IX

COMPOSITION THEOREMS OF HADAMARD TYPE

One of the most important theorems in the theory of the detection of singularities of a Taylor series, important in itself, as well as a very useful tool, is Hadamard's theorem on "the multiplication of singularities" discovered in 1898 (see for instance [**6**]).

Since we shall have later on in this chapter the opportunity to give this theorem in its most general form, we shall state it now in a form which is true only if the singularities are supposed to be isolated and not critical (not branch points), although many authors, by mistake, considered this statement as being the general one.

If E_1 is the set of the singularities of the function represented by $\sum a_n z^n$, E_2 the set of the singularities of the function represented by $\sum b_n z^n$, and E_3 the set of the singularities of the function represented by $\sum a_n b_n z^n$, then to each point e_3 of E_3 there correspond a point e_1 of E_1 and a point e_2 of E_2 such that $e_3 = e_1 e_2$.

Translated into Taylor-D series this theorem gives the following statement:

If γ is a singularity of the function represented by $\sum a_n b_n e^{-ns}$, there exists a singularity α of the function represented by $\sum a_n e^{-ns}$ and a singularity β of the function represented by $\sum b_n e^{-ns}$ such that $\gamma = \alpha + \beta$.

We have supposed in these statements that the radii of convergence of both series $\sum a_n z^n$, $\sum b_n z^n$ are finite, or, what amounts to the same thing, that the abscissa of convergence of the Taylor-D series is not $+\infty$.

The statement on Taylor-D series is not true for general Dirichlet series. In other words, the function represented by $\sum a_n b_n e^{-\lambda_n s}$ may have singularities, which are not of the form $\alpha+\beta$ where α is a singularity of the function $f(s)$ represented by $\sum a_n e^{-\lambda_n s}$ and β a singularity of the function $\varphi(s)$ represented by $\sum b_n e^{-\lambda_n s}$. This may occur even if each of the two functions $f(s)$ and $\varphi(s)$ has only one singularity. For instance, if $a_n = b_n = 1$, $\lambda_n = \log n$, then each of the functions $f(s)$, $\varphi(s)$ is the $\zeta(s)$-function of Riemann, which is known to have only one singularity, of affix one, which is a simple pole. Thus, if the theorem were true, the function represented by the composite series $(\sum a_n b_n e^{-\lambda_n s})$ should have as its only possible singularity the point of affix 2, but this composite function is, in the given circumstances, itself the function $\zeta(s)$.

The fact that the statement, given above for Taylor series, is not true for general Dirichlet series does not prove that another statement, which reduces to the one above when $\lambda_n = n$, may be true for the general case. We shall give here two theorems on Dirichlet series, one, of which the statement is relatively simple, and a second of a complex nature, but which deals with Dirichlet series of a very general type. Both statements contain Hadamard's theorem as a very particular case.

Although each of the theorems given below is true, with a certain precision of the language, for multiform functions we shall suppose, for the simplification of the language, that the functions represented by the given Dirichlet series are uniform. We shall see later on how these same statements can be modified if the corresponding functions are multiform.

Let $\sum a_n e^{-\lambda_n s}$ be a Dirichlet series with the abscissa of absolute convergence $\sigma_A < \infty$ and suppose that $-\infty < \sigma_1$.

Let Δ be a region, situated in the half-plane $\sigma > \sigma_1$, containing the part of this half-plane for which $\sigma > \sigma_A$ and such that there exists a function $f(s)$ holomorphic in Δ, equal to the sum of the series $\sum a_n e^{-\lambda_n s}$ for $s \in \Delta$, $\sigma > \sigma_A$. If there exists a domain $\Delta(\sigma_1)$ which contains each domain Δ having the properties mentioned, we shall say that the function $f(s)$, holomorphic in $\Delta(\sigma_1)$ and equal to the sum of the series for $s \in \Delta(\sigma_1)$, $\sigma > \sigma_A$, is uniform in the half-plane $\sigma > \sigma_1$. The set composed of all the points of the half-plane $\sigma \geqq \sigma_1$ which are not points of $\Delta(\sigma_1)$ will be denoted by $S_f^{\sigma_1}$ and called the *singular set of* $f(s)$ *with respect to the half-plane* $\sigma > \sigma_1$. The points s such that $\sigma = \sigma_1$ thus always belong to $S_f^{\sigma_1}$. If $\Delta(\sigma_1)$ coincides with the half-plane $\sigma > \sigma_1$, only the points with $\sigma = \sigma_1$ belong to $S_f^{\sigma_1}$. Interesting theorems will be obtained only if $S_f^{\sigma_1}$ contains points different from those situated on the line $\sigma = \sigma_1$, but for the general validity of the results to come it is important to define $S_f^{\sigma_1}$ as we did. When σ_1 is given, the function defined as above in $\Delta(\sigma_1)$ will be denoted by $f(s) = \sum a_n e^{-\lambda_n s}$, although $f(s)$ is actually equal to the sum of the series only in the part of $\Delta(\sigma_1)$ for which $\sigma > \sigma_C^f$. Since, if $\sigma_1' < \sigma_1$, the function defined, as indicated above, by means of the analytical continuation of $\sum a_n e^{-\lambda_n s}$ in the regions $\Delta(\sigma_1')$ and $\Delta(\sigma_1)$ (if it is uniform in both regions) takes, in the intersection of these regions, the same values, the indication of this function by the same letter f is justified. It should be remarked that $\Delta(\sigma_1)\Delta\subset(\sigma_1')$. Thus, to say that "$f(s) = \sum a_n e^{-\lambda_n s}$ is uniform in the half-plane $\sigma > \sigma_1$" means that $\Delta(\sigma_1)$ exists, that $f(s)$ is holomorphic in $\Delta(\sigma_1)$, that this function is given in the part of $\Delta(\sigma_1)$ in which $\sigma > \sigma_C^f$ by the sum of the series $\sum a_n e^{-\lambda_n s}$, and that $f(s)$ cannot be continued analytically to a point of $S_f^{\sigma_1}$ without effecting this continuation on a path crossing the line $\sigma = \sigma_1$.

It is well known that a function represented by a general Dirichlet series does not have necessarily a singularity on the axis of convergence, contrary to what happens in the case of a Taylor-D Series. For instance, the function $f(s)$ represented by the series $\sum \frac{(-1)^n}{n^s}$ is an entire function and yet here $\sigma_C{}^f=0$ ($\sigma_A{}^f=1$). But, let H_f be the greatest lower bound of all the quantities σ' such that $f(s)$ is holomorphic if $\sigma>\sigma'$ and is given in the part of this half-plane in which $\sigma>\sigma_C{}^f$ by $\sum a_n e^{-\lambda_n s}$. This quantity H_f is called the *abscissa of holomorphism of* $f(s)=\sum a_n e^{-\lambda_n s}$, [**3**]. Obviously

$$H_f \leqq \sigma_C{}^f \leqq \sigma_A{}^f.$$

The line represented by $\sigma=H_f$ is called the axis of holomorphism of the series. The quantity

$$H_f(\sigma_1)=\underset{s\,\epsilon\, S_f^{\sigma_1}}{\text{l. u. b.}}\ \sigma$$

shall be called the *abscissa of holomorphism of* $f(s)=\sum a_n e^{-\lambda_n s}$ *in the half-plane* $\sigma>\sigma_1$. Obviously $H_f(\sigma_1)=H_f$ if $\sigma_1 \leqq H_f$, and $H_f(\sigma_1)=\sigma_1$ if $\sigma_1>H_f$.

To say that the *only possible singularities of* $\sum a_n e^{-\lambda_n s}$ *in the half-plane* $\sigma>\sigma_1$ *are points of the closed set* E will mean that, on denoting by $P(\sigma_1)$ this half-plane and by $(P(\sigma_1)-E)$ the part of the difference of sets $P(\sigma_1)-E$ which constitutes a region[1] containing a half-plane $\sigma>\sigma_2$ in which $\sum a_n e^{-\lambda_n s}$ converges, the function $f(s)$ is holomorphic in $(P(\sigma_1)-E)$ and is given in the part of this region for which $\sigma>\sigma_2$ by the sum of the series.

It should be remarked that, if $\sigma_1'>\sigma_1$, $S_f^{\sigma_1'}$ does not necessarily contain all the points of $S_f^{\sigma_1}$. This is the reason why we use, in the definition of $S_f^{\sigma_1}$, the expression "with respect to the half-plane $\sigma>\sigma_1$," instead of "in the half-plane $\sigma>\sigma_1$."

[1]A region is a *connected* set of interior points.

Let $f(s)=\sum a_n e^{-\lambda_n s}$ be uniform in the half-plane $\sigma>\sigma_1$, and let us set

$$S_f^{\sigma_1}(\epsilon)=\bigcup_{s\in S_f^{\sigma_1}} C(s,\epsilon),^1$$

where $C(s,\epsilon)$ is the open circle of center s and radius ϵ. If to every $\epsilon>0$ there corresponds a constant $K(\epsilon)(=K(\epsilon,\sigma_1))$ such that in the closed region $\Delta(\sigma_1,\epsilon)=\overline{(P(\sigma_1)-S_f^{\sigma_1}(\epsilon))}$,[2] $|f(s)|<K(\epsilon)$, we shall say that $f(s)$ is *bounded in the half-plane* $\sigma>\sigma_1$, *except for singularities*, or "bounded e. f. s." It is obvious that each Taylor-D series, $f(s)=\sum a_n e^{-ns}$, which is uniform in the half-plane $\sigma>\sigma_1$, is bounded e. f. s. in this half-plane. This is, however, not true for general Dirichlet series. For instance, $\zeta(s)=\sum\frac{1}{n^s}$ is not bounded e. f. s. in the half-plane $\sigma>1-\delta$, if $\delta>0$.

