IV

FUNDAMENTAL THEOREMS ON THE ESTIMATION OF COEFFICIENTS

If $f(s)$ is given by the series (1), with an abscissa of absolute convergence $\sigma_A^f < \infty$, Theorem IX allows an immediate estimation of the coefficients a_n by means of numbers by which $|f(s)|$ is bounded on a line: $\sigma = \sigma_1 > \sigma_A^f$.

If $M(t_0, \sigma_1) = \underset{t_0 \leqq t}{\text{l. u. b.}} |f(\sigma_1 + it)|$, formula (15) gives readily:

$$|a_n| e^{-\lambda_n \sigma_1} \leqq M(t_0, \sigma_1).$$

The same estimate still holds if $f(s)$ is holomorphic and bounded for $\sigma \geqq \sigma_1$, $t \geqq t_0$, since by Cauchy's theorem it follows immediately from (15) that this formula still holds with σ_1 and t_0 just defined (supposing of course that $\sigma_A^f < \infty$).

In this chapter we shall show that a very useful estimate of the coefficients can be obtained if an estimate of the maximum of $|f(s)|$ is known in a circle. But this is true only if certain conditions bearing on the distribution of the elements of $\{\lambda_n\}$ are supposed to be satisfied. The radius of such a circle has to be of a certain length which depends, again, on the distribution of the λ_n. As a matter of fact, for such an estimate of the coefficients we shall not even be obliged to suppose that the series (1) converges. It will be sufficient to suppose that the series (1) represents $f(s)$ in a certain asymptotic manner in a part of the s-plane. Considerations of this type permit us not only to prove important theorems on the distribution of singularities of a function represented by a Dirichlet series as well as theorems on the distribution of the values taken by $f(s)$ in certain regions, but they allow us also to prove important theorems on quasi-analyticity. These considerations thus allow us to form a theory which unites in one principle both theories: that of the study of the general properties of a Dirichlet series (singularities, theorems of Picard's type, etc.) and that of quasi-analyticity. Although we give here the general theorem on the estimation of coefficients, we shall use it here only for Dirichlet series: for the other problems related to quasi-analyticity we shall refer the reader to our paper [**11**].

The number $N(x)$, $(x>0)$, of quantities λ_n smaller than x, will be called the *distribution function*, or for short, the *distribution* of $\{\lambda_n\}$. We shall suppose in this chapter that $\lambda_1>0$. Thus $N(x)=0$ for $x\leqq\lambda_1$. The quantity

$$D=\limsup_{x=\infty}\frac{N(x)}{x}$$

will be called the *upper density of* $\{\lambda_n\}$. The essential hypothesis in all the results of this chapter will be that the *upper density of* $\{\lambda_n\}$ *is finite*: $0\leqq D<\infty$. If $N(x)=Dx+n(x)$, the function $n(x)$ will be called the *excess distribution function*, or *excess distribution* of the sequence $\{\lambda_n\}$. In order to characterize the sequence more precisely than by its upper density we shall introduce an analytic function, the growth of which is closely related to the behavior of the excess distribution.

Let us set

$$\Lambda(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)=\sum(-1)^k c_k z^{2k}. \tag{42}$$

This product converges uniformly in each closed region and represents therefore an entire function of $z=x+iy=re^{i\varphi}$. We have $\operatorname*{Max}_{|z|=r}|\Lambda(z)|=\Lambda(ir)=\prod\left(1+\frac{r^2}{\lambda_n^2}\right)$. And, by integration by parts, we get

$$\log \operatorname*{Max}_{|z|=r} |\Lambda(z)| = \log \Lambda(ir) = \sum \log\left(1+\frac{r^2}{\lambda_n^2}\right)$$
(43)
$$= \int_0^\infty \log\left(1+\frac{r^2}{x^2}\right) dN(x) = 2r^2 \int_0^\infty \frac{N(x)dx}{x(x^2+r^2)}.$$

From this equality it follows readily that

(44) $$\limsup_{r=\infty} \frac{\log \Lambda(ir)}{r} \leqq \pi D.$$

We shall set

(45) $$\Lambda^*(u) = \int_0^\infty e^{-\pi(D+u)r} \Lambda(ir) dr.$$

From (44) it follows that $\Lambda^*(u)$ is defined by (45) for $u>0$.

