II

WHAT ARE WAVES?

IT is not easy to give a general definition of waves which would be precise and would at the same time include all cases presenting a character which our intuition attributes to waves. When we compare various kinds of phenomena which in some respect or other are connected with waves, and try to extract that which they have essentially in common, we perceive that certain circumstances, which at first sight seem characteristic, tend to efface themselves, so that one might be tempted to include (as Lagrange once did) all motion compatible with the laws of dynamics.

However, if we content ourselves with grasping an essential constituent (without pretending to include all cases which have some aspect suggestive of waves) we are led to adopt as typical the characterization enunciated in the fifteenth century by Leonardo da Vinci in the following form: "The impetus is much quicker than the water, for it often happens that the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in its place."

To render more precise this intuitive idea of Leonardo we will say that a motion of a material medium proceeds in waves, or that it is a progressive undulatory motion, when the actual displacements of the particles of the medium are accompanied by the much more accentuated and rapid motion of some striking constituent of the phenomenon in question: for example, the propagation in the medium of a certain perturbation (of which we shall recall, in §1, one of the best known aspects); or of some form of energy, or again (to consider the typical case of waves in liquids) of a free surface, that is to say, the surface of separation between water and air (§12), or finally, of a surface of separation between two different regimes of motion of a material medium (§15).

With this understanding, I shall try to present to you a summary of these different aspects, beginning, for the sake of greater clearness, with some quite classical generalities concerning vibrating strings. I shall permit myself to dwell a little on the elementary kinematic premises (§§1 to 8), in view of the fact that they will permit me to bring in the notion of a group of waves (§9), and its consequences for dispersive media (§10), in a manner which is more direct and illuminating than the usual one.

After making certain summarizing observations (§11) concerning plane waves in other classes of phenomena (acoustic, elastic, electrical, optical) and especially concerning the expression for the velocity of propagation as a function of the parameters which physically characterize the cases in question, I shall consider waves in the ordinary sense of the word, that is, waves in liquids, giving an account especially of ancient (§12) and modern (§13) conclusions bearing upon rectilinear propagation in canals or in the open sea. It is thus that we reach a point where, thanks to numerical comparisons ingeniously conceived and carried out by M. H. Favre, professor at the École Polytechnique Fédérale of Zürich, it is possible to justify in a satisfactory manner the use of Gerstner waves, as they are now used in hydrography and in the theory of naval design, in spite of the serious inconvenience of requiring for their formation entirely artificial non-conservative forces. It is thus explained quite simply how in ordinary circumstances it is possible, with as close approximation as may be desired, to replace the waves studied by myself in 1924, which fulfill exactly all physical requirements, with the much more elementary and tractable type discovered by von Gerstner about a century ago.

In §14 we limit ourselves to the mere mention of certain collateral questions; while §15 is devoted to waves of discontinuity, following the outline of Hugoniot and the very fruitful theory of Hadamard; and in §16 there is proposed a mathematical explanation of the dualism between waves and corpuscles which dominates modern physics.

I think it is well to draw attention to the fact that this lecture follows, in its essential lines, the article entitled "Onde—Teoria matematica dei fenomeni ondosi," published by Professor U. Amaldi and myself in the Enciclopedia Italiana.

I. VIBRATING STRINGS

In order to fix the ideas, we consider the phenomenon of transverse vibrations of a string. More precisely we examine the typical case of a segment of a string, homogeneous, flexible and sensibly inextensible, of length l, which is initially situated in coincidence with the rectilinear segment OA; and then by the intervention of suitable influences, vibrates in such a way that its various material elements perform small oscillations, in a well-determined plane passing through the line OA, and in a direction perpendicular to this line. Denoting by s the abscissa of a generic material element of the string in its initial rectilinear configuration, and by t the time, the transverse displacement which the element undergoes will be a certain function $\eta(s, t)$ of s and t; and from the theory of small oscillations of flexible inextensible strings (d'Alembert) it follows that the function

$\eta(s, t)$ must satisfy the second-order partial differential equation

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 \eta}{\partial s^2}, \tag{1}$$

where the constants ρ and T represent respectively the linear density of the string (mass per unit of length) and the value of the tension in the initial situation. Besides satisfying this *indefinite* equation (which is to be valid at every instant and for every element of the string) the displacement must satisfy, as the case may be, depending on the connection of the string with other bodies, suitable boundary conditions (that is, relative to the end points O and A), which, in the particular case where the end points are fixed, reduce to the condition that at O and A the displacement is always zero, in other words, for *every* possible value of t, $\eta(0, t) = \eta(l, t) = 0$. Finally, from the very nature of the physical phenomenon, the maximum amplitude of the displacement η of each particle must be small in comparison with the length of the string; and this essential fact is introduced into the mathematical treatment of the problem by means of the hypothesis that the ratio η/l may be regarded as a quantity of the first order, that is to say, one of which the second and higher powers may be neglected. We take account of this hypothesis by supposing—as we may without contradicting the indefinite equation (1) and the more common types of boundary conditions—that the function contains an arbitrarily small constant factor ϵ, which fixes its order of magnitude.

In the sequel, in order to avoid developments which would not conform to the limitations of this lecture, we shall study exclusively the consequences which derive from the validity of the indefinite equation (1), admitting for consideration once and for all only those solutions which satisfy the boundary conditions which, in each case, may be imposed by the problem treated. For this reason all that is to be said here will be valid unconditionally (that is, for every possible pair of values of s and t) in the case of an infinite string, for which no boundary conditions are present, but it will be valid only for intervals suitably delimited whenever it is necessary to take account of physical circumstances, representable in boundary conditions, as it would be in the case of a string, with reflection effects at the ends, and, more generally, with any phenomena of reflection or refraction in optics or any other field.

