III

TOPOLOGY WITHOUT POINTS

1. THE FOUNDATIONS OF THE CONCEPT OF SPACE IN TOPOLOGY

THE three foundations of the concept of space in topology start with the concept of limit, the concept of neighborhood, and the concept of closure.

As a limit class Fréchet denotes a set of elements in which certain denumerable sequences of elements are distinguished. Each distinguished sequence is called convergent, and with each convergent sequence of elements exactly one element is associated, called the limit of the convergent sequence. *It is assumed that each subsequence of a convergent sequence* is convergent, and that each sequence all of whose elements are equal to one and the same element, is convergent and has this element as its limit. Instead of describing a limit class as a set in which certain sequences are distinguished, we can, in a more formal way, speak of a limit class if a set L and a subset C of d^L are given, where d^L denotes the set of all the denumerable sequences of elements of L, and the elements of C are called distinguished sequences. The two properties of the convergent sequences can be expressed as properties of the set C.

As a neighborhood class (or as he says, a topological space) Hausdorff denotes a set T (whose elements are called points) in which certain distinguished subsets are associated with the points. Each distinguished subset associated with a point x is called a neighborhood of the point x. It is assumed that

1. Each point x of T is element of at least one neighborhood of x, and each neighborhood of x contains the point x.
2. If U_x and V_x are two neighborhoods of x, then there exists a neighborhood of x which is a subset both of U_x and of V_x.
3. If y is a point contained in the neighborhood U_x, then there exists a neighborhood of y which is a subset of U_x.
4. For each pair of distinct points x and y, there exist two neighborhoods, U_x and U_y, which have no point in common.

Again it is possible to describe the distinction of certain subsets of T as neighborhoods by saying that besides the set T there is given a subset of the set of all subsets of T.

The space T is said to satisfy the second denumerability axiom if there exists a denumerable set of neighborhoods which is a basis for the open subsets of the space, i.e., such that each open set is sum of sets belonging to the basis.

As a closure class (or as he says a class $\bar{A}$) Kuratowski denotes a set in which with each subset A there is associated a set $\bar{A}$, called the closure of A. It is assumed that

1. The closure of the sum of two sets is equal to the sum of the closures of the two sets.
2. The closure of a set consisting of one point, consists of this one point.
3. The closure of the vacuous set is the vacuous set.
4. The closure of the closure of a set is equal to the closure of the set.

These three foundations of topology have in common that they are what may be called point set theoretical. By this we mean that each of them considers the space as a set of elements. Of course, it is a set with special properties distinguishing the space from an abstract set, viz., a set in which certain sequences of elements are distinguished, or a set in which subsets are associated with elements or with subsets in a certain way. But in all three cases it is assumed that the elements of the space, the points, are somehow given individually, and that the space character of the set consists in relations between, and properties of, certain sets of these elements. The same holds for other more or less related ways of introducing the concept of space in topology as well as for the fundamental concept of a metric space in which distance numbers are associated with the pairs of elements.

2. THE CLASSICAL FOUNDATIONS OF THE CONCEPT OF THE STRAIGHT LINE

In arithmetic, three ways of introducing the set of all real numbers, or, geometrically speaking, the straight line, on the basis of the theory of the rational numbers, have been known since the 70's of the last century. They are somewhat analogous to the three introductions of the concept of space in modern topology.

The one which is related to the concept of limit class is Cantor's definition of a real number as a convergent sequence (or, as he said, fundamental sequence) of rational numbers. A sequence of rational numbers $r_1, r_2, \cdots$ is convergent if for each natural number n there exists an index k_n such that $|r_i - r_j| < 1/n$, provided that $i \geq k_n$ and $j \geq k_n$. Two numbers $r_1, r_2, \cdots$ and $s_1, s_2, \cdots$ are said to be equal if $r_1, s_1, r_2, s_2, \cdots$ is a number. A $<$-relation and a limit concept can be introduced for the real numbers so defined. The straight line is the set of all these real numbers.

The way related to the concept of neighborhood class is the definition of a real number as a nested sequence of rational intervals whose lengths approach 0. By a rational interval, we mean a pair $I=(r_1, r_2)$ of rational numbers for which $r_1<r_2$. As the length of the interval I, we denote the rational number r_2-r_1. The interval $I=(r_1, r_2)$ is said to be contained in the interval $J=(s_1, s_2)$ if $s_1 \leq r_1 < r_2 \leq s_2$. We say that I is completely contained in J if $s_1<r_1<r_2<s_2$. A sequence $I_1, I_2, \cdots$ of intervals is said to be strictly decreasing or nested if each interval I_k contains the following interval I_{k+1} completely. Two real numbers $I_1, I_2, \cdots$ and $J_1, J_2, \cdots$ are said to be equal if each I_k contains a J_l (and consequently almost all J_l completely), and each J_k contains an I_m (and consequently almost all I_m completely). Again an order relation and a limit concept can be introduced for the real numbers so defined. The set of all these numbers is called a straight line.

A way somewhat related to the concept of closure class is Dedekind's definition of a real number as a cut or upper section in the set of rational numbers. A set of rational numbers is called an upper section if together with each rational number, it contains all greater rational numbers. If for an upper section there exists a rational number (belonging or not belonging to the upper section) which is smaller than all the (other) numbers of the upper section, but greater than all the (other) numbers not belonging to the upper section, then we say that the upper section is a rational real number. If for an upper section there does not exist a rational number with these properties, then we say the upper section is an irrational real number. The set of all upper sections is called a straight line.

