VII

ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

We shall suppose in this chapter that $\sigma_C = -\infty$, in other words, that the series (1) converges for each value of s. It represents then an entire function $F(s)$, that is to say, a function holomorphic in the whole plane. Since $D < \infty$, we have also $\sigma_A = -\infty$, and the function is bounded on each straight line $\sigma = \sigma_1$.

Let us set

$$\mathcal{M}(\sigma) = \underset{-\infty < t < \infty}{\text{l. u. b.}} |F(\sigma + it)|.$$

J. Ritt [**18**] has introduced the notion of an order of an entire function represented by a Dirichlet series, which is different from the order of this function, when it is considered independently of the fact that it is given by a Dirichlet series. $\mathcal{M}(\sigma)$ being given as above, the quantity

$$\rho = \limsup_{\sigma = -\infty} \frac{\log_2 \mathcal{M}(\sigma)^{1}}{-\sigma} \tag{93}$$

shall be called *the order* (R) *of* $F(s)$. The order (R) of $F(s)$ is not to be confused with the order of $F(s)$ when it is considered merely as a function of s, without taking into special account the fact that it is given by a Dirichlet series. The order of $F(s)$ is, as is well known, the quantity

$$\delta = \limsup_{r = \infty} \frac{\log_2 M_1(r)}{\log r},$$

where $M_1(r) = \underset{|s| = r}{\text{Max}} |F(s)|$. For instance, if $F(s) = e^{-s}$ then

[1] $\log_2 a = \log(\log a)$.

$\rho=0$, $\delta=1$. Obviously if the order (R) of $\sum a_n e^{-ns}$ is ρ, then ρ is the order (in the classical sense, δ) of $\sum a_n z^n$. The following theorem is due to Ritt [**18**]:

THEOREM XXIII. *If* $F(s)=\sum a_n e^{-\lambda_n s}$, *with*

$$\sigma_A{}^F=-\infty, \liminf_{n=\infty} \frac{\lambda_n}{\log n}>0,$$

then a necessary and sufficient condition for $F(s)$ *to be of order* (R) *equal to* ρ *is that*

$$\limsup_{n=\infty} \frac{\log|a_n|}{\lambda_n \log \lambda_n}=-\frac{1}{\rho}, \tag{94}$$

$\left(\textit{with } -\frac{1}{\rho}=-\infty \textit{ if } \rho=0\right)$.

Let $F(s)$ be of a *finite* order (R) equal to ρ. Since for each σ:

$$|a_n| \leqq \mathcal{M}(\sigma)e^{\lambda_n\sigma},$$

and since, $\delta_1>0$ given, we have, by (93) for $-\sigma$ sufficiently large:

$$\log \mathcal{M}(\sigma)<e^{-(\rho+\delta_1)\sigma},$$

we see that, for $-\sigma$ sufficiently large:

$$\log|a_n| \leqq \log \mathcal{M}(\sigma)+\lambda_n\sigma<e^{-(\rho+\delta_1)\sigma}+\lambda_n\sigma.$$

Since the right-hand expression in this inequality takes its minimum at

$$\sigma=-(\rho+\delta_1)^{-1}\log\frac{\lambda_n}{\rho+\delta_1},$$

which tends to $-\infty$ as n tends to ∞, we see that, for n sufficiently large:

$$\log|a_n| \leqq \min_{-\infty<\sigma<\infty}(e^{-(\rho+\delta_1)\sigma}+\lambda_n\sigma)$$

$$=\frac{\lambda_n}{\rho+\delta_1}\left(1-\log\frac{\lambda_n}{\rho+\delta_1}\right),$$

which gives immediately:

$$\limsup_{n=\infty}\frac{\log|a_n|}{\lambda_n\log\lambda_n} \leqq -\frac{1}{\rho+\delta_1},$$

and since δ_1 is positive, arbitrary, it could be dropped in this inequality, that is to say:

(95) $$\limsup_{n=\infty} \frac{\log|a_n|}{\lambda_n \log \lambda_n} \leqq -\frac{1}{\rho}.$$