2. PERMANENT PROGRESSIVE WAVES

In order to render precise the wave aspects presented in the vibrations of the string, let us recall that the most general solution of the indefinite equation (1) is of the form

$$\eta(s, t) = \eta_1(s - Vt) + \eta_2(s + Vt), \tag{2}$$

where η_1 and η_2 denote two arbitrary functions of the respective arguments $s - Vt$, $s + Vt$ while V is a constant connected with the physical parameters ρ and T by the relation

$$V^2 = \frac{T}{\rho}; \tag{3}$$

and let us fix our attention on a solution (corresponding to the hypothesis that η_2 is identically zero) of the form

$$\eta = \eta_1(s - Vt), \tag{4}$$

making note of the fact that, in accord with the requirements of the preceding section, we must understand that the function η_1 contains as a factor an arbitrarily small constant ϵ. In this case the two variables s and t (element of the string and the time) do not influence the phenomenon independently of each other, but only by way of the combination $s-Vt$, so that, if they vary in such a way that this binomial remains constant, then the corresponding displacement will remain unaltered. Now the condition

$$s-Vt=\text{const.}, \quad \text{or} \quad s=Vt+\text{const.}$$

is none other than the equation of uniform motion along the line OA, with velocity V; whence we conclude that, during the vibration of the string governed by the equation (4), any observer, moving along OA with the constant velocity V, is confronted at every instant by a constant displacement η. In other words, it is as if the configuration assumed by the string at a given instant were displaced, without change, from O toward A with the velocity V. If then we take account of the fact that this velocity $V=\sqrt{T/\rho}$ is quite independent of the actual transverse velocity of every particle, given by $\partial\eta/\partial t$ and therefore of the same order of magnitude as η, namely, that of the factor ϵ, which is to be kept small, we realize that we are confronted with a typical wave phenomenon. To be precise, waves of this type are, for obvious reasons, said to be *permanently progressive;* the binomial $s_1=s-Vt$ on which the phenomenon depends exclusively, is called the phase, and the velocity V, which is not connected with any material element, although dependent on the physical parameters ρ and T by means of (3), is called the *velocity of propagation of the wave* (and also, sometimes, the *velocity of phase*).

It is clear then that another solution of (1), which has, so to speak, the complementary form

$$\eta=\eta_2(s+Vt),$$

defines a permanent wave; and this wave differs from that corresponding to the solution (4) only in the sense of propagation, inasmuch as the respective velocity is given, not by V, but by $-V$ (retrograde wave).

3. PERMANENTLY PROGRESSIVE WAVES IN MANY DIMENSIONS

In the case of phenomena in three dimensions also, whether acoustic, elastic, or electromagnetic, we come to the determination of waves, in seeking a particular class of solutions (of differential systems, or systems of partial differential equations, defining the phenomena in question) which depend on a single argument which is a linear function of the three space coördinates x_1, x_2, x_3 and of the time t, that is, of the type

$$\sigma = c_1x_1 + c_2x_2 + c_3x_3 + c_0\, t,$$

where the c's are constants to be suitably determined. Assuming that we are dealing with solutions which depend in some manner on the space coördinates (that is, are not functions of t alone) we must keep at least one of the quantities c_1, c_2, c_3 distinct from zero; that is, the vector $\mathbf{c}$ having c_1, c_2, c_3 as components along the coördinate axes must not be null. Thus we may regard as a single argument, on which the solutions in question depend, the expression $s_1 = \sigma/c$, where c denotes the length of the vector $\mathbf{c}$; and if we observe that the ratios $\alpha_i = c_1/c$ $(i=1, 2, 3)$ are direction cosines (of the vector $\mathbf{c}$), and if we write $V = -c_0/c$, the expression s_1 thus takes the form $s_1 = s - Vt$, where $s = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ is an abscissa in the direction $(\alpha_1, \alpha_2, \alpha_3)$. We are therefore dealing with *plane waves*, in the sense that the vibratory state depends, at every instant t, on s alone and is therefore identical at all points of a given plane $s = \text{const.}$; moreover, as in the case of the string, the phenomenon has a stationary character with respect to any observer moving in the direction (α_1, α_2, α_3) so as to satisfy the condition $s_1 = s - Vt = \text{const.}$, that is, with the constant velocity V. A more general situation is obtained if we assume that s is any function (other than linear) of x_1, x_2, x_3, and suppose that the determining parameters of the phenomenon depend not only on $s_1 = s - Vt$ but also on another argument of purely spatial character. In this last type are included the so-called *spherical waves.*

But still more general types of waves, conceived in this manner, have been studied, from a different point of view, by H. Bateman (*Electrical and Optical Wave Motion*, Cambridge, 1915) and by G. A. Maggi (*Rend. Lincei*, ser. 5, XXIX, 1920, pp. 371–378).

4. BUNDLE OF WAVES

In the schematization of notable physical phenomena there arises the case of those permanent waves, for which the displacement η is (either rigorously or at least sensibly) different from zero only in a small neighborhood, of amplitude 2δ, of a certain value of the phase, e.g., $s_1 = 0$. From this it follows that at a generic instant t, the interval in which the vibration is sensible, lies between $s = Vt - \delta$ and $s = Vt + \delta$, so that its center is displaced in a straight line through the vibrating medium, with velocity V. It is this that is called, in one dimension, a *bundle of waves;* and it has its analogue also in two or three dimensions.

5. SINUSOIDAL OR HARMONIC PERMANENT WAVES

This important type of permanent waves is obtained if we suppose η to be of either of the two forms

$$\eta = \epsilon \sin\left(\frac{2\pi}{\lambda} s_1 + \alpha\right), \quad \eta = \epsilon \cos\left(\frac{2\pi}{\lambda} s_1 + \beta\right),$$

where λ, α, β denote constants, *a priori* undetermined. But these two expressions for η are reducible one to the other by putting $\beta=\alpha-\pi/2$; and on the other hand in either of them the constants α, β may be reduced to zero by changing the origin of space or time; hence it will suffice to consider the case

$$\eta=\epsilon\sin\left(\frac{2\pi}{\lambda}s_1\right)=\epsilon\sin\left(\frac{2\pi}{\lambda}[s-Vt]\right). \tag{5}$$

If we write, for brevity, $T=\lambda V^{-1}$, or

$$\lambda=TV, \tag{6}$$

and

$$x=2\pi\left(\frac{s}{\lambda}-\frac{t}{T}\right), \tag{7}$$

the equation (5) takes the simpler form

$$\eta=\epsilon\sin x. \tag{5'}$$

From the fact that $\sin x$ is a periodic function of x, of period 2π, it follows that (5) is a periodic function of the phase s_1, with period 1, and therefore periodic in s, likewise in t, with the periods T and λ, respectively. This means that, if we fix our attention on a fixed position (that is, if we assign a fixed value to s, e.g., $s=0$), we have for the transverse vibration of the corresponding particle a sinusoidal time equation which admits the period T, called the *period* of the wave motion in question (Fig. 1). If instead