These three ways of introducing the straight line or the set of all real numbers do not presuppose the concept of a point or of a real number. On the basis of a denumerable set (the set of rational numbers or rational intervals) for whose elements certain relations are assumed to be given, (a $<$-relation for the rational numbers, a relation of containing for the rational intervals) they introduce the individual real numbers or points. The set of all of them is formed in order to enable us to talk, if necessary, about the straight line as a whole.

3. POINTS OF A GENERAL SPACE INTRODUCED AS SEQUENCES OF LUMPS

Huntington's foundation of Euclidean geometry is given in terms of spheres and a relation of inclusion.[1] Points are introduced as spheres which do not include any other spheres.

In his Paris thesis in 1923, Nicod[2] in continuation of ideas of Whitehead outlined a theory starting with "volumes," and a relation of "contained in the interior of." As an "abstractive class of volumes," he denotes a class of volumes such that, for any two volumes belonging to the class, one is contained in the interior of the other while no volume belonging to the class is contained in the interior of all the volumes of the class. If A and B are two such abstractive classes, then A is said to cover B if each volume belonging to the class A contains in its interior a volume belonging to the class B. As a point, Nicod defines an abstractive class A covered by every abstractive class which A covers. Like Huntington's elaborate theory, the rather sketchy developments of Nicod aim to lay a foundation of Euclidean geometry by introducing a definition of congruency for pairs of points.

In my book *Dimensionstheorie*[3] I pointed out the desirability of an introduction of the general concept of space in topology which is not point set theoretical in the sense of Section 1, but rather analogous to the introductions of the straight line in arithmetic, outlined in Section 2. I especially aimed to introduce the points of a space as nested sequences of what may be called pieces or lumps—analogous to the introduction of real numbers as nested sequences of rational intervals, and related to Hausdorff's concept of a neighboring class.

Such a topology of lumps seems to me to be closer to the physicist's concept of space than is the point set theoretical concept. For naturally all the physicist can measure and observe are pieces of space, and the individual points are merely given as the result of approximations.

Also, such a topology of lumps probably comes closer to the ideas of some philosophers than does the point set theoretical concept of space. It is true, in the philosophical literature the critique of the concept of the continuum as a set of points was most frequently directed against the introduction of the linear continuum, i.e., the straight line, although just in this case the introduction of points as nested sequences of rational intervals conforms with at least some of the philosophical exactions. But on somewhat better grounds some of the critical ideas might be directed against the traditional introductions of the general concept of space. To supply this latter concept with a foundation whose logical dignity equals that of the basis of the concept of the straight line, is the purpose of our theory.

We start with a partially ordered system of lumps which we shall denote by $U, V, \cdots$. They correspond to the rational intervals of arithmetic and to the neighborhoods of Hausdorff. We call the order relation defined for these lumps "completely contained in" and denote it by $\subset\subset$. The reason we speak of complete containing rather than containing, and use the symbol $\subset\subset$ rather than $\subset$ will be clear later, when we shall see that two lumps U and V

for which $U \subset\subset V$, correspond to two open sets in the topological space such that the closure of U is a subset of V —a relation described in topology by the symbol $U \subset\subset V$.[4]

About the partially ordered set of the lumps $U, V, \cdots$, we make the following assumption: Two elements U and V are identical if and only if for each element W the relations $W \subset\subset U$ and $W \subset\subset V$ are equivalent (that is to say, either both hold or both do not hold).

Now as points we define certain sequences of lumps $U_1, U_2, \cdots$, which are strictly decreasing; that is to say, for which $U_{k+1} \subset\subset U_k$ for each k. The points $U_1, U_2, \cdots$ and $V_1, V_2, \cdots$ will be called equal if and only if each U_i contains one V_j (and consequently almost all V_j) completely, and if each V_i contains one U_k (and consequently almost all U_k) completely. The point $U_1, U_2, \cdots$ is said to lie in the lump U if we have $U_i \subset\subset U$ for one (and thus for almost all) integers i.

It is clear that if we consider a strictly decreasing sequence $U_1, U_2, \cdots$ of open sets in a topological space, four cases can occur:

(1) *The sequence contracts to a point* p; that is to say, only the point p is contained in all open sets of the sequence, and each neighborhood of p (i.e., each open set containing p) contains a set U_i and consequently almost all U_i. E.g., this is the case for the strictly decreasing sequences of rational intervals of the straight line whose lengths converge toward 0, if by a rational interval we mean the set of all real numbers between two rational numbers.

(2) *The sets of the sequence have only one point in common without contracting to this point.* As an example, let us consider a space obtained from the straight line by omitting the point 1, and denote by U_n the open set consisting of the intervals $(-1/n, 1/n)$ and $(1-1/n, 1+1/n)$. The open

sets $U_1, U_2, \cdots$ have only the point 0 in common without contracting to the point 0. If we admit unbounded open sets, e.g., the ray consisting of all numbers $>n$ which we shall denote by (n, ∞), then even the ordinary straight line contains a decreasing sequence of open sets which have only one point in common without contracting to it, e.g., the sequence $U_1, U_2, \cdots$ where U_n denotes the sum of the interval $(-1/n, 1/n)$ and the ray (n, ∞).

(3) *The sets of the sequence have more than one point in common.* E.g., this is the case for a strictly decreasing sequence of rational intervals of the straight line whose diameters do not approach 0.

