continuous first derivatives, that the function

$$\phi(x, y) - C\psi_n(x, y)$$

approaches zero as x approaches y to the same order as $y-x$ or higher order.

The function g which satisfies the integral equation

$$\phi(x, y) = C\psi_n(x, y) = \int_x^y C\psi_n(x, \xi)\, g(\xi, y)\, d\xi \tag{2}$$

is finite and continuous, since $\psi_n(x, x) = 0$. We can then write explicitly

$$f(x, y) = \sqrt[n]{C}\left\{\theta + \frac{1}{n}\overset{*}{\theta}\overset{*}{g} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1.2}\overset{*}{\theta}\overset{*}{g^2} + \dots\right\}. \tag{3}$$

We see immediately that we have

$$f(x, y) = \frac{G(x, y)}{(y-x)^{\frac{n-1}{n}}},$$

where $g(x, y)$ is of the first order and such that

$$G(x, x) = \frac{\sqrt[n]{\phi(x, x)}}{\Gamma\left(\frac{1}{n}\right)}.$$

And since we can take any one of n values for the nth root of C we can obtain by means of the procedure (3) n solutions.

15. Let ϕ be reduced to the canonical form F. Then θ may be obtained from (2) Chap. II by writing

$$\lambda = \eta^{\frac{1}{n} - 1}. \tag{4}$$

In fact we shall have

$$\theta = \frac{1 + (y-x)\int_0^1 u^{\frac{1}{n} - 1}\, \Phi((y-x)\, u \,|\, x, y)\, du}{(y-x)^{\frac{n-1}{n}}},$$

* V. Volterra, loc. cit., Chap. XI, p. 3.

in which the numerator is of the first order. It will be differentiable if Φ is differentiable, and thus will have determinate derivatives provided that F has such, up to and including the third order. In this case we shall have

$$C=\Gamma^{-n}\left(\frac{1}{n}\right).$$

If instead of (4) we write the equation

$$\lambda=\eta^{\frac{1}{n}-1}\,\mu(\eta),$$

where $\mu(\eta)$ is an analytic function which does not vanish for $\eta=0$, we shall obtain another formula for θ which may be used in the formulae (5) and (3); and thus a θ may be determined in an infinite variety of ways.

It may be asked if in this way we obtain always merely the same solutions upon substitution in (3). At present we content ourselves with the observation that all such solutions are permutable among themselves.

16. The formulae which we have given lead us necessarily to extend the notion of order.

If $\phi(x, y)$ is of the first order, and if

$$f(x, y)=(y-x)^{\alpha}\,\phi(x, y),$$

the function f will be said to be of determinate order $\alpha+1$. Thus the functions θ and f of the preceding sections are of order $1/n$. In the above definition, the function $\phi(x, y)$ is said to be the *characteristic* of $f(x, y)$, and $\phi(x, x)$ its *diagonal.*

If we have a function

$$f(x, y)=(y-x)^{\alpha}\,\phi(x, y),$$

in which $\phi(x, y)$ is finite and continuous, and further, $\phi(x, x)=0$, we say that f is of *order higher than* $\alpha+1$. In this way we may obtain functions however which are of no determinate order whatever; for example the function

$$(y-x)^{n-1}\log(y-x)\,\phi(x, y),$$

where $\phi(x, y)$ is of the first order, is of order higher than

$n-\epsilon$ where ϵ is arbitrarily small, and yet has no determinate order.

If $\psi(x, y)$ and $\phi(x, y)$ do not have determinate orders, but the function

$$\frac{\psi(x, y)}{(y-x)^{\alpha}\phi(x, y)}$$

is always less than some determinate number, with α positive, it will be said that $\psi(x, y)$ with respect to $\phi(x, y)$ is of order not less than α. The operations of composition will be applicable to functions whose order is greater than a positive number, and we shall consider such functions.

In order to obtain a function of determinate order η belonging to a group of permutable functions, it is sufficient to substitute in (2), § 8:

$$\lambda(\eta)=\eta^{r-1}\mu(\eta),$$

where μ is bounded and does not have 0 as a limiting value as η approaches 0.

If two functions are of determinate orders α and β their resultant is of order $\alpha+\beta$. In fact if ϕ_1 and ϕ_2 are of the first order, we may write

$$f_1(y-x)^{\alpha-1}\phi_1(x, y),$$
$$f_2(y-x)^{\beta-1}\phi_2(x, y),$$

$$\overset{*}{f}_1\overset{*}{f}_2=(y-x)^{\alpha+\beta-1}\times$$
$$\int_0^1\eta^{\alpha-1}(1-\eta)^{\beta-1}\phi_1(x, x+(y-x)\eta)\,\phi_2(x+(y-x)\eta, y)\,d\eta,$$

whence if we substitute

$$\psi(x, y)$$
$$=\int_0^1\eta^{\alpha-1}(1-\eta)^{\beta-1}\phi_1(x, x+(y-x)\eta)\,\phi_2(x+(y-x)\eta, y)\,d\eta,$$

we shall have

$$\overset{*}{f}_1\overset{*}{f}_2=(y-x)^{\alpha+\beta-1}\psi(x, y),$$

in which

$$\psi(x, x) = \phi_1(x, x)\, \phi_2(x, x)\, \frac{\Gamma(\alpha)\, \Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

so that ψ will be of the first order. By the same procedure, it may be shown that *if one of the functions is of order higher than α, and if the other is of order β or higher, then the resultant is of order higher than $\alpha+\beta$*

The function ψ may be differentiated to whatever order ϕ_1 and ϕ_2 are both differentiable. It may further be noted that the theorem does not demand that f_1 and f_2 be permutable.

If a function is of order r, its nth power will be of order nr; and if we denote the respective characteristics of the function and its power by $G(x, y)$ and $L(x, y)$ we shall have

$$L(x, x) = \frac{\Gamma^n(r)}{\Gamma(n\, r)}\, G^n(x, x).$$

If the function were of order higher than r its nth power would be of order higher than nr.

17. If

$$\phi(x, y) = f(x, y)\, (y-x)^{\alpha-1},$$

with $|f(x, y)| < M$ and $\alpha > 0$, we obtain in the case that m is an integer, the inequality

$$|\overset{*}{\phi}{}^m| < M^m \frac{\Gamma^m(\alpha)}{\Gamma(m\alpha)}\, (y-x)^{m\alpha-1}.$$

Hence *if the series*

$$\sum_1^\infty a_m z^m$$

is convergent for values of z of absolute value less than a certain quantity, the series

$$\sum_1^\infty a_m \overset{*}{\phi}{}^m$$

is convergent whatever may be the modulus of the function $f(x, y)$ provided that it is finite (compare § 4).

18. Given f of order $n+\alpha$ with $0<\alpha<1$, and n a positive integer, and given ψ of order $n+\alpha+\beta$, or higher order, with $\beta>0$, we proceed to treat the problem of calculating ϕ so that the equation

(5) $$\overset{*}{f}\overset{*}{\phi}=\psi$$

will be satisfied. This problem can be solved by a method analogous to that which I gave in my "Leçons sur les équations intégrales et intégro-différentiélles", Chap. II, 3, p. 60.*

In fact, if we write

$$(y-x)^{-\alpha}=\theta(x, y),$$

we shall have

$$\overset{*}{\theta}\overset{*}{f}\overset{*}{\phi}=\psi.$$

Now θ is of order $1-\alpha$, and so $\overset{*}{\theta}\overset{*}{f}=g$ will be of order $n+1$, and $K=\overset{*}{\theta}\overset{*}{\psi}$ will be of order $n+1+\beta$ or of higher order.

The equation

$$\overset{*}{g}\overset{*}{\phi}=K$$

is solved at once by differentiating it $n+1$ times with respect to x, and thus reducing it to an equation of the second kind. Evidently ϕ will result of order β or less than β. In order to apply the method it is necessary to admit when f and ψ are of determinate orders, that their characteristics should have finite derivatives of the $n+1$st order.

It is not necessary that the given functions f and ϕ should be permutable. If, however, they are permutable it may be deduced that ϕ will be permutable with them. If they are not permutable, the equation

(5′) $$\overset{*}{\phi}\overset{*}{f}=\psi$$

is distinct from (5), and can be solved by forming the integral equation

$$\overset{*}{\phi}\overset{*}{f}\overset{*}{\theta}=\overset{*}{\psi}\overset{*}{\theta},$$

* Paris, Gauthier-Villars, 1913.

in which $\overset{*}{f}\overset{*}{\theta}$ will be of order $n+1$. By means of $n+1$ differentiations it could be reduced to an equation of the second kind.

The equations (5) and (5′) each admit a single solution.

Consider the equation

$$\overset{*}{f_1}\overset{*}{\phi_1}=\psi_1, \tag{6}$$

where

$$f_1=f+\overset{*}{f}\overset{*}{x},\quad \psi_1=\psi+\overset{*}{\psi}\overset{*}{\rho},$$

and assume that f and ψ have the same properties as before, while x and ρ are functions of higher order than some positive number.

The solution ϕ_1 of (6) in which f_1 and ψ_1 are supposed given, can be solved by first solving (5), and then taking

$$\phi_1=\phi+\overset{*}{\phi}\overset{*}{\rho}-(\overset{*}{\phi}+\overset{*}{\phi}\overset{*}{\rho})\overset{*}{\chi}+(\overset{*}{\phi}+\overset{*}{\phi}\overset{*}{\rho})\overset{*}{\chi}^2-\ldots,$$

which series will always be uniformly convergent. *In this case also there will be a unique solution.*

The functions f_1 and ψ_1 are respectively of the same orders as the functions f and ψ, but it is not necessary that they satisfy the conditions imposed on f and ψ with respect to the differentiability of their characteristics.

19. Suppose that we are given a function

$$\phi(x,y)=(y-x)^{\alpha-1}\psi(x,y)$$

of determinate order α, and that we wish to calculate the function f, such that

$$\overset{*}{f^n}=\phi.$$

By virtue of the preceding considerations it is possible to extend to this case the procedure followed (§ 14) in solving the analogous equation (1) in which ϕ is of the first order.

In fact, if we suppose that the group to which ϕ belongs is first reduced to the canonical form and if we calculate the function

$$\theta = \frac{1 + (y-x)\int_0^1 u^{\frac{\alpha}{n}-1}\,\Phi[(y-x)\,u\,|\,x,y]\,du}{(y-x)^{1-\frac{\alpha}{n}}},$$

$\overset{*}{\theta}{}^n$ will be of order α and its diagonal will be

$$\frac{\Gamma^n\left(\frac{\alpha}{n}\right)}{\Gamma(\alpha)}.$$

Now if we solve the equation

$$\phi - \frac{\Gamma(\alpha)}{\Gamma^n\left(\frac{\alpha}{n}\right)}\overset{*}{\theta}{}^n = \frac{\Gamma(\alpha)}{\Gamma^n\left(\frac{\alpha}{n}\right)}\overset{*}{\theta}{}^n\overset{*}{g},$$

regarding g as unknown and assuming the existence of the derivatives of ϕ and Φ, of the orders demanded by the preceding theorems, f will be given by the formula·

$$f = \frac{\sqrt[n]{\Gamma(\alpha)}}{\Gamma\left(\frac{\alpha}{n}\right)}\left(\theta + \frac{1}{n}\overset{*}{\theta}\overset{*}{g} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\,.\,2}\overset{*}{\theta}\overset{*}{g}{}^2 + \dots\right).$$

We shall thus obtain n solutions, since $\sqrt[n]{\Gamma(\alpha)}$ contains an nth root of unity as an indeterminate factor.

These solutions are all permutable with each other and with ϕ. We have to determine, as in § 15, whether it is possible to find other solutions permutable with these.

20. If f_1 and f_2 are two permutable functions of determinate orders and if the characteristic of each possesses a finite and determinate derivative of an order equal to the integer next larger than the order of the respective function, and if

$$\overset{*}{f_1}{}^n = \overset{*}{f_2}{}^n,$$

then we shall have

$$f_1 = \epsilon f_2,$$

where ϵ is an nth root of unity. In fact f_1 and f_2 will be necessarily of the same order, and if we represent their characteristics by ϕ_1 and ϕ_2, we shall have necessarily

$$\phi_1^n(x, x) = \phi_2^n(x, x),$$

and therefore

$$\phi_1(x, x) = \epsilon\, \phi_2(x, x).$$

But

$$O = \overset{*}{f_1}{}^n - \overset{*}{f_2}{}^n = (\overset{*}{f_1} - \epsilon_1 \overset{*}{f_2})\ (\overset{*}{f_1} - \epsilon_2 \overset{*}{f_2}) \ldots (\overset{*}{f_1} - \epsilon_n \overset{*}{f_2}),$$

where $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ are the nth roots of unity.

If we assume $\epsilon = \epsilon_1$, the binomial expressions $f_1 - \epsilon_2 f_2$, $f_1 - \epsilon_n f_2$ will be of the same order as f_1 and f_2 and consequently, by the results obtained in § 18, it follows that

$$f_1 = \epsilon\, f_2.$$

21. The question which we raised previously (§§ 15, 19) is now answered in view of this proposition, that is, by changing λ (η) in the manner indicated we obtain always the same solution of (1) since the results are always functions of determinate order $\frac{1}{n}$ whose characteristics possess derivatives and whose nth powers of composition are equal to each other. The same is true of the solutions of (8).

22. If f_1 and f_2 are two permutable functions of determinate order and

$$\overset{*}{f_1}{}^n = \overset{*}{f_2}{}^m,$$

it follows that

$$\overset{*}{f_1}{}^{nq} = \overset{*}{f_2}{}^{mq},$$

q being any integer whatever. Conversely, if the last equation is satisfied, the equation

$$\overset{*}{f_1}{}^n = \epsilon \overset{*}{f_2}{}^m$$

will be true, ϵ being one of the qth roots of unity. We shall write

$$f_1 = \overset{*}{f}{}^{\frac{m}{n}},$$

and obviously, in writing this equation we shall include in the symbol f_1 an undetermined root of unity.

Given f_2, in order to calculate f_1 it will be sufficient to calculate first, by the rules given in §§ 14, 15 and 19 the function

$$\overset{*}{f_2}{}^{1/n},$$

from which we obtain

$$(\overset{*}{f_2}{}^{1/n})^m.$$

The whole of the ordinary algebra of fractional powers can be applied without change to fractional powers of composition.