Suppose now that the function $f(s)=\sum a_n e^{-\lambda_n s}$, still uniform in the half-plane $\sigma>\sigma_1$, is such that there exists a non-negative constant m having the following property: to each $\epsilon>0$ there corresponds a constant $K(\epsilon,m)(=K(\epsilon,m,\sigma_1))$ such that in $\Delta(\sigma_1,\epsilon)$: $|f(s)|<K(\epsilon,m)|t|^m$, if $|t|$ is sufficiently large. Let μ be the greatest lower bound of such quantities m. Thus to each $\delta>0$ and each $\epsilon>0$ there corresponds a constant $N(\epsilon,\delta)(=N(\epsilon,\delta,\sigma_1))$ such that

$$|f(s)|<N(\epsilon,\delta)|t|^{\mu+\delta}$$

in $\Delta(\sigma_1,\epsilon)$, for $|t|$ sufficiently large, and no quantity smaller than μ has this property. We shall say that $f(s)$ is *of order* μ *except for singularities*, or of order μ e. f. s., *in the half-plane* $\sigma>\sigma_1$.

Let us denote by $Q(\sigma_1,t_0)$ the "quarter-plane" given by $\sigma>\sigma_1$, $t>t_0$. Suppose that the series $\sum a_n e^{-\lambda_n s}$ has the fol-

[1] $\bigcup_{a\in A} L(a)$ means the union of all the sets $L(a)$ (depending on the parameter a) as a takes all the possible values in the set A.

[2] $\Delta(\sigma_1,\epsilon)$ is therefore also connected, closed, and contains a half-plane $\sigma>\sigma_2$.

lowing properties: let a region Δ and a quarter-plane $Q(\sigma_2, t_0)$ be such that $Q(\sigma_1, t_0)\supset\Delta\supset Q(\sigma_2, t_0)$ and such that there exists a function $f(s)$ holomorphic in Δ, given in $Q(\sigma_2, t_0)$, by the sum of the series; we shall suppose that there exists a region $D(\sigma_1, t_0)$ which contains all the regions Δ having the properties just described. We shall then say that $f(s)=\sum a_n e^{-\lambda_n s}$ is *uniform in the "quarter-plane"* $Q(\sigma_1, t_0)$. We shall denote by $S_f^{\sigma_1, t_0}$ the set of all the points s satisfying the inequalities $\sigma \geqq \sigma_1$, $t>t_0$ which are not points of $D(\sigma_1, t_0)$. The set $S_f^{\sigma_1, t_0}$ shall be called *the singular set of $f(s)$ with respect to the quarter-plane* $Q(\sigma_1, t_0)$. We shall set

$$S_f^{\sigma_1, t_0}(\epsilon) = \bigcup_{s \in S_f^{\sigma_1, t_0}} C(s, \epsilon).$$

If to every $\epsilon>0$ there corresponds a constant

$$K(\epsilon)\,(=K(\epsilon, \sigma_1, t_0))$$

such that in the closed region $\Delta(\sigma_1, t_0, \epsilon)=\overline{(Q(\sigma_1, t_0)-S_f^{\sigma_1, t_0}(\epsilon))}$, which is a part of the closure of $Q(\sigma_1, t_0)-S_f^{\sigma_1, t_0}(\epsilon)$ connected with a quarter-plane $Q(\sigma_2, t_1)$, we have $|f(s)|<K(\epsilon)$, we shall say that $f(s)$ is *bounded in $Q(\sigma_1, t_0)$ except for singularities* (e. f. s.). Obviously if $f(s)$ is uniform and bounded e. f. s. in the half-plane $\sigma>\sigma_1$, it is also uniform and bounded e. f. s. in every quarter-plane $Q(\sigma_1, t_0)$ with t_0 arbitrary. For each t_0 such that $f(s)$ is holomorphic, uniform, and bounded in $Q(\sigma_1, t_0)$, (σ_1 fixed), let us set

$$\sigma_1^*(t_0) = \underset{s \in S_f^{\sigma_1, t_0}}{\text{l. u. b.}}\ \sigma,$$

and let us also set

$$H_f^+(\sigma_1) = \lim_{t_0=\infty} \sigma_1^*(t_0).$$

Obviously $H_f^+(\sigma_1)$ exists and $H_f^+(\sigma_1)\geqq\sigma_1$. This quantity $H_f^+(\sigma_1)$ shall be called the *ultimate (+) abscissa of holomorphism of $f(s)$ in the half-plane $\sigma>\sigma_1$.*

If A and B are two sets of complex numbers, we shall call the set composed of all the numbers $\alpha+\beta$, where $\alpha \in A$, $\beta \in B$, the *sum-composite* set of the sets A and B.

It should be remarked that, even if both sets A and B are closed sets, their sum-composite is not necessarily closed.

Since the first of the theorems given in this chapter bears on functions represented by Dirichlet series which are bounded e. f. s. either in a half-plane or in a quarter-plane, it is important to show that Dirichlet series, which are not Taylor-D series (nor series of the form $\sum a_n e^{-kns}$ with k constant), exist which have such properties. Let us suppose that $f(s) = \sum a_n e^{-\lambda_n s}$ with $\sigma_A{}^f < 0$, the λ_n being arbitrary, and let $T(z) = \sum c_n z^n$ be an entire function. Consider the function

$$F(s) = T\left(f(s) + \frac{1}{1-e^{-s}}\right) = T(\sum a_n e^{-\lambda_n s} + \sum e^{-ns})$$
$$= \sum_m c_m (\sum_n a_n e^{-\lambda_n s} + \sum_n e^{-ns})^m.$$

Since for $\sigma > 0$ the series $\sum_m |c_m| (\sum_n |a_n| e^{-\lambda_n \sigma} + \sum_n e^{-n\sigma})^m$ converges, we see that $F(s)$ is represented for $\sigma > 0$ by a Dirichlet series

$$F(s) = \sum d_n e^{-\nu_n s},$$

where each ν_n is of the form $\nu_k = \sum_1^p \alpha_i \lambda_i + \beta$, where the α_i and β are non-negative integers. On the other hand, in the half-plane $\sigma > \sigma_A{}^f$ the function $f(s) + \frac{1}{1-e^{-s}}$ has only simple poles of affixes $2k\pi i$. In the region $\sigma > \sigma_A{}^f + \epsilon$, outside the circles $C(2k\pi i, \epsilon)$, this function is bounded. Therefore $F(s)$ is bounded there. But inside each such circle the values taken by $f(s) + \frac{1}{1-e^{-s}}$ cover a neighborhood of infinity, and by Picard's theorem $F(s)$ takes in these circles each value except at most one, infinitely many times. Thus, the points of affixes $2k\pi i$ are isolated essential singularities. $F(s)$ has therefore the required properties.

The theorem we shall now prove contains Hadamard's theorem on Taylor series, but it deals with far less general series than Theorem XXIII.

THEOREM XXXII. *Suppose that* $f(s)=\sum a_n e^{-\lambda_n s}$, *with* $\sigma_A{}^f<\infty$, *is uniform and bounded e.f.s. in the quarter-plane* $\sigma>\sigma_1$, $t>t_0$, *and suppose that* $\varphi(s)=\sum b_n e^{-\lambda_n s}$ *with* $\sigma_A{}^\varphi<\infty$, *is uniform and bounded e.f.s. in the half-plane* $\sigma>\sigma_2$. *Let* $S_f^{\sigma_1,t_0}$ *be the singular set of* $f(s)$ *with respect to the quarter-plane* $\sigma>\sigma_1$, $t>t_0$, *and* $S_\varphi^{\sigma_2}$ *the singular set of* $\varphi(s)$ *with respect to the half-plane:* $\sigma>\sigma_2$. *Let* $H_f^+(\sigma_1)$ *be the ultimate* (+) *abscissa of holomorphism of* $f(s)$ *in* $\sigma>\sigma_1$ *and* $H_\varphi(\sigma_2)$ *the abscissa of holomorphism of* $\varphi(s)$ *in* $\sigma>\sigma_2$.

The series $\sum a_n b_n e^{-\lambda_n s}$ *has an abscissa of absolute convergence not greater than* $\sigma_A{}^f+\sigma_A{}^\varphi$, *the function* $F(s)=\sum a_n b_n e^{-\lambda_n s}$ *is uniform in the half-plane* $\sigma>H_f^+(\sigma_1)+\sigma_2$ *and the only possible singularities of* $F(s)$ *in this half-plane are the points of the sum-composite set of the sets* $S_f^{\sigma_1,t_0}$, *the limit-points of this set, and the points of* $S_\varphi^{\sigma_2}$.

