FUNDAMENTAL THEOREMS CONCERNING POINT SETS[1]

I

FOUNDATIONS OF A POINT SET THEORY OF SPACES IN WHICH SOME POINTS ARE CONTIGUOUS TO OTHERS

IN my article "Foundations of Plane Analysis Situs"[2] and in my book *Foundations of Point Set Theory*,[3] a study was made of spaces satisfying certain sets of axioms in terms of the undefined notions "point" and "region." In the present treatment, "point," region" and "contiguous to" (used as a relation of one point to another) are undefined. The notion *limit point* of a point set is defined exactly as before. That is to say, the point P is said to be a *limit point* of a point set M if every region that contains P contains at least one point of M distinct from P. As before, the point set M is said to be *closed* if it contains all of its limit points, it is said to be *perfect* if it is closed and every point that belongs to it is a *limit point* of it, and it is said to be *compact* if every infinite subset of it has at least one limit point, not necessarily belonging to it. But, in the book referred to, the point sets H and K are said to be *mutually separated* if

[1] An elaboration and extension of material presented in a series of three lectures delivered at the Rice Institute in November, 1932, by Robert Lee Moore, Ph.D., Professor of Pure Mathematics at the University of Texas.

[2] *Trans. Am. Math. Soc.*, XVII, April, 1916, pp. 131–164.

[3] *Am. Math. Soc. Colloquium Publications*, XIII, 1932, New York. In the present treatment the abbreviation P. S. T. will be used to designate this book.

(1) they are mutually exclusive and (2) neither of them contains a limit point of the other one. *Here* they will be said to be *mutually separated* if, in addition to (1) and (2), they satisfy the third requirement that *no point of H be contiguous to any point of K.* In terms of mutual separatedness, connectedness is defined here just as there. That is to say, the point set M is said to be *connected* if and only if it is not the sum of two mutually separated sets.[1] But since mutual separatedness has a new significance here, so does connectedness.

If K is a proper subset of the connected point set M and $M-K$ is not connected, then M is said to be *disconnected by the omission of K* or to be *disconnected by K* or to be *separated by K*, and K is called a cut set of M; and if K is a point it is called a *cut point* of M and, if it is a continuum, it is called a *cut continuum* of M.

DEFINITION. If A and B are two distinct points a *simple continuous* arc from A to B is a closed, connected and compact point set which contains A and B and which is disconnected by the omission of any one of its points except A and B. The statement "AB is an arc" is to be interpreted as meaning that AB is an arc from A to B.

This definition is worded precisely as in P. S. T. But the word "connected" occurs in two places and connectedness having been defined in terms of mutual separatedness, it has a different significance here. *Here* the point set consisting of two contiguous points is an arc. So is the point set consisting of three distinct points A, B and C such that B is contiguous both to A and to C but A and C are not contiguous to each other. But *there* every arc necessarily contains uncountably many points.

[1] Cf. N. J. Lennes, "Curves in Non-Metrical Analysis Situs with Applications in the Calculus of Variations," *Am. Jour. of Math.*, XXXIII (1911), and *Bull. Amer. Math. Soc.*, XII (1906).

The point set D is said to be a *domain* if for each point P of D there exists a region containing P and lying in D. The collection G of point sets is said to *cover* the point set M if each point of M belongs to some point set of the collection G. If a point set is denoted by a certain letter, that letter with a bar above it will denote the sum of that point set and all of its limit points. The letter S will be used to denote the set of all points.

Of the following axioms, the first three are worded precisely as in P. S. T. This set of six axioms will be called the set Σ_c.

AXIOM 0. Every region is a point set.

AXIOM 1. There exists a sequence $G_1, G_2, G_3, \cdots$ such that (1) for each n, G_n is a collection of regions covering S, (2) for each n, G_{n+1} is a subcollection of G_n, (3) if R is any region whatsoever, X is a point of R and Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X then $\bar{g}$ is a subset of $(R-Y)+X$, (4) if $M_1, M_2, M_3, \cdots$ is a sequence of closed point sets such that, for each n, M_n contains M_{n+1} and, for each n, there exists a region g_n of the collection G_n such that M_n is a subset of $\bar{g}_n$, then there is at least one point common to all the point sets of the sequence $M_1, M_2, M_3, \cdots$.