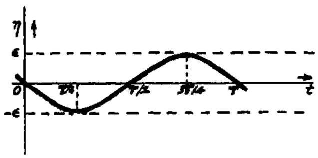

Fig. 1

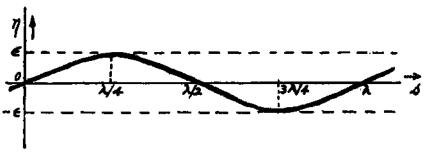

Fig. 2

we fix our attention on a given instant t (e.g., $t=0$), the law by which the transverse displacements of the various particles are governed in the vibrating medium, at this instant, is also sinusoidal, with period λ, which is called the *wave-length,* since the displacements are reproduced identically at intervals of length λ (Fig. 2). The two periods T and λ (which have the dimensions of time and length) are bound together by the relation (6), which represents the law of uniform motion with respect to the time. With these we consider also the respective reciprocals

$$\nu=\frac{1}{T}, \quad k=\frac{1}{\lambda},$$

which are given the names *frequency* and *undulance,* and which enable us to give to the equation (5) of the sinusoidal wave the form

$$\eta=\epsilon \sin 2\pi(ks-\nu t). \tag{5''}$$

From the sinusoidal nature of the time equation of the transverse vibrations of a generic particle it follows that the corresponding displacement, in its periodic variations, reaches, at time intervals of length $T/2$, *maxima* with the value ϵ and *minima* with the value $-\epsilon$, which are designated by the common name wave-crests; and in the middle of the interval of time $T/2$, between the formation of one crest and the next, we encounter a *node,* that is an instant at which the displacement vanishes, and the particle passes through its natural position; thus $T/4$ is the time which is required for the particle to pass from a crest to the succeeding node, and vice versa. The progress of the diagram of displacements of the various particles at one and the same instant is analogous to the foregoing, with the sole difference that the distance (no longer temporal, but spatial) from a crest to the succeeding node, or vice versa, is given by $\lambda/4$. Thus, in the typical case of a vibrating string, we can give a plastic model of the phenomenon by saying that: to an observer stationed at a point of the string, crests and nodes are formed alternately at intervals of time $T/4$; and, when he happens to see before him a crest or a node, he sees at the same instant formed at a distance $\lambda/4$ ahead of him and behind him a node O, or a crest, respectively.

6. PERIODIC PERMANENT WAVES IN GENERAL

The same qualitative considerations will also be valid obviously in the case where the displacement η, instead of being sinusoidal, is an arbitrary periodic function of period 1 of the single argument

$$\frac{2\pi}{\lambda} s_1 = 2\pi\left(\frac{s}{\lambda} - \frac{t}{T}\right) = 2\pi(ks - \nu t).$$

Particularly simple examples are obtained, if we return to the sinusoidal wave (5) and restore to its place in the argument the additive constant from which we freed ourselves at the beginning of the preceding section, that is, as we may say, by giving *phase* to a sinusoidal wave. We have thus

$$\eta = \epsilon \sin\left(\frac{2\pi}{\lambda}[s_1 - s_0]\right),$$

where s_0 denotes a constant having, like s_1, the dimensions of a length. We deal here with giving phase to typical sinusoidal solutions, and thus all it amounts to is a displacement by the amount s_0 of the space origin, which is equivalent to a shift of amount s_0 (in value and sign) of the diagram of Fig. 2. In particular, if s_0 corresponds to a forward or backward shift equal to one fourth of a period, we pass from the diagram of the sine to that of the cosine or else to the negative of the cosine.

On the other hand we must call to mind that, by the theorem of Fourier, every periodic function $\eta(s_1)$ continuous and differentiable, or satisfying other conditions still less restrictive, may be represented as a sum, or, in the limit, as a series of sines and cosines of multiples of the independent variable, so that we obtain, if the period of $\eta(s_1)$ is 1,

$$\eta(s_1) = \frac{a_0}{2} + \sum_n \left[a_n \cos \frac{2\pi n}{\lambda} s_1 + b_n \sin \frac{2\pi n}{\lambda} s_1\right],$$

or

$$\eta(s_1) = \frac{a_0}{2} + \sum_n c_n \sin \left(\frac{2\pi n}{\lambda}[s_1 - \sigma_n]\right),$$

where a_0, a_n, b_n or a_0, c_n, σ_n are suitable constants. In this way, if we set aside the constant term $a_0/2$ (which corresponds to a translation in the direction of the displacement, in general incompatible with the boundary conditions), the periodic wave turns out to be generated by the superposition of sinusoidal waves having, in order, wave lengths $\lambda, \lambda/2, \lambda/3, \lambda/4, \cdots$ and periods $T, T/2, T/3, T/4, \cdots$. All of this, naturally, is valid unconditionally in the case of an infinite vibrating string; when, on the other hand, we deal with a string of given length l, there intervene, as we have already noted, boundary conditions, which may be of various kinds and may influence the course of the phenomenon in an essential manner (giving rise, for example, to partial or total reflections at the ends, assumed to be fixed, or to particular dynamical conditions if an end is free). One who wishes, if only from an elementary viewpoint, to go further into these considerations, which must here be passed over, but which are particularly important sometimes in concrete cases (elastic and, in particular, acoustical phenomena) may consult any of the treatises indicated in the bibliography.

7. PRINCIPLE OF SUPERPOSITION

When a phenomenon has to do with differential equations, or partial differential equations, which are linear and homogeneous, as in fact (1) is, the sum of two solutions is also a solution; and in this consists the so-called "principle of superposition," of which a particular case has been met with in the preceding section. But if we preserve a maximum of generality it is to be observed that, when we take account of all the characteristics of a given phenomenon, and, in particular, not only those of the indefinite equation, but also those of the boundary conditions, it happens, in general, that the sum of two solutions satisfying those conditions is no longer a solution of this kind, so that the principle of superposition must be applied with due caution. If, moreover, we consider, more particularly, problems relating to waves, it is necessary to note that, even in the favorable case in which it is true that, for the phenomenon in question, the sum of two solutions each corresponding to a propagation of waves is also a solution, it happens generally that the undulant character presented by each of these two solutions is no longer present in the sum; and a particularly striking example of this possibility is given by the general integral of (1)

$$\eta(s, t) = \eta_1(s - Vt) + \eta_2(s + Vt),$$

the terms of which represent two trains of waves, the first progressive, the second retrogressive, while the sum, by virtue of representing the most general vibratory motion of the string, does not possess in fact the character that distinguishes wave motion.

8. STANDING WAVES

A vibratory state, which deserves particular consideration, arises in correspondence with those solutions of (1) which are represented as products of two functions, one of s alone, the other of t alone,

$$\eta(s, t) = \tau(t) f(s).$$

In such a case the values of s which reduce $f(s)$ to zero also reduce η to zero, *whatever t may be*, so that the vibratory phenomenon has *fixed nodes*. Moreover, since we have

$$\frac{\partial \eta}{\partial s} = \tau(t) f'(s),$$

it follows clearly that those values of s, for which $f'(s)$ vanishes, also reduce $\partial\eta/\partial s$ to zero for *every t*. Thus the maxima or minima, if any, of the displacement η are *fixed*, and the crests and troughs likewise. A simple case, in which this situation is realized, is obtained when we superimpose two sinusoidal waves of equal amplitude, phase, length, and period, but one of them progressive and the other retrogressive; that is, by setting

$$\eta = \epsilon[\sin 2\pi(ks - \nu t) + \sin 2\pi(ks + \nu t)] \quad (8)$$
$$= 2\epsilon \cos 2\pi\nu t \sin 2\pi ks.$$

This vibratory state is often referred to as a typical example of a "standing wave," but as a matter of fact the motion (8), resulting from the superposition of two trains of waves propagated in opposite directions, does *not* have the character of a wave motion according to the general criterion we indicated at the beginning.