23. In the expression

$$\overset{*}{f}{}^{\frac{m}{n}},$$

if we suppose that f is of a determinate order α, *i.e.*,

$$f = (y-x)^{\alpha-1} G(x, y),$$

it then follows that $\overset{*}{f}{}^{\frac{m}{n}}$ will be of order $\frac{\alpha m}{n}$, or

$$\overset{*}{f}{}^{\frac{m}{n}} = (y-x)^{\frac{\alpha m}{n}-1} L(x, y),$$

and

$$L(x, x) = [G(x, x)]^{\frac{m}{n}} \frac{\Gamma^{\frac{m}{n}}(\alpha)}{\Gamma\left(\frac{m\alpha}{n}\right)}.$$

The fractional power $\overset{*}{f}{}^{\frac{m}{n}}$ is determined to within a factor equal to a root of unity. We shall be able to do away with this indeterminateness when the diagonals are all positive.

If f is a function of determinate order α whose diagonal is positive

$$\overset{*}{f}{}^{\frac{m}{n}} = (y-x)^{\frac{m\alpha}{n}-1} L\left(x, y \middle| \frac{m}{n}\right),$$

and the diagonal of this function is also positive.

Let us suppose that as we make the number $\frac{m}{n}$ approach a positive rational number β, $L\left(x, y \middle| \frac{m}{n}\right)$ tends uniformly toward $L(x, y \mid \beta)$; and that as $\frac{m}{n}$ approaches any irrational number $z > 0$ $L\left(x, y \middle| \frac{m}{n}\right)$ tends uniformly toward a determinate finite limit $L(x, y \mid z)$.

We shall write

$$\overset{*}{f^z} = (y - x)^{\alpha z - 1} L(x, y \mid z),$$

and refer to this function as an *irrational power of composition of order z.*

We shall have

$$L(x, x \mid z) = G(x, x)^z \frac{\Gamma^z(\alpha)}{\Gamma(\alpha z)}.$$

If $\mid G(x, y) \mid < M$, we shall have

$$\mid L(x, y \mid z) \mid < M^z \frac{\Gamma^z(\alpha)}{\Gamma(z \alpha)}.$$

All of the algebraic calculus of powers with commensurable or incommensurable positive exponents is extensible to the case of powers of composition, and consequently

$$\overset{*}{f^z} \overset{*}{f^{z_1}} = \overset{*}{f^{z+z_1}},$$
$$(\overset{*}{f^z})^{z_1} = \overset{*}{f^{z z_1}},$$

the numbers z and z_1 being any positive numbers whatever.*

24. When we know the function

$$\overset{*}{f^z} = (y - x)^{\alpha z - 1} L(x, y \mid z),$$

for any positive value of z whatever we are in a position to calculate the function $\overset{*}{\phi^z}$, where ϕ is given by the equation

$$\phi = f + \overset{*}{f} \overset{*}{\psi},$$

*The actual calculation of $\overset{*}{f^z}$ cannot be carried out without recourse to the theory of *logarithms of composition*. It is done in Lecture III, §25. We may add at the outset the following statement of a fundamental property: $\overset{*}{f^z}$ is an entire function of z.

and where ψ is any function of an order greater than a certain positive number.

We shall have in fact

$$\overset{*}{\phi}{}^z = \overset{*}{f}{}^z + z\overset{*}{f}{}^z\overset{*}{\psi} + \frac{z(z-1)}{1\,.\,2}\overset{*}{f}{}^z\overset{*}{\psi}{}^2 + \ldots,$$

and the series is always uniformly convergent (§ 17).

It is seen immediately that from the fact that $\overset{*}{f}{}^z$ is an analytic function of z it follows that $\overset{*}{\phi}{}^z$ is also analytic, and that since $\overset{*}{f}{}^z$ is an entire function so also is $\overset{*}{\phi}{}^z$.

25. As an example let us treat the case of functions which belong to the closed cycle group.

Unity belongs to the closed-cycle group and if z is positive we have

$$\overset{*}{1}{}^z = \frac{(y-x)^{z-1}}{\Gamma(z)},$$

and therefore $\overset{*}{1}{}^z$ *is an entire function of* z.

Now let $\phi\,(y-x)$ be a function of the first order possessing a derivative. If $\phi\,(0)=1$, then

$$\phi\,(y-x) = 1 + \overset{*}{1}\,\overset{*}{\phi}{}'$$

where $\overset{*}{\phi}{}'$ denotes the derivative of ϕ. Consequently

$$\overset{*}{\phi}{}^z = \overset{*}{1}{}^z + z\,\overset{*}{1}{}^z\overset{*}{\phi}{}' + \frac{z(z-1)}{1.2}\overset{*}{1}{}^z\overset{*}{\phi}{}'^2 + \cdots$$

and therefore $\overset{*}{\phi}{}^z$, thus obtained, is an entire function of z.

VITO VOLTERRA.

SECOND LECTURE

Introduction — Zero and Negative Powers of Composition — Fractions of Composition — Progressions of Composition — Logarithms of Composition — Naperian Logarithms of Composition, Extension of Logarithms of Composition.

1. *Introduction*

1. If we have the relation

$$\overset{*}{f}\,\overset{*}{\phi}=\psi$$

(in which we suppose f, ϕ, ψ to lie in the field of permutable functions) and if we consider the operation of composition as analogous to multiplication, we can write, *by analogy*,

$$(1)\quad \phi=\frac{\overset{*}{\psi}}{\overset{*}{f}},\qquad (1')\quad f=\frac{\overset{*}{\psi}}{\overset{*}{\phi}},$$

and also

$$(2)\quad \phi=\overset{*}{\psi}\,\overset{*}{f}^{-1},\qquad (2')\quad f=\overset{*}{\psi}\,\overset{*}{\phi}^{-1},$$

and we can regard the symbols (1), (2), (1′), 2′) as *representative* of the operations whereby we solve the integral equation

$$\psi=\int_x^y f(x,\xi)\,\phi(\xi,y)\,d\xi,$$

in which we are to regard ϕ and f successively as the unknown function.

We observe that if f is of order m and ϕ of order n, ψ will be of order $m+n$, m and n being positive numbers. Hence $m<m+n>n$.

If then we write the symbol

$$\frac{\overset{*}{\Phi}}{\overset{*}{F}} \text{ or } \overset{*}{\Phi}\overset{*}{F}^{-1},$$

where Φ and F are permutable functions, it will have no meaning if the order of Φ is less or equal to the order of F.

A great difficulty arises if we wish to give a meaning, in general, to the symbol in question.

But it is to be remembered that an analogous difficulty arises in the elements of arithmetic if we restrict ourselves to the field of integers. If we write

$$2 \times 3 = 6,$$

we can represent division of 6 by 3 or by 2, by the symbols

$$2 = \frac{6}{3} = 6 \times 3^{-1}, \qquad 3 = \frac{6}{2} = 6 \times 2^{-1}.$$

But until we leave the field of integers the symbols

$$\frac{3}{5}, \quad \frac{1}{4}$$

have no meaning.

In arithmetic we can introduce the number $\frac{1}{2} = 2^{-1}$ by defining multiplication of an even number by $\frac{1}{2}$ as equivalent to dividing it by 2. Similarly we could define the symbol f^{-1} by the property that

$$\overset{*}{F}\overset{*}{f}^{-1} = \phi,$$

provided that

$$\overset{*}{F} = \overset{*}{\phi}\overset{*}{f}.$$

But just as, by this procedure, we should obtain in arithmetic only the reciprocals of the integers, so, in the field of composition we should obtain only special functions of composition. Consequently, in order to obtain readily more general functions, we follow another course, the principle of which we will now explain.

2. In arithmetic we can arrive at the fractional numbers by an extension of the number field, introducing them, after the integers, as new quantities for which we define equivalence and all the operations which can be performed on them in combination with each other and with integers. Unless we depart from the field of integers these quantities have only formal significance. But all the calculations and all the propositions in which they are involved cease to be purely formal, whenever we desire, provided that we multiply by a suitable integer. They then represent actual relations between integers.

We shall follow precisely this method in order to introduce fractions of composition, and they will be formal in character, but the remark which we have just made applies to them, namely, that they can be combined by composition with a convenient function in such a way that the results cease to be formal and represent actual relations between functions.

2. *Zero and Negative Powers of Composition—Functions of Composition*

3. First we must introduce the element which corresponds to unity in arithmetic but which, in the field of composition, we do not yet possess in a perfectly clear and simple manner. Let us return, therefore, to the simpler integral equation considered in the first lecture and let us write it in the form

$$f(x, y) = \phi(x, y) + \int_x^y \phi(x, \xi)\, F(\xi, y)\, d\xi,$$

in which the given functions are assumed to be permutable. In the notation of composition we can write

$$f = \phi + \overset{*}{\phi}\, \overset{*}{F},$$

and the solution

$$\phi = f - \overset{*}{f}\,\overset{*}{F} + \overset{*}{f}\,\overset{*}{F}{}^2 - \overset{*}{f}\,\overset{*}{F}{}^3 + \ldots$$

In the lectures which I gave at the Sorbonne ("Fonctions de lignes," Chap. IX) these formulae were written

$$f = \overset{*}{\phi}\,(1 + \overset{*}{F})$$

and

$$\phi = \overset{*}{f}(1 - \overset{*}{F} + \overset{*}{F}{}^2 - \overset{*}{F}{}^3 + \ldots)$$

$$= \frac{\overset{*}{f}}{1 + \overset{*}{F}}.$$

In other words we had, by definition,

$$\overset{*}{\phi}(1 + \overset{*}{F}) = \phi + \overset{*}{\phi}\,\overset{*}{F}.$$

Unity, in this case, functioned in such a way that when combined by composition with ϕ it gave ϕ as a result.

On the other hand, if we combine ϕ with 1, giving to 1 its ordinary meaning, we have

$$\overset{*}{\phi}\,\overset{*}{1} = \int_x^y \phi(x, \xi)\, d\xi,$$

which is different from ϕ. Therefore unity sometimes means the element which composed with a given function reproduces that function and sometimes it has its ordinary significance. To avoid confusion we agree to state explicitly on each occasion which meaning we wish to attribute to unity. In order to remove all uncertainty we will use two different symbols for the two meanings.*

4. Let $f(x, y)$ be a function belonging to a group of permutable functions. We know what is meant by composition of a function of the group with f.† By composition with $\overset{*}{f}{}^{-1}$ we mean performing the inverse operation, that is, finding a function which, operated upon by f, will re-

* Evans has removed this ambiguity by another method.

† Cf. Lecture 1, § 2.

produce the given function. If then we compose the first function with f and then with $\overset{*}{f}^{-1}$, this is equivalent to leaving the function unaltered. Then

$$\overset{*}{f}\overset{*}{f}^{-1}=\overset{*}{f}^{\circ}$$

will be a new entity which we shall introduce into the group, defining it as that element which *composed with any other function of the group* leaves it unaltered. It is this element which corresponds to unity.

The properties of $\overset{*}{f}^{\circ}$ are given by

$$\overset{*}{f}\overset{*}{f}^{\circ}=f,\ (\overset{*}{f}^{\circ})^{m}=\overset{*}{f}^{\circ},\ (\overset{*}{f}^{\circ})^{-1}=\overset{*}{f}^{\circ},\ \overset{*}{f}^{\circ}=\overset{*}{\phi}^{\circ},$$

f and ϕ being functions which belong to the group. And if a is a constant

$$(a\,\overset{*}{f}^{\circ})\ (\overset{*}{f})=a\,f.$$

$$(a\,\overset{*}{f}^{\circ}+b\,\overset{*}{f})\ (c\,\overset{*}{f}^{\circ}+d\,f)=a\,c\,\overset{*}{f}^{\circ}+a\,d\,\phi+b\,c\,f+b\,d\,\overset{*}{f}\overset{*}{\phi}.$$

Hence it turns out that $a\,\overset{*}{f}^{\circ}+b\,\overset{*}{f}$ has only a formal meaning by itself but acquires an actual meaning provided that it is combined with any function of the group.

The introduction of the element f° greatly simplifies the formulae which I have given in previously published works on the theory of permutable functions (cf. loc. cit., p. 138). In addition let us consider, for example, the series

$$F\,|\,[\overset{*}{f}]\,|=\overset{*}{f}^{\circ}+f+\frac{\overset{*}{f}^{2}}{2!}+\frac{\overset{*}{f}^{3}}{3!}+\ldots,$$

which satisfies the addition theorem

$$\mathrm{F}\ |\ [\overset{*}{f}+\overset{*}{\phi}]\ |=\overset{*}{F}\ |\ [\overset{*}{f}]\ |\ \overset{*}{F}\ |\ [\overset{*}{\phi}]\ |,$$

a form of statement which is much simpler than that given on page 159 of the work cited.

Besides this we can show more clearly the period of $F \mid [\overset{*}{f}] \mid$ since we have the relation

$$F \mid [\overset{*}{f} + 2\pi i \overset{*}{f}{}^{\circ}] \mid = F \mid [\overset{*}{f}] \mid,$$

that is to say $F \mid [\overset{*}{f}] \mid$ has the period $2\pi i \overset{*}{f}{}^{\circ}$.

5. We will proceed now to the study of fractions of composition in the strict sense of the term. Let us consider a set of permutable functions of determinate orders. We will denote these functions by $f, \phi, \psi, \ldots; f_1, \phi_1, \psi_1, \ldots f_2, \phi_2, \psi_2, \ldots$ and suppose that linear combinations of them are likewise of determinate orders and also that if we take any one of them which is of higher order than a second it is always possible to find one and only one function of the group which, when composed with the second, will give the first.

For example, a set of functions of this nature would be that which could be generated by taking a function of the first order, forming its integral and fractional powers, forming products of composition of these powers and adding together constant multiples of these results.

We shall say that $\frac{\overset{*}{f}}{\overset{*}{\phi}}$ *is the fraction of composition belonging to the group and having f for its numerator and ϕ for its denominator.*

We shall say that

$$\frac{\overset{*}{f}}{\overset{*}{f}{}^{\circ}} = f,$$

and that

$$\frac{\overset{*}{f_1}}{\overset{*}{\phi_1}} = \frac{\overset{*}{f_2}}{\overset{*}{\phi_2}},$$

whenever

$$\overset{*}{f_1}\overset{*}{\phi_2} = \overset{*}{f_2}\overset{*}{\phi_1}.$$

From this we can infer the proposition that two fractions of composition which are equal to a third are equal to each other.