AXIOM 2. If P is a point of a region R there exists a non-degenerate[1] connected domain containing P and lying in R.

AXIOM A. No point is contiguous to itself.

AXIOM B. If the point A is contiguous to the point B, then B is contiguous to A.

AXIOM C. If M is a closed point set and every point of the point set H is contiguous to some point of M then no point of $S-M$ is a limit point of H.

[1] A point set is said to be *degenerate* if it consists of only one point. Otherwise it is said to be non-degenerate.

I will describe examples of spaces satisfying the six axioms of the set Σ_c.

EXAMPLE 1. In Euclidean space of three dimensions, let K denote a definite sphere, in the sense of a spherical *surface* and let α denote an infinite sequence of distinct points $P_1, P_2, P_3, \cdots$ all lying on K and such that every point of K either belongs to α or is a limit point of the set of all the points of α. There exists a sequence β of mutually exclusive spheres $K_1, K_2, K_3, \cdots$ such that, for each n, K_n is externally tangent to K at the point P_n and is of radius less than $1/n$.

Now let S denote the set whose elements are the ordinary points of K and the spheres of the sequence β. Let the elements of S be called points and let two points of S be called contiguous to each other if and only if one of them is a sphere of the sequence β and the other one is the point of α at which that sphere is tangent to K. If P is a point of K and m is a positive integer let T_{Pm} denote the set of all points of K at a distance from P less than $1/m$ and let R_{Pm} denote the set whose elements are the points, in the ordinary sense, of T_{Pm} and the points, in the new sense, which are spheres of the sequence $K_m, K_{m+1}, K_{m+2}, \cdots$ that are tangent to K at points belonging to T_{Pm}. Let a point set be called a region if it is either (1) a single element of the sequence β or (2) a set R_{Pm} for some integer m and some point P of K. For each n let G_n denote the set of all regions which are elements of the sequence β and all regions R_{Pm} for which m is greater than n. It may be verified that Axioms 2, A and B are satisfied for this interpretation of *point*, *region* and *contiguity* and that the sequence $G_1, G_2, G_3, \cdots$ satisfies all the requirements of Axiom 1. Suppose now that M is a closed subset of S and that H is a set of points each contiguous to some point of M. Suppose P is a limit point of H. The point P does not belong to the sequence β. For if it did then P itself would be a region containing P and therefore containing at least one point of H distinct from P which is absurd. Hence P is a point of K. Suppose R is a region containing P. There exist a positive integer m and a point X of K such that R is identical with R_{Xm}. There exists a number i such that T_{Pi} is a subset of T_{Xm} and such that no point other than P of the finite set $P_1, P_2, P_3, \cdots, P_m$ is at a distance from P less than $1/i$. The region R_{Pi} contains a point Y of H distinct from P. There exists a number j greater than m such that Y is identical either with P_j or with K_j. Since P_j belongs to R_{Pi} and to K it belongs to T_{Pi} and therefore to T_{Xm}. But $j>m$. Therefore K_j belongs to R_{Xm}. But either P_j or K_j belongs to M. Thus R_{Xm} contains a point of M distinct from P. Hence P is a limit point of M. Therefore it belongs to M. Thus Axiom C holds true in this example.

Example 2. In a Cartesian space let O denote the origin, let OX denote the axis of abscissas and let A and B denote the points of OX whose abscissas are 1 and -1 respectively. Let OA denote the straight line interval consisting of the points O and A and all points between them and let OB denote the straight line interval from O to B. Let S denote the set whose elements are the intervals OA and OB and the points of OX whose abscissas are either less than -1 or greater than 1. Call the elements of S points. Let two elements of S be regarded as contiguous if and only if one of them is OA and the other one is OB.