(4) *The sets of the sequence have no point in common.* E.g., this is the case if U_n denotes the set of all real numbers excluding 1 between $1-1/n$ and $1+1/n$ in the space obtained from a straight line by omitting the point 1, or if U_n denotes the ray (n, ∞) in the ordinary straight line.

In the case of the straight line, a number was associated with each rational interval, namely, its length, and the only condition that had to be imposed on a strictly decreasing sequence of rational intervals in order that it should define a real number was that the lengths of the intervals of the sequence should converge toward 0. This was necessary in order to exclude case (3). The cases (2) and (4) in the example of the rational intervals of the straight line are automatically excluded. More generally, it can easily be shown that in any compact space a sequence of open sets which have exactly one point in common, contracts to this point. Moreover, in a compact space the sets of a strictly decreasing sequence of open sets have at least one point in common. If the sequence $U_1, U_2, \cdots$ is decreasing without being strictly decreasing, then there need not exist a point common to all sets. E.g., even in the straight line the decreasing open intervals (0, 1), (0, 1/2), $\cdots$, (0, $1/n$), $\cdots$ have no point in common.

In my book I suggested the consideration of points of a space as certain decreasing sequences of lumps, without giving a criterion as to which sequences should be called points. I pointed out the desirability of formulating such a criterion in my colloquium in Vienna. In 1930 this problem was solved by A. Wald in a way that we shall outline presently.

At this point I should like only to mention that in order to obtain a space whose abstract properties are related to those of a straight line, we shall start with a system of lumps which has certain denumerability properties. On the other hand, it is clear that if we start with a set of lumps that is altogether denumerable (as I originally did), then we can not possibly get all topological types of spaces on the mere basis of the order relation for the lumps. For, the set of all partially ordered denumerable sets which are distinct (that is to say, no two of which are isomorphic) has the power of the continuum. Consequently, if we wish to describe the points of a space in terms of the order relation alone, we cannot get more than continuously many spaces —while the set of all types of topological spaces satisfying the second denumerability axiom, and even the set of all topological types of subsets of the straight line has a greater power than the continuum.

4. THE THEORY OF WALD

The main idea of Wald's theory[5] is a criterion which in a regular topological space satisfying Hausdorff's first denumerability postulate holds for a sequence in the case (1) and fails to hold for a sequence in one of the cases (2), (3), (4) mentioned in Section 3. In such a space a necessary and sufficient condition in order that the strictly decreasing sequence of non vacuous open sets U_1, U_2, $\cdots$ contract to a point is *that for each open set V which does not contain any of the U_k completely, and for each open set W which is completely contained in V, the set W is disjoint from almost all U_k.* Here two sets are called disjoint if they have no point in common. A space is called regular if for each point p lying in an open set U there exists an open set V containing p and such that $V \subset\subset U$. A space satisfies Hausdorff's first denumerability axiom if for each point p there exists a denumerable set of open sets containing p and such that each open set containing p contains one of the open sets of the denumerable set as a subset.

The necessity of the condition is easily verified. The interesting fact is its sufficiency. In order to prove that the condition is sufficient, we have to show that it does not hold in Cases (2), (3), and (4). Clearly, each sequence U_{i_1}, U_{i_2}, $\cdots$ extracted from a sequence satisfying the condition, satisfies the condition.

It is easily seen that the sets of a sequence U_1, U_2, $\cdots$ satisfying Wald's condition have at most one point in common. For, if the sets of a sequence U_1, U_2, $\cdots$ have two distinct points p and q in common, then we may choose an open set V containing p, but not q, and an open set W containing p and completely contained in V. Since q does not lie in V, the set V does not contain any of the sets U_k. Nevertheless, none of the sets U_k and W are disjoint since W contains p. Thus a sequence of open sets having more than one point in common does not satisfy Wald's condition. Hence, the condition excludes Case (3).

Next we show that the sets of a sequence U_1, U_2, $\cdots$ satisfying Wald's condition have at least one point in common. Let us assume that the sets of the sequence U_1, U_2, $\cdots$

have no point in common. Then, first of all, it is clear that if we pick out a point p_k of the set U_k, there does not exist any point p which is cluster of the sequence $p_1, p_2, \cdots$, that is to say, each neighborhood of which contains p_n for infinitely many n. For, on account of the strict decreasing of the sets $U_1, U_2, \cdots$ such a cluster point p would be common to all the sets U_k. Since the sequence $p_1, p_2, \cdots$ does not have any cluster point, it contains infinitely many distinct points. Thus, from $p_1, p_2, \cdots$, we may extract a sequence $q_1, q_2, \cdots$ of mutually distinct points. For each k, if q_k is the point p_{i_k}, we set $U_{i_k}=X_k$. Now we may choose open sets V_1, $V_3, V_5, \cdots$ such that of all the points $q_1, q_2, q_3, \cdots$ the set V_{2n-1} contains only the point q_{2n-1}. We may furthermore choose a neighborhood W_{2n-1} of q_{2n-1} which is completely contained both in V_{2n-1} and in X_{2n-1}. If now we set $V=V_1+V_3+V_5+\cdots$ and $W=W_1+W_3+W_5+\cdots$, then we see

(1) that V does not contain any of the sets X_k completely since it does not contain the points $q_2, q_4, q_6, \cdots$

(2) that W is not disjoint from any of the sets X_k since it contains the points $q_1, q_3, q_5, \cdots$

(3) that, as one can prove, the closure of W contains merely points of the closure of $W_1, W_3, W_5, \cdots$ and consequently is contained in V; that is to say, that W is completely contained in V.