In fact, if

$$\frac{\overset{*}{f}_1}{\overset{*}{\phi}_1} = \frac{\overset{*}{f}_2}{\overset{*}{\phi}_2}, \quad \frac{\overset{*}{f}_1}{\overset{*}{\phi}_1} = \frac{\overset{*}{f}_3}{\overset{*}{\phi}_3},$$

it follows that

$$(1) \quad \overset{*}{f}_1, \overset{*}{\phi}_2 = \overset{*}{f}_2 \overset{*}{\phi}_1, \qquad (2) \quad \overset{*}{f}_1 \overset{*}{\phi}_3 = \overset{*}{f}_3 \overset{*}{\phi}_1,$$

and therefore, composing both members of (1) with ϕ_3,

$$\overset{*}{f}_1 \overset{*}{\phi}_2 \overset{*}{\phi}_3 = \overset{*}{f}_2 \overset{*}{\phi}_1 \overset{*}{\phi}_3 = \overset{*}{f}_2 \overset{*}{\phi}_3 \overset{*}{\phi}_1.$$

But by equation (2)

$$\overset{*}{f}_1 \overset{*}{\phi}_2 \overset{*}{\phi}_3 = \overset{*}{\phi}_2 \overset{*}{f}_1 \overset{*}{\phi}_3 = \overset{*}{\phi}_2 \overset{*}{f}_3 \overset{*}{\phi}_1,$$

therefore

$$\overset{*}{f}_2 \overset{*}{\phi}_3 \overset{*}{\phi}_1 = \overset{*}{\phi}_2 \overset{*}{f}_3 \overset{*}{\phi}_1,$$

and from this it follows, by the general hypothesis that we made previously,

$$\overset{*}{\phi}_2 \overset{*}{f}_3 = \overset{*}{f}_2 \overset{*}{\phi}_3,$$

that is to say,

$$\frac{\overset{*}{f}_2}{\overset{*}{\phi}_2} = \frac{\overset{*}{f}_3}{\overset{*}{\phi}_3}.$$

From the definitions which we have given it is easily seen that $\frac{\overset{*}{f}}{\overset{*}{\phi}} = \frac{\overset{*}{f}\overset{*}{\psi}}{\overset{*}{\phi}\overset{*}{\psi}}$.

6. We have now to distinguish between three cases:

(i) In the expression $\frac{\overset{*}{f}}{\overset{*}{\phi}}$ suppose that f is of higher order than ϕ. Then, assuming that the conditions which we have stated (§ 5) are satisfied, we can calculate a function ψ such that

$$f = \overset{*}{\phi} \overset{*}{\psi},$$

and we thus have

$$\frac{\overset{*}{f}}{\overset{*}{\phi}}=\frac{\overset{*}{\psi}}{\overset{*}{\phi}{}^{\circ}}=\psi.$$

(ii) Again, if f is of lower order than ϕ, then (supposing always that the aforesaid conditions are satisfied)

$$\phi=\overset{*}{f}\,\overset{*}{\psi}$$

and therefore

$$\overset{*}{\phi}\,\overset{*}{\psi}{}^{\circ}=\overset{*}{f}\,\overset{*}{\psi},$$

whence it follows that

$$\frac{\overset{*}{f}}{\overset{*}{\phi}}=\frac{\overset{*}{\psi}{}^{\circ}}{\overset{*}{\psi}}.$$

(iii) Finally suppose that f and ϕ are of the same order. The ratio of their characteristics will be constant. If we denote this constant by a, the function

$$\psi=f-a\,\phi$$

will have a definite order greater than that of ϕ. Then we can write

$$\psi=\overset{*}{\phi}\,\overset{*}{\theta},$$

and therefore

$$f=a\,\phi+\overset{*}{\phi}\,\overset{*}{\theta}.$$

Consequently

$$\overset{*}{f}\,\overset{*}{\phi}{}^{\circ}=(a\,\overset{*}{\phi}{}^{\circ}+\theta)\,\overset{*}{\phi},$$

and finally

$$\frac{\overset{*}{f}}{\overset{*}{\phi}}=a\,\overset{*}{\phi}{}^{\circ}+\theta.$$

7. *Two or more fractions of composition can always be reduced to a common denominator which is a function whose order is not less than the order of any of the denominators.*

In fact, if we are given

$$\frac{\overset{*}{f_1}}{\overset{*}{\phi_1}},\quad\frac{\overset{*}{f_2}}{\overset{*}{\phi_2}},$$

and if ϕ is of higher order than ϕ_1 and ϕ_2 then

$$\phi = \overset{*}{\phi}_1 \overset{*}{\psi}_1 = \overset{*}{\phi}_2 \overset{*}{\psi}_2,$$

and therefore

$$\frac{\overset{*}{f}_1}{\overset{*}{\phi}_1} = \frac{\overset{*}{f}_1 \overset{*}{\psi}_1}{\overset{*}{\phi}}, \quad \frac{\overset{*}{f}_2}{\overset{*}{\phi}_2} = \frac{\overset{*}{f}_2 \overset{*}{\psi}_2}{\phi}$$

If the order of ϕ were equal to that of one or both of the given denominators, we should have

$$\phi = \overset{*}{\phi}_1 \,(a\, \overset{*}{\phi}{}^{\circ} + \psi_1)$$

$$\phi = \overset{*}{\phi}_2 \,(b\, \overset{*}{\phi}{}^{\circ} + \psi_2)$$

where a and b are constants, one of which might be zero. Therefore

$$\frac{\overset{*}{f}_1}{\overset{*}{\phi}_1} = \frac{a \overset{*}{f}_1 + \overset{*}{f}_1 \overset{*}{\psi}_1}{\overset{*}{\phi}},$$

$$\frac{\overset{*}{f}_2}{\overset{*}{\phi}_2} = \frac{b \overset{*}{f}_2 + \overset{*}{f}_2 \overset{*}{\psi}_2}{\overset{*}{\phi}}.$$

A method of reducing several fractions of composition

$$\frac{\overset{*}{f}_1}{\overset{*}{\phi}_1}, \quad \frac{\overset{*}{f}_2}{\overset{*}{\phi}_2}, \quad \frac{\overset{*}{f}_3}{\overset{*}{\phi}_3}, \ \ldots,$$

to the same denominator is to write the equivalent fractions

$$\frac{\overset{*}{f}_1 \overset{*}{\phi}_2 \overset{*}{\phi}_3 \ldots}{\overset{*}{\phi}_1 \overset{*}{\phi}_2 \overset{*}{\phi}_3 \ldots}, \quad \frac{\overset{*}{f}_2 \overset{*}{\phi}_1 \overset{*}{\phi}_3 \ldots}{\overset{*}{\phi}_2 \overset{*}{\phi}_1 \overset{*}{\phi}_3 \ldots}, \quad \frac{\overset{*}{f}_3 \overset{*}{\phi}_1 \overset{*}{\phi}_2 \ldots}{\overset{*}{\phi}_3 \overset{*}{\phi}_1 \overset{*}{\phi}_2 \ldots}, \quad \ldots.$$

8. If we reduce several fractions of composition to a common denominator and form a fraction of composition which has this denominator and whose numerator is obtained from the various numerators by the operations of addition or subtraction, the fraction obtained is independent of the choice of the denominator, *according to the definition of equivalence which has been given.*

In fact, if we have

$$\frac{\overset{*}{f}_1}{\overset{*}{\phi}_1} = \frac{\overset{*}{f}_2}{\overset{*}{\phi}_2}, \quad \frac{\overset{*}{\psi}_1}{\overset{*}{\phi}_1} = \frac{\overset{*}{\psi}_2}{\overset{*}{\phi}_2},$$

then

$$\frac{\overset{*}{f}_1 \pm \overset{*}{\psi}_1}{\overset{*}{\phi}_1} = \frac{\overset{*}{f}_2 \pm \overset{*}{\psi}_2}{\overset{*}{\phi}_2},$$

since

$$\overset{*}{f}_1 \overset{*}{\phi}_2 \pm \overset{*}{\psi}_1 \overset{*}{\phi}_2 = \overset{*}{f}_2 \overset{*}{\phi}_1 \pm \overset{*}{\psi}_2 \overset{*}{\phi}_1.$$

The operation here indicated is called the addition or subtraction of fractions of composition.

Thus it is seen that all the rules of arithmetic relative to the addition or subtraction of fractions are extensible to fractions of composition.

9. *The multiplication of a fraction of composition by a constant consists in multiplying the numerator by the constant, leaving the denominator unaltered.*

The composition of several fractions of composition signifies the formation of a fraction of composition which has for its numerator the resultant of their numerators and for its denominator the resultant of their denominators.

The associative and commutative properties hold in the case of composition of fractions, and it is seen that the results remain equivalent if equivalent fractions are substituted for the fractions composed.

A function f is equivalent to the fraction

$$\frac{\overset{*}{f}}{\overset{*}{f^\circ}},$$

hence

$$\frac{\overset{*}{f}\overset{*}{\psi}}{\overset{*}{\phi}} = \frac{\overset{*}{f}}{\overset{*}{f^\circ}} \cdot \frac{\overset{*}{\psi}}{\overset{*}{\phi}} = \frac{\overset{*}{f}\overset{*}{\psi}}{\overset{*}{f^\circ}\overset{*}{\phi}} = \frac{\overset{*}{f}\overset{*}{\psi}}{\overset{*}{\phi}}. \tag{3}$$

We obtain in this way the composition of a function with a fraction of composition and we see immediately that for this kind of composition also the commutative properly holds.

10. If we compose m equivalent fractions $\frac{\overset{*}{f}}{\overset{*}{\phi}}$ the result is written in the form

$$\left(\frac{\overset{*}{f}}{\overset{*}{\phi}}\right)^m \text{ and we have evidently } \left(\frac{\overset{*}{f}}{\overset{*}{\phi}}\right)^m = \frac{\overset{*}{f}^m}{\overset{*}{\phi}^m}.$$

The same formula will be extended, by definition, to the case in which m is equal to a fraction or to an incommensurable number.

11. From (3) we obtain

$$\overset{*}{f}\frac{\overset{*}{\psi}}{\overset{*}{f}} = \frac{\overset{*}{f}\overset{*}{\psi}}{\overset{*}{f}} = \frac{\overset{*}{\psi}}{\overset{*}{f}^{0}} = \psi.$$

Therefore, by composing a fraction of composition with its denominator we obtain the numerator, or, *every fraction of composition may be regarded as the result of the operation inverse to composition applied to the numerator by means of the denominator*, and if we adopt the exponent -1 to indicate, as we have done previously, the inverse operation, we have

$$\frac{\overset{*}{\psi}}{\overset{*}{f}} = \overset{*}{\psi}\overset{*}{f}^{-1},$$

and, by making an extension of the property that composition by $\overset{*}{f}^{0}$ does not alter the element upon which it operates, we can write

$$\frac{\overset{*}{f}^{0}}{\overset{*}{f}} = \overset{*}{f}^{-1}.$$

Now (see § 8)

$$(\overset{*}{f}^{-1})^m = \left(\frac{\overset{*}{f}^{0}}{\overset{*}{f}}\right)^m = \frac{\overset{*}{f}^{0}}{\overset{*}{f}^{m}} = (\overset{*}{f}^{m})^{-1},$$

which we can also write in the form

$$\overset{*}{f}^{-m}.$$

In general $(\overset{*}{f}{}^{h})^m = \overset{*}{f}{}^{hm}$ whether h, or m, is positive, negative or zero.

12. We wish now to find the fraction of composition which, when composed with $\dfrac{\overset{*}{\psi}}{\overset{*}{\theta}}$, will give $\dfrac{\overset{*}{f}}{\overset{*}{\phi}}$. We have evidently as a solution

$$\frac{\overset{*}{f}\overset{*}{\theta}}{\overset{*}{\phi}\overset{*}{\psi}},$$

or

$$\left(\frac{\overset{*}{f}}{\overset{*}{\phi}}\right)\left(\frac{\overset{*}{\psi}}{\overset{*}{\theta}}\right)^{-1} = \frac{\overset{*}{f}}{\overset{*}{\phi}}\cdot\frac{\overset{*}{\theta}}{\overset{*}{\psi}},$$

and also

$$\frac{\left(\dfrac{\overset{*}{f}}{\overset{*}{\phi}}\right)}{\left(\dfrac{\overset{*}{\psi}}{\overset{*}{\theta}}\right)} = \frac{\overset{*}{f}\overset{*}{\theta}}{\overset{*}{\phi}\overset{*}{\psi}}.$$

13. These results are summarized in the statement that the arithmetic theory of fractions can be carried over to the field of composition.

The elements

$$\frac{\overset{*}{f}}{\overset{*}{\psi}},\quad \overset{*}{f}{}^{-1},\quad \overset{*}{f}{}^{\circ},$$

can be included in the field of a group of permutable functions. *They no longer have the significance of functions in the ordinary sense, but all the operations together with their associative, commutative and distributive properties can be extended to these elements.* However for this reason they can be called *functions belonging to the given group of permutable functions*, and we are able to extend to these elements also the concept of order, that is to say, if $\overset{*}{f}$ is of order m, and

$$\overset{*}{\phi}\overset{*}{f}$$

is of order $n<m$ we shall say that ϕ has the negative order

$$n-m.$$

Evidently if ϕ is of positive order p, $\overset{*}{\phi}{}^{-1}$ will be of order $-p$ and the theorem, that the order of the resultant of two functions of given orders is equal to the sum of the orders of the components, is extended to the case of negative orders We also extend easily the concept of an order greater than a given negative order to the case in which the order is not determined, that is, if $\overset{*}{\phi}\,\overset{*}{f}$ is not of determined order but is of an order greater than $n<m$ we shall say that ϕ is of higher order than $n-m$.

14. As we have said (§ 1), it might seem as if we have constructed in this way a purely formal theory; but this is not the case in view of the fact that the elements which have been introduced, formal though they are, cease to be such by acquiring the significance of ordinary functions whenever they are composed with a function of sufficiently high order. Thus, for example, if we have the sum

$$\overset{*}{f}+\overset{*}{f}{}^{-m}+\frac{\overset{*}{f}}{\overset{*}{\phi}}+\frac{\overset{*}{\psi}{}^{h}}{\overset{*}{\theta}{}^{h}},$$

it is sufficient to compose it with $\overset{*}{f}{}^{m}\,\overset{*}{\phi}\,\overset{*}{\theta}{}^{h}$ in order to convert it into an ordinary function.

3. *Progressions of Composition — Logarithms of Composition*

15. Let $\phi(x, y)$ be a function of determinate finite order and let us consider the sequence

$$\ldots\ldots\overset{*}{\phi}{}^{-3},\ \overset{*}{\phi}{}^{-2},\ \overset{*}{\phi}{}^{-1},\ \overset{*}{\phi}{}^{\circ},\ \phi,\ \overset{*}{\phi}{}^{2},\ \overset{*}{\phi}{}^{3},\ \ldots\ldots$$

We shall say that this constitutes a *progression of composition* having the ratio ϕ.

The exponents will be called the *logarithms of composition* of the various powers of composition and ϕ will be called the *base*. We shall write

$$n = \overset{*}{\log}_\phi \overset{*}{\phi}^n,$$

n being positive or negative.