Let c and t_1 be chosen such that $c>\sigma_A{}^f$, $t_1>t_0$ and consider

$$F_T(z)=\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)\varphi(z-s)ds,\ (T>t_1), \tag{103}$$

the path of integration being on the line $\sigma=c$. $F_T(z)$ is holomorphic if $x=\mathcal{R}(z)>c+\sigma_A{}^\varphi$, since then $z-s$, s varying on the path of integration, lies in the half-plane $\sigma>\sigma_A{}^\varphi$. For such a value of z we have

$$F_T(z)=\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)\varphi(z-s)ds$$

$$=\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)\sum b_n e^{-\lambda_n(z-s)}ds=\sum b_n e^{-\lambda_n z}\cdot\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)e^{\lambda_n s}ds.$$

But by Theorem IX:

$$\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)e^{\lambda_n s}ds=a_n+\epsilon_n(T), \tag{104}$$

with $\lim_{T=\infty}\epsilon_n(T)=0$. Therefore

$$F_T(z)=\sum a_n b_n e^{-\lambda_n z}+\sum \epsilon_n(T)b_n e^{-\lambda_n z}.$$

The series $\sum a_n b_n e^{-\lambda_n z}$ converges absolutely for $x > \sigma_A{}^{\varphi} + c$, since

$$\sum_{p+1}^{\infty} |a_n b_n| e^{-\lambda_n x} = \sum_{p+1}^{\infty} |b_n| e^{-\lambda_n x} \left| \lim_{T=\infty} \frac{1}{iT} \int_{c+it_1}^{c+iT} f(s) e^{\lambda_n s} ds \right|$$
$$\leqq \mathcal{M}(c) \sum_{p+1}^{\infty} |b_n| e^{-\lambda_n (x-c)},$$

where

$$\mathcal{M}(c) = \underset{-\infty < t < \infty}{\text{l. u. b.}} |f(c+it)| \leqq \sum |a_n| e^{-\lambda_n c},$$

and $\sum |b_n| e^{-\lambda_n (x-c)} < \infty$ for $x - c > \sigma_A{}^{\varphi}$. Thus $\sum a_n b_n e^{-\lambda_n s}$ converges absolutely for $\sigma \geqq \sigma_A{}^f + \sigma_A{}^{\varphi}$.

For $x > \sigma_A{}^{\varphi} + c$ we have then, for each p:

$$|F_T(z) - F(z)| = \left| \sum_1^{\infty} b_n e^{-\lambda_n z} (a_n + \epsilon_n(T)) - \sum_1^{\infty} a_n b_n e^{-\lambda_n z} \right|$$
$$\leqq \sum_1^{\infty} |b_n| \, |\epsilon_n(T)| e^{-\lambda_n x}.$$

It follows from the definition of $\epsilon_n(T)$ that (see (104)):

$$|\epsilon_n(T)| < 2\mathcal{M}(c) e^{\lambda_n c},$$

hence, for each p:

$$|F_T(z) - F(z)| \leqq \sum_1^{p} |b_n| \, |\epsilon_n(T)| e^{-\lambda_n x} + 2\mathcal{M}(c) \sum_{p+1}^{\infty} |b_n| e^{-\lambda_n (x-c)},$$

which proves, since (when p is fixed) $\epsilon_n(T) \to 0$ as $T \to \infty$ $(n = 1, \cdots p)$, and $\sum_{p+1}^{\infty} |b_n| e^{-\lambda_n (x-c)} \to 0$ with $p \to \infty$, $(x - c > \sigma_A{}^{\varphi})$, that $\lim_{T=\infty} F_T(z) = F(z)$, as $x > \sigma_A{}^{\varphi} + c$.

Let us now suppose that $c > \max(\sigma_A{}^f, \sigma_1)$ and let us pave the strip $\sigma_1 \leqq \sigma \leqq c$ with squares of side $\epsilon = \dfrac{c - \sigma_1}{q}$ (q integer), and let $\mathcal{D}(\epsilon)$ be the set composed of all the squares which neither contain (in their closure) a point of $S_f{}^{\sigma_1, t_0}$, nor are contiguous to such a square. If ϵ is sufficiently small (if q is sufficiently large), all the squares bordering the line $\sigma = c$ belong to $\mathcal{D}(\epsilon)$. We shall suppose that ϵ is chosen in such a way. Let then $\mathcal{D}^*(\epsilon)$ be the greatest region belonging to $\mathcal{D}(\epsilon)$ and containing the squares bordering the line $\sigma = c$, and $\mathcal{D}^*(\epsilon, t_1)$ be the part of $\mathcal{D}^*(\epsilon)$ for which $t > t_1$. By the properties of $f(s)$ this function is holomorphic and bounded in $\mathcal{D}^*(\epsilon, t_1)$. Let $C(\epsilon)$ be the boundary of $\mathcal{D}^*(\epsilon, t_1)$. Let $\{\alpha_m\}$ be a sequence of quantities increasing to infinity with m, such that $\alpha_m > t_1$, $(m \geqq 1)$, and let us denote by $\mathcal{D}^*(\epsilon, t_1, m)$ the part of $\mathcal{D}^*(\epsilon, t_1)$ for which $t < \alpha_m$, and by $C(\epsilon, t_1, m)$ the part of its frontier which does not contain the points of $\sigma = c$. From Cauchy's theorem it follows that, if $x > \sigma_A{}^\varphi + c$ (the integral being taken in an appropriate sense):

$$(105) \qquad F_{\alpha_m}(z) = \frac{1}{i\alpha_m} \int_{C(\epsilon, t_1, m)} f(s)\varphi(z-s)ds,$$

and thus, still for $x > \sigma_A{}^\varphi + c$:

$$F(z) = \lim_{m=\infty} \frac{1}{i\alpha_m} \int_{C(\epsilon, t_1, m)} f(s)\varphi(z-s)ds.$$

Let l_m be the set of the points forming the segments of $C(\epsilon, t_1, m)$ for which $t = \alpha_m$, and L the set of the points forming the segments of $C(\epsilon, t_1, m)$ for which $t = t_1$. Since the total length of each of those sets of segments for a given m, and that of L, is not larger than $c - \sigma_1$, and since on these segments $|f(s)|$ and $|\varphi(z-s)|$ (if $x > \sigma_A{}^\varphi + c$) are bounded, we see that, on denoting by $C'(\epsilon, t_1, m)$ the set $C(\epsilon, m) - l_m - L$, we have also:

$$(106) \qquad F(z) = \lim_{m=\infty} \frac{1}{i\alpha_m} \int_{C'(\epsilon, t_1, m)} f(s)\varphi(z-s)ds,$$

where t_1 is fixed and satisfies only the inequality $t_1 > t_0$.

Let now Δ be a bounded region in the z-plane such that it contains points with $x > \sigma_A{}^\varphi + c$, such that for each $z \in \Delta$: $x > H_f{}^+(\sigma_1) + \sigma_2 - 3\epsilon$, and such that, on denoting by $S_{f,\varphi}$ the sum-composite of $S_f{}^{\sigma_1, t_0}$ and $S_\varphi{}^{\sigma_2}$, the inequality $|z - \gamma| > 6\epsilon$ holds for each $z \in \Delta$ and each $s \in S_{f,\varphi}$. If t_1 is chosen sufficiently large, then for each $s \in C(\epsilon, t_1, m)$, $z \in \Delta$, $\beta \in S_\varphi^{\sigma_2}$ the inequalities

$$\mathcal{R}(z-s)=x-\sigma>\sigma_2, \quad |z-s-\beta|>\epsilon$$

are satisfied. Indeed, if t_1 is sufficiently large, we have for each $s \in C(\epsilon, t_1, m)$: $\sigma < H_f^+(\sigma_1)+3\epsilon$, and, if $z \in \Delta$: $x-\sigma>\sigma_2$; on the other hand, if for a $z' \in \Delta$, an $s' \in C(\epsilon, t_1, m)$ and a $\beta' \in S_\varphi^{\sigma_2}$ we had had $|z'-s'-\beta'|<\epsilon$, since to each s' of $C(\epsilon, t_1, m)$ there corresponds an $\alpha' \in S_f^{\sigma_1, t_0}$ such that $|s'-\alpha'|<2\sqrt{2}\epsilon$, we would have $|z'-\alpha'-\beta'| \leqq |z'-s'-\beta'| + |s'-\alpha'| < (2\sqrt{2}+1)\epsilon$, contrary to the supposition that $|z'-\alpha'-\beta'|>6\epsilon$. Thus, with our choice of t_1 if $s \in C(\epsilon, t_1, m)$, $z \in \Delta$, the variable $u=z-s$ lies in the part of the half-plane $\sigma>\sigma_2$ which is connected with the half-plane $\sigma>\sigma_A^\varphi$, and in which the function $\varphi(s)$ is holomorphic, uniform, and bounded. The functions $F_{\alpha_m}(z)$ defined by (105) constitute thus a bounded family of holomorphic functions in Δ. But in a closed part of Δ situated in the half-plane $x>\sigma_A^\varphi+c$, the series $F_{\alpha_m}(z)$ converges uniformly towards $F(z)=\sum a_n b_n e^{-\lambda_n z}$, and by a classical theorem of Stieltjes-Vitali [**15**] $F_{\alpha_m}(z)$ converges uniformly to a holomorphic function in each closed part of Δ. This limit is the analytic continuation of $F(z)$. We have therefore proved that the series $\sum a_n b_n e^{-\lambda_n z}$ can be continued analytically in each region Δ described above, and this proves the theorem.