Since we have:

$$\Lambda(ir) = \sum_0^\infty c_k r^{2k},$$

where the coefficients c_k are positive, $c_0=1$, we see from (45) that

(46) $$\Lambda^*(u) = \int_0^\infty e^{-\pi(D+u)r} \left(\sum c_k r^{2k}\right) dr = \sum_0^\infty \frac{(2k)!\, c_k}{[\pi(D+u)]^{2k+1}}.$$

Thus $\Lambda^*(u)$ is holomorphic and is represented by (46) for u complex with $|D+u|>D$, but we shall only use this function for $u>0$. It is positive and increases as u decreases to zero. From (43) it follows that

$$\log \Lambda(ir) = \pi D r + 2r^2 \int_0^\infty \frac{n(x)dx}{x(x^2+r^2)},$$

and (45) gives immediately:

$$\Lambda^*(u) = \int_0^\infty e^{-\pi u r + 2r^2 \int_0^\infty \frac{n(x)dx}{x(x^2+r^2)}} dr.$$

Thus, only the function $n(x)$ enters in the definition of $\Lambda^*(u)$, and this is the reason we shall call $\Lambda^*(u)$ the *growth* *function of the excess distribution* $(n(x))$ *of the sequence* $\{\lambda_n\}$. Let us also set

$$\Lambda_n = \frac{1}{\lambda_n |\Lambda'(\lambda_n)|}, \quad (n \geqq 1), \tag{47}$$

(where $\Lambda'(z)$ is the derivative of $\Lambda(z)$). If we set

$$\Lambda_j(z) = \frac{\Lambda(z)}{\left(1 - \frac{z^2}{\lambda_j^2}\right)} = \sum (-1)^k c_k^{(j)} z^{2k}, \tag{48}$$

we see, since $\Lambda(\lambda_j) = 0$, that

$$\Lambda_j(\lambda_j) = \lim_{z=\lambda_j} \frac{\Lambda(z)}{\left(1 - \frac{z^2}{\lambda_j^2}\right)} = \frac{-\Lambda'(\lambda_j)\lambda_j}{2},$$

that is to say:

$$\Lambda_j = \frac{1}{2|\Lambda_j(\lambda_j)|}. \tag{49}$$

The sequence $\{\Lambda_n\}$ given by (47), or, what amounts to the same thing, by (49), shall be called the *sequence associated with the sequence* $\{\lambda_n\}$.

Let now $\{\varphi_n(x)\}$ be a sequence of non-negative continuous functions defined for $x \geqq 0$, each of these functions being non-decreasing, with $\varphi_1(0) = 0$, and satisfying, for every n, the inequality

$$\varphi_{n+1}(x) = o(\varphi_n(x)), \tag{50}$$

as x tends to 0. The function $\varphi(x) = \text{g. l. b.}_{1 \leqq n}\, \varphi_n(x)$ shall be called the *envelope of the sequence* $\{\varphi_n(x)\}$. Since $\varphi_n(0) = 0$ for $n \geqq 1$ (by $\varphi_1(0) = 0$ and (50)), we have $\varphi(0) = 0$. The function $\varphi(x)$ is the limit of non-negative, non-decreasing functions $\varphi_n^*(x) = \min_{1 \leqq k \leqq n} \varphi_k(x)$ tending to $\varphi(x)$ monotonically (decreasing), and therefore the integral $\int_a^b \log \varphi(x)dx$, where $0 < a < b < \infty$, has either a finite value or the value $-\infty$.

This integral can have the value $-\infty$ only if for a certain c with $a \leqq c$, $\varphi(x)=0$, for $0<x<c$. A sequence $\{\varphi_n(x)\}$ having the properties just described shall be called *an asymptotic sequence.*

We shall denote by $C(w,R)$ the open circle $|s-w|<R$, and by $S(v,R)$ the region $|t-t'|<R$, $\sigma>\sigma'$, where $v=\sigma'+it'$. The region $S(v,R)$ will be called a *horizontal strip of width* $2R$. If L is a Jordan arc, we shall call the union of circles $\bigcup_{s'\epsilon L} C(s',R)$ *a channel of width* $2R$. The curve L will be called the *central line of the channel.* If v and w are the extremities of L and if $R_0 \leqq R$, $R_0 \leqq R'$, we shall say *that the* corresponding *channel of width* $2R_0$ *connects the circle* $C(w,R)$ *to the horizontal half strip* $S(v,R')$.

If a function $F(s)$, holomorphic in a horizontal half-strip $S(v,R)$, and two sequences $\{a_n\}$ of complex quantities, and $\{\lambda_n\}$ with $0<\lambda_1<\lambda_2, \cdots \lim \lambda_n=\infty$, together with an asymptotic sequence $\{\varphi_n(x)\}$ satisfy the inequalities:

$$\left|F(s)-\sum_1^n a_m e^{-\lambda_m s}\right| \leqq \varphi_n(e^{-\sigma}),$$

where $s \epsilon S(v,R)$, with $\sigma>\sigma_1$, and where $n \geqq 1$ (σ_1 independent of n), we shall say that $\sum a_n e^{-\lambda_n s}$ *represents* $F(s)$ *in* $S(v,R)$ *asymptotically with respect to the sequence* $\{\varphi_n(x)\}$.