Thus, the sequence $X_1, X_2, \cdots$ does not satisfy Wald's condition. From the assumption that the sets $U_1, U_2, \cdots$ have no point in common we derived that the sequence $U_1, U_2, \cdots$ contains a subsequence $X_1, X_2, \cdots$ not satisfying Wald's condition. Thus it does not satisfy the condition itself, and hence the condition excludes Case (4).

Finally, it is easily seen that Wald's condition also excludes Case (2).

In addition to the concept of complete containing, Wald's criterion refers also to the concept of two disjoint open sets. Two open sets are called disjoint if they have no point in common. Assuming that each open set does contain a point, we can define two open sets U and V to be disjoint by saying that there is no open set W which is completely contained in both.

On the basis of this criterion Wald solved the problem mentioned in Section 3 by formulating the following definition of a point on the basis of a partially ordered set of lumps: *A point is a strictly decreasing sequence of lumps,* $U_1, U_2, \cdots$ *such that for any lump* V *which does not completely contain any of the* U_k*, and any* W *completely contained in* V*, the lump* W *is disjoint from almost all* U_k*.* Here two lumps are called disjoint if there does not exist any lump completely contained in both.

As in Section 3, a point $U_1, U_2, \cdots$ is said to lie in the lump U if U contains completely a U_k (and consequently almost all U_k). Moreover, the point $U_1, U_2, \cdots$ is said to belong to the closure of the lump V if V is not disjoint from any of the U_k. By virtue of this definition it is clear that if W is completely contained in V, then each point of the closure of W is a point of V.

In order to prove that the set of all points defined as above form a regular topological space, it is necessary to assume that also conversely if W is not completely contained in V, there exists at least one point of the closure of W which is not a point of V. Under these assumptions one can prove that for each lump U the set of all points lying in U is an open set in the topological space obtained, and that for two lumps V and W, the closure of W is a subset of V if and only if the lump W is completely contained in the lump V in the sense of the partially ordered set. This is the reason we use the words "completely contained" and the symbol $\subset\subset$ rather than the word "contained" and the symbol $\subset$, to denote the order relation in the set of lumps.

We get a topological space satisfying Hausdorff's second denumerability axiom by assuming that the basic set of lumps consists of a denumerable system of lumps and lumps which in a symbolic way are sums of some of the lumps of the denumerable system. The points of the space can be defined as sequences of lumps belonging to the denumerable system; but in order to constitute points, a sequence must satisfy Wald's condition with regard to any two lumps V and W of the whole undenumerable set of lumps. (In proving that in a regular topological space Wald's criterion excludes Case (4) for a strictly decreasing sequence of open sets, we had to form sets V and W as sums of denumerably many open sets.) The reader interested in further details of Wald's theory is referred to his original paper.[5] The necessity of basing a theory which is to yield all regular topological spaces satisfying the second denumerability axiom on a non-denumerable set of lumps, follows from the remarks at the end of Section 3.

Here I should like to add to Wald's theory the following remark. In a compact space satisfying the denumerability axiom, that is to say, admitting a denumerable system of open sets which is equivalent to the system of all open sets, we can say: A strictly decreasing sequence $U_1, U_2, \cdots$ of open sets belonging to the denumerable system contracts to a point if and only if for each open set V *belonging to the denumerable system* which does not contain completely any of the U_k, and for each open set W *belonging to the denumerable system* which is completely contained in V, the set W is disjoint from almost all U_k. Moreover for each point of the space there exists a strictly decreasing sequence of open sets belonging to the denumerable system which satisfies the condition mentioned and contracts to the point. This remark shows that the theory of compact spaces can be developed on the basis of a denumerable partially ordered set of lumps. We simply have to define points as strictly decreasing sequences of lumps of the denumerable system, satisfying Wald's criterion with respect to the lumps of the denumerable system.

It seems certain to me that not only compact spaces, but also spaces of certain more general properties can be derived from a denumerable partially ordered set of lumps, and probably in a similar way. Of course, from the consideration at the end of Section 3 it follows that one cannot even hope to introduce all the spaces of the type in consideration unless their totality has a power not surpassing that of the continuum. It would be quite interesting to carry out this idea for Borelian and analytic spaces. Another question is what we can get by admitting as points transfinite sequences of lumps.

5. THE THEORY OF MOORE

Referring to previous lectures, R. L. Moore in 1935 published a foundation of the topology of the plane in terms of the concepts "piece" and "embedded in." We shall indicate the main points of contact and the main differences existing between the theory of Moore and the ideas outlined in the preceding sections, referring the reader to Moore's paper in Vol. 25 of the Fundamenta Mathematicae for the details of his development.

Moore starts with a partially ordered set of pieces. He assumes that for each integer n there exists a system of pieces with certain properties. A suggestive description of Moore's assumptions is obtained by calling "basic pieces of n^{th} degree of smallness" or briefly "pieces of n^{th} degree" the pieces which are members of the n^{th} system. By using this notation we do not mean to indicate that any metric assumptions about these pieces are explicitly made which would enable us to speak about a numerical upper bound of their size. But ultimately the pieces of the n^{th} system correspond to open sets whose diameters approach 0 with increasing n.