This theorem was proved by the author in 1929 [**8**], in a different wording and in a somewhat more general form. In 1940 S. Bochner [**4**] proved a very similar theorem in which the sets $S_f^{\sigma_1, t_0}$, S_φ, $S_{f,\varphi}$ are replaced by the boundaries of the stars of the corresponding functions.[1] The boundary of a star of a function $f(s)$ represented by $\sum a_n e^{-\lambda_n s}$ is the

[1]S. Bochner mentions in his paper Theorem XXXIII which was reproduced in Bernstein's book [**3**], but he does not seem to have had knowledge of Theorem XXXII. It should be remarked that Bochner states his theorem also for general almost-periodic functions.

set of the points composed of all the half-lines $t=t^*$, $\sigma \leqq \sigma^*$, which have the following property: in the region remaining after the extraction of these half-lines the function represented by $\sum a_n e^{-\lambda_n s}$ can be continued analytically, this being not true if one of the half-lines is replaced by $t=t^*$, $\sigma<\sigma_1^*$ with $\sigma_1^*<\sigma^*$. There is nothing to change in the proof of Theorem XXXII if in its statements the sets $S_f^{\sigma_1, t_0}$, $S_\varphi^{\sigma_2}$ are, respectively, replaced by $E_f^{\sigma_1, t_0}$, $E_\varphi^{\sigma_2}$ where $E_f^{\sigma_1, t_0}$ is the set composed of the part of the boundary of the star of $f(s)$ situated in the quarter-plane $\sigma>\sigma_1$, $t>t_0$ and the half-line $\sigma=\sigma_1$, $t>t_0$, and where $E_\varphi^{\sigma_2}$ is the boundary of the part of the star of $\varphi(s)$ situated in the half-plane $\sigma>\sigma_2$. The introduction of the stars allows us to avoid the introduction of the uniformity of the function. It should, however, be remarked that, on introducing stars (at least for uniform functions), the region in which it can be asserted that the composite function is holomorphic is diminished. On the other hand, in order to be able to state Theorem XXXII as well, for multiform functions as for uniform ones, curvilinear stars can be introduced. In other words, the set $E_\varphi^{\sigma_2}$, for instance, can be replaced by a set of simple Jordan arcs which connect the line $\sigma=\sigma_2$ to different points in the half-plane $\sigma>\sigma_2$ and which are such that in the remaining part of this half-plane $\sum a_n e^{-\lambda_n s}$ can be continued analytically. The reader will see for himself how $E_f^{\sigma_1, t_0}$ could be conveniently replaced. The arcs, for each function $f(s)$ and $\varphi(s)$, respectively, should be chosen in such a way that they do not intersect each other, except if one of these arcs is a part of another. The fact that these arcs can be taken in a very arbitrary way allows the determination of a very general region of holomorphism for the composite function.

Theorem XXXII contains Hadamard's theorem, since for Taylor-D series σ_1 and σ_2 can be taken arbitrarily (if the functions are uniform; for multiform functions the introduction of the arcs described above is essential). The fact that t_0 of the statement can be taken arbitrarily does not improve the theorem for the Taylor-D case, since the functions $f(s)$ and $\varphi(s)$ are periodic with period $2\pi i$. But the possibility of an arbitrary choice of t_0 does play an important rôle in the applications of Theorem XXXII [**5**]. It should be remarked that for two Taylor-D series the sum-composite of $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$ is closed, since the points of both sets $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$ are distributed periodically with period $2\pi i$.

Before we begin to deal with Dirichlet series of a very general character, let us give the following definition. Let $\{\mu_n\}$ be a sequence of positive quantities strictly increasing to infinity, let $f(s) = \sum a_n e^{-\lambda_n s}$, and let k be a positive integer. Let n_0 be the smallest integer n such that $\mu_n > \lambda_1$, and let us set:

$$a_n^{(k)} = 0 \quad (0 < n < n_0) \quad (\text{if } n_0 > 1)$$
$$a_n^{(k)} = \sum_{\lambda_m < \mu_n} (\mu_n - \lambda_m)^k a_m, \quad (n \geqq n_0).$$

The quantity $a_n^{(k)}$ shall be called the n^{th} *coefficient of $f(s)$ with respect to the sequence $\{\mu_n\}$, of order k.*

THEOREM XXXIII. *Suppose that $f(s) = \sum a_n e^{-\lambda_n s}$ with $\sigma_A{}^f < \infty$, is uniform and of order ν e. f. s., in the half-plane $\sigma > \sigma_1$, and that $\varphi(s) = \sum b_n e^{-\mu_n s}$, with $\sigma_A{}^\varphi < \infty$, is uniform and of order μ e. f. s. in the half-plane $\sigma > \sigma_2$.*

Let $S_f^{\sigma_1}$ be the singular set of $f(s)$ with respect to the half-plane $\sigma > \sigma_1$, and $S_\varphi^{\sigma_2}$ the singular set of $\varphi(s)$ with respect to the half-plane $\sigma > \sigma_2$. Let $H_f(\sigma_1)$ be the abscissa of holomorphism of $f(s)$ in $\sigma > \sigma_1$.

If k is an integer such that $k > \nu + \mu$, the series $\sum a_n^{(k)} b_n e^{-\mu_n s}$, where $a_n^{(k)}$ is the n^{th} coefficient of $f(s)$ with respect to the sequence $\{\mu_n\}$ of order k, has an abscissa of absolute convergence not larger than max $(\sigma_A{}^\varphi, \sigma_A{}^f + \sigma_A{}^\varphi)$, *the function $F_k(s) = \sum a_n^{(k)} b_n e^{-\mu_n s}$*

is uniform in the half-plane $\sigma > \text{Max}(H_f(\sigma_1), 0) + \sigma_2$, *and the only possible singularities of* $F_k(s)$ *in this half-plane are the points of the sum-composite of the sets* $S_f^{\sigma_1}$ *and* $S_\varphi^{\sigma_2}$, *the limit-points of this set, and the points of* $S_\varphi^{\sigma_2}$.

Let $c > \max(0, \sigma_A{}^f)$, and consider:

$$F(z) = \frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{f(s)\varphi(z-s)}{s^{k+1}}\,ds, \tag{107}$$

the path of integration being the line $\sigma = c$. $F(z)$ is holomorphic if $x = \mathcal{R}(z) > c + \sigma_A{}^\varphi$, since then $z - s$, s varying on the path of integration, lies in the half-plane $\sigma > \sigma_A{}^\varphi$. For such values of z we may write:

$$\begin{aligned}(108)\qquad &\frac{k!}{2\pi}\int_{c-i\infty}^{c+i\infty} f(s) \sum b_n e^{-\mu_n(z-s)} \frac{ds}{s^{k+1}} \\ &= \sum b_n e^{-\mu_n z}\cdot \frac{k!}{2\pi}\int_{c-i\infty}^{c+i\infty} f(s) e^{\mu_n s}\frac{ds}{s^{k+1}} \\ &= \sum_n b_n e^{-\mu_n z}\frac{k!}{2\pi}\int_{c-i\infty}^{c+i\infty} \sum_m a_m e^{(\mu_n-\lambda_m)s}\frac{ds}{s^{k+1}},\end{aligned}$$

and by Theorem X

$$F(z) = \sum_{n=n_0}^{\infty} b_n e^{-\mu_n z} \sum_{\lambda_m < \mu_n} a_m(\mu_n - \lambda_m)^k = \sum_{1}^{\infty} a_n^{(k)} b_n e^{-\mu_n z}, \tag{109}$$

n_0 being the smallest integer n such that there exists a quantity λ_m smaller than μ_n.

Obviously the last series converges for $x > c + \sigma_A{}^\varphi$. But in (108) all the equalities hold if $f(s)$ is replaced by $\sum |a_n| e^{-\lambda_n s}$, $\varphi(s)$ by $\sum |b_n| e^{-\mu_n s}$.

Therefore the series

$$\sum_{n=n_0}^{\infty} \sum_{\lambda_m < \mu_n} |a_m| (\mu_n - \lambda_m)^k b_n e^{-\mu_n x}$$

converges for $x > c + \sigma_A{}^\varphi$, and so does the series $\sum |a_n^{(k)} b_n| e^{-\mu_n x}$, which proves that the abscissa of absolute convergence of $\sum a_n^{(k)} b_n e^{-\lambda_n z}$ is not greater than $\max(\sigma_A{}^\varphi, \sigma_A{}^f + \sigma_A{}^\varphi)$, since c has only to be such that

$$c > \max(0, \sigma_A{}^f).$$

The equality (109) proves that $F(z) = F_k(z)$.

Let us now set, moreover, $c > \max(0, \sigma_1, \sigma_A{}^f)$, and, as for the proof of Theorem XXXII, let us pave the strip $\sigma_1 \leqq \sigma \leqq c$ with squares of side $\epsilon = \frac{c - \sigma_1}{q}$ (q integer). Let $\mathcal{D}_1(\epsilon)$ be the set composed of all the squares which contain (in their closure) neither a point of $S_f^{\sigma_1}$ nor the point $s = 0$, and which are not contiguous to such squares.

If ϵ is sufficiently small, all the squares bordering on the line $\sigma = c$ will belong to $\mathcal{D}_1(\epsilon)$. Let us choose ϵ so that this is the case, and let $\mathcal{D}_1^*(\epsilon)$ be the greatest region belonging to $\mathcal{D}_1(\epsilon)$ and containing the squares bordering the line $\sigma = c$. The function $f(s)$ is holomorphic in $\mathcal{D}_1^*(\epsilon)$, and to each $\delta_1 > 0$ there corresponds a constant $N_1(\delta_1)(= N_1(\delta_1, \epsilon, \sigma_1))$ such that, in $\mathcal{D}_1^*(\epsilon), |f(s)| < N_1(\delta_1)|t|^{\mu + \delta_1}$, for $|t|$ sufficiently large.