We shall prove the following fundamental theorem:

THEOREM XV. *Let* $F(s)$ *be a holomorphic and bounded function in a region* Δ *composed of a circle* $C(s_1,\pi d)$, *of a horizontal half-strip* $S(s_2,\pi d_2)$ *and of a channel connecting them, of width* $2\pi d_1$, ($d_1 \leqq d$, $d_1 \leqq d_2$). *Suppose that* $F(s)$ *is represented in* $S(s_2,\pi d_2)$ *by* $\sum a_n e^{-\lambda_n s}$ *asymptotically with respect to an asymptotic sequence* $\{\varphi_n(x)\}$, *where* $\{\lambda_n\}$ *is of finite upper density* D.

If $D<d_1$, *and if, on setting* $\omega=(2(d_2-D))^{-1}$ *and, on denoting by* $\varphi(x)$ *the envelope of* $\{\varphi_n(x)\}$ *and by* $\Lambda^*(u)$ *the growth*

function of the excess distribution of $\{\lambda_n\}$, *one of the following conditions I, II is satisfied*:

(I) *there exists a positive constant* p *such that*

(51) $$\Lambda^*(pu^{\omega})\varphi(u) = O(1)\,(u \to 0)$$

(52) $$\liminf_{t=0+} \int_t^1 \log(\Lambda^*(pu^{\omega})\varphi(u))u^{\omega-1}du = -\infty,$$

(II) *there exists a constant* $\omega' > \omega$ *such that*

(53) $$\int_0^1 \log \varphi(u)u^{\omega'-1}du = -\infty,$$

then

(54) $$|a_n| \leqq 2\pi d\Lambda^*(d-D)M(s_1)\Lambda_n e^{\lambda_n \sigma_1}, \quad (n \geqq 1),$$

where $M(s_1)$ *is the maximum of* $|F(s)|$ *in* $C(s_1, \pi d)$, *where* $\{\Lambda_n\}$ *is the sequence associated with the sequence* $\{\lambda_n\}$ *and where* σ_1 *is the real part of* s_1.

It should be remarked that (52) can hold only if

$$\int_0^1 \log \varphi(u)u^{\omega-1}du = -\infty,$$

since the function $\Lambda^*(u)$ increases as u tends to $0 (u > 0)$, and $\int_0^1 \log \Lambda^*(pu^{\omega})u^{\omega-1}du > -\infty$. Therefore the condition (53) is much more restrictive than the condition (52).

For the proof of the theorem we need to prove some lemmas.

LEMMA III. *If* $\Phi(s)$ *is holomorphic in a circle* $C(s', \pi c)$ *and satisfies there the inequality* $|\Phi(s)| < M$, *if* $\Lambda(z)$ *and* $\Lambda_j(z)$ *are respectively defined by* (42) *and* (48), *where the upper density of* $\{\lambda_n\}$ *satisfies the inequality* $D < c$, *then the series*

$$\sum (-1)^k c_k^{(j)} \Phi^{(2k)}(s)$$

converges uniformly in every circle $C(s', \pi(c-D-u))$, *with* $0 < u < c-D$, *and represents there a holomorphic function* $\Phi_j(s)$ *satisfying the inequality*

$$|\Phi_j(s)| < \pi c M \Lambda^*(u),$$

where $\Lambda^*(u)$ *is the growth function of the excess distribution of* $\{\lambda_n\}$.

By Cauchy's theorem we have for $s \in C(s', \pi(c-D-u))$:

$$|\Phi^{(2k)}(s)| \leqq \frac{(2k)!}{2\pi}\left|\oint_{|w-s'|=\pi c'} \frac{\Phi(w)dw}{(w-s)^{2k+1}}\right| \leqq \frac{\pi(2k)!Mc'}{[\pi(c'+D+u-c)]^{2k+1}},$$

where c' is any quantity such that $c-D-u<c'<c$. Thus

$$(55) \qquad |\Phi^{(2k)}(s)| \leqq \frac{\pi(2k)!Mc}{[\pi(D+u)]^{2k+1}}.$$

On the other hand, it is clear from the definition of the quantities c_k and $c_k^{(j)}$ ((42), (48)), that $0<c_k^{(j)} \leqq c_k (k \geqq 0)$, and therefore, by (55) and (46):

$$|\Phi_j(s)| \leqq \sum_0^\infty c_k |\Phi^{(2k)}(s)| \leqq \pi c M \sum \frac{(2k)!c_k}{[\pi(D+u)]^{2k+1}} = \pi c M \Lambda^*(u).$$

We prove now a well known statement.