The whole of the arithmetic theory of logarithms is evidently extensible to the logarithms of composition now introduced. Thus *the logarithm of the resultant of several functions is the sum of the logarithms of the component functions, etc.*

The progression of composition possesses properties analogous to those of geometrical progressions. In particular

$$\overset{*}{\phi}^\circ + \phi + \overset{*}{\phi}^2 + \ldots + \overset{*}{\phi}^n = \frac{\overset{*}{\phi}^\circ - \overset{*}{\phi}^{n+1}}{\overset{*}{\phi}^\circ - \overset{*}{\phi}}.$$

16. By inserting means we can pass to fractional logarithms. Let us insert, by calculating $\overset{*}{\phi}^{1/m}$, m means between two elements of the progression; we obtain the sequence

$$\ldots \overset{*}{\phi}^{-1} \ldots \overset{*}{\phi}^{-\frac{2}{m}}, \overset{*}{\phi}^{-\frac{1}{m}}, \overset{*}{\phi}^\circ, \overset{*}{\phi}^{\frac{1}{m}}, \overset{*}{\phi}^{\frac{2}{m}}, \ldots \phi, \overset{*}{\phi}^{\frac{m+1}{m}}, \ldots, \overset{*}{\phi}^2, \ldots,$$

which can be regarded as a new progression of composition having $\overset{*}{\phi}^{1/m}$ as its ratio, and we can write

$$\frac{h}{m} = \overset{*}{\log}_\phi \overset{*}{\phi}^{\frac{h}{m}},$$

h and m being whole numbers.

In general, if ν is a positive or negative number, rational or irrational, we can write

$$(1) \qquad \nu = \overset{*}{\log}_\phi \overset{*}{\phi}^\nu,$$

and to these logarithms of composition we can extend the properties of arithmetic logarithms.

17. It will be convenient to write powers of composition in the form

$$\overset{*}{\phi}^{\nu} = \phi^{\nu \overset{*}{\phi}{}^{\circ}},$$

and therefore to write instead of (1)

$$\nu \overset{*}{\phi}{}^{\circ} = \overset{*}{\log}_{\phi} \overset{*}{\phi}^{\nu \overset{*}{\phi}{}^{\circ}} = \overset{*}{\log}_{\phi} \overset{*}{\phi}^{\nu}.$$

In this way, starting with a base ϕ and considering all of its real powers, we can express the logarithms as real numbers multiplied by $\overset{*}{\phi}{}^{\circ}$; but unless we take a further step in the theory it will not be possible to obtain

$$\overset{*}{\log}_{\phi} \psi,$$

except when ψ, besides being permutable with ϕ, is a power of ϕ. This further step cannot be taken until we have introduced the *fundamental concept of the naperian base.* The complete construction of the theory is thus less easy than it may have seemed at first sight. It is necessary to introduce into the field of composition the fruitful concept, discovered three hundred years ago by Lord Napier, of the base of natural logarithms, in order to extend the theory within the limits in which it can be formulated effectually.

4. *Naperian Logarithms of Composition — Extension of the Theory of Logarithms of Composition*

18. Consider the function

$$e \overset{*}{f}{}^{\circ} = \overset{*}{e}.$$

We have

$$\overset{*}{e}^{z} = e^{z} \overset{*}{f}{}^{\circ},$$

and therefore

$$\frac{d}{dz} \overset{*}{e}^{z} = e^{z} \overset{*}{f}{}^{\circ} = \overset{*}{e}^{z}.$$

On account of this property we shall call $\overset{*}{e}$ the *base of naperian logarithms of composition.*

We have evidently

$$\overset{*}{e}{}^{z}=\overset{*}{f}{}^{\circ}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\right),$$

and, by virtue of the convention made in the preceding section,

$$\overset{*}{e}{}^{z}=\overset{*}{e}{}^{z\overset{*}{f}{}^{\circ}}=(z\overset{*}{f}{}^{\circ})+z\overset{*}{f}{}^{\circ}+\frac{(z\overset{*}{f}_{0})^2}{2!}+\frac{(z\overset{*}{f}{}^{\circ})^3}{3!}+\ldots$$

19. Analogously, *by definition*, we can write

$$(1)\qquad \overset{*}{e}{}^{\Phi}=\overset{*}{\Phi}{}^{\circ}+\Phi+\frac{\overset{*}{\Phi}{}^2}{2!}+\frac{\overset{*}{\Phi}{}^3}{3!}+\ldots=\Psi,$$

and we shall write

$$\Phi=\overset{*}{\log}_{\overset{*}{e}}\Psi$$

or, more simply

$$\Phi=\overset{*}{l}\,\Psi.$$

With the introduction of this new concept we shall be able to solve the problem proposed in § 17.

20. For this purpose we begin by solving the problem: given Ψ to determine Φ satisfying the preceding equation.

We shall have

$$\overset{*}{e}{}^{z\Phi}=\overset{*}{\Psi}{}^{z},$$

whence, differentiating with regard to z,

$$\frac{d\,\overset{*}{\Psi}{}^{z}}{dz}=\overset{*}{e}{}^{z\Phi}\,\overset{*}{\Phi}=\overset{*}{\Psi}{}^{z}\,\overset{*}{\Phi},$$

and therefore

$$(2)\qquad \overset{*}{\Psi}{}^{-z}\,\frac{d\,\overset{*}{\Psi}{}^{z}}{dz}=\Phi=\overset{*}{l}\,\Psi.$$

We have arrived therefore at the very simple formula (2) for finding $\overset{*}{l}\,\Psi$ when $\overset{*}{\Psi}{}^{z}$ is known. However, as we have said before, the validity of the formula is subject to the knowledge *a priori* that $\overset{*}{l}\,\Psi$ exists.

21. But let us suppose now that we are given any function ψ whatever of the group, and let us assume, for simplicity, that it has a determinate order α.

We suppose also (see Lecture I, § 23) that, having calculated $\overset{*}{\Psi}{}^z$, the result is expressed by

(3) $$(y-x)^{\alpha z-1}\, G(x, y \mid z),$$

where G is an analytic function of z.*

Let us calculate

$$\frac{d\overset{*}{\psi}{}^z}{dz}.$$

We obtain

$$\alpha(y-x)^{\alpha z-1}\, G(x, y \mid z) \log (y-x) + (y-x)^{\alpha z-1}\, G'(x, y \mid z),$$

where G' denotes the derivative of G with respect to z.†

It follows that

$$\frac{d\overset{*}{\psi}{}^z}{dz}$$

is not of determinate order; we therefore consider the fraction of composition

(2′) $$\overset{*}{\psi}{}^{-z}\frac{d\overset{*}{\psi}{}^z}{dz} = \Theta.$$

It is easy to show that Θ is independent of z. In fact, we have

$$\overset{*}{\psi}{}^{z+\epsilon} - \overset{*}{\psi}{}^z = \overset{*}{\psi}{}^z(\overset{*}{\psi}{}^\epsilon - \overset{*}{\psi}{}^\circ).$$

Now $\overset{*}{\psi}{}^\epsilon - \overset{*}{\psi}{}^\circ$ is independent of z; our statement is therefore proved.

From (2′) it follows that

$$\frac{d\overset{*}{\psi}{}^z}{dz} = \overset{*}{\psi}{}^z\, \Theta,$$

*This is always the case in consequence of a general theorem to which we have referred previously in Lecture I, § 23. See Lecture III, § 25.

† To avoid misunderstanding we recall that by $\log A = lA$ we mean the naperian logarithm of the number A.

therefore

$$\frac{d^2 \overset{*}{\psi}{}^z}{d z^2} = \overset{*}{\psi}{}^z \Theta^2,$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{d^h \overset{*}{\psi}{}^z}{d z^h} = \overset{*}{\psi}{}^z \Theta^h.$$

But the series

$$\overset{*}{\psi}{}^z + \frac{d \overset{*}{\psi}{}^z}{d z} + \frac{1}{2!} \frac{d^2 \overset{*}{\psi}{}^z}{d z^2} + \dots + \frac{1}{h!} \frac{d^h \overset{*}{\psi}{}^z}{d z^h} + \dots,$$

if convergent in the unit circle with center at z, has for its sum

$$\overset{*}{\psi}{}^{z+1},$$

hence

$$\overset{*}{\psi}{}^{z+1} = \overset{*}{\psi}{}^z \left(\overset{*}{\Theta}{}^\circ + \Theta + \frac{\overset{*}{\Theta}{}^2}{2!} + \frac{\overset{*}{\Theta}{}^3}{3!} + \dots \right),$$

and consequently

$$\psi = \overset{*}{\Theta}{}^\circ + \Theta + \frac{\overset{*}{\Theta}{}^2}{2!} + \frac{\overset{*}{\Theta}{}^3}{3!} + \dots,$$

from which it follows that

$$\Theta = \overset{*}{l} \psi.$$

In order, therefore, to conclude that $\overset{*}{l} \psi$ exists and can be obtained from the formula (2′) it is sufficient to know that $\overset{*}{\Psi}{}^z$ can be put in the form (3) and that this expression represents an entire function of z.

22. As an example we will calculate

$$\overset{*}{l}\, 1,$$

recalling the fact that unity belongs to the closed-cycle group (cf. Lecture I, § 11).

We found (Lecture I, § 25)

$$\overset{*}{1}{}^z = \frac{1}{\Gamma(z)} (y - x)^{z-1}.$$

But it is well known that $\frac{1}{\Gamma(z)}$ is an entire function; hence we can apply the foregoing procedure and we shall have

$$(4)\quad \frac{d\overset{*}{1}^z}{dz} = \frac{1}{\Gamma(z)}(y-x)^{z-1}\log(y-x) - \frac{1}{\Gamma^2(z)}(y-x)^{z-1}\Gamma'(z) = \Theta(x, y\,|\,z),$$

and

$$(5)\quad \overset{*}{l}1 = \overset{*}{\Theta}\,\overset{*}{1}^{-z}.$$

By the preceding theorem the right-hand member of (5) is independent of z. We therefore put $z=1$; and obtain

$$\Theta = \log(y-x) - \Gamma'(1).$$

But

$$C = -\Gamma'(1) = -\int_0^\infty e^{-x}\log x\,dx = 0.57721\ldots$$

(Euler's constant),

therefore

$$(6)\quad \overset{*}{l}1 = \frac{\overset{*}{\overbrace{\log(y-x)+C}}}{\overset{*}{1}}.$$

23. The general theorem of § 21 enables us to see that the expression (5) must be independent of z. But *a priori* the fact cannot be inferred immediately on seeing the expression (5) for the first time. For the sake of the interesting and curious nature of the result we will verify it directly. The expression cannot be put in an analytical form; in fact, as we have noted, it has a purely symbolic meaning; we can however apply the general principle (cf. § 14) by means of which any expression of this kind loses its purely symbolic meaning and becomes an ordinary function. Although

$$\overset{*}{\Theta}\,\overset{*}{1}^{-z}$$

has a purely symbolic meaning,

$$(\overset{*}{\Theta}\,\overset{*}{1}^{-z})\overset{*}{1}$$

is an ordinary function. It will be sufficient then to show directly that this ordinary function is independent of z.

Now
$$(\overset{*}{\Theta}\,\overset{*}{1}^{-z})\overset{*}{1} = \overset{*}{\Theta}\,\overset{*}{1}^{1-z}.$$

But
$$\Theta = \frac{1}{\Gamma(z)}(y-x)^{z-1}\log(y-x) - \frac{\Gamma'(z)}{\Gamma(z)}\overset{*}{1}^{z};$$

and therefore
$$\overset{*}{\Theta}\,\overset{*}{1}^{1-z} = \frac{1}{\Gamma(z)\,\Gamma(1-z)}\int_x^y \frac{\log(\xi-x)\,d\xi}{(\xi-x)^{1-z}\,(y-\xi)^z} - \frac{\Gamma'(z)}{\Gamma(z)}.$$

We have
$$\int_x^y \frac{\log(\xi-x)\,d\xi}{(z-x)^{\beta}\,(y-\xi)} = -\frac{\partial}{\partial\beta}\int_x^y \frac{d\xi}{(y-\xi)^{\alpha}\,(\xi-x)^{\beta}}$$
$$= -\frac{\partial}{\partial\beta}\left[\frac{1}{(y-x)^{\alpha+\beta-1}}\frac{\Gamma(\alpha)\,\Gamma(1-\beta)}{\Gamma(1-\alpha|-\beta)}\right].$$

This expression can be calculated without difficulty. Thus, putting $\alpha = z$, $\beta = 1-z$, we obtain
$$\int_x^y \frac{\log(\xi-x)\,d\xi}{(y-\xi)^z\,(\xi-x)^{1-z}}$$
$$= \Gamma(z)\,\Gamma(1-z)\left\{\log(y-x) + \frac{\Gamma'(z)}{\Gamma(z)} - \Gamma'(1)\right\},$$

from which, finally, we find that
$$\overset{*}{\Theta}\,\overset{*}{1}^{1-z} = \log(y-x) + C,$$

which coincides with the result (6) obtained before.*

24. We now show that, if we substitute for Φ
$$\overset{*}{l}\,1 = \Phi$$

in the series (1), we obtain unity as the sum of the series. In fact, if we put
$$\overset{*}{1}^{z} = \frac{1}{\Gamma(z)}(y-x)^{z-1} = \Theta(x, y \mid z)$$

* This result is met with in the calculation which is carried out in an entirely different manner in the last lecture, § 23.

we obtain

$$\frac{d\Theta}{dz} = \overset{*}{\Theta}\,\overset{*}{\Phi}$$

$$\frac{d^2\Theta}{dz^2} = \overset{*}{\Theta}\,\overset{*}{\Phi}{}^2$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{d^h\Theta}{dz^h} = \overset{*}{\Theta}\,\overset{*}{\Phi}{}^h,$$

$$\cdots\cdots\cdots\cdots$$

and therefore

$$\overset{*}{\Theta}\left(\overset{*}{\Phi}{}^\circ + \Phi + \frac{\overset{*}{\Phi}{}^2}{2!} + \frac{\overset{*}{\Phi}{}^3}{3!} + \ldots\right) = \Theta + \frac{d\Theta}{dz} + \frac{1}{2!}\frac{d^2\Theta}{dz^2} + \ldots$$

Now Θ is an *entire function;* consequently

$$\overset{*}{\Theta}\left(\overset{*}{\Phi}{}^\circ + \Phi + \frac{\overset{*}{\Phi}{}^2}{2!} + \ldots\right) = \Theta(x, y \mid z+1),$$

whence

$$\overset{*}{\Phi}{}^\circ + \Phi + \frac{\overset{*}{\Phi}{}^2}{2!} + \ldots = \frac{\Theta(x, y \mid z+1)}{\Theta(x, y \mid z)} = 1.$$

25. In § 19 we stated the definition of $\overset{*}{l}\,\Psi$. It is easy to verify the following properties:

(7) $\overset{*}{l}(\overset{*}{\Psi}\,\overset{*}{\Theta}) = \overset{*}{l}\,\Psi + \overset{*}{l}\,\Theta,$ (8) $\overset{*}{e}{}^{\phi+\chi} = (\overset{*}{e}{}^{\phi})\,(\overset{*}{e}{}^{\chi}),$

(7′) $\overset{*}{l}\left(\frac{\overset{*}{\Psi}}{\overset{*}{\Theta}}\right) = \overset{*}{l}\,\Psi - \overset{*}{l}\,\Theta,$ (8′) $\overset{*}{e}{}^{\phi-\chi} = \frac{\overset{*}{e}{}^{\phi}}{\overset{*}{e}{}^{\chi}},$

(7″) $\overset{*}{l}(\overset{*}{\Psi}{}^m) = m\,\overset{*}{l}\,\Psi,$ (8″) $(\overset{*}{e}{}^{\phi})^m = \overset{*}{e}{}^{m\phi},$

(7‴) $\overset{*}{l}(c\,\Theta) = \overset{*}{l}\,c\,\overset{*}{\Theta}{}^\circ + \overset{*}{l}\,\Theta,$ (8‴) $\overset{*}{e}{}^{c\Phi^\circ+\Phi} = e^c\,\overset{*}{e}{}^{\Phi},$

where m is an integer or a fraction; and in the limit these formulae will be valid for incommensurable values of m. c denotes a constant.