9. GROUP OF PERIODIC WAVES AND GROUP VELOCITY

In the schematic case in one dimension (e.g., the vibrating string), any vibratory state is called, for obvious reasons, a "group of waves" if it is determined by the superposition of several permanent waves of the type considered in §2, and therefore corresponds to a displacement of the form

$$\eta = \eta_1 + \eta_2 + \cdots + \eta_n,$$

with

$$\eta_i(s, t) = \eta_i(k_i s - \nu_i t),$$

$i = 1, 2, \cdots, n$, where the η_i are all periodic functions, of period 1, of the respective arguments $s_i = k_i s - \nu_i t$; and, in general, the undulances k_i and frequencies ν_i vary with the index i. As we have noted in the case of a single component wave (§7) the phenomenon does not have in general a wave character; but if we fix our attention more particularly on the case $n = 2$, that is if we consider a binary group, which we will write, for simplicity,

$$\eta = \eta_1(ks - \nu t) + \eta_2(k's - \nu' t),$$

we can define a certain constant velocity U, the so-called *group velocity*, which is such that, with respect to an observer moving with this velocity in the direction of propagation, the phenomenon, without being properly undulant, turns out to be *periodic with respect to the time*. To show this, we consider first an observer Ω, moving with a constant generic velocity U; and, supposing that at the instant $t = 0$ he occupies the position $s = 0$, we denote by ξ the abscissa with respect to Ω of the generic particle of the vibrating system which has the absolute abscissa s. We have, for each instant, $s = \xi + Ut$, and, if in η_1, η_2 we let the relative abscissa ξ appear instead of s, the respective arguments become

$$k\xi-(\nu-kU)t,\quad k'\xi-(\nu'-k'U)t. \tag{9}$$

Since each of the two functions, η_1, η_2, is periodic, with period 1, with respect to its own argument, it is clear that if we select U so that the coefficients of t in the two arguments (9) are equal, the two functions will be periodic with respect to t, with one and the same period, equal to the reciprocal of the common value of the two coefficients of t. Thus we are led to set $\nu-kU=\nu'-k'U$; whence we obtain

$$U=\frac{\nu-\nu'}{k-k'}, \tag{10}$$

and this is the expression for the "group velocity."

This term seems quite justified if we consider the case in which both component waves are sinusoidal and the constants k', ν' are approximately equal to k, ν, respectively. In this case we have

$$\begin{aligned}\eta&=\epsilon\{\sin[2\pi(ks-\nu t)]+\sin[2\pi(k's-\nu't)]\}\\&=2\epsilon\cos\{\pi[(k-k')s-(\nu-\nu')t]\}\sin\{\pi[(k+k')s-(\nu+\nu')t]\},\end{aligned}$$

or, introducing the relative abscissa ξ of the observer Ω, moving with velocity U, and taking account of the expression (10) for this velocity,

$$\eta=2\epsilon\cos[\pi(k-k')\xi]\sin\left[\pi\left\{(k+k')\xi+2\frac{k'\nu-k\nu'}{k-k'}t\right\}\right]. \tag{11}$$

In order to recognize the salient characteristic of this vibratory phenomenon with respect to the moving observer

Ω, it is to be noted that at every point whose relative abscissa is equal to

$$\xi_n = \frac{2n+1}{2(k-k')},$$

where n is any integer (positive or negative), we have rigorously $\eta=0$, and in the neighborhood of any one of these points the displacement η is very small, so that we have what amounts to a region which is sensibly quiet (so-called "region of still water," if these generalities are applied to waves in canals, of which we shall speak a little later). These tranquil regions are separated by intervals of constant length $1/|k-k'|$, a quantity which, when k' is approximately equal to k, is much greater than either of the (practically equal) wave lengths $1/k$, $1/k'$ of the two component waves. In the interval extending between two consecutive regions of quiet the variation of η, which, inasmuch as it depends on two *distinct* arguments which are linear in ξ and t, is not undulant, gives place to a diagram which may wriggle in an arbitrary manner varying with respect to the time which would be referred to as "rippled," if we were dealing with the surface of a body of water. We see thus that the group velocity is that with which these successive wrinkled intervals, together with the intervening regions of tranquillity, are displaced with respect to a fixed observer.

10. DISPERSIVE PHENOMENA

In the preceding generalities, kinematic in character, we have not taken account of possible relations between wave-length and period, or, what amounts to the same thing, between undulance and frequency. When the two constants of one of these pairs are not independent of each other, so that one of them is a function of the other, the phenomenon is said to be *dispersive*, inasmuch as, according to (6), the velocity of propagation V turns out to depend on the frequency $\nu = 1/T$. In such cases the formula (10), giving the group velocity, acquires a particularly interesting significance. Taking as independent argument the undulance k, we can express as a function of this number, not only the frequency $\nu = 1/T$, but also the velocity of propagation $V = \lambda/T = \nu/k$. Under the particular hypothesis of two progressive periodic trains of waves having approximately equal undulances and frequencies, we can take $k' - k$ as a differential dk and $\nu' - \nu$ as a differential $d\nu$ of the corresponding function, so that, by virtue of (10), we have for the group velocity the expression

$$U = \frac{d\nu}{dk},$$

and from this, remembering that $\nu = kV$, where V is the phase velocity—that is the velocity of propagation of the first wave train and, therefore, approximately that of the second also—we deduce

$$U = V + k\frac{dV}{dk} = V - \lambda\frac{dV}{d\lambda}.$$

When we deal with light phenomena, it is useful to introduce in the second term of this last expression for U, in place of V, the index of refraction $n = c/V$, where c denotes the velocity of light in empty space; and we arrive at the *formula of Rayleigh*

$$U = V\left(1 + \frac{\lambda}{n}\,\frac{dn}{d\lambda}\right),$$

whence we deduce that, since, in general, the index of refraction decreases as the wave-length increases (normal dispersion)—so that we have $dn/d\lambda < 0$—in this case the group velocity is smaller than the phase velocity.