LEMMA IV. *If the function $F(z)$ is holomorphic and bounded in the half-plane $x \geqq a > -\infty$ and if $F(z)$ is not identically zero, then, for each $b>0$, the integral*

$$\int_{-\infty}^{\infty} \frac{\log|F(a+iy)|}{b^2+y^2} dy$$

converges to a finite value.

Let us set $F_1(z) = \dfrac{F(z+a-\frac{1}{2})}{M}$, where M is such that $|F(z)|<M$ in $x \geqq a$. $F_1(z)$ is holomorphic for

$$x \geqq \frac{1}{2}, \quad |F_1(z)|<1\left(x \geqq \frac{1}{2}\right), \quad F_1(z) \not\equiv 0.$$

If $0<l<1$, let us denote by c_l the circle

$$(56) \qquad \left|\frac{1-z}{z}\right| = l.$$

The point $z=1$ is inside of every C_l, C_l itself being inside the half-plane $x>\frac{1}{2}$. The center of C_l is the point $z_l = \dfrac{1}{1-l^2}$; its radius is $R_l = \dfrac{l}{1-l^2}$. We shall choose only such values of l that $F_1(z)$ has no zeros on the circumference C_l. There exists obviously an infinity of such l tending increasingly to 1, since, if this were not true, the zeros of $F_1(z)$ would have a limit point in $x \geqq \frac{1}{2}$ (different from infinity), and would be identically zero, contrary to the hypotheses. If $z=1$ is a zero for $F_1(z)$ of order m_0, we shall write

(57) $$F_1(z) = \Phi_l(z)(z-1)^{m_0}(z-\alpha_1)^{m_1}\cdots(z-\alpha_{n(l)})^{m_{n(l)}},$$

where the α_i are the other zeros (of order m_i) of $F_1(z)$ in C_l. If $F_1(1) \neq 0$, $m_0 = 0$.

The function $\Phi_l(z)$ is holomorphic in C_l (closed), without zeros in this closed circle. On putting $p_l = z_l - 1 = \frac{l^2}{1-l^2}$ we have, by Poisson's integral formula, and by (56):

$$\log|\Phi_l(1)| = \frac{1}{2\pi}\int_0^{2\pi} \log|\Phi_l(z_l + R_l e^{i\varphi})| \frac{R_l^2 - \rho_l^2}{R_l^2 + 2R_l\rho_l\cos\varphi + \rho_l^2}$$

(58) $$= \frac{l}{2\pi}\oint_{C_l} \log|\Phi_l(z)| \frac{|dz|}{|1-z|^2} = \frac{1}{2l\pi}\oint_{C_l}\log|\Phi_l(z)|\frac{|dz|}{|z|^2}.$$

But, if $\alpha \epsilon C_l$, then $\left|\frac{1-\alpha}{\alpha}\right| < l$, and :

(59)

$$\oint_{C_l} \frac{\log|z-\alpha|}{|z|^2}|dz| = 2\pi(l\cdot\log l + l\log|\alpha|) \geqq 2\pi l\cdot\log|1-\alpha|.$$

The formulas (57), (58), and (59) give thus immediately:

$$\oint_{C_l}\log|F_1(z)|\frac{|dz|}{|z|^2} = \oint_{C_l}(\log|\Phi_l(z)| + m_0\log|z-1|$$
$$+\sum_1^n m_i\log|z-\alpha_i|)\frac{|dz|}{|z|^2} \geqq 2\pi l\log|\Phi_l(1)| + 2\pi m_0 l\log l$$
$$+2\pi l\sum_1^n m_i\log|1-\alpha_i| = 2\pi l(m_0\log l + \log|F_2(1)|),$$

where $F_2(z) = \frac{F(z)}{(z-1)^{m_0}}$. Obviously $F_2(1) \neq 0$. Since $0 < l < 1$, we see that:

$$\oint_{C_l}\log|F_1(z)|\frac{|dz|}{|z|^2} > A > -\infty,$$

where A is a constant, independent of l. Since $|F_1(z)|<1$ in $x \geqq \frac{1}{2}$,

$$\int_L \log|F_1(z)| \frac{|dz|}{|z|^2} > A,$$

where L is an arbitrary arc of C_l. On the other hand, the arbitrary positive quantities T, ϵ given, there exists a number l $(0<l<1)$ such that each point of an arc of C_l is at a distance smaller than ϵ from a point of the segment $\left(x=\frac{1}{2},\ |y| \leqq T\right)$, which proves that

$$\int_{-T}^{T} \log\left|F_1\left(\frac{1}{2}+iy\right)\right| \frac{dy}{\left(\frac{1}{2}+iy\right)^2} \geqq A. \tag{60}$$

Since (60) holds for each T (A being fixed), our lemma follows then easily.