It is evident that

$$\overset{*}{e}^{2\pi n i \overset{*}{f}^\circ} = \overset{*}{f}^\circ,$$

when n is an integer. It follows that $\overset{*}{l} f$ is determined except for an additive term $2\pi n i \overset{*}{f}^\circ$ in the same way that the naperian logarithm of a number A is determined except for the term $2\pi n i$. Consequently the formulae (7), (7′), (7″), (7‴) and the corresponding ones opposite wil! be interpreted in a manner similar to that in which the analogous formulae are interpreted in the ordinary theory of logarithms.

Suppose now that m is comolex. We will extend the formula (8″), *by definition*, to this case; and since the second member, by virtue of (1), is represented by the series

$$(m\overset{*}{\phi})^\circ + m\,\phi + \frac{(m\overset{*}{\phi})^2}{2!} + \ldots,$$

which has a definite meaning, the definition of the first member is established.

Suppose now that $\overset{*}{e}^\phi = \Psi$; then

$$\overset{*}{\Psi}^m$$

is defined for complex values of m, and we have

$$\overset{*}{\Psi}^m = \overset{*}{e}^{m\overset{*}{l}\Psi}.$$

Evidently, inasmuch as $\overset{*}{e}^\phi$ is unchanged by adding to ϕ the term $2\pi n i \overset{*}{\phi}^\circ$, $(\overset{*}{e}^\phi)^m$ is determined except for the factor $e^{2\pi m n i}$.

By definition we establish also the following formula:

$$(\overset{*}{e}^\phi)^\theta = \overset{*}{e}^{\overset{*}{\phi}\overset{*}{\theta}},$$

under the hypothesis that ϕ and θ are permutable functions. Since the second member has a known meaning, the same is true of the first member; and if

$$\overset{*}{e}^\phi = \Psi,$$

then the definition of

$$\overset{*}{\Psi}{}^{\theta}$$

is established, and we have precisely

$$\overset{*}{\Psi}{}^{\theta}=\overset{*}{e}{}^{\overset{*}{\theta}\overset{*}{l}\Psi}. \tag{9}$$

Also in this case, analogously with the preceding one, $\overset{*}{\Psi}{}^{\theta}$ is determined except for the factor of composition $\overset{*}{e}{}^{2\pi ni\theta}$.

If we write

$$\overset{*}{\Psi}{}^{\theta}=\chi, \tag{10}$$

we can write, by way of definition,

$$\theta=\overset{*}{\log}_{\Psi}\chi, \tag{11}$$

and we shall call θ the *logarithm of composition of* χ *to base* Ψ.

From (9) and (10) it follows that

$$\overset{*}{\theta}\,\overset{*}{l}\,\Psi=\overset{*}{l}\,\chi,$$

and therefore, by (11),

$$\overset{*}{\log}_{\Psi}\chi=\frac{\overset{*}{l}\chi}{\overset{*}{l}\Psi}. \tag{12}$$

Of course it is necessary, in order that the second member may have a meaning, not only that it be possible to find the naperian logarithms of composition of χ and Ψ, but also that the fraction which appears in the second member have a meaning. If these conditions are satisfied, then the problem proposed in § 17 is solved.

26. If

$$\Psi=\overset{*}{f}{}^{0}+f,$$

where f is a function whose order is positive and greater than a certain given number, then the series

$$f+\frac{\overset{*}{f}{}^{2}}{2}+\frac{\overset{*}{f}{}^{3}}{3}+\dots$$

is convergent (see Lecture I, § 17), and we have

$$\overset{*}{l}\,\Psi = f + \frac{\overset{*}{f^2}}{2} + \frac{\overset{*}{f^3}}{3} + \dots$$

This can be verified immediately by returning to formula (1).

If

$$\Psi = a\,\overset{*}{f^\circ} + f,$$

where a is a constant, we have

$$\overset{*}{l}\,\Psi = l\,a\,.\,\overset{*}{f^\circ} + \frac{f}{a} + \frac{\overset{*}{f^2}}{2\,a^2} + \dots,$$

and, if

(13) $$\psi = \theta^\alpha\,(a\,\overset{*}{f^\circ} + \overset{*}{f}),$$

then

(14) $$\overset{*}{l}\,\Psi = a\,\overset{*}{l}\,\theta + l\,a\,\overset{*}{f^\circ} + \frac{f}{a} + \frac{\overset{*}{f^2}}{2\,a^2} + \dots$$

It follows that, *if we know the naperian logarithm of composition of a function θ of a group, then, by using formula* (14) *we can calculate that of every function of the group of the form* (13). These form a very extensive class of functions (cf. Lecture II, § 5).

27. We will now give an application of this result:

To calculate the naperian logarithm of composition of of any function belonging to the closed-cycle group, possessing a derivative and having the order 1.

Let $F(y-x)$ be this function, and suppose, for simplicity, that $F(0) \doteq 1$.

We have

$$F = \overset{*}{1}(\overset{*}{F^\circ} + \overset{*}{F'}),$$

where F' denotes the derivative of F.

Therefore, applying formulae (14) and (16), we obtain

$$\overset{*}{l}\,F = \overbrace{(\log(y-x)+C)}^{*}\,\overset{*}{1}^{-1} + F' + \frac{\overset{*}{F'^2}}{2} + \frac{\overset{*}{F'^3}}{3} + \dots$$

28. Let us suppose now that it is desired to obtain the logarithm of composition of F to base 1.

Formally we can write:

$$\overset{*}{\log}_1 F = \frac{\overset{*}{l} F}{\overset{*}{l}_1} = \frac{\overset{*}{1}\,\overset{*}{l} F}{\overset{*}{1}\,\overset{*}{l}\, 1} = \frac{\overset{*}{1}\,\overset{*}{l} F}{\overbrace{(\log(y-x)+C)}^{*}}$$

$$= \overset{*}{F}{}^{\circ} + \frac{\overset{*}{F}{}'\,\overset{*}{1} + \dfrac{\overset{*}{F}{}^{12}\,\overset{*}{1}}{2} + \dfrac{\overset{*}{F}{}^{13}\,\overset{*}{1}}{3} + \ldots}{\overbrace{(\log(y-x)+C)}^{*}}.$$

This brings us, consequently, to the solution of an integral equation of the first kind having as its kernel $\log(y-x)+C$. (Cf. § 6.)

We are thus led to a new class of integral equations which it will be convenient to study and which will form the subject of the sections which follow.

29. The preceding problem is only a particular case of a much more general question.

From (12), if ψ and χ are permutable functions, we have

$$\overset{*}{\log}_\psi \chi = \frac{\overset{*}{l} \chi}{\overset{*}{l} \psi};$$

and, if $\overset{*}{\psi}{}^z$ is given by formula (2),

$$\overset{*}{\log}_\psi \chi$$

$$= \frac{\overset{*}{l} \chi\, \overset{*}{\psi}{}^z}{\overbrace{[\alpha(y-x)^{\alpha z-1} G(x,y\,|\,z)\log(y-x) + (y-x)^{\alpha z-1} G(x,y\,|\,z)]}^{*}};$$

and we shall therefore have to solve new integral equations of the first kind whose kernels involve logarithmic terms.

30. As we foresaw, especially in the last section, we are confronted with the necessity of solving certain integral equations (of Volterra type) whose kernels contain a term having a logarithmic factor.

This is a new class of integral equations to the solution of which we cannot apply directly the known methods of procedure. We shall not stop to solve the general problem but treat a single example to indicate the method which is convenient to use.

Let us try then to solve the integral equation

$$(15) \qquad \int_0^x f(\xi)\,[\log(x-\xi)+C]\,d\xi = \phi(x),$$

where C denotes Euler's constant.

Multiplying both members by $\nu(x, y)$, a finite continuous function, and integrating from 0 to y, we have

$$(16) \quad \int_0^y \nu(x,y)\,dx \int_0^x f(\xi)\,[\log(x-\xi)+C]\,d\xi$$
$$= \int_0^x f(\xi)\,d\xi \int_\xi^y [\log(x-\xi)+C]\,\nu(x,y)\,dx$$
$$= \int_0^y \phi(x)\,\nu(x,y)\,dx.$$

If we can obtain $\nu(x, y)$ in such a manner that

$$\int_x^y [\log(x-\xi)+C]\,\nu(x-y)\,dx = (y-\xi)^\alpha,$$

where α is any positive number, the problem will be solved, for the equation (16) can be written

$$\int_0^x f(\xi)\,(y-\xi)^\alpha\,d\xi = \int_0^y \phi(x)\,\nu(x,y)\,dx,$$

and this equation belongs to a well-known class of integral equations. Now, if we take the well-known formula in the theory of the Euler Integral:

$$\int_\xi^y \frac{(x-\xi)^{\alpha-1}\,(y-x)^{\beta-1}}{\Gamma(\alpha)\,\Gamma(\beta)} = \frac{(y-\xi)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

we observe that the second member is a function of α and β.

If we integrate with respect to β and differentiate with respect to α, we obtain

$$\int_{\xi}^{y} \frac{\partial}{\partial \alpha}\left(\frac{(y-\xi)^{\alpha-1}}{\Gamma(\alpha)}\right) d\xi \int_{\beta'}^{\infty} \frac{(y-x)^{\beta-1}}{\Gamma(\beta)}\, d\beta = -\frac{(y-\xi)^{\alpha+\beta'-1}}{\Gamma(\alpha+\beta)},$$

and if we put $\alpha=\beta'=1$, we have

$$\int_{\xi}^{y}(\log(x-\xi)+C)\,d\xi \int_{0}^{\infty} \frac{(y-x)^{\beta}}{\Gamma(\beta+1)}\, d\beta = -(y-x).$$

We can therefore take for $\nu(x, y)$ the function

$$\nu(x, y) = -\int_{0}^{\infty} \frac{(y-x)^{\beta}}{\Gamma(\beta+1)}\, d\beta.$$

The formula giving the solution of the integral equation (15) is therefore:

$$f(x) = -\frac{d^2}{dx^2}\int_{0}^{x}\phi(\xi)\, d\xi \int_{0}^{\infty} \frac{1}{\Gamma(1+z)}\,(x-\xi)^{\zeta}\, d\zeta.$$

In a similar manner the integral equations are treated upon which depend various problems in the determination of logarithms of composition.

31. As an application let us consider the solution of the problem proposed in § 28, that is, to determine

$$\overset{*}{\log}_1 \overset{*}{F}$$

where F is a function which belongs to the closed-cycle group and $F(0)=1$.

We found

$$\overset{*}{\log}_1 F = \overset{*}{F}{}^{\circ} + \frac{\overset{*}{F'}\,\overset{*}{1} + \dfrac{\overset{*}{F'^2}\,\overset{*}{1}}{2} + \dfrac{\overset{*}{F'^3}\,\overset{*}{1}}{3} + \dots}{\overset{*}{\overbrace{(\log(y-x)+C)}}}.$$

Hence if we put

$$\overset{*}{F'}\,\overset{*}{1} + \frac{\overset{*}{F'^2}\,\overset{*}{1}}{2} + \frac{\overset{*}{F'^3}\,\overset{*}{1}}{3} + \dots = \phi(y-x),$$

we shall have to solve the integral equation

$$\int_{x}^{y} f(\xi-x)\,(\log(y-\xi)+C)\, d\xi = \phi(y-x),$$

or, changing variables,

$$\int_0^x f(\xi)\,(\log(x-\xi)+C)\,d\xi=\phi(x).$$

If we observe that $\phi(0)=0$, then

$$\phi'=F'+\frac{\overset{*}{F}{}'^{2}}{2}+\frac{\overset{*}{F}{}'^{3}}{3}+\ldots,$$

and therefore, applying formula (1′), we have

$$f(y-x)=-\frac{\partial}{\partial y}\int_x^y\Big[F'(\xi-x)+\frac{\overset{*}{F}{}'^{2}(\xi-x)}{2}+\frac{\overset{*}{F}{}'^{3}(\xi-x)}{3}+\ldots\Big]\,d\xi\int_0^\infty\frac{(y-\xi)^\zeta}{\Gamma(1+\zeta)}\,d\zeta,$$

and finally

$$\overset{*}{\log}_1 F=\overset{*}{F}{}^{\circ}-\frac{\partial}{\partial y}\int_x^y\Big[F'(\xi-x)+\frac{\overset{*}{F}{}'^{2}(\xi-x)}{2}+\frac{\overset{*}{F}{}'^{3}(\xi-x)}{3}+\ldots\Big]\,d\xi\int_0^\infty\frac{(y-\xi)^\zeta}{\Gamma(1+\zeta)}\,d\zeta.$$

Therefore, by using the preceding formula, *we can obtain the logarithm of composition to base unity of any differentiable function of the first order which has its characteristic equal to unity and belongs to the closed-cycle group.*

VITO VOLTERRA.

THIRD LECTURE

INTRODUCTION — FUNCTIONS OF COMPOSITION — DERIVATIVES AND INTEGRALS OF COMPOSITION — APPLICATION OF INTEGRATION OF COMPOSITION TO LOGARITHSM OF COMPOSITION AND TO POWERS OF COMPOSITION.