11. VELOCITY OF PROPAGATION IN CERTAIN IMPORTANT CASES

Analogously to what we have seen in the case of the transverse waves of a vibrating string (§2), as in every other type of physical phenomenon, when we seek those solutions of the corresponding differential equations which have the character of waves, we are led to express the velocity of propagation by means of the salient physical constants of the phenomenon. Thus, if first we turn to the case of a string, supposing it to be elastic, and consider its *longitudinal vibrations*, that is, no longer normal, but parallel to the direction of propagation, we find

$$V^2 = \frac{E}{\rho},$$

where ρ is as before the linear density, while E denotes the so-called *longitudinal modulus of elasticity of Young*, that is the ratio between a longitudinal pull, to which the cord is subjected, and the corresponding unit elongation.

For sound waves in air, considered as a perfect gas in an adiabatic system, we have

$$V^2 = \frac{\gamma p}{\rho},$$

where $\gamma = 1.41$ is the ratio of the two specific heats of air, corresponding to constant pressure and constant volume, and p, ρ represent the pressure and density of air when unperturbed by sound vibrations. As is well known, V in normal conditions turns out to be about 330 m./sec.

In the case of elastic waves in an isotropic medium we find, denoting by λ and μ the so-called *elastic constants of Lamé*, either

$$V^2=\frac{\lambda+2\mu}{\rho}, \text{ or } V^2=\frac{\mu}{\rho},$$

according to whether we are dealing with longitudinal or transverse waves. Since the constants λ and μ are positive, we see that the velocity of the longitudinal wave is always greater than that of the transverse one; and it is worthy of note that both of these velocities turn out to be higher than those of sound in air. Inasmuch as λ and μ, while being in general markedly different, have on the other hand the same order of magnitude, the ratio of the first to the second velocity cannot be much inferior to 2; for example, in steel the two velocities are, respectively, 6100 and 3200 m./sec. It is perhaps worth while to add that, for isotropic bodies, Young's modulus is connected with the Lamé constants by the relation

$$E=\frac{(3\lambda+2\mu)\mu}{\lambda+\mu}.$$

In an elastic medium presenting a free surface, as happens, for example, in a semi-space (elastic stratum), waves may be propagated which are quickly reduced in intensity from the free surface toward the interior; these are called *superficial waves*. Considered in the first place by Rayleigh, they have been studied or generalized, with particular regard to seismic applications by L. De Marchi, C. Somigliana, R. Einaudi (in *Rend. Lincei*, ser. 5, XXV, 1916; XXVI, 1917, XXVII, 1918, ser. 6, XIX, 1934). The velocity of propagation of these waves is about equal to 95/100 of that of the transverse waves.

Finally, in a homogeneous isotropic dielectric, for which ϵ and μ are respectively the dielectric constant and the magnetic permeability, the velocity of propagation of electromagnetic and, in particular, of light waves is given by the celebrated *formula of Maxwell*

$$V^2=\frac{c^2}{\epsilon\mu},$$

where c represents as usual the velocity of light in vacuo (to which, as we see, the above velocity for electromagnetic waves reduces when $\epsilon=\mu=1$). If however the dielectric is anisotropic, while conserving magnetic isotropy, plane waves in all directions are still possible, but their velocities of propagation will vary from one direction to another. In so-called *biaxial media*, where we denote by V_1, V_2, V_3, the velocities of propagation of the waves normal to the optical axes, the velocity V, with which waves can be propagated normal to a generic direction having cosines α_1, α_2, α_3, is defined by the equation

$$\frac{\alpha_1^2}{V^2-V_1^2}+\frac{\alpha_2^2}{V^2-V_2^2}+\frac{\alpha_3^2}{V^2-V_3^2}=0,$$

and from this we obtain for the velocity a striking geometrical construction, by having recourse to the *Fresnel surfaces*, that is, those algebraic surfaces of order 4 (and of class 4), which admit with respect to these same axes the equation

$$\frac{V_1^2x_1^2}{V_1^2-r^2}+\frac{V_2^2x_2^2}{V_2^2-r^2}+\frac{V_2^2x_3^2}{V_3^2-r^2}=1,$$

where $r^2=x_1^2+x_2^2+x_3^2$.

12. WAVES IN LIQUIDS AND WAVES IN CANALS

We consider a rectilinear canal of rectangular section, with a horizontal bottom and vertical sides; and we suppose that the liquid contained in the canal—which we shall speak of hereafter as water—moves parallel to the banks and has the same motion in all longitudinal sections, that is, in all vertical planes parallel to the banks. Lagrange, introducing for simplicity the hypothesis that the vertical accelerations of the separate particles of liquid are negligible in comparison with the acceleration of gravity g, considers *progressive waves of permanent type*, that is, those motions of the water in which the free surface l is displaced without alteration of form, progressing with a velocity V, while the separate particles, instead of being animated by a velocity comparable with V, execute only small oscillations with a local velocity having a mean value zero.

Now, in relation to the form of the free surface, the types most studied reduce to *periodic waves* (called by the French "la houle") and to *solitary waves*. It is quite clear that, in the first case the line l consists merely of consecutive equal arcs, repeated at constant intervals λ (wave-length); and the qualitative behavior of the curve is that of a sinusoid (Fig. 3). On the other hand, a solitary wave, such as might be produced by striking by suitable means, a mass of water in a canal initially at rest, consists of a single swelling,

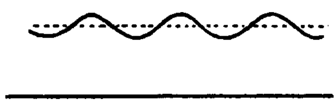

Fig. 3

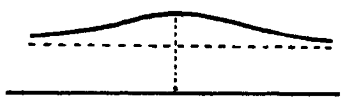

Fig. 4

symmetrical with respect to the maximum ordinate (Fig. 4). This was studied experimentally by Scott Russell (1844) and theoretically, in a wholly independent way, by J. Boussinesq (1871) and by Lord Rayleigh (1876).

A rigorously periodic type of waves is given by the so-called *trochoidal waves* discovered, independently, by F. J.

von Gerstner and W. J. M. Rankine. In these the curve of the free surface has exactly the form of a trochoid (or modified cycloid) and the separate fluid particles describe small circles which become rapidly smaller toward the bottom. The velocity of propagation of the waves is given in this case by

$$V^2 = \frac{g\lambda}{2\pi},$$

where, as usual, g denotes the gravitational acceleration and λ the wave-length. This solution of von Gerstner, by reason of the expressive simplicity of its characteristics, is widely used in hydraulic and nautical applications, as a theoretical basis for a first approach to the problem of estimating in some way the manifold perturbing influences which arise in practice; but it presents a certain inconvenience, whereby the application of the theoretical schema seems to be justified only provisionally, in the absence of something better (cf. §13); and this inconvenience consists in the *vortical* or *rotational* character of the corresponding vibrations of the liquid particles, whereas in a perfect liquid, under the action of conservative forces, it is possible to set up only *irrotational* motions.