The next lemma is a corollary of Lemma IV.

LEMMA V. *If the function $\Phi(s)$ is holomorphic and bounded in the region $R(\alpha,\beta)$ defined by*

$$|t| < \frac{\pi}{2}\,(1-\alpha e^{-\sigma}),\quad \sigma \geqq \beta,$$

where $0 \leqq \pi\alpha < 2e^{\beta}$, and is not identically zero, then

$$\sigma + i\cos^{-1} e^{\beta-\sigma} \in R(\alpha,\beta)$$

for $\sigma > \beta$ (here $0 \leqq \cos^{-1} e^{\beta-\sigma} < \frac{\pi}{2}$) and

$$\int_{\beta}^{\infty} \log|\Phi(\sigma+i\cos^{-1} e^{\beta-\sigma})|\, e^{-\sigma} d\sigma > -\infty. \tag{61}$$

If s belongs to the boundary of $R(\alpha,\beta)$, and if $\sigma>\beta$, then

$$\cos t = \sin\left(\frac{\pi}{2}-|t|\right) \leqq \frac{\pi}{2}-|t| = \frac{\pi\alpha}{2e^{\sigma}} < e^{\beta-\sigma},$$

and therefore, if $\cos t = e^{\beta-\sigma}$, $\sigma \geqq \beta$, then $s=\sigma+it \in R(\alpha,\beta)$, which is equivalent to the assertion that

$$s=\sigma+i\cos^{-1}(e^{\beta-\sigma}) \in R(\alpha,\beta)$$

for $\sigma \geqq \beta$. The curve C, in the s-plane, on which $\cos t = e^{\beta-\sigma}$ $(\sigma \geqq \beta)$, and the region bounded by this curve, in which $\cos t > e^{\beta-\sigma}$ $\left(|t| < \frac{\pi}{2}, \sigma > \beta\right)$, are mapped by the transformation $z = x+iy = e^s$ into the straight-line $x = e^\beta = \gamma$ and the half-plane $x > \gamma$. The function of z, $f(z) = \Phi(s)$, is then, by hypothesis, holomorphic and bounded for $x \geqq \gamma$, and is not identically zero.

It follows then from Lemma IV that

$$\int_0^\infty \frac{\log|f(\gamma+iy)|}{1+y^2}\,dy > -\infty,$$

and from $y = e^\sigma \sin t$, $dy = (e^\sigma \sin t + e^\beta \operatorname{cotn} t)d\sigma$, as $s \in C$, it follows that

(62)

$$\int_{\beta+1}^\infty \log|\Phi(\sigma + i\cos^{-1}e^{\beta-\sigma})|(e^\sigma \sin t + e^\beta \operatorname{cotn} t)\frac{d\sigma}{1+e^{2\sigma}\sin^2 t} > -\infty.$$

From the obvious relationship

$$(e^\sigma \sin t + e^\beta \operatorname{cotn} t)(1+e^{2\sigma}\sin^2 t)^{-1} \backsim e^{-\sigma},$$

as s, on C, tends to infinity $\left(\sigma \text{ tends then to } +\infty \text{ and } t \text{ to } \frac{\pi}{2}\right)$, and from (62) follows then (61).

We now proceed to the proof of Theorem XV.

Let L be the central line of the channel connecting $C(s_1, \pi d)$ to $S(s_2, \pi d_2)$, $(s_2 = \sigma_2 + it_2)$, and belonging to Δ. Let $F(s) = \sum(-1)^k c_k^{(j)} F^{(2k)}(s)$, where the $c_k^{(j)}$ are given by (48). By the hypotheses of the theorem and by Lemma III, $F_j(s)$ is holomorphic in each of the circles $C(s_1, \pi(d-D-u))$, $C(s', \pi(d_1-D-u))(s' \in L)$, $C(\sigma+it_2, \pi(d_1-D-u))(\sigma_2 < \sigma \leqq \sigma_2 + \pi d_2)$, $C(\sigma+it_2, \pi(d_2-D-u))(\sigma > \sigma_2 + \pi d_2)$, where u is such that $0 < u < d_1 - D$.

We have by the same Lemma III:

$$|F_j(s_1)| < \pi d M(s_1)\Lambda^*(u),$$

where u is such that $0<u<d-D$. Therefore, since $\Lambda^*(u)$ is continuous for $u>0$:

(63) $$|F_j(s_1)| \leqq \pi d M(s_1)\Lambda^*(d-D).$$

Let us now write

(64) $$\Phi_n(s) = F(s) - \sum_1^n a_m e^{-\lambda_m s}.$$

By hypotheses, there exists a number σ' such that $\Phi_n(s)(n \geqq 1)$ is holomorphic in every circle $C(\bar{\sigma}+it_2, \pi d_2)$ with $\bar{\sigma} \geqq \sigma'$, and in every such circle:

(65) $$|\Phi_n(s)| \leqq \varphi_n(e^{-\bar{\sigma}+\pi d_2}), \quad (n \geqq 1),$$

(σ' is independent of n).