If n is a positive integer and P is the interval OA, let R_{Pn} denote the point set whose elements are P and the points of OX whose abscissas, in the ordinary sense, lie between 1 and $1+1/n$. If P is OB let R_{Pn} denote the set whose elements are P and the points of OX whose abscissas are between -1 and $-1-1/n$. If P is a point of S distinct from OA and from OB, let R_{Pn} denote the set of all points of S distinct from OA and from OB and at a distance from P less than $1/n$. In this illustration a point set will be called a region if and only if it is a set R_{Pn} for some positive integer n and some point P of S. The set of axioms Σ_c is satisfied by this interpretation of point, contiguity and region. If, for each n, G_n denotes the set of all regions R_{Pn} for all points P of S, the resulting sequence $G_1, G_2, G_3, \cdots$ fulfills all the requirements of Axiom 1.

It is to be noted that if, in this example, H denotes the point set whose elements are OA and the points of OX whose abscissas are greater than 1 and K denotes the set consisting of OB and the points of OX whose abscissas are less than -1, then H and K together make up the whole of S but they are mutually exclusive and no region contains a point of both of them and therefore neither of these sets contains a limit point of the other one. This, however, does not imply that S is not connected. For the point OA of H is contiguous to the point OB of K and therefore H and K are not mutually separated.

EXAMPLE 3. Let S denote the set whose elements are the intervals of OX whose endpoints are consecutive numbers of the set composed of zero and the integers. Consider a space whose points are the elements of S. Let two points of S be regarded as contiguous if and only if they are abutting intervals of OX. Let a point set be regarded as a region if and only if it is a single point of S. The axioms of Σ_c are satisfied by this interpretation. If, for each n, G_n denotes the set of *all* regions, the sequence $G_1, G_2, G_3, \cdots$ satisfies the requirements of Axiom 1. In this space no point has a limit point. Nevertheless, S is infinite and connected.

Definition. The point P is said to be a boundary point of the point set M if either (1) P belongs to one of the sets M and $S-M$ and is a limit point of the other one or (2) P is a point of $S-M$ which is contiguous to some point of M.

If the proposition P can be proved on the basis of Axioms 0, 1 and 2 alone, *mutual separatedness* and *boundary* and notions partly or wholly dependent on one or both of these being defined in accordance with the treatment given in P. S. T., and on the basis of the axioms of the set Σ_c it is possible to prove a proposition P_C which is worded precisely as is proposition P, the notions referred to being *now* defined in accordance with the present treatment; then the proposition P will be said to "hold here just as in ordinary point set theory."

For example, let P denote the proposition that if AB is a simple continuous arc from A to B and $AB-O$ is the sum of two mutually separated point sets H and K, then one of these sets contains A and the other one contains B. This proposition is proved in P. S. T. on the basis of Axioms 0 and 1, the terms "*simple continuous arc* from A to B" and "*mutually separated* point sets H and K" being defined as indicated there. *Here* these terms have a different significance but if P_C denotes a proposition worded precisely as is P but in which these two terms are given this new meaning, this new proposition with the old wording can be proved on the basis of the set of Axioms 0, 1, A, B and C. The proposition P_C is, then, said to hold here just as in ordinary point set theory.

An example of a proposition that holds in ordinary point set theory but which can not be proved here is the statement that if AB is a simple continuous arc then $AB-(A+B)$ is connected. Here, if A and B are contiguous to each other, $AB-(A+B)$ does not even exist.

Consider the very first proposition of P. S. T.: No point of a region is a boundary point of that region. *There* this proposition means that if P is a point of a region R then not every region that contains P contains both a point of R and a point not belonging to R. This is obviously true since R itself is a region containing P and certainly it contains no point not belonging to itself. *Here* this proposition means that if P is a point of a region R then not only is the above obviously fulfilled condition satisfied but P is not a point of $S-R$ which is contiguous to a point of R. This condition also is obviously fulfilled. So the proposition in question holds here just as in ordinary point set theory.

On the other hand, in ordinary point set theory it is true that no point of a region R is a boundary point of the complement of R. But there exists a space satisfying Σ_c and consisting of just two points R and E these points being contiguous to each other. In this example, R is necessarily a region and E is its complement and not only is it true that R *contains* a boundary point of its complement but it *is* the *entire* boundary of its complement.