If, on the other hand, this inequality holds with $\rho < \infty$, then to each $\delta_1 > 0$ there corresponds a quantity $B(\delta_1)$ such that $|a_n| < B(\delta_1) e^{-\frac{1}{\rho+\delta_1}\lambda_n \log \lambda_n} = B(\delta_1) \lambda_n^{-\frac{\lambda_n}{\rho+\delta_1}}$, and

(96) $$\begin{aligned} \mathcal{M}(\sigma) &\leqq \sum |a_n| e^{-\lambda_n \sigma} \leqq B(\delta_1) \sum \lambda_n^{-\frac{\lambda_n}{\rho+\delta_1}} e^{-\lambda_n \sigma} \\ &\leqq B(\delta_1) \operatorname*{Max}_{n \geqq 1} e^{-\frac{1}{\rho+2\delta_1}\lambda_n \log \lambda_n - \lambda_n \sigma}. \ \sum e^{-a\lambda_n \log \lambda_n}, \text{ where} \end{aligned}$$

$a = \dfrac{\delta_1}{(\rho+\delta_1)(\rho+2\delta_1)}$. But from $\liminf\limits_{n=\infty} \dfrac{\lambda_n}{\log n} > 0$, it follows that $\lambda_n > b \log n$ ($b > 0$, $n \geqq 1$) and

$$\sum_2^\infty e^{-a\lambda_n \log \lambda_n} < \sum e^{-ab \log n \log \lambda_n} = \sum n^{-ab \log \lambda_n} < \infty.$$

Thus, by (96), for $-\sigma$ sufficiently large:

$$\begin{aligned} \mathcal{M}(\sigma) &\leqq C(\delta_1) \operatorname*{Max}_{n \geqq 1} e^{-\frac{1}{\rho+2\delta_1}\lambda_n \log \lambda_n - \lambda_n \sigma} \\ &\leqq C(\delta_1) \operatorname*{Max}_{x \geqq 0} e^{-\frac{x \log x}{\rho+2\delta_1} - x\sigma} = C(\delta_1)\ e^{-(k+2\sigma)}\ e^{-\left(1+\frac{\sigma}{k}\right)}, \end{aligned}$$

where $k = \dfrac{1}{\rho+2\delta_1}$. Thus for $-\sigma$ large:

$$\begin{aligned} \log \mathcal{M}(\sigma) \leqq \log C(\delta_1) - (k+2\sigma) e^{-\left(1+\frac{\sigma}{k}\right)} &\leqq e^{-\left(\frac{1}{k}+\delta_1\right)\sigma} \\ &= e^{-(\rho+3\delta_1)\sigma}, \end{aligned}$$

and

$$\limsup_{\sigma=-\infty} \frac{\log_2 \mathcal{M}(\sigma)}{-\sigma} \leqq \rho + 3\delta_1.$$

But since $\delta_1 > 0$ is arbitrary:

(97) $$\limsup_{\sigma=-\infty} \frac{\log_2 \mathcal{M}(\sigma)}{-\sigma} \leqq \rho.$$

We have thus proved that from (95) with $\rho < \infty$ follows (97) and from (97) with $\rho < \infty$ follows (95). This is obviously equivalent to our statement (even in the case where $\rho = \infty$).

We shall now introduce the notion of an order in a horizontal strip. Let $S(R)$ be a horizontal strip of width $2R$ that is to say, the set of points s with $|t-t_0| < R$, where t_0 is fixed. If $F(s)$ is holomorphic in $\overline{S}(R)$ ($S(R)$ closed) we shall set for each σ:

$$M_S(\sigma) = \operatorname*{Max}_{|t-t_0| \leqq R} |F(\sigma+it)|,$$

and the quantity:

$$\limsup_{\sigma=-\infty} \frac{\log_2 M_S(\sigma)}{-\sigma} = \rho_S$$

shall be called *the order of $F(s)$ in the strip S.*

We shall prove the following theorem:

THEOREM XXIV. *If $F(s) = \sum a_n e^{-\lambda_n s}$, with*

$$\sigma_C^F = -\infty, \liminf (\lambda_{n+1} - \lambda_n) > 0,$$

and if the order (R) of $F(s)$ is positive, then the order of $F(s)$ in each horizontal strip $S(\pi a)$, with $a > D$, is equal to the order (R) of $F(s)$.