In this notation, Moore assumes:

(1) In each piece for each n there is embedded a basic piece of n^{th} degree of smallness.

(2) If the piece W is embedded in the piece V then there is an integer n such that each of the pieces of n^{th} degree which intersects W is embedded in V. (Moore says that two pieces U_1 and U_2 intersect if they are not disjoint in the sense of the theories outlined in the preceding sections; i.e., if there exists a piece embedded both in U_1 and U_2.)

(3) For each piece U and each n the set of the pieces of n^{th} degree embedded in U is coherent, i.e., cannot be split into two sets of pieces such that no piece of one of the sets intersects any of the pieces of the other set.

(4) For each piece U there exists a finite number of pieces of n^{th} degree, say $V_1 \cdots, V_k$, such that for a certain number m each piece of m^{th} degree intersecting U is embedded in one of the pieces $V_1 \cdots, V_k$.

As a point Moore defines *a decreasing sequence of pieces* $U_1, U_2, \cdots$ *such that for each* n, U_n *is one of the pieces of* n^{th} *degree.*

While the assumptions mentioned above represent merely a part of Moore's postulates, they are sufficient for a comparison of his ideas with the theories outlined in the preceding sections.

Driving toward a foundation of the topology of the plane, Moore makes rather special assumptions at the start. E.g., assumption (3) mentioned above essentially means that each piece is connected, thus making connected and locally connected the space derived from the set of pieces. Assumption (4) amounts to a Borel-covering theorem for the pieces thus making the closures of the pieces compact.

Clearly, Moore might have adopted the course of first drawing conclusions from the mere facts that the pieces form a partially ordered set having some general properties and gradually introducing more special assumptions. But even then his treatment would differ from the theory outlined in the preceding sections on account of his introduction of a topological degree of smallness, i.e., of the system of pieces of n^{th} degree for $n=1, 2 \cdots$. It is these systems that enable him to exclude Case (3) mentioned in section 3, i.e., decreasing sequences of pieces whose members have more than one point in common. With their help he can define points simply as decreasing sequences of pieces of increasing degrees. We are reminded of the classical introduction of a real number as a nested sequence of rational intervals whose lengths approach 0. One of the main advantages of the theory outlined in the preceding section was a definition of points without the help of any (metric or topological) criterion of smallness—exclusively in terms of the relation "contained in."

Moreover, Moore defines as a point *each* nested sequence of pieces which get indefinitely small in his topological sense, and discards the possibility of Case (4), mentioned in section 3, viz., the possibility of a nested sequence of pieces whose elements have no point in common. Since this case can occur even in a regular topological space satisfying the second denumerability axiom, Moore in his definition implicitly assumes local compactness of the space to be derived, a procedure which again is justified by his aim, the topology of the plane. But just the exclusion of nested sequences without common point by an intrinsic criterion in terms of the relation "contained in" presented the main difficulty in introducing the general concept of space in the preceding sections.

Aside from these differences there are, of course, important similarities between the two approaches. In addition to the fundamental idea of describing the relation of "completely contained in" for open sets, and of defining points as strictly decreasing sequences, several details are common to the two theories. For instance, Moore's definition of a point pertaining to a piece *U* if one of the pieces defining the point is embedded in *U* (and consequently almost all of the pieces are), is the same as that of a point lying in a lump given in the preceding sections.

6. THE THEORY OF STONE

In the following sections we shall outline some recent theories which likewise avoid starting with given points, and instead introduce points as classes of given entities with given relations. But they move in directions different from the theories introducing points as nested sequences of lumps. Since they are better known than these latter theories our exposition will be brief.

M. H. Stone[6] starts with a Boolean ring or a distributive lattice with complementation, that is to say, with a system of entities for which two associative, commutative, totally linear, and distributive operations are defined, denoted by $+$ and $\cdot$, admitting indifferent elements V and U such that for each element A there exists a "complementary" element A' satisfying the conditions $A \cdot A' = V$ and $A + A' = U$.

Stone calls a "point" a set of elements, $\mathscr{A}$, which contains at least one element and does not contain all elements and has the following properties:

(S_1) *If A belongs to $\mathscr{A}$, and $C \subset A$ (i.e., $C + A = A$), then C belongs to $\mathscr{A}$.*

(S_2) *If A and B belong to $\mathscr{A}$, then $A + B$ belongs to $\mathscr{A}$.*

(S_3) *If $A \cdot B$ belongs to $\mathscr{A}$, then at least one of the two elements A, B belongs to $\mathscr{A}$.*

A point so defined is said to be associated with the element A if the element A belongs to the set defining the point. The set of all points defined in this way is made a topological space in such a way that for each element of the Boolean ring, the corresponding point set, i.e., the set of all associated points, is an open subset of the space, and that the point sets associated with all the elements of the Boolean ring form a basis of the open subsets of the space. However, in the space so obtained the closure of a set consisting of exactly one point may contain other points. In order to exclude this possibility Stone has to make the following assumption about the Boolean ring:

(S_4) *No point $\mathscr{A}$ is proper subset of any other point $\mathscr{B}$,* the points considered as sets with the properties (S_1), (S_2), (S_3).

For two elements A and B of the Boolean ring the two associated point sets are complementary subsets of the space if and only if A and B are complementary elements of the Boolean ring. Since for each element the Boolean ring contains a complementary element whose corresponding point set is likewise open, it follows that each of the open sets of the basis of the topological space has a complementary set which is likewise open. From this we readily conclude that the topological space derived from the Boolean ring is totally disconnected. Moreover it can be shown to be bicompact.