NOTATION. If M and N are point sets the notation $M\cdot N$ is used to denote the common part of M and N that is to say the set of all points that belong both to M and to N.

DEFINITIONS. As in ordinary point set theory, a subset K of a point set M is said to be an *open* subset of M if for each point P of K there exists a region R containing P such that $R\cdot M$ is a subset of K and a point set M is said to be *locally compact* if, for each point P of M there is a compact open subset of M containing P. An open subset of a point set is sometimes called a *domain with respect to* that point set.

The sequence of domains D_1, D_2, D_3, $\cdots$ is said to *close down on the point set* M if (1) M is the common part of all the domains of this sequence, (2) for every n, $\overline{D}_{n+1}$ is a subset of D_n and (3) if D is a domain containing M there exists a number n such that $\overline{D}_n$ is a subset of D.

The point P is said to be a *sequential limit point* of the sequence of points $P_1, P_2, P_3, \cdots$ if for every region R containing P there exists a natural number δ such that if $n > \delta$ then P_n lies in R. If P is a sequential limit point of the sequence α then α is said to *converge to* P and it is said to be a *convergent* sequence.

There is a large body of propositions which are consequences of Axioms 0 and 1 and in whose statement there occurs neither the term *mutually separated* nor the term *boundary* nor any other term which is defined either wholly or partly in terms of one of them. It is clear that all such propositions hold true here. For convenience of reference I will list a few of these many propositions as numbered theorems, referring to P. S. T. for proofs or indications of proofs.

THEOREM 1. *If P is a point there exists an infinite sequence of regions closing down on P.*

THEOREM 2. *If P is a limit point of the point set M then P is the sequential limit point of some infinite sequence of points of M all distinct from each other and from P.*

DEFINITION. The collection G of point sets is said to be monotonic provided it is true that if x and y are two point sets of the collection G then either x contains y or y contains x.

THEOREM 3. *If G is a monotonic collection of closed and compact point sets, there exists at least one point common to all the point sets of the collection G and their common part is closed.*

THEOREM 4. *No locally compact and countable point set is perfect.*

THEOREM 5. *Every closed and compact point set has the Borel-Lebesgue property.*[1]

THEOREM 6. *If M and N are two mutually exclusive closed point sets and one of them is compact then there exist two domains D_M and D_N containing M and N respectively and such that $\overline{D}_M$ and $\overline{D}_N$ are mutually exclusive.*

DEFINITIONS. If α is a sequence of point sets $M_1, M_2, M_3, \cdots$ then by the *limiting set* of α is meant the set of all points P such that if R is a region containing P then there exist infinitely many natural numbers n such that M_n contains a point of R.

The point set M is said to be the *sequential limiting set* of the sequence α of point sets if M is the limiting set of every infinite subsequence of α. Under these conditions α is said to converge to M.

THEOREM 7. *If the limiting set of a sequence α is compact then some subsequence of α has a sequential limiting set.*

THEOREM 8. *Every closed and compact point set is a metric space.*[2]

Let us now proceed to a consideration of propositions involving the notion of contiguity.

THEOREM 9. *If H and K are mutually exclusive closed point sets and H is compact there do not exist infinitely many points of H each contiguous to some point of K.*

Proof. Suppose there exists an infinite subset L of H

[1] The point set M is said to have the Borel-Lebesgue property if for every collection of domains covering M there is a finite subcollection of that collection that also covers M.

[2] The point set M is said to be a metric space if with every pair of points X and Y belonging to M there is associated a number $d(X, Y)$, called the distance from X to Y, such that (1) $d(X, Y)=d(Y, X)\geqq 0$, (2) $d(X,Y)=0$ if and only if $X=Y$, (3) if A, B and C are any three points of M then $d(A, B)+d(B, C)\geqq d(A, C)$, (4) the point P of M is the sequential limit point of the sequence of points $P_1, P_2, P_3, \cdots$ of M if and only if $d(P_n, P)$ approaches 0 as a limit as n increases indefinitely. Cf. M. Frechet, *Rend. Circ. Mat. di Palermo*, XXII (1906), p. 17. Under these conditions the point set M will be said to be metric with respect to the distance function $d(X, Y)$.

such that each point of L is contiguous to some point of K. Since H is compact the point set L has at least one limit point P. Since H is closed P belongs to H and, by Axiom C, it belongs to K. Hence H and K have a point in common, contrary to hypothesis.