On denoting by $\mathcal{D}_1^*(\epsilon, T)$ the set of points of $\mathcal{D}_1^*(\epsilon)$ for which $|t| < T$, let $C_1^*(\epsilon, T)$ be the part of its boundary which does not contain the points of $\sigma = c$. For $x > \sigma_A{}^\varphi + c$ we have, by Cauchy's theorem

$$\int_{C_1^*(\epsilon, T)} f(s)\varphi(z - s)\frac{ds}{s^{k+1}} = \int_{c - iT}^{c + iT} f(s)\varphi(z - s)\frac{ds}{s^{k+1}},$$

the first integral being taken, in an appropriate sense, on $C_1^*(\epsilon, T)$, the second on the straight-line segment joining the two points $c - iT$ and $c + iT$. If L^+ and L^- are the set of segments of $C_1^*(\epsilon, T)$ on which we have, respectively, $t = T$, $t = -T$, we see immediately that the integrals extended over these sets of segments tend to zero as $T \to \infty$. This proves that, for $x > \sigma_A{}^\varphi + c$:

$$F_k(z) = \frac{k!}{2\pi i}\int_{C_1^*(\epsilon)} f(s)\varphi(z - s)\frac{ds}{s^{k+1}}, \tag{110}$$

where $C_1^*(\epsilon)$ is the part of the boundary of $\mathcal{D}_1^*(\epsilon)$ which does not contain the points of $\sigma = c$.

Let now Δ be a bounded region of the z-plane, such that it contains points with $x > \sigma_A^\varphi + c$, such that for each $z \in \Delta$: $x > \max(H_f(\sigma_1), 0) + \sigma_2 + 2\epsilon$, and such that, on denoting by $S_{f\varphi}^*$ the union of the sum-composite of the two sets $S_f^{\sigma_1}$, $S_\varphi^{\sigma_2}$ and of the set $S_\varphi^{\sigma_2}$ ($S_{f\varphi}^*$ is therefore the sum-composite of two sets, one of which is composed of the points of $S_f^{\sigma_1}$ and the point $s = 0$, the other being the set $S_\varphi^{\sigma_2}$), the inequality $|z - \gamma| > 6\epsilon$ holds for each $z \in \Delta$, and each $\gamma \in S_{f\varphi}^*$. The inequalities $\mathcal{R}(z - s) = x - \sigma > \sigma_2$, $|z - s - \beta| > \epsilon$ are then satisfied for each $z \in \Delta$, $s \in C_1^*(\epsilon)$, and $\beta \in S_\varphi^{\sigma_2}$. This can be proved in a manner analogous to that in which we have proved a similar fact in proving Theorem XXXII. Thus, if $s \in C_1^*(\epsilon)$, $z \in \Delta$, the variable $u = z - s$ lies in the part of the half-plane $\sigma > \sigma_2$ which is connected with the half-plane $\sigma > \sigma_A^\varphi$ and in which the function $\varphi(s)$ is holomorphic, uniform, and is such that to each $\delta_2 > 0$ there corresponds a constant $N_2(\delta_2)\,(= N_2(\delta_2, \epsilon, \sigma_2))$ with the property that $|\varphi(s)| < N_2(\delta_2)|t|^{\nu+\delta_2}$, for $|t|$ sufficiently large. Since the region Δ is bounded, we see thus that, if $s \in C_1^*(\epsilon)$, $z \in \Delta$, then $|\varphi(z - s)| < N_3(\delta_2)|t|^{\nu+\delta_2}$ for $|t|$ sufficiently large. Since $\nu + \mu < k$, the constants δ_1 and δ_2 can be chosen in such a manner as to have $\mu + \nu + \delta_1 + \delta_2 < k$, and then the integral (110) converges uniformly with respect to z as z varies in any fixed closed region $\overline{\Delta}_1 \subset \Delta$. Therefore $F_k(z)$ is holomorphic in Δ. This obviously proves our theorem.

This theorem was proved by the author, in the same paper as Theorem XXXII [**8**] in a somewhat more general and precise form. The statement given here is somewhat easier to prove than the original one. Some simplifications of the details are taken from the proof of the author's theorem given by V. Bernstein [**3**]. Let us remark that, in using the integral (110), introduced by the author, D. Widder proved a particular case of Theorem XXXIII [**21**].

Hadamard's theorem is contained in Theorem XXX, at least when the Taylor series in question are supposed to represent uniform functions. Let indeed

$$\psi(s) = \sum_{2}^{\infty} d_n e^{-ns}, \quad \varphi(s) = \sum_{1}^{\infty} b_n e^{-ns},$$

and let $\alpha = e^{s_0}$ be one of the singularities of $\psi(s)$ on the axis of convergence. $\psi_1(s) = \sum d_n \alpha^n e^{-ns} = \sum A_n e^{-ns}$ admits the point $s=0$ as a singularity. Let us now set $f(s) = (1-e^{-s})^2 e^s \psi_1(s) = (1-e^{-s})^2 e^s \sum_{2} A_n e^{-ns} = \sum_{1} a_n e^{-ns}$. We have, as is readily seen,

$$A_n = (n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1} = \sum_{m<n} (n-m)a_m, \ (n \geqq 2).$$

The singularities of $f(s)$ are those of $\psi_1(s)$ except perhaps the points $2k\pi i$, which are singular for $\psi_1(s)$, but may be regular for $f(s)$. Since for the Taylor-D series, $f(s)$, $\varphi(s)$, the quantities σ_1 and σ_2 can be taken as negative numbers, arbitrarily large in absolute value, and since, with any such choice of σ_1 and σ_2, k can be taken equal to unity, we see, by Theorem XXX, that the only possible singularities of $F(s) = \sum a_n^{(1)} b_n e^{-ns}$, where $a_n^{(1)} = \sum_{m<n} a_m(n-m) = A_n \ (n>1)$, $a_1^{(1)} = 0$, are the points of the sum-composite $S_{f\varphi}$ of the sets S_f and S_φ and the points of S_φ, where S_f is the set of singularities of $f(s)$ and S_φ the set of singularities of $\varphi(s)$,[1] in the whole plane. If we denote by S_{ψ_1} the set of singularities of $\psi_1(s)$, since this function admits as singularities all the points of S_f and the point $s=0$, we see that the sum-composite of S_{ψ_1} and S_φ is the union of the two sets $S_{f\varphi}$ and S_φ. In other words, the only possible singularities of $F(s) = \sum A_n b_n e^{-ns} = \sum d_n b_n e^{ns_0} e^{-ns}$ are the points of the sum-composite of the sets S_{ψ_1} and S_φ. On denoting by S_ψ the set of singularities of $\psi(s)$, it is obvious that each point of

[1]For a uniform function the set of singularities is, by the definition we have adopted in various circumstances, the complement to the region of existence of the function (and not only the boundary of this region).

S_ψ is obtained by adding $-s_0$ to an affix of a point of S_{ψ_1}. Therefore, the only possible singularities of $\Phi(s)=\sum d_n b_n e^{-ns}$ are those of the sum-composite of the sets S_ψ and S_φ. This constitutes the most precise form of Hadamard's theorem, at least when the functions are supposed to be uniform. If the functions are multiform, the statement of Theorem XXX can easily be adapted, on introducing, as we have said above in speaking of Theorem XXXII, simple arcs of Jordan which lead to uniform functions.

Returning to Theorem XXX, we notice that if $\sigma_1<0$, and if $s=0$ does not belong to $S_f^{\sigma_1}$, then for ϵ sufficiently small the boundary $C_1^*(\epsilon)$ contains a square (of side 3ϵ) which contains inside the point $s=0$, this square being separated from the other points of $C_1^*(\epsilon)$. Let $c(\epsilon)$ be the boundary of this square. The part of $F_k(z)$ given by the integration on $c(\epsilon)$ furnishes, for x sufficiently large (by the theorem of residues):

$$(111)\qquad \begin{aligned} I_k(z) &= \frac{k!}{2\pi i}\int_{c(\epsilon)} f(s)\varphi(z-s)\frac{ds}{s^{k+1}} = \frac{\partial^k[f(s)\varphi(z-s)]}{\partial s^k}_{(s=0)} \\ &= (-1)^k[f(0)\varphi^{(k)}(z) - C_k^1 f'(0)\varphi^{(k-1)}(z) \\ &\qquad + \cdots + (-1)^k f^{(k)}(0)\varphi(z)]. \end{aligned}$$

On the other hand, it follows readily from the proof of Theorem XXX that the only singularities of the function

$$F_k(z) - I_k(z) = \frac{k!}{2\pi i}\int_{C_2^*(\epsilon)} f(s)\varphi(z-s)\frac{ds}{s^{k+1}},$$

where $C_2^*(\epsilon)=C_1^*(\epsilon)-c(\epsilon)$, are the points of the sum-composite of $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$. From the form of $I_k(z)$ it follows readily that the only singularities of this function are those of $\varphi(z)$. Theorem XXXIII can therefore be stated in the following more precise form:

THEOREM XXXIV. *With the same hypotheses as in Theorem XXXIII, if $s=0$ is a regular point for $f(s)$, the function $F_k(s)=\sum a_n^{(k)} b_n e^{-\mu_n s}$ is uniform in the half-plane $\sigma>H_f(\sigma_1)+\sigma_2$,* *and the only possible singularities, in this half-plane, of the function*

$$F_k(s) - I_k(s),$$

where

$$I_k(s) = (-1)^k[f(0)\varphi^{(k)}(s) - C_k^1 f'(0)\varphi^{(k-1)}(s) + \cdots + (-1)^k f^{(k)}(0)\varphi(s)],$$

are the points of the sum-composite set of the sets $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$ and their limit points.

The fact that in the general Theorem XXXIII (or Theorem XXXIV), the two sequences $\{\lambda_n\}$, $\{\mu_n\}$ can be chosen as entirely unrelated sequences allows us a great variety of interesting applications. For instance, the distribution of singularities of functions represented by Taylor series is much better known than the distribution of singularities of functions represented by general Dirichlet series. Real progress would therefore be achieved if the problem of the location of singularities of a Dirichlet series could be reduced to the problem of the location of singularities of a relatively simple Dirichlet series and that of a Taylor series, of which the coefficients are simply related to the sequences of the exponents and coefficients of the given Dirichlet series.