It follows then from Lemma III that the function

$$\Phi_{n,j}(s) = \sum_{n=0}^{\infty}(-1)^k c_k^{(j)}\Phi_n^{(2k)}(s) = \sum_k (-1)^k c_k^{(j)} F^{(2k)}(s) -$$

(66) $$\sum_k (-1)^k c_k^{(j)} \sum_{m=1}^{n} \lambda_m^{2k} a_m e^{-\lambda_m s} = F_j(s) -$$

$$\sum_{m=1}^{n} a_m e^{-\lambda_m s} \sum_n (-1)^k c_k^{(j)} \lambda_m^{2k} = F_j(s) - \sum_{m=1}^{n} a_m \Lambda_j(\lambda_m) e^{-\lambda_m s}$$

is holomorphic in every circle $C(\bar{\sigma}+it_2, \pi(d_2-D-u))$, $\bar{\sigma} \geqq \sigma'$ with $0<u<d_2-D$, and in this circle:

(67) $$|\Phi_{n,j}(s)| \leqq \pi d_2 \varphi_n(e^{-\bar{\sigma}+\pi d_2})\Lambda^*(u).$$

Since $\Lambda_j(\lambda_m)=0$ for $m \neq j$, we see that for $n \geqq j$:

$$\sum_{m=1}^{n} a_m \Lambda_j(\lambda_m) e^{-\lambda_m s} = a_j \Lambda_j(\lambda_j) e^{-\lambda_j s},$$

and we may write, for $n \geqq j$ (by (66) and (67)):

(68) $$\Phi_{n,j}(s) = \Psi_j(s) = F_j(s) - a_j \Lambda_j(\lambda_j) e^{-\lambda_j s},$$

(69) $$|\Psi_j(s)| \leqq \pi d_2 \varphi_n(e^{-\sigma+\pi d_2})\Lambda^*(u)$$

$(s=\sigma+it,\ |t-t_2| \leqq \pi(d_2-D-u),\ 0<u<d_2-D,\ \sigma>\sigma',\ j \leqq n)$.

From $\varphi_{n+1}(x) = o(\varphi_n(x))(x \to 0)$ it follows that for σ sufficiently large and j fixed $\varphi_k(e^{-\sigma+\pi d_2}) \geqq \varphi_j(e^{-\sigma+\pi d_2})$, $1 \leqq k \leqq j$. Therefore there exists a quantity σ^* independent of n such that (69) holds for $|t-t_2| \leqq \pi(d_2-D-u)$, $\sigma \geqq \sigma^*$, $0<u<d_2-D$, $n \geqq 1$. We have hence, for the same values of s,

(70) $$|\Psi_j(s)| \leqq \pi d_2 \Lambda^*(u) \underset{n\geqq 1}{\text{g. l. b.}}\ \varphi_n(e^{-\sigma+\pi d_2}) = \pi d_2 \Lambda^*(u)\varphi(e^{-\sigma+\pi d_2}).$$

We shall now prove that $\Psi_j(s)$ is identically zero. For that purpose we shall consider the function

$$\theta_j(\nu;\, s) = \Psi_j\left(\frac{s}{\nu} + it_2\right),$$

where $\nu \geqq \omega$. It follows from (70) that, in the region

$$|t| \leqq \pi\nu(d_2 - D - u) = \frac{\pi}{2}\left(\frac{\nu}{\omega} - 2\nu u\right),\ \sigma \geqq \nu\sigma^*$$

with $0 < u < d_2 - D$, the following inequality holds:

(71) $$|\theta_j(\nu;\, s)| \leqq \pi d_2 \Lambda^*(u)\varphi(e^{-\frac{\sigma}{\nu}+\pi d_2}).$$

Let q be an arbitrary positive quantity and let us set $\alpha = \alpha(q) = 2q\omega e^{\pi d_2 \omega}$, $\beta = \beta(q) > \max\left(\nu\sigma^*, \log\frac{\pi\alpha}{2}\right)$. We have then $\beta > \nu\sigma^*$, $2e^\beta > \pi\alpha > 0$. Since in (71) u can be taken arbitrarily from the interval $(0, d_2 - D)$, we set for $\sigma \geqq \beta$: $u = \frac{1}{2\omega} - \frac{1}{2\nu} + \frac{\alpha}{2\nu e^\sigma}$. The condition $0 < u < d_2 - D = \frac{1}{2\omega}$ is then satisfied, since $\nu > \omega$ and $1 - \alpha e^{-\beta} > 0$. It follows from (71) that $\theta_j(\nu;\, s)$ is holomorphic in the region $R(\alpha, \beta)$ given by $|t| < \frac{\pi}{2}(1 - \alpha e^{-\sigma})$, $\sigma \geqq \beta$, and satisfies there the inequality:

(72) $$|\theta_j(\nu;\, s)| \leqq \pi d_2 \Lambda^*\left(\frac{1}{2\omega} - \frac{1}{2\nu} + \frac{\omega}{\nu} q e^{\pi d_2 \omega - \sigma}\right)\varphi\left(e^{-\frac{\sigma}{\nu}+\pi d_2}\right).$$

This inequality shows first that, if $\nu > \omega$, $\theta_j(\nu;\, s)$ is bounded in $R(\alpha, \beta)$. Indeed, as $\sigma \to \infty$, the right member of (72) tends to $\pi d_2 \Lambda^*\left(\frac{1}{2\omega} - \frac{1}{2\nu}\right)\varphi(0) = 0$. But, if, moreover, (51) of the condition (I) of the theorem is satisfied, then $\theta_j(\omega;\, s)$ is also bounded in $R(\alpha, \beta)$ provided that in the definition of α and β, given above, q takes the value p involved in (51).

Indeed, by (72), we have, in the region $R(\alpha, \beta)$ so defined $(\alpha=\alpha(p),\ \beta=\beta(p))$:

(73) $$|\theta_j(\omega; s)| \leqq \pi d_2 \Lambda^* \left(pe^{-\omega\left(\frac{\sigma}{\omega}-\pi d_2\right)}\right) \varphi\left(e^{-\left(\frac{\sigma}{\omega}-\pi d_2\right)}\right),$$

and by (51) we see that $|\theta_j(\omega; s)| < N < \infty$ in $R(\alpha, \beta)$. Thus we have proved that $\theta_j(\nu; s)$ is bounded in $R(\alpha(q), \beta(q))$ with $q>0$, arbitrary, and that $\theta(\omega; s)$ is bounded in $R(\alpha(p), \beta(p))$. If $\Psi_j(s)$ were not identically zero, $\theta_j(\nu; s)$ would be not identically zero for ν arbitrary, and, by Lemma V we would have:

(74) $$\int_\beta^\infty \log|\theta_j(\nu; \sigma+i\cos^{-1}e^{\beta-\sigma})|e^{-\sigma}d\sigma > -\infty,$$

with $\beta=\beta(1)$, if $\nu>\omega$, and with $\beta=\beta(p)$, if $\nu=\omega$. From (74) and (72), both with $\nu>\omega$, $q=1$, it follows that for $k>0$ sufficiently large:

(75) $$\begin{aligned}&\log\Lambda^*\left(\frac{1}{2\omega}-\frac{1}{2\nu}\right)e^{-k}+\int_k^\infty \log\varphi(e^{-\frac{\sigma}{\nu}+\pi d_2})e^{-\sigma}d\sigma\\ &\geqq \int_k^\infty \log\left[\Lambda^*\left(\frac{1}{2\omega}-\frac{1}{2\nu}+\frac{\omega}{\nu}e^{\pi d_2\omega-\sigma}\right)\varphi\left(e^{-\frac{\sigma}{\nu}+\pi d_2}\right)\right]e^{-\sigma}d\sigma > -\infty.\end{aligned}$$

If condition (II) is satisfied, and if we choose $\nu=\omega'$, we shall have by (75), on putting $e^{-\frac{\sigma}{\omega'}+\pi d_2}=u$:

$$\int_0^K \log\varphi(u)u^{\omega'-1}du > -\infty \quad (K \text{ constant} > 0),$$

which is in contradiction with (53). Thus if condition (II) is satisfied, $\Psi_j(s)\equiv 0$.

From (74) and (73) it follows, on putting $e^{-\frac{\sigma}{\omega}+\pi d_2}=u$, that

$$\liminf_{t=0+}\int_t^1 \log[\Lambda^*(pu^\omega)\varphi(u)]u^{\omega-1}du > -\infty,$$

which is in contradiction with (52). Thus if condition (I) is satisfied we still have $\Psi_j(s)\equiv 0$. In other words: if the conditions of the theorem are satisfied we have

(76) $$F_j(s) \equiv a_j\Lambda_j(\lambda_j)e^{-\lambda_j s}.$$

It follows then from (63) that

(77) $$|a_j\Lambda_j(\lambda_j)|e^{-\lambda_j\sigma_1} \leqq \pi d M(s_1)\Lambda^*(d-D),$$

which is equivalent to (54). Theorem XV is then completely proved.