Conversely, each given totally disconnected bicompact space can be obtained in this way.

Of the topological spaces homeomorphic with subsets of Euclidean spaces or the Hilbert space, those which are homeomorphic with a closed subset of Cantor's discontinuum and only those are included in Stone's theory.

7. THE THEORY OF WALLMAN

Instead of a Boolean ring, H. Wallman[7] starts with a mere distributive lattice without requiring that the lattice contain a complementary element of each of its elements. As a "point" Wallman defines each system of elements with the following properties:

(W_1) *Any finite number of elements of the system have a product which is* $\neq V$.

(W_2) *The system cannot be increased without losing property* (W_1), that is to say, for each element A which does not belong to the system there is a finite number of elements in the system whose product and A have the product V.

Comparing Stone's definition of a point with that of Wallman, we see that the latter is a special case of the former if we interchange the operations $+$ and $\cdot$, and, consequently, the relations $\subset$ and $\supset$. If we perform this change in Stone's definition it deals with a system $\mathscr{A}$ with the following properties:

(S_1') *If* A *belongs to* $\mathscr{A}$ *and* $C \supset A$, *then* C *belongs to* $\mathscr{A}$.

(S_2') *If* A *and* B *belong to* $\mathscr{A}$, *then* $A \cdot B$ *belongs to* $\mathscr{A}$.

(S_3') *If* $A+B$ *belongs to* $\mathscr{A}$, *then at least one of the two elements* A, B *belongs to* $\mathscr{A}$.

Now we readily see that a point in the sense of Wallman has these three properties, and in addition the property (S_4).

Wallman, like his predecessors, associates a point with the element A of the basic system (lattice) if the element

A belongs to the set of elements defining the point. He makes the set of all points a topological space in such a way that for each element of the lattice the corresponding point set, i.e., the set of all points associated with the element, is a closed subset of the space. Moreover, the point sets associated with all the elements of the lattice, form a basis for the closed subsets of the space, that is to say, each closed subset of the space can be represented as the product of closed sets of the basis.

However, the basis of the closed subsets of the space so obtained (with point set addition and multiplication) is not necessarily isomorphic with the lattice. A point set belonging to the basis may correspond to different elements of the lattice. In order to exclude this possibility, Wallman has to make the following assumption about the lattice:

(W_3) *For each pair of elements of the lattice, A and B, such that A not $\subseteq B$, there exists an element C of the lattice such that $A \cdot C \neq V$ and $A \cdot C = V$.*

The topological space so obtained can be shown to be bicompact. Conversely, each bicompact space can be obtained in this way.

8. THE THEORY OF MILGRAM

Stone starts with a Boolean ring, Wallman with a distributive lattice thus dispensing with complementation but retaining the two lattice operations. A. N. Milgram in a recent theory[8] starts with a mere partially ordered set satisfying some simple conditions thus dispensing with the lattice operations altogether.

A subset, S, of a partially ordered set P is called an upper section of P if for each element x belonging to S any element of P which is $>x$, belongs to S. Now let P be a partially ordered set containing a largest element u, for which there exists a system of upper sections, $\mathcal{S}$, with the following properties:

(M_1) *For any two elements x and y of P such that x not* $\leq y$, *the system* $\mathcal{S}$ *contains an upper section containing x but not containing y.*

(M_2) *No upper section belonging to* $\mathcal{S}$ *is a proper subset of any upper section belonging to* $\mathcal{S}$.

(M_3) *If S is an upper section belonging to* $\mathcal{S}$, *and x and y are two elements of P which are not contained in S, then P contains an element which is* $\geq x$ *and* $\geq y$ *and is not contained in S.*

Then Milgram calls "point" *each upper section of P belonging to* $\mathcal{S}$, and shows that T, the set of all these points, is a closure class whose closed sets have a basis isomorphic with P.

For each element x of P we form the set of those points (i.e., upper sections of P belonging to $\mathcal{S}$) to which the element x belongs. We shall denote this point set by $[x]$. For instance, $[u]$ is the set of all points, thus $[u] = T$. We further call $[P]$ the set of the point sets $[x]$ formed for all the elements x of P. By virtue of the condition (M_1) the set $[P]$, when partially ordered by the subset relation, is isomorphic with the partially ordered set P, that is to say, $[x]$ is a proper subset of $[y]$ if and only if $x < y$.

If now A is any point set, i.e., any subset of T, then Milgram calls "closure of A" the intersection of all point sets $[x]$ which contain A as a subset. There is at least one such set which does contain A as a subset, viz., $[u] = T$. If the space T contains more than one point, then the closure defined in this way satisfies Kuratowski's four postulates mentioned in section 1. The last one, i.e., that the closure of the closure of a set is equal to the closure of the set, is an immediate consequence of the definition of the closure.

The same is true for one half of Kuratowski's first postulate, viz., for the statement that the sum of the closures of two sets is a subset of the closure of their sum. The other half, viz., that the closure of the sum of two sets is a subset of the sum of their closures, is a consequence of the assumption (M_3). Also one half of Kuratowski's second postulate, viz., that a set consisting of exactly one point is a subset of its closure, immediately follows from the definition of closure. The other half, viz., that the closure of a set consisting of one point does not contain any other point, is a consequence of the assumption (M_2). Kuratowski's third postulate that the closure of the vacuous set is vacuous, means that the intersection of all point sets $[x]$ is vacuous. If the space contains more than one point, then this is a consequence of Kuratowski's second postulate and the definition of closure.