We shall show that in many cases an indication of the location of singularities of $\sum a_n e^{-\lambda_n s}$ can be obtained when the affixes of the singularities of $\sum e^{-\lambda_n s}$ and those of a Taylor series of which the coefficients are formed with the quantities a_n and λ_n are known.

Let $\{\lambda_n\}$ be a sequence having the usual properties and, moreover, such that $\lambda_1 > 1$, let $\{a_n\}$ be a sequence of complex numbers, and let $k > 0$ be an integer. If there exists a sequence $\{d_n\}$ such that for each $n \geqq 1$:

$$a_n = \sum_{m < \lambda_n} (\lambda_n - m)^k d_m,$$

we shall say that $\{a_n\}$ admits a *generatrix-sequence* $\{d_n\}$ *of order k, with respect to* $\{\lambda_n\}$.

Generally a sequence $\{a_n\}$ has no generatrix-sequence of a given order, with respect to a given sequence $\{\lambda_n\}$. But conditions bearing on $\{\lambda_n\}$ can easily be given in order that each sequence $\{a_n\}$ will have a generatrix-sequence of a given order, with respect to $\{\lambda_n\}$. Let us denote by e_n the greatest integer smaller than λ_n. The following lemma is immediate.

LEMMA XII. *If for an infinite sequence $\{n_i\}$, $n_i<\lambda_{n_i}\leqq n_i+1$ and if the determinant Δ_{n_i} of order n_i—of which the term in the p^{th} row and the q^{th} column is $(\lambda_p-q)^k$ (if $1\leqq p\leqq n_i$, $q\leqq e_p$) and 0 (if $1\leqq p<n_i$, $e_p<q\leqq e_{n_i}=n_i$)—is different from zero, then each sequence $\{a_n\}$ has a generatrix-sequence of order k with respect to $\{\lambda_n\}$.*

Indeed, each system of equations (with the d_n as unknowns),

$$\sum_{m<\lambda_1}(\lambda_1-m)^k d_m=a_1$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\sum_{m\leqq n_i}(\lambda_{n_i}-m)^k d_m=a_{n_i},$$

admits a system of solutions. If n_{i_1} and n_{i_2} belong both to the sequence $\{n_i\}$ and if $n_{i_1}<n_{i_2}$, the quantities $d_1^{(i_2)}$, $d_2^{(i_2)}$, $\cdots$ $d_{n_{i_1}}^{(i_2)}$, which are the first n_{i_1} terms of the solution of the system (S_{i_2}), constitute also the system of solutions of the system (S_{i_1}). The union of the solutions of all the systems (S_i), $(i\geqq 1)$, constitutes therefore a sequence $\{d_n\}$ (to each n there corresponds only one d_n), which satisfy all the systems of equations:

$$\sum_{m<\lambda_1}(\lambda_1-m)^k d_m=a_1$$
$$\cdots\cdots \qquad (n\geqq 1)$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\sum_{m<\lambda_n}(\lambda_n-m)^k d_m=a_n.$$

LEMMA XIII. *If* $n < \lambda_n \leqq n+1$ $(n \geqq 1)$, *each sequence* $\{a_n\}$ *has a generatrix-sequence of any order* $k > 0$, *with respect to* $\{\lambda_n\}$.

Here $n_i = i$, and obviously

$$\Delta = [(\lambda_1 - 1)(\lambda_2 - 2) \cdots (\lambda_n - n)]^k \neq 0;$$

the conclusion follows then from Lemma XII.

From Theorem XXXIII follows immediately [**7**]:

THEOREM XXXV. *Let* $F(s)$ *be represented by* $\sum a_n e^{-\lambda_n s}$, $(\lambda_1 > 1)$. *Suppose that* $\varphi(s) = \sum e^{-\lambda_n s}$ *is uniform and of order* $\mu < \infty$ *e. f. s. in a half-plane* $\sigma > \sigma_2$, *and that* $\{a_n\}$ *admits a generatrix-sequence* $\{d_n\}$ *of an integral order* $k > \mu$, *with respect to* $\{\lambda_n\}$. *If*

$$\limsup_{n=\infty} \frac{\log |d_n|}{n} = a < \infty,$$

and if $f(s) = \sum d_n e^{-ns}$ *is uniform, then* $F(s)$ *is uniform and the only possible singularities of* $F(s)$ *in the half-plane* $\sigma > \text{Max}(a, 0) + \sigma_2$ *are the points of the sum-composite of* S_f *and* $S_\varphi^{\sigma_2}$, *their limit-points, and the points of* $S_\varphi^{\sigma_2}$, *where* S_f *and* $S_\varphi^{\sigma_2}$ *are, respectively, the set of singularities of* $f(s)$ *in the whole plane and the singular set of* $\varphi(s)$ *with respect to the half-plane* $\sigma > \sigma_2$.

This is a particular case of Theorem XXXIII with $\{d_n\}$ playing the rôle of $\{a_n\}$ of Theorem XXXIII, the b_n of that theorem being here all equal to unity, σ_1 being an arbitrary quantity, $H_f(\sigma_1) = a$ for $\sigma_1 < a$, a_n of the statement of Theorem XXXV being nothing but the n^{th} coefficient of $f(s) = \sum d_n e^{-ns}$ with respect to the sequence $\{\lambda_n\}$, of order k.

It is seen by Theorem XXXV that each time that the sequence $\{\lambda_n\}$ satisfies the conditions of Lemma XII (or, in particular, of Lemma XIII) then only the behavior of the function $\varphi(s) = \sum e^{-\lambda_n s}$ has to be known, so far as Dirichlet series, different from Taylor-D series, are concerned.

REMARK. If the sequence $\{\lambda_n\}$ is such that to each positive integer k there corresponds at most one λ_n, such that $k < \lambda_n \leqq k+1$, in other words, with the notations given above, if $(e_{n+1} - e_n) \geqq 1$, $(n \geqq 1)$, (with $\lambda_1 > 1$), then the series $\sum a_n e^{-\lambda_n s}$ can be written in the form $\sum a'_p e^{-L_p s}$, where $L_{e_n} = \lambda_n$, and where L_p for p distinct from each e_n is chosen arbitrarily in the interval $(p, p+1]$, where $a'_{e_n} = a_n$ and where $a'_p = 0$, for p distinct from each e_n. The quantities L_p are now such that $p < L_p \leqq p+1$. Thus, by Theorem XXXV, the study of the singularities of $F(s) = \sum a_n e^{-\lambda_n s}$ can still be reduced to that of the singularities of a function represented by a Dirichlet series of which all the coefficients are equal to unity, $\sum e^{-L_n s}$, and a Taylor-D series $\sum d'_n e^{-ns}$, where $\{d'_n\}$ is the generatrix-sequence of $\{a'_n\}$, with respect to $\{L_n\}$. The order of the generatrix-sequence depends, of course, on the order e. f. s. of $\varphi_1(s) = \sum e^{-L_n s}$ (if this function is of finite order e. f. s. in a certain half-plane $\sigma > \sigma_2$).

It is of course important to give composition theorems, in which the order e. f. s. of the composed functions is not involved, even if these functions are not bounded e. f. s. in certain half-planes. This can be achieved if the sets $S_f^{\sigma_1}$, $S_\varphi^{\sigma_2}$ are replaced by sets which contain, respectively, the sets $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$ but which for Taylor-D series are reduced to the sets $S_f^{\sigma_1}$, $S_\varphi^{\sigma_2}$. We shall give here an example of such a theorem, proved by the author [**7**]. But first let us make the following remarks: Let $f(s) = \sum a_n e^{-\lambda_n s}$ be uniform in a half-plane $\sigma > \sigma_1$. We shall denote by $E_f(a, \sigma_1)$ the set of points in the half-plane $\sigma > \sigma_1$ in which $f(s)$ takes the value a. In other words, if $\rho \in E_f(a, \sigma_1)$, then $\mathcal{R}(\rho) > \sigma_1$ and $f(\rho) = a$. Let now $\varphi(s) = \sum b_n e^{-\mu_n s}$ be uniform in the half-plane $\sigma > \sigma_2$, and let, as usual, $S_\varphi^{\sigma_2}$ be the singular set of $\varphi(s)$ with respect to the half-plane $\sigma > \sigma_2$. We shall denote by $P[f, \sigma_1; \varphi, \sigma_2]$ the set of points having the following property: a necessary and sufficient condition that $\gamma \in P[f, \sigma_1; \varphi, \sigma_2]$ is that if a