It is therefore advantageous to seek other progressive waves of permanent type, which, while approximating as closely as possible to the simplicity of those of von Gerstner, will be due to irrotational vibrations of the liquid particles. Such indeed are the waves of Lagrange, to which we referred a little earlier, which have in fact particular importance in the study of tides; but the restrictive hypothesis to which their existence is subject (the negligibility in comparison with g, of the vertical accelerations of the fluid particles) is not in good accord with the physical facts. An elementary type of periodic waves, permanent and irrotational, which lends itself better to applications, is that of the *simple sinusoidal waves* of G. B. Airy. Even these are approximate, but the approximation in this case corresponds to the physical nature of the phenomenon, in that it amounts to the assumption that it is possible to regard as a quantity of the first order the ratio of the maximum velocity of the particles to the velocity of the propagation V. The value of V is defined by *Airy's formula*

$$\frac{V^2}{gh} = \frac{\tanh \alpha}{\alpha},$$

where, letting λ and g have their usual meanings and denoting by h the depth of the canal, we have put

$$\alpha = \frac{2\pi h}{\lambda},$$

tanh α denoting the "hyperbolic tangent of α," that is, the quotient sinh α: cosh α. For the so-called *long waves*, that is, those having a length λ which is very large in comparison with the depth h (normally tidal waves are of this kind), α is small and $\tanh \alpha/\alpha$ may be regarded as unity, so that we obtain for the velocity V the expression

$$V^2 = gh,$$

already found by Lagrange in his approximation. With this same expression is to be reconciled that of the velocity of propagation of the single wave which is given by

$$V^2 = g(h+a),$$

where a denotes the height of the wave (height of the top of the swelling above the undisturbed level); and the analogy between the two expressions seems in a certain sense justified, in that the single wave may be considered as a limiting case of a periodic wave, when the length λ becomes infinite. If, instead, we deal with *short waves* (and in practice we can regard as such all those for which the length is not greater than twice the depth of the canal) we can take $\tanh \alpha$ as unity, and there results

$$V^2 = \frac{gh}{\alpha} = \frac{g\lambda}{2\pi},$$

that is, we find again the expression which is valid for the velocity of propagation of trochoidal waves; and this result is not surprising, if we take account of the fact that the waves of von Gerstner, while being rotational, must in any case be considered as short, inasmuch as they have to do with a depth which is infinitely great.

For the simple waves of Airy, which, as we have said, correspond to a solution of first approximation, the principle of superposition (§7) is valid. In particular, we can consider the motion of the water which results from the superposition of two simple waves, having the same length, period, and amplitude, but directed in opposite senses and which gives rise, as we have seen in §8, to "standing waves," those having fixed nodes and crests. Of this type is the "clapotis" of the French.

In the case of waves in canals, besides the general property that the absolute motion of the liquid particles is small in amount compared with the velocity of propagation, the important mechanical fact is verified that in the deeper strata there is, in the mean, no global transfer of liquid: if the wave propagation does give rise to some transfer of matter, this is confined exclusively to the surface strata. Thus in the typical case of the simple waves of Airy, it is as if there were, in the mean, a transfer of liquid with a velocity γ (very small compared with V) given by

$$\frac{\gamma}{V} = \frac{1}{2}\left(\frac{a}{h}\right)^2 \frac{gh}{V^2},$$

with the usual meanings for g, h, a; and from this it follows that the intensity of transfer varies directly as the square of the height of the wave and inversely as the velocity of propagation and the depth of the canal.

13. RIGOROUS SOLUTION OF THE PROBLEM

In the face of results as striking, though approximate, as these, it was natural to pose the mathematical problem of the *rigorous* determination of permanent and irrotational periodic waves. G. G. Stokes and Lord Rayleigh were prompted in this way to obtain further approximations; and the actual existence of an exact solution was proved, in the case of very deep canals, by T. Levi-Civita (1925), who gave also the formulæ suitable for actual calculation, and found for the velocity of propagation the expression

$$V^2 = \frac{g\lambda}{2\pi}\left(1 - \alpha^2 + \alpha^3 - \tfrac{3}{4}\alpha^4 + \tfrac{25}{12}\alpha^5 + \cdots\right),$$

where we have set

$$\alpha = \frac{2\pi a}{\lambda},$$

a being the maximum height of the wave above the mean level of the canal.

When a is negligible in comparison with the wave-length λ we are led back naturally to the short waves of Airy.

Following the researches of Levi-Civita, inquiry has been extended by D. J. Struik (1926) to the case of canals whose depth is not extremely great, but well determined, while A. Weinstein (1926) has carried further the approximations of Boussinesq-Rayleigh for solitary waves. Particularly interesting on account of its theoretico-practical implications is a recent memoir by Professor H. Favre (*Publications of the Laboratory of Hydraulics of the Federal Polytechnic School of Zurich*, 1936), in which an accurate numerical comparison has been carried out between the exact solutions, namely, the irrotational ones (of Levi-Civita) and the vortical ones of von Gerstner, assuming equal values of λ (wave-length) and a (height) in the two cases. Favre has succeeded thus in establishing the fact that, for $a/\lambda \leq 1/20$, which is almost always the case for the types of waves which actually arise in practice, there is no sensible difference in the behavior of the lines of flow, nor in the other properties which can be tested experimentally.

This shows why the trochoidal waves of von Gerstner, justly appreciated for their geometrical simplicity and elegance, but used hitherto without sufficient justification (on the contrary, in spite of the explicit counter-indications arising from their vortical character), constitute an adequate representation, i.e., sufficiently approximate, of the physical phenomenon in the various cases, hydraulic, hydrographic, and nautical, in which it is customary to use them. This then is the complete explanation of a success, already achieved several decades ago, which, however, remained obscure, according to the standards of general mechanical principles, i.e., principles which are reasonably incontestable.

14. VARIOUS PROBLEMS OF WAVE MOTION

The limits of this lecture do not permit us to speak at length of other types of waves, however noteworthy, nor of their more complex characteristics, nor of the physical problems of which these are the basis. We limit ourselves therefore to a simple enumeration: in one dimension, waves produced at the surface of separation of two liquids of different densities (J.-M. Burgers, N. Kotchine), and the influence on the production and propagation of waves exercised by capillary and by dissipative phenomena (friction, viscosity, turbulence, etc.); in two or three dimensions, waves induced by the motion of ships, small motions in very deep canals (so-called *waves of Poisson-Cauchy*), oscillations in locks, "seiches" in lakes, and, leaving the field of liquid motion, the whole theory of sound and wave phenomena on a large scale, oceanic and atmospheric, which are affected by the rotation of the earth.