1. *Introduction*

1. At the beginning of the first lecture we considered a particular function of composition which we expressed in the form

$$S(x, y) = -F(x, y) + \overset{*}{F}{}^2(x, y) - \overset{*}{F}{}^3(x, y) + \ldots,$$

and which, we said, belonged to the general class of functions which depend on other functions, *i.e.*, to the class of *functions of lines*. We can obtain a more complete formula, according to the notation which we have been using, by adding a term to the right-hand member and writing instead of S the function

$$\frac{\overset{*}{F}{}^\circ}{F^\circ + F} = \overset{*}{F}{}^\circ - F + \overset{*}{F}{}^2 - F^3 + \ldots$$

We shall extend this concept and examine in this lecture *general functions of composition.*

2. *Functions of Composition — Derivatives and Integrals of Composition*

2. We shall develop in particular those ideas which are connected with the subject which we have been treating, *i.e.* with the subject of powers and logarithms of composition,

reserving for another memoir a more thorough and complete development of the subject in general.

3. If we have an analytic element

$$(1) \qquad \sum_0^\infty {}_m \, a_m z^m,$$

with center $z=0$, convergent within a circle radius R. Then, if the function $F(x, y)$ is finite and continuous, the function

$$\sum_0^\infty {}_m \, a_m \overset{*}{F}{}^m$$

is also finite and continuous and permutable with F. We shall call it a *rational entire function of composition of* F.

Now consider the expression

$$\sum_1^p {}_m \, b_m z^{-m}.$$

This has a pole of order p if $b_p \gtrless 0$. If we assume that F possesses a derivative, then, according to the definition given in the preceding lecture, the expressions

$$\sum_0^n {}_m \, b_m$$

and

$$\sum_1^\infty {}_m \, a_m \overset{*}{F}{}^m + \sum_0^p {}_m \, b_m \overset{*}{F}{}^{-m}$$

have a meaning. The latter will be called a *rational function of composition having a pole of order* p.

If we suppose n to be positive we can calculate also the expression

$$\sum_1^\infty {}_m \, a_m \overset{*}{F}{}^{\frac{m}{n}} + \sum_0^p {}_m \, b_m \overset{*}{F}{}^{\frac{m}{n}}.$$

This is an irrational function of composition if it contains at least one term having a fractional exponent.

Let us consider

$$\overset{*}{l} F.$$

We do not yet know whether, corresponding to any finite continuous function F, the function $\overset{*}{l}\,F$ exists (cf. the preceding lecture). However, if the logarithm of composition exists when F belongs to a certain functional field we shall call it a *logarithmic function of composition.*

The sums, the resultants of composition and the ratios of composition of several functions of composition will be regarded as *new functions of composition.*

4. We shall represent the various functions of composition by means of the symbol

$$\overset{*}{\Phi}(F).$$

F will be called the *argument of the function* Φ.

If $\overset{*}{\Psi}\,(F)$ is a function of composition, and $\overset{*}{F}(\Phi)$ is another function of composition, we can obtain

$$\overset{*}{\Psi}(\overset{*}{F}(\Phi)),$$

which will be called a *function of a function of composition.*

5. We thus have the means of defining various classes of functions of composition. We should be able to obtain new functions as uniform limits of those previously obtained by making the parameters which they contain approach given values. But we will proceed now to establish a general definition of the term *function of composition* which will include as particular cases all these classes. The method which we shall adopt to attain this end consists in stating two fundamental properties common to all the functions hitherto examined, and in assuming these properties as those which define, in general, all functions of composition.

The first of these properties is this, that *all the functions previously examined are permutable with the function which constitutes the argument;* we proceed to formulate in the following paragraph the second fundamental property.

6. Let us return to the rational entire function of composition

$$\overset{*}{\Phi}(F) = \sum_{1}^{\infty}{}_m\, a_m \overset{*}{F}{}^m.$$

If we form the expression

$$\overset{*}{\Phi}(F+\epsilon f) - \Phi(F),$$

where ϵ is a number and f is a function permutable with F, then as ϵ approaches zero the expression approaches zero; moreover

$$\frac{\overset{*}{\Phi}(F+\epsilon f) - \overset{*}{\Phi}(F)}{\epsilon \overset{*}{f}}$$

approaches a limit which is easily calculated, namely

$$\sum_{1}^{\infty}{}_m\, m\, a_m \overset{*}{F}{}^{m-1}.$$

This expression is thus independent of f. It is called *the derivative of composition of Φ with respect to F.*

The rule for calculating the derivative consists in differentiating the series (1) with respect to z and substituting for the powers of z powers of composition of F. Hence the rule is the same as the one for calculating the ordinary derivative provided that instead of the ordinary powers we use powers of composition.

It is easy, in this manner, to extend the concept of the derivative of composition to rational functions of composition having poles, to irrational functions of composition and to all those which can be obtained from these by the operations of addition, composition, forming ratios of composition and forming functions of functions of composition. The rules for finding derivatives of composition are the same as those which are applied in ordinary differentiation except that instead of the ordinary operations of multiplication, raising to powers and forming ratios, we substitute the corresponding operations of composition.

We can extend this concept of the derivative also to logarithmic functions of composition. If we wish to obtain the derivative of composition of $\overset{*}{l} F$, it is sufficient to observe that

$$\overset{**}{e}{}^{lF} = F,$$

and thus we find that

$$\frac{d \overset{*}{l} F}{d \overset{*}{F}} = \overset{*}{F}{}^{-1}.$$

We can therefore generalize the foregoing rule for finding the derivative of composition also to expressions containing logarithms of composition.

Evidently, in all the cases considered, if the increment given to the function F is ϵf, a function permutable with F, and if ϵ is made to approach zero, then the limit of the ratio of composition which has for its numerator the increment of the function of composition and for its denominator ϵf will be *independent of* f. This is the *second fundamental property* which we were seeking.

To represent the derivative of composition of the function Φ of the argument F we shall use the symbol

(1) $$\frac{d \overset{*}{\Phi}}{d \overset{*}{F}}.$$

7. With these introductory notions let us proceed to state the general definition of the term function of composition.

Let Φ $(x; y)$ be a function which *depends on all the values of the function* $F(\xi \eta)$, $x \leq \xi \leq \eta \leq y$, in the sense in which these terms are used in the *theory of functions of lines*, so that we can write, in the notation of that theory

$$\Phi = \Phi \left| [F(\overset{\eta}{\underset{x}{\xi}}, \overset{y}{\underset{\xi}{\eta}})] \right|,$$

and let us suppose that $F(x, y)$ can vary in a certain functional field for which it is assumed that, if f and ϕ are contained in it; $a\,f+b\,\phi$ is also contained in it, a and b being parameters which are independent of x and y and which vary over a certain interval. *Let the functions* Φ *and* F *be permutable.* We will further assume that they are *continuous*, that is to say, if $\Phi(x, y)$ corresponds to $F(x, y)$ and $\Phi_1(x, y)$ corresponds to $F_1(x, y)$, Φ approaches Φ_1 uniformly when F approaches F_1 uniformly.

Besides this we will assume that, in the sense in which the operation is carried out in the *theory of functions of lines*, Φ can be differentiated with respect to F, so that it will be possible to obtain its successive derivatives to any order that it may be necessary to consider.

Let us substitute for $F(\xi, \eta)$

$$F(\xi, \eta)+\epsilon f(\xi, \eta),$$

which is permutable with $F(\xi, \eta)$, and indicate by $\Phi'(x, y)$ the corresponding value of $\Phi(x, y)$, and let us form the ratio of composition

$$\frac{\overset{*}{\Phi}{}'-\overset{*}{\Phi}}{\epsilon \overset{*}{f}},$$

and let ϵ approach zero. *If there exists a limit of this ratio, independent of* f, *we shall say that* Φ *is a function of composition of the argument* F and we shall represent it always by the symbol $\overset{*}{\Phi}(F)$ which we adopted previously. The limit in question will be called the *derivative of composition* and will be indicated by the symbol, already introduced,

$$\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}.$$

Like the ordinary derivative the derivative of composition can be regarded as an *actual ratio of composition,* and the numerator and denominator constitute respectively the *differential of the function and the differential of the argument.*

The two fundamental properties of functions of composition are therefore: (1) *The function is permutable with its argument;* (2) *the derivative is independent of the differential of the argument* (cf. § 10).

8. Let us prove now that *the derivative is also a function of composition of F.*

If we denote by Ψ the derivative $\dfrac{d\overset{*}{\Phi}}{d\overset{*}{F}}$ we can state first that

$$\overset{*}{\Psi}=\Psi \mid [F(\overset{\eta}{\underset{x}{\xi}}, \overset{y}{\underset{\xi}{\eta}})] \mid.$$

We will show that Ψ possesses the two fundamental properties which serve to characterize a function of composition.

The first of these fundamental properties holds for the derivative Ψ, since it is permutable with F.

Let us proceed to prove that the second property also holds. It is to be noted, then, that we have

$$\overset{*}{\Phi}(F+\epsilon_1 f_1)-\overset{*}{\Phi}(F)=\epsilon_1 \overset{*}{f}\overset{*}{\Psi}+\epsilon_1 h_1,$$

where f is permutable with F and h_1 approaches zero with ϵ_1.

Let f_2 be a function also permutable with F_1 and form the expression

$$\overset{*}{\Phi}(F+\epsilon_1 f_1+\epsilon_2 f_2)-\overset{*}{\Phi}(F+\epsilon_1 f_1)-\overset{*}{\Phi}(F+\epsilon_2 f_2)+\overset{*}{\Phi}(F)$$
$$=\epsilon_1 \overset{*}{f_1}\,\{\overset{*}{\Psi} \mid [F+\epsilon_2 f_2] \mid -\overset{*}{\Psi} \mid [F] \mid \}+\epsilon_1(h_1'-h_1''),$$

where h_1' and h_1'' become zero with ϵ_1 and ϵ_2 respectively.

Since Ψ, by the hypothesis which we have made, is differentiable according to the rule for functions of lines we can write

(2) $$\Psi \mid [F+\epsilon_2 f_2] \mid - \Psi \mid [F] \mid =\epsilon_2\, \theta_1+\epsilon_2 h_2$$

where h_2 approaches zero with ϵ_2. If then we assume the existence of the third differential, in the sense of the theory of functions of lines, we can write

$$\overset{*}{\Psi}(F+\epsilon_1 f_1+\epsilon_2 f_2)-\overset{*}{\Phi}(F+\epsilon_1 f_1)-\overset{*}{\Phi}(F+\epsilon_2 f_2)+\overset{*}{\Phi}(F)$$
$$=\epsilon_1\,\epsilon_2\,\overset{*}{f_1}\,\theta_1+\epsilon_1\,\epsilon_2\,h'',$$

where h'' approaches zero with ϵ_1 and ϵ_2.

Hence

$$(3)\quad \lim_{\substack{\epsilon_1=0\\ \epsilon_2=0}} \frac{\overset{*}{\Phi}(F+\epsilon_1 f_1+\epsilon_2 f_2)-\overset{*}{\Phi}(F+\epsilon_1 f_1)-\overset{*}{\Phi}(F+\epsilon_2 f_2)+\overset{*}{\Phi}(F)}{\epsilon_1\,\epsilon_2}$$
$$=\overset{*}{f_1}\,\overset{*}{\theta_1}.$$

But the first member is symmetrical with respect to f_1 and f_2; it can therefore be put in the form $\overset{*}{f_2}\,\overset{*}{\theta_2}$, and therefore

$$\overset{*}{f_1}\,\overset{*}{\theta_1}=\overset{*}{f_2}\,\theta_2.$$

Assuming f_1 and f_2 to be of determinate orders it necessarily follows that

$$(4)\qquad \theta_1=\overset{*}{f_2}\,\overset{*}{\theta}_{12}$$

$$(4')\qquad \theta_2=\overset{*}{f_1}\,\overset{*}{\theta}_{12}$$

so that the limit (3) will be equal to

$$\overset{*}{f_1}\,\overset{*}{f_2}\,\theta_{12}.$$

Now Ψ is independent of f_1, therefore θ_1 is also independent of f_1. On account of (4) it follows that θ_{12} is independent of f_1. By the same reasoning, on account of (4′), θ_{12} must be independent of f_2. Hence θ_{12} is independent of f_1 and f_2.

Now consider again equation (2). In view of (4) we can write

$$\Psi\,|\,[F+\epsilon_2 f_2]\,|\,-\Psi\,|\,[F]\,|=\epsilon_2\,\overset{*}{f_2}\,\overset{*}{\theta}_{12}+\epsilon_2\,h_2,$$

and consequently

$$\lim_{\epsilon_2=0}\frac{\Psi\,|\,[F+\epsilon_2 f_2]\,|-\Psi\,|\,[F]\,|}{\epsilon_2\,\overset{*}{f_2}}=\theta_{12},$$

θ_{12} being independent of f_2. We have thus shown that

$\Psi = \frac{d\overset{*}{\Phi}}{d\overset{*}{F}}$ possesses also the second fundamental property and it is therefore proved that Ψ is a function of composition of F.

When the successive derivatives of composition of Φ exist they will be denoted by

$$\frac{d^2\overset{*}{\Phi}}{d\overset{*}{F}{}^2},\quad \frac{d^3\overset{*}{\Phi}}{d\overset{*}{F}{}^3},\ \ldots$$

and *they will all be functions of composition of F in the sense already stated.*

9. We will proceed now to set forth certain observations which will serve to make clear and complete the concepts which we have so far formulated.

We will begin by noting that it is always desirable to specify the functional field of variation for the argument of the function of composition which is being examined.

Thus, for example, consider the expression

$$\int_x^y F(x, \xi)\,\Omega(y-\xi)\,d\xi, \tag{5}$$

and regard it as dependent on all the values of $F(\xi, \eta)$, $x \leq \xi \leq \eta \leq y$, in the sense of the theory of functions of lines. Provided that the function F belongs to the closed-cycle group the expression (5) represents a function of composition of F; such is not the case otherwise.

In fact, if we take

$$F(x, \xi) = F(\xi - x),$$

the function (5) will be permutable with F; whereas, in general, if $F(x, \xi)$ does not belong to the closed-cycle group the condition of permutability is not satisfied. Therefore the first fundamental property (§ 7) holds only if F belongs to the closed-cycle group and therefore the argument will have to be supposed to range over this field.

10. The question might be asked whether, in defining a function of composition (§ 7), it was necessary to state explicitly the condition which we have imposed upon the derivative: *that it be independent of the differential of the argument* (second condition), or whether this condition might be, on the other hand, a necessary consequence of the condition of the *permutability of the function with its argument* (first condition); but it easy to see that the two properties are *independent*, in the sense that it is possible for a function to be permutable with its argument without the necessity of its derivative being independent of the differential of the argument. As an example let us consider the expression

$$\int_x^y F(x, \xi)\, \Omega(\xi - x)\, d\xi \tag{6}$$

as dependent on F.