It follows from the hypotheses on $\{\lambda_n\}$ that

$$\liminf_{n=\infty} \frac{\lambda_n}{\log n} > 0,$$

and $D < \infty$.

If ρ is the order (R) of $F(s)$ in the plane, then, by Theorem XXIII, since the conditions of that theorem are satisfied:

$$\text{(98)} \qquad \limsup_{n=\infty} \frac{\log|a_n|}{\lambda_n \log \lambda_n} = -\frac{1}{\rho}.$$

If $S(\pi a)$ is the strip $|t-t_0| < \pi a$, and if

$$M_1(\sigma_j + it_0) = \operatorname{Max}|F(s)|$$

when $s \in C(\sigma_j + it_0, \pi a)$, we see by Theorem XVI that, for σ_j arbitrary:

$$\text{(99)} \qquad \log M_1(\sigma_j + it_0) \geqq \log|a_n| - \log \Lambda_n - \lambda_n \sigma_j - K, \ (n \geqq 1),$$

where K is a constant. By (98), $\delta_1 > 0$ given, there exists a sequence $\{n_j\}$ such that:

(100) $$\log|a_{n_j}| > -\left(\frac{1}{\rho}+\delta_1\right)\lambda_n \log \lambda_{n_j}, \quad (j \geqq 1),$$

and by Theorem XVII:

$$\limsup_{n=\infty} \frac{\log \Lambda_n}{\lambda_n} < \infty,$$

that is to say:

(101) $$\log \Lambda_n < P\lambda_n, \quad (n \geqq 1),$$

where P is a constant.

Let us denote by $s_j^* = \sigma_j^* + it_0^*$ a quantity such that $M_1(\sigma_j+it_0) = |F(s_j^*)|$. It is obvious that $M_S(\sigma_j^*) \geqq M_1(\sigma_j+it_0)$ and that $\sigma_j^* = \sigma_j + k_j$, where $|k_j| \leqq \pi a$. If we choose σ_j such as to have $\sigma_j = -\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j}$, where $\{n_j\}$ is a sequence such that (100) holds, the inequality (99) allows us to write

$$\log M_S(\sigma_j^*) = \log M_S(\sigma_j+k_j) = \log M_S\left(-\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j} + k_j\right)$$
$$\geqq \log M_1\left(-\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j} + it_0\right) \geqq \log|a_{n_j}| - \log \Lambda_{n_j}$$
$$+\left(\frac{1}{\rho}+2\delta_1\right)\lambda_{n_j}\log \lambda_{n_j} - K \geqq -\left(\frac{1}{\rho}+\delta_1\right)\lambda_{n_j}\log \lambda_{n_j} - P\lambda_{n_j}$$
$$+\left(\frac{1}{\rho}+2\delta_1\right)\lambda_{n_j}\log \lambda_{n_j} - K = \delta_1\lambda_{n_j}\log \lambda_{n_j} - P\lambda_{n_j} - K.$$

We have therefore:

$$\rho_S = \limsup_{\sigma=-\infty} \frac{\log_2 M_S(\sigma)}{-\sigma} \geqq \limsup_{j=\infty} \frac{\log_2 M_S(\sigma_j^*)}{-\sigma_j^*}$$
$$\geqq \limsup_{j=\infty} \frac{\log(\delta_1\lambda_{n_j}\log \lambda_{n_j} - P\lambda_{n_j} - K)}{\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j} - K_j} = \frac{1}{\frac{1}{\rho}+2\delta_1},$$

and, since $\delta_1 > 0$ is arbitrary, we have $\rho_S \geqq \rho$. But obviously $\rho_S \leqq \rho$, and therefore $\rho_S = \rho$.

This theorem was proved by the author [**10**] and, in weaker form, in the paper already mentioned above by

Gergen and the author [**14**]. There, as we said above, the rôle of D is played by $\frac{1}{h}$ where $h=\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)$, and, as we have said, $\frac{1}{h} \geqq D$.