Each partially ordered set P satisfying the conditions (M_1), (M_2), (M_3) is thus isomorphic with a basis of the closed subsets of some closure class.

Conversely, let T be any given closure class, and P any basis of the set of all closed subsets of T, partially ordered by the subset relation. For each point p of T, let S_p be the set of all sets belonging to P which contain p. Obviously, for each point p the set S_p is an upper section of P. If we denote by $\mathscr{S}$ the set of the sections S_p formed for all points p of T, then the system $\mathscr{S}$ obviously satisfies the assumptions (M_1) and (M_2). It also satisfies the assumption (M_3). For if S_p is the upper section consisting of all the closed sets of the basis P which contain the point p, and if x and y are two closed sets of the basis P which do not belong to S_p, that is to say, do not contain p, then the set $x+y$ is a closed set not containing p although not necessarily belonging to P. But since P is a basis of the system of all closed subsets of T, the set $x+y$ is the intersection of closed sets belonging to P, and since $x+y$ does not contain p, not all sets of P containing $x+y$ contain p. Hence the basis P contains a set $\geq x$ and $\geq y$ and not containing p, thus not belonging to S_p.

It follows that (M_1), (M_2), (M_3) are both necessary and sufficient conditions for a partially ordered set to be isomorphic with a basis of the system of all closed subsets of some closure class.

This theory is entirely free of any reference to operations and deals with the order relations alone. If we look upon the theories of Stone and Wallman (the former dualized as in section 7), we see that they likewise introduce points as upper sections. If a Boolean ring satisfies Stone's condition (S_4), then his points are upper sections satisfying Milgram's conditions (M_1), (M_2), (M_3). The same is true for Wallman's points if the lattice satisfies condition (W_3).

By choosing a general partially ordered set as the basis of his theory Milgram not only provided the concepts of point and of space with a foundation free of irrelevant elements but obtained a means of deriving important topological theorems independent of the existence of points, that is to say, for entities more general than ordinary spaces. For in order to derive a space from a partially ordered set P he has to assume the existence of a system of upper sections of P, satisfying the three conditions (M_1) to (M_3) while important parts of topology go through without the last two of these conditions, provided that the system of upper sections satisfying (M_1) is denumerable or has a power not surpassing some preassigned cardinal number.

What is necessary in order to derive large parts of topology in a partially ordered set P is the existence of a system of upper sections of P containing for any two elements x and y

of P such that x not $\leq y$, an upper section to which x but not y belongs, and such that the power of the system lies under a preassigned cardinal number N. Such a system of upper sections Milgram calls a *strong separating system* (of a power $\leq N$) of the partially ordered set P. For many purposes it is even sufficient to know that there exists a system of upper sections of a power $\leq N$ which for any two elements x and y such that $x > y$, contains an element to which x but not y belongs. Such a system is simply called a *separating system* (of a power $\leq N$) of P. Clearly in these assumptions the accent lies on the bounded power of the separating systems. For the existence of a separating system in P whose power is equal to that of P is certain without any assumption. The set of upper sections of P formed for all the elements of P is an example of a strong separating system of P whose power equals that of P.

Let U_1, U_2, $\cdots$ be the elements of a separating system of P ordered in some way. If the separating system is denumerable, then we can order its elements in a sequence whose indices are natural numbers. If the separating system is non-denumerable, then the sequence U_1, U_2, $\cdots$ will necessarily be transfinite. Now let x_0 be a given element of P. If U_1 contains an element $>x_0$, then choose one of these elements and call it x_1. Otherwise, set $x_1 = x_0$. If U_2 contains an element $>x_1$, then choose one of them and call it x_2. Otherwise, set $x_2 = x_1$. Proceeding in this way, we get a sequence of elements x_1, x_2, $\cdots$ (which is denumerable if the separating system is denumerable, or transfinite otherwise) which has the following property: Any element y, which is $\geq x_n$ for each index n, is what we may call an upper bound above x_0; that is to say, it is $\geq x_0$ and P does not contain any element $>y$.

If P is what is called *inductive* of the power of the separating system, that is to say, if each increasing sequence $z_1, z_2, \cdots$ of elements of P whose power does not surpass that of the separating system, contains an element z such that $z \geq z_n$ for each element z_n of the sequence, then for each element x_0 there exists in P an upper bound above x_0.

If P is a set of closed (or open) subsets of a regular topological space satisfying the second denumerability axiom then there exists a denumerable separating system of P. If the $<$-relation means "is subset of" and if P is inductive of denumerable order, then by virtue of the last theorem each element of P is contained in an element of P which is not a proper subset of any element of P. Such an element is said to be saturated in P. Its existence constitutes Brouwer's so-called saturation theorem. Thus this theorem follows from Milgram's theorem (denumerable case).

Now let P be a set of subsets of a separable space and $<$ mean "contains as a subset." If P is inductive of denumerable order, and x_0 is an element of P, then x_0 contains an element of P, which does not contain any element of P as a proper subset. Such an element is said to be irreducible in P. Its existence constitutes Brouwer's so-called reduction theorem. If, for instance, P is the set of all sub-continua of a Euclidean space which contain two distinct points p and q, and $<$ means the subset relation, then p is inductive of denumerable order, by virtue of the theorem that the product of a sequence of continua containing p and q is a continuum containing p and q. The reduction theorem, in this case, yields the existence of an irreducible continuum joining p and q within each continuum joining p and q, a continuum being called irreducible between p and q if it has no proper subset which is a continuum containing p and q.