15. WAVES OF DISCONTINUITY

In the present state of mechanics and physics one can say also that many natural phenomena (and, in the macroscopic field, almost all of them) find their mathematical representation in systems S of partial differential equations which involve a certain number of unknown functions φ_1, $\varphi_2, \cdots, \varphi_m$ (characteristic physical parameters of the phenomenon), and in the case of three dimensions, four independent variables (three spatial coördinates x_1, x_2, x_3 and the time t); and these systems S are *normal,* that is, they contain as many equations as there are unknown functions and, as a result, they are solvable with respect to certain derivatives of maximum order. Each particular determination of the phenomenon, that is to say, every solution of the system S, remains uniquely individualized, when, after a three-dimensional variety (or hypersurface) Σ has been chosen arbitrarily in the four-dimensional space x_1, x_2, x_3, t, we preassign, for that matter arbitrarily, the functions to which the unknown φ_i and their derivatives are to reduce, up to a certain order dependent on the nature of the system S. This is valid for a generic Σ; but we may have particular hypersurfaces Γ, called characteristics of the system S, for which it happens that the arbitrary elements just mentioned (functions to which the φ_i and their derivatives to a certain order reduce, on Γ) are no longer sufficient to determine a solution of S, inasmuch as there exist *infinitely many* solutions, all satisfying the preassigned conditions on Γ. If in such a case we fix, by means of further suitable conditions, two of these infinitely many solutions, e.g., φ_i and φ_i^*, mutually distinct, it results in general that these are defined on both sides of the hypersurface Γ; but if we limit ourselves to considering φ_i on one side and φ_i^* on the other, we have two solutions of the system S which are *in agreement* with each other only *partially* (that is, only to those orders reached by the arbitrary character of the initial elements on a generic surface Σ), while beyond those orders the two solutions present, with respect to each other, across Γ, certain discontinuous characters. Now, if we turn from the four-dimensional space x_1, x_2, x_3, t, to the ordinary space x_1, x_2, x_3, restoring to the fourth variable t its rôle as time parameter, the hypersurface gives place to a surface σ_t, variable with respect to the time; and the two solutions φ_i, φ_i^* define for the phenomenon in question two distinct regimes, valid, respectively, on opposite sides of σ_t, and this surface, instant by instant, marks a frontier of partial discontinuity between these two regimes. As time passes, this frontier σ_t is displaced and, in general, is deformed; and the consequent propagation of the discontinuity within the space (and in the continuous medium which may be involved in the phenomenon) presents the fundamental characteristics of a wave, which, in contrast with those considered hitherto, is called precisely a *wave of discontinuity.*

To make the motion precise, let a point P be fixed on σ_t relative to the instant t together with the corresponding normal, n, to the surface. When we pass to the instant $t+dt$, σ_t assumes a new configuration σ_{t+dt}, cutting n at a point Q close to P, which determines on n the direction of advancement of the wave and, with respect to σ_t (or, better, with respect to a sufficiently small region about P), a face, called the *wave front.*

This schematization, which goes back to Hugoniot and has been systematically developed by J. Hadamard, has made it possible to reduce to algebraic calculations, substantially, the study of the propagation of acoustic, elastic, electromagnetic, and other waves (but not those in liquids). It is not possible here to dwell upon the developments, for that matter relatively elementary, which permit us to deduce from a normal system S the definition of the *characteristic varieties* which it may have, nor upon the criteria, which give the procedure in the evolution of the wave surface from its initial configuration. A few observations will suffice.

Concerning the characteristic varieties of a normal system S, it is an important fact that in every case these are defined by a single partial differential equation (E), which is obtained from S in every instance by a well determined and valid process; but it may happen that it admits no real solution. For example, to Laplace's equation

$$\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}+\frac{\partial^2\varphi}{\partial t^2}=0,$$

there corresponds as the equation (E) defining the respective characteristic varieties,

$$\left(\frac{\partial z}{\partial x_1}\right)^2+\left(\frac{\partial z}{\partial x_2}\right)^2+\left(\frac{\partial z}{\partial x_3}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2=0,$$

which, in the real domain, is not satisfied by any function except $z=\text{const.}$; however, for the so-called *canonical equation of small motions* (of an incompressible fluid)

$$\frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x_1^2}-\frac{\partial^2\varphi}{\partial x_2^2}-\frac{\partial^2\varphi}{\partial x_3^2}=0,$$

($V=\text{const.}$) the equation (E) is given by

$$\text{(E)} \qquad \frac{1}{V^2}\left(\frac{\partial z}{\partial t}\right)^2-\left(\frac{\partial z}{\partial x_1}\right)^2-\left(\frac{\partial z}{\partial x_2}\right)^2-\left(\frac{\partial z}{\partial x_3}\right)^2=0,$$

which admits real solutions possessing the maximum generality consistent with its order. Generally speaking, we may add that the equation (E) of the characteristic varieties certainly has real solutions (and therefore there exist waves of discontinuity) for all normal systems of the so-called *hyperbolic type*, while the contrary is the case for those of so-called *elliptic type;* and there is presented a whole series of intermediate cases of systems which are neither completely hyperbolic nor completely elliptic.

As for the various classes of phenomena in which the concept of waves of discontinuity has found fruitful applications, it will suffice to consider two cases of particular interest:

(i) If we are dealing with a phenomenon relating to a continuous *homogeneous* medium, it is proved that, under suitable specifications which express the physical fact that any other concomitant causes that there may be are independent of position, the equation (E), governing the evolution of the surface σ_t of the wave of discontinuity, is such that this surface, if initially plane, will remain plane at any other time; that is, homogeneous media admit in every case *plane waves of discontinuity.*

(ii) Turning to the case of an arbitrary phenomenon, we consider the possibility that the initial configuration σ_0 of σ_t is a very small *closed* surface, surrounding a given point P, and that across σ_t there are set up partially, in the manner sketched above, an interior solution different from zero and the solution identically zero in the exterior. We have here a representation of those phenomena which are initiated by a perturbation which is restricted to the immediate neighborhood of a given point, called the *epicenter;* in the course of time the wave front spreads and the perturbation is confined to the interior. When the physical characteristics of the phenomenon are constant, as happens in the propagation of sound or light in homogeneous media, the surfaces σ_t relative to successive instants are all homothetic with respect to the epicenter; and, in particular, if we are dealing with the propagation of light in a biaxial medium, the epicentral surfaces σ_t are homothetic to the surfaces of Fresnel (§11). Even more interesting is the case in which the perturbed region (non-zero solution) instead of penetrating the whole interior of σ_0 and hence also of σ_t, is always restricted to a small lamina contained between σ_0 and another characteristic surface interior to it and everywhere close to it; this corresponds to the ordinary case of a sound or light signal of short duration.