It should be remarked that an order can also be defined in a curvilinear strip extending to $-\infty$ at the left. If $\Sigma=\Sigma(s(u), R)$ is such a strip, we shall set

$$M_\Sigma(\sigma)=\operatorname{Max}|F(\sigma+it)|,$$

the maximum being taken when σ is fixed, the point $s+it$ belonging to Σ and $\rho_\Sigma=\limsup_{\sigma=-\infty}\frac{\log_2 M(\sigma)}{-\sigma}$. In the statement of Theorem XXIV "horizontal strip $S(\pi a)$" can be replaced by "curvilinear strip horizontal at the right and extending to $-\infty$ at the left, of width $2\pi a$," and there is nothing to change in the proof of Theorem XXIV in order to establish this new version of its statement.

G. Julia proved the following theorem [**15**]. If $f(z)$ is an entire function, there exists a straight line issuing from the origin such that in each angle with vertex at the origin, and containing this line, $f(s)$ takes infinitely many times each value except at most one. Such a line is called "*line of Julia*" or *line J*. For a Taylor-D series this can be translated in the following manner: If $\Sigma a_n e^{-ns}$ represents an entire function, $F(s)$, then in each horizontal (closed) strip of width 2π there exists a line $t=\text{const.}$, such that in each horizontal strip containing this line $F(s)$ takes infinitely many times each value except at most one.

We shall generalize the notion as well as the theorem of Julia to general Dirichlet series. Let $F(s)$ be holomorphic in a horizontal strip S containing the line $t=t_0$. This line shall be called *line J*, if in each horizontal strip containing this line, and contained in S, $F(s)$ takes infinitely many times each value, except at most one.

We may define horizontal lines with even more specific properties than those of a line $\overline{J}$. Let us suppose, once more, that $F(s)$ is holomorphic in the horizontal strip $S(R)$ of width $2R$ defined by the inequality $|t-t_0|<R$, and let us consider the family $\mathcal{J}$ of functions of z defined by $f(z, \sigma)=F(\sigma+it_0+z)$, $(-\infty<\sigma<\infty)$. Each of these functions is holomorphic in $C(0, R)$, that is to say, the circle $|z|<R$. If the family $\mathcal{J}$ is not normal in $C(0, R)$ there exists, in $C(0, R)$, at least one irregular point [**15**], $z_0=x_0+iy_0$, that is to say, a point such that for no $\epsilon>o$ is the family $\mathcal{J}$ normal in $C(z_0, \epsilon)\equiv(|z-z_0|<\epsilon)$. The straight line $t=t_0+y_0$ will be called a *line* J^0. Clearly a line J^0 *is a line* $\overline{J}$, since, if J^0 were not a line $\overline{J}$ there would exist an $\epsilon>0$ such that in $C(z_0, \epsilon)$ the functions of the family $\mathcal{J}$ would take only a finite number of times two distinct values a and b $(a\neq b)$, and it would then be normal, contrary to the definition of a line J^0. In other words, a line J^0 has more precise properties than a line $\overline{J}$. A corresponding remark was already made with respect to lines J [**15**].

We shall now prove a theorem of a more specific nature than Theorem XXII, or even more specific than the statement which is contained in the remark at the end of Chapter VI. It is true that the conditions on $\{\lambda_n\}$ have to be those of Theorem XXIV, the function $F(s)$ has to be entire, and the strip horizontal, the width of which depends as well on D as on the order (R) of the function $F(s)$.

Theorem XXV. *If* $F(s)=\sum a_n e^{-\lambda_n s}$ *with* $\sigma_C{}^F=-\infty$, $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)>0$, *and if the order* (R) *of* $F(s)$ *is* $\rho>0$, *then in each horizontal strip* $S(\pi a)$ *of width* $2\pi a$, *with*

$a > \max\left(D, \frac{1}{2\rho}\right)$, *there exists a line* J^0 (*which is therefore a line* $\bar{J}$).

The proof of this theorem depends on Theorems XXII, XXIV, and on an interesting theorem of Valiron [**20**] which generalizes Picard's theorem to an angle. We shall state this result (here Lemma XI) for a horizontal strip.