The following lemma will allow us to consider Dirichlet series with an abscissa of absolute convergence ($\sigma_A < \infty$) as representing the principal branch of the function represented by this series, asymptotically with respect to a simple asymptotic sequence $\varphi_n(x)$, for which both conditions (I) and (II) of Theorem XV are automatically satisfied.

LEMMA VI. *If $\sum a_n e^{-\lambda_n s}$ has an abscissa of absolute convergence $\sigma_A < \infty$, then the principal branch of the function $F(s)$ represented by this series is represented asymptotically by this series with respect to the asymptotic sequence*

$$\varphi_n(x) = A(\sigma')e^{\lambda_n\sigma'}x^{\lambda_n}, \quad A(\sigma') = \sum |a_n|e^{-\lambda_n\sigma'}, \quad \sigma' > \sigma_A,$$

in each horizontal half-strip $S(v, R)$ with $v = \sigma_1 + it_1$, $\sigma_1 \geqq \sigma' > \sigma_A$, t_1 and R arbitrary.

We have indeed for $\sigma > \sigma'$:

$$\left|F(s) - \sum_{m=1}^{n} a_m e^{-\lambda_m s}\right| \leqq \sum_{n+1}^{\infty} |a_m|e^{-\lambda_m\sigma} = e^{-\lambda_n\sigma}\sum_{n+1}^{\infty} |a_m|e^{-(\lambda_m-\lambda_n)\sigma}$$

$$\leqq e^{-\lambda_n\sigma}\sum_{n+1}^{\infty} |a_m|e^{-(\lambda_m-\lambda_n)\sigma'} \leqq \left[\sum_{m=1}^{\infty} |a_m|e^{-(\lambda_m-\lambda_n)\sigma'}\right]e^{-\lambda_n\sigma}$$

$$= A(\sigma')e^{\lambda_n\sigma'}e^{-\lambda_n\sigma}.$$

Theorem XV together with Lemma VI gives immediately the following theorem.

THEOREM XVI. *Let $F(s)$ be holomorphic in a region composed of a circle $C(s_1, \pi d)$, of the half-plane $\sigma > \sigma_0$ and of a channel of width $2\pi d_1$, the central line of which has one extremity in the half-plane $\sigma > \sigma_0$, the other extremity being the point s_1. Let $F(s)$ be given for $\sigma > \sigma_0$ by*

$$F(s) = \sum a_n e^{-\lambda_n s},$$

the upper density of $\{\lambda_n\}$ being D. If $D < d_1 \leqq d$ then

(78) $$|a_n| \leqq 2\pi d \Lambda^*(d-D) M(s_1) \Lambda_n e^{\lambda_n \sigma_1},$$

where Λ^*, $M(s_1)$, Λ_n, σ_1 *have the same meaning as in Theorem XV.*

Since $D<\infty$, we see, by Theorem VII, that $\sigma_A{}^F \leqq \sigma_0 < \infty$, and by Lemma VI, $F(s)$ is represented asymptotically in $S(s_2, \pi d_2)$, with $s_2=\sigma_2+it_2$, $\sigma_2 \geqq \sigma' > \sigma_A{}^F$, $d_2=d$, by $\sum a_n e^{-\lambda_n s}$ with respect to $\{\varphi_n(x)\}$, where $\varphi_n(x)=A(\sigma')e^{\lambda_n\sigma'}x^{\lambda_n}$. But obviously $\varphi(x)=\underset{1\leqq n}{\text{g. l. b.}}\ \varphi_n(x)=0$ for $x<e^{-\sigma'}$, and therefore both conditions (I) and (II), each of which is sufficient for the conclusion of Theorem XV, are satisfied. The conclusion of Theorem XVI, which is that of Theorem XV, is therefore satisfied.

We have seen that Theorem XVI follows immediately from Theorem XV, but Theorem XVI can be proved much more simply than Theorem XV itself.

In order to prove Theorem XVI directly it is sufficient, after having proved Lemmas III and VI (Lemmas IV and V are here unnecessary), to proceed with the proof exactly as for Theorem XV, with $\varphi_n(x)=A(\sigma')e^{\lambda_n\sigma'}x^{\lambda_n}$, until the inequality (69) is established. From this inequality it follows immediately that:

(69′) $$|\Psi_j(s)| \leqq \pi d_2 \Lambda^*(u)\ \underset{j\leqq n}{\text{g. l. b.}}\ \varphi_n(e^{-\sigma+\pi d_2})=0,$$

for $\sigma>\pi d_2+\sigma'$. This gives then (76) and (77). In other words, all the reasoning which is extended from the inequality (69) to the equality (76) is replaced by the establishment of (69′) as an immediate consequence of (69).

The theorems of this chapter were established by the author [**11**].