But the generalized saturation and reduction theorems are derived without the assumption of the existence of points.[9] In the same way Milgram derives the covering theorems of topology as a part of the theory of the partially ordered system of all subsets of a space without assuming that the space is a point set in the sense of the concepts discussed in Section 1. These are examples of a topology without points.

9. CONCLUDING REMARKS

The idea of dealing with the topological relations between the subsets or certain subsets of the space without considering them as point sets is of course related to the program which with regard to projective and affine geometry has been carried out by the algebras of these elementary geometries treated in the second lecture in this pamphlet. It is the same shift from the consideration of individual points or sets of points to the study of the mutual relations of spatial figures—in topology mainly of closed and open sets, in the elementary geometries of flats of different dimensions.

Another remark should clear up the relation between the theories discussed in sections 3-5, and those studied in sections 6-8. The former ones introduce points as nested sequences of lumps after the model of the introduction of real numbers as sequences of rational intervals—the latter ones introduce points as sets of lumps which if applied to the case of the straight line would yield a definition of a real number as the set of all open rational intervals containing the number.

A transition from the "sequence" definition to the "set" definition of point can be made by forming, for any sequence of lumps defining a point, the set of all lumps which contain at least one lump of the sequence. A transition from the "set" definition to the "sequence" definition can be made if the underlying partially ordered set is denumerable.

Compared with the earlier sequence theories, the method of Stone and Wallman has the advantage of being applicable to spaces in which the former theories (at least in their original form, dealing with denumerable sequences) would not work, and Milgram's theory even comprises entities which are not point spaces at all. Obviously, the sequence theories work only in spaces satisfying Hausdorff's first denumerability axiom. While this condition holds in the most important spaces, in particular, in all metrizable spaces, it is desirable for a foundation of topology to comprise still more general spaces. Of course, by admitting transfinite sequences of lumps we should get sequence theories which would also comprise more general spaces, but probably at the expense of simplicity. Moreover, from the algebraist's point of view, the set definition of points may seem preferable because of the analogy it establishes with the theory of ideals—in fact, Stone and Wallman use the terminology of the theory of numbers in formulating their definitions of points. However, Milgram's theory, as well as the sequence theories, seems to indicate that essentially the definition of point is based on the relation of partial order rather than on the lattice operations from which the analogy with the ideals is derived.

On the other hand, in those cases in which the sequence definition of points works, it operates with a minimum of logical machinery. It is true that it has to complement the definition of a point by a definition of the identity of two points, but the set definitions of points need equivalent completeness postulates (S_4), (W_2), and (M_2). From the point of view of reducing the set theoretical character of the foundation, the sequence definition of points is obviously preferable to any set definition.

At any rate, there have been various attempts in the topological literature essentially towards the same end, namely, that of eliminating points as basic concepts, and yet so far these attempts have been quite unrelated. For some time it has seemed desirable to me to coordinate these ideas. The present lecture is intended to carry out this synthesis.

In concluding, I should like to point out that even the introduction of points as nested sequences of lumps somehow transcends what can be observed in nature. For, by a lump, we mean something with a well defined boundary. But well defined boundaries are themselves results of limiting processes rather than objects of direct observation. Thus, instead of lumps, we might use at the start something still more vague—something perhaps which has various degrees of density or at least admits a gradual transition to its complement. Such a theory might be of use for wave mechanics.

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NOTES

[1]*Math. Ann.*, LXXIII (1913), p. 522.

[2]*La géométrie dans le monde sensible*, Paris, Alcan, 1924. An English translation in J. Nicod, *Foundations of Geometry and Induction*, New York, Harcourt, Brace & Co., 1930.

[3]*Dimensionstheorie*, Leipzig, Teubner, 1928, p. 15.

[4]In my book I defined points as decreasing sequences of open sets, not as strictly decreasing sequences, as is stated in this lecture in anticipation of Wald's method.

[5]*Ergebnisse e. Math. Kolloquiums*, Vienna, 3, 1932, p. 6.

[6]*Trans. Amer. Math. Soc.*, Vol. 40 (1936), p. 37, and Vol. 41 (1937), p. 374.

[7]*Annals of Math.*, Vol. 39 (1937), p. 112.

[8]*Reports of Math. Colloquium*, Notre Dame, Ind., 2[d] ser., No. 1 (1939), p. 18; No. 2 (1940), p. 1; *Proc. Nat. Acad. Sci.*, Vol. 26 (1940), p. 291.

[9]Milgram also gives a proof without the use of transfinite induction (although, of course, using the axiom of choice). While Kuratowski's method of elimination of transfinite numbers [*Fundamenta Math.*, III (1922), p. 76] does not yield Milgram's theorem about partially ordered sets, the latter contains Kuratowski's theorem as a special case. Milgram's theorem that a partially ordered set which is inductive for *all* powers, contains for each element x_0 an upper bound above x_0, contains also the maximum principle formulated for algebraic purposes by M. Zorn in his paper, "A Remark on Method in Transfinite Algebra," *Bull. Am. Math. Soc.*, Vol. 41 (1935), p. 667.

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