LEMMA XI. *If* $F(s)$ *is holomorphic in a horizontal strip containing a closed horizontal strip* $\bar{S} = \bar{S}\left(\frac{\pi}{\gamma}\right)$, *of width* $\frac{2\pi}{\gamma}$, *in which the order of* $F(s)$ *is larger than* $\frac{\gamma}{2}$, *then* $F(s)$ *takes in* $\bar{S}$ *infinitely many times each value, except at most one finite value.*

We shall only indicate that the proof of this lemma depends on the well known theorem of Schottky by which if a function $f(z) = c_0 + c_1 z + \cdots$ is holomorphic in $|z| < 1$ and does not take, in this circle, the two values 0 and 1, then there exists a function $K(u, v) < \infty$ such that, in

$$|z| < \theta \ (0 < \theta < 1), \ |f(z)| \leqq K(c_0, \theta).$$

A precision of this theorem, which is also needed for Valiron's proof, was given by Landau [**20**]: $K(u, v) < \frac{K_1(u)}{1 - v}$, where $K_1(u) < \infty$ depends only on u.

In order to prove Theorem XXV it is then sufficient to make the following remarks: Suppose that the strip $S(\pi a)$ is given by $|t - t_0| < \pi a$, and let us choose $\epsilon > 0$ such that $a - \epsilon > \max\left(D, \frac{1}{2\rho}\right)$. By Theorem XXIV, the order ρ_{S_1} of $F(s)$ in the strip S_1 given by $|t - t_0| < \pi(a - \epsilon)$ is equal to ρ. On putting $\gamma = \frac{1}{a - \epsilon}$, we see that $\rho > \frac{\gamma}{2}$, and, by Lemma XI, $F(s)$ takes in S_1 infinitely many times each value except at most one. The condition b) of Theorem XXII cannot therefore be satisfied in $S(\pi a)$, hence, by the remark at the end of Chapter VI, the family $\mathcal{J}$ of functions $f(z, \sigma) = F(\sigma + it_0 + z)$ is not normal in $|z| < \pi a$, and there exists an irregular point $z_0 = x_0 + iy_0$ with $|z_0| < \pi a$ (obviously, since $D < a$, the conditions of Theorem XXII are satisfied if the conditions of Theorem XXV are satisfied). The line given by $t = t_0 + y_0$ is a line J^0 which is situated in $S(\pi a)$.

For functions of positive order, Theorem XXV contains Julia's theorem on entire functions. Indeed if $F(s) = \sum a_n e^{-ns}$, $(\sigma_C^F = -\infty)$, is of order (R) equal to $\rho > 0$, since here $D \leqq 1$ we see that if k is an integer such that $k > \max\left(1, \frac{1}{2\rho}\right)$, $F(s)$ has a line $\bar{J}$ in the horizontal strip $|t| < k\pi$. But since $F(s)$ is periodic of period $2\pi i$, this function has a line $\bar{J}$ in the strip $|t| \leqq \pi$. That is to say, $\varphi(z) = \sum a_n z^n$ has a line J.

In one of his papers A. Bloch (see [**17**]) has suggested, without proof whatsoever, the following principle: if a property (P) bearing on sequences of complex numbers is such that from the fact that $\{a_n\}$ has this property and from $\limsup \sqrt[n]{|a_n|} = 1$ it follows that the point of affix $e^{i\theta}$ is a singularity for the series $\sum a_n z^n$, then from the fact that $\{b_n\}$ satisfies the property (P) and from $\lim \sqrt[n]{b_n} = 0$ should follow that the entire function given by $\sum b_n z^n$ admits the line of argument θ issuing from the origin as a line J.