If we suppose F to belong to the closed-cycle group, that is to say, $F(x, \xi) = F(\xi - x)$, the preceding expression becomes

$$\int_0^{y-x} F(\eta)\Omega(\eta) d\eta,$$

and we have here a function which belongs to the closed-cycle group. It is therefore permutable with F. *The first fundamental property* (§ 7) *is therefore satisfied.*

Now let the increment $\epsilon f(y-x)$ be added to $F(y-x)$. The increment of the expression (6) is

$$\epsilon \int_0^{y-x} f(\eta)\Omega(\eta) d\eta.$$

In order to calculate the ratio of composition of this increment and the increment of the argument F it is sufficient to solve the integral equation

$$\epsilon \int_0^{y-x} f(\eta)\Omega(\eta)\, d\eta = \epsilon \int_x^y f(y-\xi)\, \psi(\xi - x)\, d\xi.$$

The solution ψ will be the required ratio.

Now, differentiating both members with regard to y, we have

$$f(y-x)\,\Omega(y-x)=f(0)\,\psi(y-x)+\int_x^y f'(y-\xi)\,\psi(\xi-x)\,d\,\xi,$$

or

$$f(x)\,\Omega(x)=f(0)\,\psi(x)+\int_0^x f'(x-\xi)\,\psi(\xi)\,d\,\xi,$$

and it is evident that the solution ψ depends on f.

It follows that the second fundamental condition of § 7 (namely that the derivative of composition be independent of the differential of the argument) *is not satisfied*, although the first condition, that of the permutability of the function with its argument, is satisfied.

11. The importance of this second condition consists in its being *invariant to the successive operations of passing from the function to its successive derivatives:* in other words, if the condition is satisfied in passing from the function to its first derivative it will also be satisfied in the successive passages to the other derivatives. This is the significant fact concerning the thorem of § 8.

In recognition of its importance we will give another proof of this fact for the particular case in which the functional field of the argument is that of the closed cycle.

Suppose then that to every function F of the closed cycle another function Φ, also belonging to the closed-cycle group, is made to correspond in such a way that, according to the general definition of § 7, Φ is a function of composition $\Phi(F)$.

Consider the function

$$\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}=\psi;$$

ψ belongs also the closed-cycle group and is independent of $d\,F$ (according to the fundamental property of § 7).

According to the theory of functions of lines,

$$\delta \Phi = \int_0^x \Phi'(x, \xi)\, \delta F(\xi) d\xi = \int_0^x \psi(x-\xi) \delta F(\xi) d\xi,$$

and therefore

$$\Phi'(x, \xi) = \psi(x-\xi).$$

If we pass to the second functional derivative, we obtain

$$\delta \Phi'(x, \xi) = \delta \psi(x-\xi) = \int_0^x \Phi''(x-\xi, \eta)\, \delta F(\eta)\, d\eta.$$

But by virtue of the symmetry of the second derivative (see loc. cit., "Leçons sur les fonctions de lignes," Chap. II, § 4), it follows that

$$\Phi''(x-\xi, \eta) = \Phi''(x-\eta, \xi),$$

or, assuming the existence of the derivatives of Φ'',

$$\frac{\partial \Phi''}{\partial x} = -\frac{\partial \Phi''}{\partial \xi} = -\frac{\partial \Phi''}{\partial \eta},$$

and consequently

$$\Phi''(x-\xi, \eta) \equiv \Phi''(x-\eta, \xi) \equiv \Phi''(x-\xi-\eta);$$

therefore

$$\delta \psi(x-\xi) = \int_0^x \Phi''(x-\xi-\eta)\, \delta F(\eta)\, d\eta.$$

It follows that

$$\frac{\partial}{\partial x}\int_0^x \Phi''(x-\xi-\eta)\, \delta F(\eta) d\eta + \frac{\partial}{\partial \xi}\int_0^x \Phi''(x-\xi-\eta)\, \delta F(\eta)\, d\eta = 0,$$

and therefore, performing the differentiation,

$$\Phi''(-\xi)\, \delta F(x) = 0 \qquad (\xi > 0).$$

We have then

$$\Phi''(x-\xi-\eta) = 0$$

when η lies between $x-\xi$ and x, from which it follows that

$$\delta \psi(x-\xi) = \int_0^{x-\xi} \Phi''(x-\xi-\eta)\, \delta F(\eta)\, d\eta;$$

and this shows that our theorem is true, since the equation can be written

$$\delta \psi = \overset{*}{\Phi}''\, \delta \overset{*}{F},$$

and therefore

$$\frac{d\overset{*}{\psi}}{d\overset{*}{F}}=\Phi'';$$

that is to say, the ratio

$$\frac{d\overset{*}{\psi}}{d\overset{*}{F}}$$

is independent of the differential appearing in the denominator.

12. If Φ is a function of composition of F the operation *differentiation of composition* can be carried out by means of differentiation, as applied in the ordinary sense.

In fact,

$$\lim_{\epsilon=0}\frac{\overset{*}{\Phi}(F+\epsilon f)-\overset{*}{\Phi}(F)}{\epsilon \overset{*}{f}}$$

must be independent of f, since this function is permutable with F. If we take $f=\overset{*}{F^\circ}$, we have

$$\frac{\overset{*}{\Phi}(F+\epsilon \overset{*}{F^\circ})-\overset{*}{\Phi}(F)}{\epsilon \overset{*}{F^\circ}}=\frac{\overset{*}{\Phi}(F+\epsilon \overset{*}{F^\circ})-\Phi(F)}{\epsilon},$$

therefore

$$\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}=\lim_{\epsilon=0}\frac{\overset{*}{\Phi}(F+\epsilon \overset{*}{F^\circ})-\overset{*}{\Phi}(F)}{\epsilon}=\left(\frac{d\overset{*}{\Phi}(F+z\overset{*}{F^\circ})}{dz}\right)_{z=0};$$

and by this formula differentiation of composition is reduced to ordinary differentiation.

13. Let us consider the expression

$$\overset{*}{\Phi}(F_2)-\overset{*}{\Phi}(F_1),$$

where F_1 and F_2 are permutable. By Lagrange's formula we can write

$$\overset{*}{\Phi}(F_2)-\overset{*}{\Phi}(F_1)=\left(\frac{d\,\Phi\,[F_1+z(F_2-F_1)]}{dz}\right)_{z-\theta};$$

where θ is a number which lies between 0 and 1.

By this formula it follows that

(7) $$\overset{*}{\Phi}(F_2)-\overset{*}{\Phi}(F_1)=(\overset{*}{F}_2-\overset{*}{F}_1)\left(\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}\right)_{F=F_1+\theta(F_2-F_1)},$$

or, if we put $F_2=F_1+f$,

$$\overset{*}{\Phi}(F_1+f)=\overset{*}{\Phi}(F_1)+\overset{*}{f}\left(\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}\right)_{F=F_1+\theta f},$$

where F_1 and f are permutable.

There follows a formula analogous to Taylor's formula in which the existence of the successive derivatives of composition is assumed, namely

$$\overset{*}{\Phi}(F_1+f)=\overset{*}{\Phi}(F_1)+\overset{*}{f}\left(\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}\right)_{F=F_1}+\frac{\overset{*}{f^2}}{2!}\left(\frac{d^2\overset{*}{\Phi}}{d\overset{*}{F^2}}\right)_{F=F_1}+\dots$$
$$+\dots+\frac{\overset{*}{f^m}}{m!}\left(\frac{d^m\overset{*}{\Phi}}{d\overset{*}{F^m}}\right)_{F=F_1}+\frac{\overset{*}{f^{m+1}}}{(m+1)!}\left(\frac{d^{m+1}\overset{*}{\Phi}}{d\overset{*}{F^{m+1}}}\right)_{F=F_1+\theta_m f},$$

f being permutable with F_1 and $\theta\ m$ lying between 0 and 1.

14. From formula (7), assuming the continuity of $\dfrac{d\overset{*}{\Phi}}{d\overset{*}{F}}$,

$$\lim_{F_1=F_2}\frac{\overset{*}{\Phi}(F_2)-\overset{*}{\Phi}(F_1)}{\overset{*}{F}_2-\overset{*}{F}_1}=\left(\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}\right)_{F=F_1},$$

where the functions F_1 and F_2 are supposed to be permutable.

Let us suppose that $F(x, y \mid s)$ depends on the parameter s in such a way that for all values of s lying in a certain interval the function F obtained belongs to a certain group of permutable functions. We can then consider

$$\Phi(F(x, y \mid s))$$

as a function of s. By equation (7) we have

$$\overset{*}{\Phi}(F(x, y \mid s_0+h))-\overset{*}{\Phi}(F(x, y \mid s_0))$$
$$=\frac{\overset{*}{F}(x, y \mid s_0+h)-\overset{*}{F}(x, y \mid s_0)}{h}\left(\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}\right)_{F=F_1+\theta(F_2-F_1)},$$

where it is supposed that $F_1=F(x, y \mid s_0)$, $F_2=F(x, y \mid s_0+h)$ and θ lies between 0 and 1.

Assuming the existence of the derivative of F with regard to S, and passing to the limit as h tends toward zero, we obtain

$$\frac{d\overset{*}{\Phi}(F)}{d\,s}=\frac{d\overset{*}{F}}{d\,s}\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}. \tag{8}$$

15. From formula (7) it follows that if, for every function F of the field we are considering,

$$\frac{d\overset{*}{\Phi}}{d\overset{*}{F}}=0,$$

then Φ is independent of F: that is to say, Φ is equal to a fixed determinate function belonging to the group of permutable functions, which contains the field of functions over which the argument F ranges. Consequently, if $\overset{*}{\Phi}_1(F)$ and $\overset{*}{\Phi}_2(F)$ have the same derivative of composition they can differ only by a fixed determinate function which belongs to the group of permutable functions to which F belongs.

16. We proceed now to the subject of *integration of composition*. Let there be given a function of composition $\overset{*}{\Phi}(F)$, and let us consider the function

$$F(x,y\mid s)$$

such that, for all values of s lying between certain limits a and b, $F(x,y\mid s)$ belongs always to the same group of permutable functions. Let us suppose that Φ and F are of positive order.

We now form the expression

$$\overset{*}{\Phi}(F(x,y\mid s))\,\frac{\partial\overset{*}{F}(x,y\mid s)}{\partial s}$$

by composition; and, assuming that as a function of s, it is integrable, we calculate the integral

$$\int_a^b\overset{*}{\Phi}(F(x,y\mid s))\,\frac{\partial\overset{*}{F}(x,y\mid s)}{\partial s}\,d\,s, \tag{9}$$

This is obtained by dividing the interval $a\ b$ into n parts, $h_1, h_2, \ldots h_n$, then forming

$$F_\nu = F(x, y \mid a + h_1 + h_2 + \ldots + h_\nu), \quad F_\circ = F(x, y \mid a)$$

$$\lim \overset{*}{\sum_{0}^{n-1}}{}_{\nu} \overset{*}{\Phi}(F_\nu)\ (\overset{*}{F}_{\nu+1} - \overset{*}{F}_\nu), \tag{10}$$

and passing to the limit in the final sum by making all the intervals $h_1, h_2, \ldots, h_n$ tend toward zero, at the same time increasing the number of intervals indefinitely.

17. Let us put $F(x, y \mid a) = F_A$, $F(x, y \mid b) = F_B$. We will write the integral (9) in the form

$$\int_s \overset{*}{\Phi}(F)\, d\, \overset{*}{F} = \int_a^b \overset{*}{\Phi}(F)\, d\, \overset{*}{F},$$

or in the form

$$\int_{F_A}^{F_B} \overset{*}{\Phi}(F)\, d\, \overset{*}{F}. \tag{11}$$

To justify the notation (11) it is necessary to prove that, if we take another function

$$F'(x, y \mid s'),$$

which for the totality of values of s' lying between a' and b' represents a set of functions belonging to the same group of permutable functions as before, then, provided that $F'(x, y \mid a') = F_A$, $F'(x, y \mid b') = F_B$, we obtain for the integral

$$\int_{a'}^{b'} \overset{*}{\Phi}(F'(x, y \mid s'))\, \frac{\partial\, \overset{*}{F}'(x, y \mid s')}{\partial\, s'}\, d\, s', \tag{9'}$$

the same result (9)*.

For this purpose consider

$$F(x, y \mid u, v),$$

and let us regard u and v as the coördinates of points of a plane. For all the values of u and v corresponding to points

* For an example of two such functions $F(x, y \mid s)$ and $F'(x, y \mid s')$ we can take

$$F(x, y \mid s) = 1 + s(y-x) + s^2(y-x)^2$$

$$F'(x, y \mid s') = 1 + (s'-2)\ (s'-1)\ \cos\ (y-x) + (y-x)^2(s'-1)^3 + (y-x)\ (s'-1),$$

and $A=0$, $b=1$, $a'=1$, $b'=2$. We then have

$$F\ (x, y \mid 0) = F^1(x, y \mid 1) = 1;\ F\ (x, y \mid\ ) = F'\ (x, y \mid 2) = 1 + (y-x) + (y-x)^2.$$

of a certain area σ and of its contour S let us suppose that F is a function belonging to a given group of permutable functions. Let us form the integral

$$\int_S \overset{*}{\Phi}(F)\, d\overset{*}{F} = \int_S \overset{*}{\Phi}(F) \frac{d\overset{*}{F}}{dS} dS$$

$$= \int_S \overset{*}{\Phi}(F) \left(\frac{\partial \overset{*}{F}}{\partial u}\frac{du}{dS} + \frac{\partial \overset{*}{F}}{\partial v}\frac{dv}{dS}\right) dS.$$

If no singularity exists in the interior of the region σ, by virtue of formula (8), we shall have

$$\int_S \overset{*}{\Phi}(F) \left(\frac{\partial \overset{*}{F}}{\partial u}\frac{du}{dS} + \frac{\partial \overset{*}{F}}{\partial v}\frac{dv}{dS}\right) dS$$

$$= \int_\sigma \left\{\frac{\partial}{\partial v}\left(\overset{*}{\Phi}(F)\frac{\partial \overset{*}{F}}{\partial u}\right) - \frac{\partial}{\partial u}\left(\overset{*}{\Phi}(F)\frac{\partial \overset{*}{F}}{\partial v}\right)\right\} d\sigma$$

$$= \int_\sigma \left\{\frac{\partial}{\partial v}\overset{*}{\Phi}(F)\frac{\partial \overset{*}{F}}{\partial u} - \frac{\partial}{\partial u}\overset{*}{\Phi}(F)\frac{\partial \overset{*}{F}}{\partial v} d\sigma\right.$$

$$= \int_\sigma \frac{d\overset{*}{\Phi}}{d\overset{*}{F}}\left(\frac{\partial \overset{*}{F}}{\partial v}\frac{\partial \overset{*}{F}}{\partial u} - \frac{\partial \overset{*}{F}}{\partial u}\frac{\partial \overset{*}{F}}{\partial v}\right) d\sigma = 0.$$

It follows therefore that

$$\int_S \overset{*}{\Phi}(F)\, d\overset{*}{F} = 0. \tag{12}$$

From this formula we deduce the result that, if it is possible to pass from $F(x, y \mid s)$ to $F'(x, y \mid s')$ continuously without allowing F and $\overset{*}{\Phi}(F)$ to traverse any singularities, then the two integrals (9) and (9′) lead to the same result.