In every case a fundamental character of the phenomenon which completes quantitatively the notion of the wave front—corresponding to any element $d\sigma_t$ of the surface σ_t, relative to a generic instant—is the velocity, with respect to the trihedral of reference (for a fixed observer), with which the element of surface is displaced in the direction of its normal. This is the so-called *velocity V of advance* of the element of wave surface. If we are concerned with a specifically mechanical phenomenon, that is, involving along with the motion of a surface of discontinuity, that of a material medium (air or some other gas or a solid body), there is to be considered besides the velocity V, the analogous velocity W, with which the element is displaced with respect to the medium. This is usually called the *velocity of propagation* and, by the principle of relative motion, is connected with V by the relation

$$V = W + V_n,$$

where V_n denotes the normal component (to the surface σ_t in the direction of propagation) of the velocity with which the material particle moves which is situated on the element $d\sigma_t$ at the instant t.

The distinction between the velocity of advance and that of propagation, which is essential for phenomena which are typically mechanical, has no *raison d'être* for phenomena which, like electromagnetic ones, do not imply the existence of a material medium; and in such cases we have usually a single velocity, which is strictly that of advance, but which, in the absence of any danger of ambiguity, is designated indifferently by the name velocity of propagation.

If we consider a given category of phenomena, with possible boundary conditions, on the one hand there may or may not exist progressive waves, and on the other hand (according to the type of the corresponding normal system) one may or may not have waves of discontinuity. Whenever waves of both kinds are possible at one time, it is found, at least in the cases studied hitherto, that the velocity of advance V of the wave of discontinuity admits the same expressions in terms of the physical parameters of the phenomenon which are valid for the corresponding progressive waves (§11).

But there are cases in which there exists only one of the two types of waves. For example, liquids are endowed solely with progressive waves (differential systems of elliptic type, or, at any rate, more nearly elliptic than hyperbolic); however, in homogeneous elastic media of general structure (depending on twenty-one structural constants) plane progressive waves are not possible (E. Beltrami), but there exist plane waves of discontinuity (G. Lampariello).

16. WAVES AND CORPUSCLES

The theory of waves of discontinuity, just referred to, opens the way for a remark of a general order on the dualism between the corpuscular and undulatory conceptions which, in the development of physics, after repeated vicissitudes in which one or the other has held the advantage, today tend toward agreement, as we said at the very outset, in a more advanced common point of view.

In the field of optics we usually attribute to Newton the corpuscular conception of light and to Huygens the undulatory conception, though in reality, from certain points of view at least, these are much older, and Newton himself, though giving preference to the emission theory, sometimes made use of undulatory models. As we have already pointed out, the great contest was decided when, following Young and Fresnel, it became possible to frame all phenomena known up to that time in an undulatory scheme, furnished first by an elastic model and later by the electromagnetic equations of Maxwell. But after all no one succeeded in reconciling in a simple manner the undulatory theory with the observed facts concerning the photoelectric phenomenon, which is due to H. Hertz; so that in order to represent these facts it became necessary to turn to the corpuscular point of view with the quantum hypothesis of A. Einstein, according to which any beam of light of given frequency ν is to be regarded as made up of a swarm of photons (or quanta of light), small portions E of energy, each proportional to the respective frequency according to the formula $E=h\nu$, where h is the well known constant of Planck. Still later the same hypothesis made it possible to account for the Compton effect (1923), the complex nature of which turned out to be satisfactorily explained, as shown by Compton himself, P. Debye, E. Fermi, and E. Persico, when one adjoins to the hypothesis of Einstein not only the principle of the conservation of energy but also that of the conservation of the quantity of motion.

A corresponding change, but in the opposite direction, is presented in the theory of electrons. Whereas at the end of the nineteenth century, chiefly on the basis of the behavior of cathode rays and the celebrated experiments carried out mainly by J. J. Thomson, W. Kaufmann, H. A. Wilson, and R. A. Millikan, electrons had remained completely characterized as purely electric charges, all equal to each other, this point of view, exclusively corpuscular, was shown to be insufficient to account for the phenomenon of the diffraction of electrons in crystals, discovered in 1927 by G. Davisson and L. H. Germer, and confirmed by later experiments, due to E. Rupp and G. P. Thomson; hence here also it became necessary to have recourse to a complementary treatment of undulatory type.

This dualism, whereby the most notable facts of modern physics require the simultaneous intervention of corpuscles and waves, was recognized and regarded as a general law of nature, even before the beautiful confirmation of the diffraction of electrons by L. de Broglie, who sought from the first to give a more concrete form to his conception by associating with every moving corpuscle a well defined group or bundle of waves. But he also recognized the difficulties involved in such an association.

Now, the considerations pointed out in the preceding section concerning waves of discontinuity are amenable to the formulation of a sufficiently broad mathematical schema to reconcile the two aspects, undulatory and corpuscular, of one and the same phenomenon, whenever it is adequately represented in a normal differential system. In fact, with the partial differential equation (E) of the first order which defines the wave surfaces, there is intrinsically associated (a classical result of Cauchy) a determinate ordinary differential system of equations which defines, in ordinary space, ∞^6 motions. Any one of these is represented in the space-time variety x_1, x_2, x_3, t by a line (time path), which constitutes a *bicharacteristic* of the equation (E) and, thereby, of the normal system governing the phenomenon. In this way, there are associated simultaneously with the phenomenon under consideration an undulatory aspect (wave of discontinuity) and a corpuscular aspect which seems to be schematized in the motions of real or fictitious particles along the bicharacteristics with the time law which is determined for each of them by the canonical system (C). There is, in short, for any physical theory (expressible in a normal differential system), the possibility of waves of discontinuity, resolvable mathematically into the motion of discrete particles.

This point of view is doubtless abstract in character and, from a physical point of view, agnostic; and, in particular it gives no criterion for a concrete linking of the two distinct kinds of phenomena. Manifold attempts in this direction carried out by eminent scientists, beginning with de Broglie himself, as we have said, have led to the conclusion that it is not possible to establish a one-to-one correspondence between waves—or bundles of waves—and corpuscles, without violating the so-called principle of indetermination of Heisenberg. One can, however, broaden the question and seek to lay down a law of correspondence, not between individual elements but between trains of waves on the one hand and swarms of corpuscles on the other. In this way we succeed in establishing a correlation of a statistical type, on the basis of which, under suitable restrictions, we find and extend to more general cases the celebrated formula of de Broglie, which prescribes (in the absence of other perturbing phenomena) a well determined wave length λ to every type of electronic radiation, regarded from the corpuscular point of view. This is given by the formula

$$\lambda = \frac{h}{mv},$$

where h is Planck's constant and m and v denote, respectively, the material mass of the material particles constituting the radiation and their (mean) velocity.

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