takes any complex value, except at most one, to every $\epsilon>0$ there corresponds a point $\rho_a=\rho_a(\epsilon)\in E_f(a,\sigma_1)$, and a point $\beta=\beta(\epsilon)\in S_\varphi^{\sigma_2}$ such that $|\gamma-\rho_a-\beta|<\epsilon$. If both series, the series representing $f(s)$ and that representing $\varphi(s)$, are Taylor-D series, then the set $P[f,\sigma_1;\ \varphi,\sigma_2]$ is a subset of the sum-composite of the sets $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$. Indeed, let $\gamma\in P[f,\sigma_1;\ \varphi,\sigma_2]$, and let $\beta_n\in S_\varphi^{\sigma_2}$, $\rho_{\omega_n}\in E_f(\omega_n,\sigma_1)$, with $\omega_n\to\infty$ as $n\to\infty$, be such that $\lim_{n=\infty}(\rho_{\omega_n}+\beta_n)=\gamma$, which is possible by the definition of γ. Since in our case $f(s)$ and $\varphi(s)$ are periodic with period $2\pi i$, on setting $\rho_{\omega_n}=\rho'_{\omega_n}+i\rho''_{\omega_n}$, we can suppose that $0\leqq\rho''_{\omega_n}\leqq 2\pi$, and then there exist two integers k_1, k_2 such that, on setting $\beta_n=\beta'_n+i\beta''_n$, $2k_1\pi\leqq\beta''_n\leqq 2k_2\pi$, all these equalities and inequalities hold for $n\geqq 1$. On the other hand, let σ' be such that $\sigma'>\sigma_A{}^f$; if for $n>n_0$ $|\omega_n|>\operatorname{Max}_{\sigma=\sigma'}|f(s)|$, then all the $\rho_{\omega_n}(n>n_0)$, are situated in the rectangle Δ, defined by $\sigma_1\leqq\sigma\leqq\sigma'$, $0\leqq t\leqq 2\pi$, and thus all the β_n with $n>n_0$ are also situated in a bounded rectangle Δ_2. There exist therefore a sequence $\{n_i\}$, a point ρ belonging to Δ, and a point β belonging to Δ_2, such that $\lim_{i=\infty}\rho_{\omega_{n_i}}=\rho$, $\lim_{i=\infty}\beta_{n_i}=\beta$, and $\gamma=\rho+\beta$. But obviously, $\rho\in S_f^{\sigma_1}$ and $\beta\in S_\varphi^{\sigma_2}$, which proves our assertion.

It is also easy to see by Picard's theorem that if $\alpha_1\in S_f^{\sigma_1}$, and if α_1 (with $\mathcal{R}(\alpha_1)>\sigma_1$) is an isolated essential singularity of $f(s)$, then, for each β belonging to $S_\varphi^{\sigma_2}$: $\alpha_1+\beta\in P[f,\sigma_1;\varphi,\sigma_2]$. At any rate, if $S_{f\varphi}$ denotes the sum-composite of $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$, $f(s)$ and $\varphi(s)$ being represented by Taylor-D series, then the closure of the union of the two sets $S_{f\varphi}\cup P[f,\sigma_1;\varphi,\sigma_2]$ is equal to the set $S_{f\varphi}$. But, for general Dirichlet series, it cannot even be asserted that this closure is equal to the closure of $S_{f\varphi}$. In order to have composition-theorems bearing on general Dirichlet series, without being obliged to take into account the order e. f. s. of the functions, the introduction of sets of the type $P[f, \sigma_1; \varphi, \sigma_2]$ seems natural and indispensable. If $\sigma_1 = \sigma_2 = -\infty$, the set $P[f, \sigma_1; \varphi, \sigma_2] = P[f, -\infty; \varphi, -\infty]$ will be denoted simply by $P[f; \varphi]$. Let us remark that $P[f; \varphi]$ is not equal generally to $P[\varphi; f]$.

THEOREM XXXVI. [7] *Let $F(s)$ be represented by $\sum a_n e^{-\lambda_n s}(\lambda_1 > 1)$. Suppose that $\varphi(s) = \sum e^{-\lambda_n s}$ is uniform in the whole plane, and that $\{a_n\}$ admits a generatrix-sequence $\{d_n\}$ of an integral order k with respect to $\{\lambda_n\}$. If*

$$\limsup_{n=\infty} \frac{\log|d_n|}{n} < \infty,$$

and if $f(s) = \sum d_n e^{-ns}$ is uniform, then $F(s)$ is uniform and the only possible singularities of $F(s)$ are the points of the sum-composite of S_f and S_φ, the points of $P[\varphi; f]$, the limit-points of all these points, and the points of S_φ.

We shall first suppose that $\varphi(s)$ takes all the possible values except at most one. As for the proof of Theorem XXXIII we can show, on writing formulas analogous to (107) and (108), that $\sigma_A{}^F \leqq \sigma_A{}^f + \sigma_A{}^\varphi$, $\left(\sigma_A{}^f = \limsup_{n=\infty} \frac{\log|d_n|}{n}\right)$. We have to prove that $F(s)$ is uniform and holomorphic in the part Δ of the complement (with respect to the whole plane) of the set $\overline{S_{f\varphi} \cup S_\varphi \cup P[\varphi; f]}$ which constitutes a region containing a half-plane $\sigma > \sigma_1'$. Here $S_{f\varphi}$ is the sum-composite of S_f and S_φ. Let L be a closed simple arc of Jordan belonging to Δ having one of its extremities z^* in the half-plane $\sigma > \sigma_A{}^f + \sigma_A{}^\varphi$, and let z_0 be a point of L. There exists a positive quantity, $\delta > 0$, and two distinct quantities a and b, $a = a(z_0)$, $b = b(z_0)$ such that for each $\beta \in S_\varphi$: 1) $|z_0 - \beta| > \delta$, and for each $\rho_a \in E_\varphi(a, -\infty)$, each $\rho_b \in E_\varphi(b, -\infty)$, and each $\alpha \in S_f$: 2) $|z_0 - \alpha - \rho_a| > \delta$, 3) $|z_0 - \alpha - \rho_b| > \delta$, 4) $|z_0 - \beta - \alpha| > \delta$.

It is obvious that there exists a quantity $\delta_1 > 0$ such that

1) and 4) hold with α, β having their precise meaning and with $\delta=\delta_1$, since if this were not true, z_0 would belong either to S_φ or to $\overline{S}_{f,\varphi}$. If, on the other hand, there were not two quantities a and b and a $\delta_2>0$ such that 2) and 3) hold with $\delta=\delta_2$, then for each $\delta>0$ the inequality $|z-\beta-\rho_c|\leqq\delta$ would hold for every c, with a certain $\rho_c \in E_\varphi(c, -\infty)$, with a certain β, except for at most one value $c=c(\delta)$. But obviously, for δ sufficiently small, $c(\delta)$ (if it does exist) will have the same value, and z_0 would belong to $\overline{P[\varphi;f]}$. The smaller of the two quantities δ_1 and δ_2 is the desired quantity δ. It follows then from the established inequalities that to every $z_0 \in L$ there corresponds an $\epsilon>0$, $\epsilon=\epsilon(z_0)$, such that, if $z \in \overline{C(z_0, \epsilon)}$, then

$$|z-\beta|>\epsilon,\ |z-\alpha-\rho_a|>\epsilon,\ |z-\alpha-\rho_b|>\epsilon,\ |z-\beta-\alpha|>\epsilon.$$

The quantity $\epsilon=\epsilon(z_0)$ depends on z_0, but, if we denote by $\epsilon'(z_0)$ the least upper bound of all the quantities $\epsilon(z_0)$ (for $z_0 \in L$ fixed), having the properties defined above, we see, by classical reasoning (L is bounded, since it is a closed arc of Jordan), that $\underset{z_0 \in L}{\text{g. l. b.}}\ \epsilon'(z_0)=\eta>0$. In other words, there exists a constant $\eta>0$, independent of z_0, such that for $a=a(z_0)$, $b=b(z_0)$, every $\rho_a \in E_\varphi(a(z_0), -\infty)$, $\rho_b \in E_\varphi(b(z_0), -\infty)$, $\alpha \in S_f$, $\beta \in S_\varphi$:

$$(112)\quad |z-\beta|>\eta,\ |z-\alpha-\rho_a|>\eta,\ |z-\alpha-\rho_b|>\eta,\ |z-\beta-\alpha|>\eta,$$

when $z \in \overline{C(z_0, \eta)}$, with $z_0 \in L$.

Let us choose a finite number of points $z_0^{(1)}, z_0^{(2)}, \cdots z_0^{(k)}$ such that $z_0^{(m+1)} \in C\left(z_0^{(m)}, \frac{\eta}{2}\right)$, $(1\leqq m\leqq k-1)$, $z_0^{(j)} \in L$, $(1\leqq j\leqq k)$, $z_0^{(1)}=z^*$, the other extremity z_1^* of L being such that $z_1^* \in \overline{C\left(z_0^{(k)}, \frac{\eta}{2}\right)}$.

Obviously $\varphi(s)\to 0$, uniformly with respect to t as $\sigma\to\infty$ (since $\lambda_1>0$, here $\lambda_1>1$). But it is easily seen (this is a well known fact for each Dirichlet series with $\lambda_1 > 0$) that for σ sufficiently large $\varphi(s) \neq 0$. Indeed, for σ large,

$$\varphi(s) = \sum_1^\infty e^{-\lambda_n s} = e^{-\lambda_1 s}(1 + \sum_2^\infty e^{-(\lambda_n - \lambda_1)s}),$$

and, since $\sum_2^\infty e^{-(\lambda_n - \lambda_1)s} \to 0$, as $\sigma \to \infty$, uniformly with respect to t, we see that $\varphi(s) \neq 0$, for σ large. Therefore there exists a σ_0 such that $\mathcal{R}(\rho_{a(z_0^{(j)})}) < \sigma_0$, $\mathcal{R}(\rho_{b(z_0^{(j)})}) < \sigma_0$, and if $\sigma_1 + \epsilon$ is negative, numerically sufficiently large, and if $s = \sigma_1 + \epsilon + it$, then, for each $z \in \overline{C(z_0^{(j)}, \eta)}$, $(1 \leqq j \leqq k)$, each $\beta \in S_\varphi$, and each $a = a(z_0^{(j)})$, $b = b(z_0^{(j)})$, $(1 \leqq j \leqq k)$:

(113) $\quad |z - s - \rho_a| > \eta, \quad |z - s - \rho_b| > \eta, \quad |z - s - \beta| > \eta.$

Let now $C_1^*(\epsilon)$ have the same meaning as in formula (110).[1] From what we have seen (inequalities (112) and (113)), if ϵ is sufficiently small, the inequalities (113) hold if s is any point of $C_1^*(\epsilon)$ (σ_1 is chosen negative, numerically sufficiently large), moreover (still if ϵ is sufficiently small), we shall have $\mathcal{R}(z^* - s) \geqq \mathcal{R}(z^*) - \sigma_A^f - 2\epsilon > \sigma_A^\varphi + \epsilon$. If s is a fixed point of $C_1^*(\epsilon)$, the point $u = z - s$, when z varies on L, varies on a curve $L(s)$, which is obtained from L by translation. One extremity $u^* = z^* - s$, of $L(s)$ is in the half-plane $\sigma > \sigma_A^\varphi + \epsilon$; we have thus $|\varphi(u^*)| = |\varphi(z^* - s)| \leqq M = \sum e^{-\lambda_n(\sigma_A^\varphi + \epsilon)}$, for each $s \in C_1^*(\epsilon)$.