This principle, necessarily somewhat vague as to the character of the property (P), was justified some years later by many interesting theorems. The first important contributions in that direction were made by G. Pólya [**17**]. Let $\{\lambda_n\}$ be a sequence of increasing positive integers. $N(x)$ being the distribution function of $\{\lambda_n\}$, Pólya introduced the notion of maximum density $\mathcal{D}$ which is defined in the following manner: let us set for $0 \leqq \xi \leqq 1$:

$$\mathcal{D}(\xi) = \limsup_{x=\infty} \frac{N(x) - N(\xi x)}{x(1-\xi)}.$$

This expression tends to a limit as ξ tends increasingly to 1, and

(102) $$\mathcal{D} = \lim_{\xi=1-0} \mathcal{D}(\xi).$$

It is easily seen that $\mathcal{D} \geqq D$. Indeed, $\epsilon > 0$ given, from the definition of D it follows that there exists a sequence $\{x_i\}$ tending to infinity, such that $N(x_i) > (D-\epsilon)x_i (i \geqq 1)$ and such that for $0 < \xi < 1$, fixed, $N(\xi x_i) < (D+\epsilon)\xi x_i$, hence

$$\frac{N(x_i) - N(\xi x_i)}{x_i(1-\xi)} > D - \frac{\epsilon(1+\xi)}{1-\xi},$$

and

$$\mathcal{D}(\xi) \geqq \limsup_{i=\infty} \frac{N(x_i) - N(\xi x_i)}{x_i(1-\xi)} \geqq D - \frac{\epsilon(1+\xi)}{1-\xi}.$$

Since $\epsilon > 0$ is arbitrary we see that $\mathcal{D}(\xi) \geqq D$, and also $\mathcal{D} \geqq D$. It is obvious, since the λ_n are integers, that $\mathcal{D} \leqq 1$. Pólya proved that, if the Taylor series $\sum a_n z^{\lambda_n}$ (λ_n integers) represents an entire function $f(z)$ of order $\delta = \infty$, then $f(z)$ admits a line J in each angle with the vertex at the origin and of opening equal to $2\pi\mathcal{D}$.

Pólya's theorem is the first to justify Bloch's remark, since by a theorem also due to Pólya [**17**], if such a Taylor series instead of representing an entire function has a finite radius of convergence, equal say to unity, then it has a singularity on each arc of length $2\pi\mathcal{D}$ of the circle of convergence.

It is seen that Theorem XXV is more general than Pólya's theorem for the following reasons: in Theorem XXV, 1^0) a Dirichlet series is considered instead of a Taylor series; 2^0) the order is supposed to be only positive instead of being supposed to be equal to infinity; 3^0) if the order is equal to infinity Theorem XXV gives immediately the following one:

THEOREM XXVI. *If* $F(s)=\sum a_n e^{-\lambda_n s}$ *with* $\sigma_C{}^F=-\infty$, $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)>0$, *and if the order* (R) *of* $F(s)$ *is* $\rho=\infty$, *then in each horizontal strip of width* $2\pi a$, *with* $a>D$, *there exists a line* J^0.

Thus, this statement, even if the λ_n are, moreover, supposed to be integers, is more general than Pólya's statement, since $\mathcal{D}\geqq D$.

Another remark which seems to be of a fundamental character can be made. For theorems concerning singularities the notion of maximum density, $\mathcal{D}$, seems to be inherent in the subject, as far as the study of singularities *on the axis of convergence* is concerned, since, for instance, in Pólya's theorem mentioned above, and concerning the singularities on the circle of convergence, the quantity $\mathcal{D}$ can certainly not be replaced by D. Therefore it seems that, even if a profound analogy does exist between the study of singularities and that of lines J (or $\bar{J}$), the second subject is of a more elementary character, at least the conditions required for the existence of lines J are simpler than those for the existence of singularities on the axis of convergence (or on segments of this axis of specified length).

As a corollary of Theorem XXVI let us state the following one.

THEOREM XXVII. *If* $F(s)=\sum a_n e^{-\lambda_n s}$ *with* $\sigma_C{}^F=-\infty$, $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)>0$, $D=0$, *and if the order* (R) *of* $F(s)$ *is* $\rho=\infty$ *then each line* $t=t_0$ *with* t_0 *arbitrary, is a line* J^0.

This theorem was proved, for Taylor series, by Pólya. Let us finally mention that Theorem XXIV was also proved by Pólya for Taylor series, however, with the quantity $\mathcal{D}$ playing the rôle of D.