18. Regarding F_A as fixed and F_B as variable, the integral

$$\int_{F_A}^{F_B} \overset{*}{\Phi}(F)\, d\overset{*}{F}$$

represents a function of composition of F_B. If we call it $\overset{*}{\psi}(F_B)$, we shall have

$$\frac{d\overset{*}{\Psi}}{d\overset{*}{F}_B} = \overset{*}{\Phi}(F_B).$$

In order to integrate the rational or irrational functions of composition which we have considered previously it is sufficient to apply the ordinary rules of integration and to substitute powers of composition for the ordinary ones.

It is possible, evidently, to consider *differential equations of composition* by examining the relations between functions of composition and their derivatives of various orders.

It is also possible, evidently, to consider functions of composition and the derivatives of functions of composition of several arguments.

19. It is easy to recognize the analogy between the theory which we have developed and the theory of functions of complex variables. The second condition imposed upon the derivative of composition of a function of composition (§ 7) corresponds evidently to the condition that the derivative of a function of the complex variable z be independent of the direction in which the point representing the variable z is displaced in the complex plane (condition of monogeneity). Each of these conditions is preserved in successive differentiations (theorem of § 8).

Furthermore, formula (12) corresponds to Cauchy's Theorem, and evidently we can state that a necessary condition that a function be a function of composition is that it satisfy formula (12), thus establishing a reciprocal theorem analogous to the well-known theorem of Morera, the converse of Cauchy's Theorem.

3. *Application of Integration of Composition to Logarithms of Composition and Powers of Composition*

20. We have seen that

$$\frac{d\overset{*}{l}F}{d\overset{*}{F}} = \overset{*}{F}^{-1};$$

hence

$$\overset{*}{l}F=\int_{\overset{*}{F^\circ}}^{F}\overset{*}{F}{}^{-1}\,d\overset{*}{F}.$$

We can also put

$$\overset{*}{l}F=\lim_{n=\infty}\frac{\overset{*}{F}-\overset{*}{F^\circ}}{n}\left\{\frac{1}{\overset{*}{F^\circ}}+\frac{1}{\overset{*}{F^\circ}+\frac{\overset{*}{F}-\overset{*}{F^\circ}}{n}}+\frac{1}{\overset{*}{F^\circ}+\frac{2(\overset{*}{F}-\overset{*}{F^\circ})}{n}}\right.$$

$$\left.+\frac{1}{\overset{*}{F^\circ}+\frac{3(\overset{*}{F}-\overset{*}{F^\circ})}{n}}+\dots+\frac{1}{\overset{*}{F}}\right\}=(\overset{*}{F}-\overset{*}{F^\circ})\int_0^1\frac{dz}{\overset{*}{F^\circ}+z(\overset{*}{F}-\overset{*}{F^\circ})};$$

or

$$\overset{*}{F}\,\overset{*}{l}F=(\overset{*}{F}-\overset{*}{F^\circ})\int_0^1\frac{\overset{*}{F}\,dz}{\overset{*}{F^\circ}+z(\overset{*}{F}-\overset{*}{F^\circ})}.$$

Now

$$\frac{\overset{*}{F}}{\overset{*}{F^\circ}+z(\overset{*}{F}-\overset{*}{F^\circ})}=\frac{1}{1-z}\left(F-\frac{z}{z-1}\overset{*}{F}{}^2+\frac{z^2}{(1-z)^2}\overset{*}{F}{}^3-\dots\right).$$

We are thus led to inquire whether the integral

$$\int_0^1\frac{dz}{1-z}\left(F-\frac{z}{1-z}\overset{*}{F}{}^2+\frac{z^2}{(1-z)^2}\overset{*}{F}{}^3-\dots\right)$$

is convergent. The proof of the convergence depends on the following theorems which we will state without proof.

21. Theorem I. Let $F(x, y)$ *be a finite continuous differentiable function*, $a \leq x \leq y \leq b$, and let

$$\frac{\partial F(x,y)}{\partial x}=F_1(x,y),\quad \frac{\partial F(x,y)}{\partial y}=F_2(x,y),\quad \frac{\partial^2 F(x,y)}{\partial x\,\partial y}=F_{12}(x,y),$$

$$F(x,x)=1,\ F_1(x,x)=F_2(x,x)=0$$

(in other words we suppose that F has the canonical form, Lecture I, § 10) and let F_1, F_2, F_{12} be *finite and continuous. Then*

(I) $$zF-z^2\overset{*}{F}{}^2+z^3\overset{*}{F}{}^3-\dots=z\,e^{-z(y-x)}+\frac{\Phi(x,y\mid z)}{z},$$

and when z *varies from* $h>0$ *to* ∞, Φ *will remain less than a certain finite number.*

Note: The condition $F_1(x, x)' = F_2(x, x) = 0$ can be removed and the theorem will still be true.

Theorem II. *If the conditions of the preceding theorem are satisfied, and if the functions*

$$\frac{\partial^2 F}{\partial x^2}, \quad \frac{\partial^2 F}{\partial y^2}, \quad \frac{\partial^3 F}{\partial x\, \partial y^2}, \quad \frac{\partial^3 F}{\partial x^2 \partial y},$$

are finite and continuous and their absolute values are less than M, then for $y > x$

$$\lim_{z=\infty} \Phi(x, y \mid z) = \frac{\partial^2 F(x, y)}{\partial x\, \partial y} + \int_x^y \frac{\partial^2 F(x, \xi)}{\partial x\, \partial \xi}\, \Psi(\xi, y)\, d\,\xi,$$

where

$$\Psi(x, y) = F_2 + \overset{*}{F}_2{}^2 + \overset{*}{F}_2{}^3 + \ldots,$$

and for $y = x$

$$\lim_{z=\infty} \Phi(x, y \mid z) = 0.$$

22. From Theorem I it follows, under the conditions imposed, that the expression

$$\lim_{z=+\infty} (z\, F - z^2\, \overset{*}{F}{}^2 + z^3\, \overset{*}{F}{}^3 - \ldots)$$

is equal to zero for $y > x$ and is infinite for $y = x$. Furthermore, if $F(x, y)$ is greater than a certain positive quantity,

$$\lim_{z=+\infty} (z\, F + z^2\, \overset{*}{F}{}^2 + z^3\, \overset{*}{F}{}^3 + \ldots) = \infty\,.$$

These properties show the close connection between the very general series (I) and the exponential series. Moreover, Theorem I (see the remark at the end of the theorem) serves to answer the question concerning the convergence of the integral (1). The fact that F is reduced to the *canonical form* (Lecture I, § 10) or even simply the fact that $F(x, x) = 1$ *suffices to show that the integral in question is convergent,* and therefore

(II) $$\overset{*}{F}\, \overset{*}{l}\, F = (\overset{*}{F} - \overset{*}{F}{}^0) \int_0^1 \frac{d\, z}{1-z} \left(\overset{*}{F} - \frac{z}{1-z}\, \overset{*}{F}{}^2 + \frac{z^2}{(1-z)^2}\, \overset{*}{F}_3 - \ldots \right).$$

23. As an example we will now apply formula (II) in order to obtain the expression, already found in another way, for $\dot{1}\,\dot{l}\,1$ (cf. Lecture II, § 22).

Putting

$$f(y-x)=\int_0^1\frac{dz}{1-z}\left(1-\frac{z}{1-z}(y-x)+\frac{z^2}{(1-z)^2}(y-x)^2-\ldots\right)$$
$$=\int_0^1 e^{-\frac{z}{1-z}(y-x)}\frac{dz}{1-z},$$

we have by virtue of (II)

$$\dot{1}\,\dot{l}\,1=(\dot{1}-\dot{1}^\circ)\dot{f}.$$

But

$$\begin{cases} f(x)=\int_0^1 e^{-\frac{z}{1-z}x}\dfrac{dz}{1-z}, \\ f'(x)=\int_0^1 -\dfrac{z}{(1-z)^2}e^{-\frac{z}{1-z}x}\,dz, \\ f(x)-f'(x)=\int_0^1 e^{-\frac{z}{1-z}x}\dfrac{z}{(1-z)^2}\,dz=\dfrac{1}{x}, \end{cases}$$

therefore

$$(17)\qquad f(x)=e^x\int_x^\infty\frac{e^{-\xi}\,d\xi}{\xi}=-\log x+e^x\int_x^\infty\log\xi\, e^{-\xi}\,d\xi,$$

from which it follows that

$$(\dot{1}-\dot{1}^\circ)\dot{f}=-\int_0^x\log\xi\,d\xi+\int_0^x e^\eta\,d\eta\int_\eta^\infty\log\xi\,e^{-\xi}\,d\xi$$
$$=\log x-\int_0^\infty\log\xi\,e^{-\xi}\,d\xi=\log x-\Gamma'(1)=\log x+C,$$

where C denotes Euler's constant. We thus find again the result

$$\dot{1}\,\dot{l}\,1=\log x+C.$$

24. Returning now to the general formula (II) we observe that it gives us a method of calculating

$$\dot{F}\,\dot{l}\,F,$$

where F, its first derivatives and its second mixed derivative are finite and continuous and also $F(x, x)=1$.

We have in fact, by reason of (I),

$$\frac{1}{1-z}\left(F-\frac{z}{1-z}\overset{*}{F}{}^{2}+\frac{z^2}{(1-z)^2}\overset{*}{F}{}^{3}-\ldots\right),$$

$$=\left(\frac{z}{1-z}F-\frac{z^2}{(1-z)^2}\overset{*}{F}{}^{2}+\frac{z^3}{(1-z)^3}\overset{*}{F}{}^{3}-\ldots\right)\frac{1}{z}$$

$$=\left(\frac{z}{1-z}e^{-\frac{z}{1-z}(y-x)}+\frac{\Phi\left(x,y\,\middle|\,\frac{z}{1-z}\right)}{\left(\frac{z}{1-z}\right)}\right)\frac{1}{z}$$

$$=\frac{1}{1-z}e^{-\frac{z}{1-z}(y-x)}+\Psi(x,y\,|\,z)\,(1-z),$$

where $\Psi\,(x, y\,|\,z)$ is always finite and continuous. Hence, making use of (16) and (17),

$$\int_0^1\frac{dz}{1-z}\left(F-\frac{z}{1-z}\overset{*}{F}{}^{2}+\frac{z^2}{(1-z)^2}\overset{*}{F}{}^{3}-\ldots\right)$$

$$=\int_0^1 e^{-\frac{z}{1-z}(y-x)}\frac{dz}{1-z}+\int_0^1\Psi(x,y\,|\,z)\,(1-z)\,dz$$

$$=-\log\,(y-x)+e^{y-x}\int_{y-x}^{\infty}\log\xi\,e^{-\xi}\,d\xi+\theta(x,y),$$

where $\theta\,(x, y)$ is finite and continuous. It follows that

$$(\overset{*}{F}-\overset{*}{F}{}^{\circ})\int_0^1\frac{dz}{1-z}\left(\overset{*}{F}-\frac{z}{1-z}\overset{*}{F}{}^{2}+\frac{z^2}{(1-z)^2}\overset{*}{F}{}^{3}-\ldots\right)$$

$$=\log\,(y-x)+e^{y-x}\int_{y-x}^{\infty}\log\xi\,e^{-\xi}d\,\xi+\theta(x,y)$$

$$+\int_x^y F(x,\xi)\left\{-\log\,(y-\xi)+e^{y-\xi}\int_{y-\xi}^{\infty}\log\eta\,e^{-\eta}\,d\,\eta+\theta(\xi,y)\right\}d\,\xi$$

$$=\log\,(y-x)+\chi(x,y),$$

where χ is finite and continuous.

We therefore have the theorem: *If $F\,(x, y)$ is such that $F\,(x, x)=1$, then*

$$\overset{*}{F}\,\overset{*}{l}\,F=\log\,(y-x)+\chi(x,y),$$

where $\chi\,(x, y)$ is a finite and continuous function.

The function χ (x, y) can be calculated by obtaining first Φ as the solution of a certain integral equation and then finding Ψ and θ in the manner indicated above.

It is unnecessary, therefore, to know $\overset{*}{F}{}^z$ in order to be able to calculate $\overset{*}{F}\,\overset{*}{l}\,F$, and *the calculation can be carried out by operating directly on the given function* F (cf. Lecture II, § 21).

25. In this connection one other fact should be added: not only is it unnecessary to know $\overset{*}{F}{}^z$ in order to obtain $\overset{*}{F}\,\overset{*}{l}\,F$, but, on the other hand, by means of the latter it is possible to calculate $\overset{*}{F}{}^z$ when $\overset{*}{F}$ is given.

In fact, when $\overset{*}{F}\,\overset{*}{l}\,F$ is known, we can obtain

$$\overset{*}{F}(\overset{*}{l}\,\overset{*}{F})^2,\ \overset{*}{F}(\overset{*}{l}\,F)^3,\ \overset{*}{F}(\overset{*}{l}\,F)^4 \ldots,$$

and therefore, by applying the formula (see Lecture II, § 25),

$$\overset{*}{F}{}^{z+1} = F + \frac{z\,\overset{*}{F}\,\overset{*}{l}\,F}{1!} + \frac{z^2\,\overset{*}{F}(\overset{*}{l}\,F)^2}{2!} + \frac{z^3\,\overset{*}{F}(\overset{*}{l}\,F)^3}{3!} + \ldots,$$

we can obtain $\overset{*}{F}{}^{z+1}$, expressed in terms of a series of powers of z by means of operations performed on F alone (see Lecture I, § 23). The power series thus obtained for $\overset{*}{F}{}^{z+1}$ is always an *entire function.*

We have here verified, in the case of powers of composition, a fact which corresponds to one which we meet with in the case of ordinary powers, namely, that in order to obtain the former in general, it is convenient to use logarithms of composition, just as in order to obtain the latter in general it is convenient to use the logarithms of ordinary algebra.

This is another confirmation of the utility of introducing logarithms of composition.

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