By Schottky's theorem, if $\Phi(z)$ is holomorphic in a circle $\overline{C(A, R)}$, if $|\Phi(A)| < N$, if $\Phi(z)$ does not take in $\overline{C(A, R)}$ two distinct values a and b, and if $0 < \theta < 1$, then

$$|\Phi(z)| < K(a, b;\ N;\ \theta) < \infty$$

in $\overline{C(A, \theta R)}$. Therefore, if for s fixed we set $u_2^* = z_0^{(2)} - s$, $\cdots$, $u_k^* = z_0^{(k)} - s$, we shall have:

$$|\varphi(u)| < K(a(z_0^{(1)}),\ b(z_0^{(1)});\ M;\ \tfrac{1}{2}) = K_1,$$

for $u \in \overline{C\left(u^*, \frac{\eta}{2}\right)}$, that is to say, also: $|\varphi(u_2^*)| < K_1$. On ap-

[1]We define here $C_1^*(\epsilon)$ with respect to the function $f(s)$ of the actual statement as we did for the function $f(s)$ of Theorem XXXIII.

plying once more Schottky's theorem we shall have

$$|\varphi(u)| < K(a(z_0^{(2)}),\ b(z_0^{(2)});\ K_1;\ \tfrac{1}{2}) = K_2,$$

for $u \in C\left(u_2^*, \dfrac{\eta}{2}\right)$, that is to say, also $|\varphi(u_3^*)| < K_3$, etc. . . .

In applying Schottky's theorem k times, we see that $|\varphi(z-s)| < B$, when z is in a channel of which L is the central line, s being an arbitrary point of $C_1^*(\epsilon)$, B being independent of z and s.

It is now easy to see that the integral in (110), once more, defines a function holomorphic on L. The remaining part of the proof is obvious.

We have supposed that $\varphi(s)$ takes all possible values except at most one. If $\varphi(s)$ does not take two distinct values then, as the reader can see easily, the proof becomes simpler, since then, if a_1 and b_1 are the two values which $\varphi(s)$ does not take, the last passage of the proof, where Schottky's theorem is used, can be applied directly with $a(z_0^{(j)}) = a_1$, $b(z_0^{(j)}) = b_1$, $(1 \leqq j \leqq k)$.

A more general theorem of the same kind can be proved, in exactly the same manner, if we consider series of the form $\sum d_n e^{-ns}$, $\sum b_n e^{-\lambda_n s}$, and $\sum a_n e^{-\lambda_n s}$ where $a_n = b_n \sum_{m<\lambda_n} (\lambda_n - m)^k d_n$. H. Brunk, in his doctoral thesis [**5**] proved theorems of a very general type in which points of a character similar to those of a set $P[f, \varphi]$ are involved.

Theorems on singularities of Taylor series can also be applied in a reverse sense, so to speak, in order to assure that a function represented by a Dirichlet series of a general type necessarily admits singularities of given affixes.

Let $f(s)$ be represented by $\sum a_n e^{-\lambda_n s}$, and let k be a positive integer. The n^{th} coefficient of $f(s)$ with respect to the sequence $\{n\}$, of order k, shall be called the n^{th} *Taylor-*

coefficient of $f(s)$, *of order k.* This quantity shall be denoted by $a_T^{(k)}(n)$.

THEOREM XXXVII. *Suppose that* $f(s)=\sum a_n e^{-\lambda_n s}$, *with* $\sigma_A^f<\infty$, *is uniform in a half-plane* $\sigma>\sigma_1$, *and that* $f(s)$ *is of order* ν *e. f. s. in this half-plane. Let* β *be a singularity of the function* $F(s)$ *represented by* $\sum a_T^{(k)}(n)e^{-ns}$, *where* $k>\nu$, *and let us suppose that* β *is not of the form* $2m\pi i$, *where m is an integer. To each* $\epsilon>0$ *there corresponds an integer p such that there exists a point* α *belonging to* $S_f^{\sigma_1}$ *with the property*:

$$|\alpha-(2p\pi i+\beta)|<\epsilon.$$

Indeed, the function $\varphi(s)=\sum e^{-ns}$ is uniform in each half-plane $\sigma>\sigma_2$, and it is bounded e. f. s. in such a half-plane. Therefore Theorem XXXIII can be applied to $f(s)$ and $\varphi(s)$, with $k>\nu$. Hence the only possible singularities of $F(s)$, in the whole plane, are the points of the sum-composite of $S_f^{\sigma_1}$ and S_φ, the limit-points of these points, and the points of S_φ. But S_φ is composed of the points of affixes $2m\pi i$, where m takes all integral values. If β is a singularity of $F(s)$, and is not of the form $2m\pi i$, that is to say, does not belong to S_φ, it must belong either to the sum-composite of $S_f^{\sigma_1}$ and S_φ or has to be a limit-point of this sum-composite. In other words, either there exist a point α_1 belonging to $S_f^{\sigma_1}$ and an integer m_1 such that $\beta=\alpha_1+2m_1\pi i$; or there exist a sequence $\{\alpha_n\}$, with $\alpha_n \in S_f^{\sigma_1}$, $(n\geqq 1)$, and a sequence of integers $\{m_n\}$ such that $\beta=\lim\limits_{n=\infty}(\alpha_n+2m_n\pi i)$. In the second case, if $\epsilon>0$ is given, for n sufficiently large the inequality $|\beta-\alpha_n-2m_n\pi i|<\epsilon$ is satisfied. In both cases the statement of the theorem is satisfied.

Theorem XXXVII allows us to associate with each theorem on Taylor series furnishing a condition in order that a given point be singular for the corresponding function, a new theorem on Dirichlet series of general type (of finite order e. f. s. in a half-plane) locating some of its singularities.

This and other theorems of this type were proved by the author [**8**].

Since these lectures were delivered, H. Brunk has proved, in his work mentioned above [**5**], many interesting and important theorems related with the subject of this chapter. He gives in fact a generalization of Theorem XXXII, in which he replaces the boundedness e. f. s. in a half-plane by a less restrictive condition, but conserving the form $\sum a_n b_n e^{-\lambda_n s}$ of the composite function.

He supposes, for instance, that in the part $\Delta(\sigma_2)$ of the half-plane $\sigma > \sigma_2$ from which the points $S_\varphi^{\sigma_2}$ are excluded $\varphi(s)$ has the following property: to each positive δ there correspond two positive quantities $L(\delta)$ and $M(\delta)$ such that if $s_1 \epsilon \Delta(\sigma_2)$, and if there exists a channel connecting s_1 to the half-plane $\sigma > \sigma_A{}^\varphi$, of width 2δ, which lies in $\Delta(\delta_2)$, and of which the central line is not larger than $L(\delta)$, then $|\varphi(s_1)| < M(\delta)$. A similar property is supposed to be satisfied by $f(s)$ in a quarter-plane: $\sigma > \sigma_1$, $t > t_0$. The conclusion of Brunk's theorem is similar to that of Theorem XXXII. He made also an extensive study of cases in which the sets $S_f^{\sigma_1, t_0}$ and $S_\varphi^{\sigma_2}$ can be replaced by larger sets, by adding, for instance, to $S_\varphi^{\sigma_2}$ the set in which $\varphi(s)$ takes two distinct values, in order to obtain composition-theorems in which neither the boundedness nor other similar conditions are any longer necessary. Such theorems generalize Theorem XXXVI.

Brunk gave also many interesting applications of Theorem XXXII, as well as of its generalizations. We shall state here a particular case of one of his theorems [**5**].

THEOREM XXXVIII. *If $f(s) = \sum a_n e^{-\lambda_n s}$ is uniform and bounded e. f. s. for $\sigma > \sigma_1$, with $\sigma_1 < \sigma_A{}^f (|\sigma_A{}^f| < \infty)$, if the only singularities of $f(s)$ in the half-plane $\sigma > \sigma_1$ are poles of affixes*

$s_0+2k\pi i$, *where* $s_0=\sigma_A{}^f+it_0$, *if there exists a quantity* M *such that the orders of these poles are all smaller than* M, *if* $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)>0$, $\lim_{n=\infty}\frac{\log|a_n|}{\lambda_n}=\sigma_A{}^f$, *then there exists an integer* $m>0$, *such that*

$$f(s)=\sum_{q=0}^{m} e^{-\lambda'_q s}T_q(s),$$

where $T_q(s)$ *is a Taylor-D series.*

For the proof of this theorem and for many other interesting results we refer the reader to Brunk's Thesis [**5**].