THEOREM 10. *Every boundary point of the common part of two point sets is a boundary point of one of them.*

THEOREM 11. *The boundary of the sum of a finite number of point sets is either vacuous or a subset of the sum of their boundaries.*

THEOREM 12. *The boundary of a point set is closed, and so is the sum of a point set and its boundary.*

Proof. Let β denote the boundary of M and let M_C denote the set of all points X of β, if there are any, such that X is contiguous to some point of M. Suppose P is a limit point of β. Then it is a limit point either of M_C or of $\beta - M_C$. That every limit point of $\beta - M_C$ belongs to β follows as in ordinary point set theory. Suppose P is a limit point of M_C. Then, since every point of M_C is contiguous to some point of the closed point set $\overline{M}$ therefore, by Axiom C, P belongs to $\overline{M}$. But M_C is a subset of $S-M$. Hence P is a limit point of $S-M$. Therefore if P belongs to M it is a boundary point of M by definition. If it does not belong to M, it belongs to $\overline{M}-M$ and therefore, again by definition, it is a boundary point of M. Thus β is closed. But $M+\beta=\overline{M}+\beta$ and $\overline{M}$ is closed. Hence $M+\beta$ is closed.

THEOREM 13. *If M is a closed and compact point set, $X_1, X_2, X_3, \cdots$ is an infinite sequence of distinct points of M, and, for each n, Y_n is a point of M which is contiguous to X_n and $d(X, Y)$ is a distance function with respect to which M is metric, then $d(X_n, Y_n) \to 0$ as $n \to \infty$.*

Proof. Suppose, on the contrary, that there exist a positive number e and an infinite sequence of positive integers

$n_1, n_2, n_3, \cdots$ such that, for each i, $d(X_{n_i}, Y_{n_i}) > e$. There exist points X and Y and a subsequence $m_1, m_2, m_3, \cdots$ of the sequence $n_1, n_2, n_3, \cdots$ such that X and Y are the sequential limit points of the sequences $X_{m_1}, X_{m_2}, X_{m_3}, \cdots$ and $Y_{m_1}, Y_{m_2}, Y_{m_3}, \cdots$ respectively. Now, for each j, the point set T_j consisting of Y and the points of the sequence $Y_{m_{j+1}}, Y_{m_{j+2}}, Y_{m_{j+3}}, \cdots$ is closed and, for each i, $X_{m_{j+i}}$ is contiguous to $Y_{m_{j+i}}$. Furthermore, since the points of the sequence $X_{m_1}, X_{m_2}, X_{m_3}, \cdots$ are all distinct, X is a limit point of the point set $X_{m_{j+1}} + X_{m_{j+2}} + X_{m_{j+3}} + \cdots$. Therefore, by Axiom C, for each j, X belongs to T_j. But Y is the only point common to all the point sets of the sequence $T_1, T_2, T_3, \cdots$. Therefore $d(X_{m_i}, Y) \to 0$ as i increases indefinitely. But so does $d(Y_{m_i}, Y)$. Therefore so does $d(X_{m_i}, Y_{m_i})$. Thus the supposition that $d(X_n, Y_n)$ does not approach 0 as n increases indefinitely leads to a contradiction.

THEOREM 14. *If the closed point set M is compact there does not exist an uncountable subset K of M such that each point of K is contiguous to some point of M.*

Proof. Suppose there exists such a set K. By Theorem 8, there exists a distance function $d(X, Y)$ with respect to which M is a metric space. For each point P of K let X_P denote some point of M which is contiguous to P. There exists a positive number e such that, for uncountably many distinct points P of K, $d(P, X_P) > e$. Let $P_1, P_2, P_3, \cdots$ denote a sequence of distinct points of K. By Theorem 13, $d(P, X_{P_n}) \to 0$ as $n \to \infty$. This involves a contradiction.

DEFINITION. The subset H of the point set M is said to be *closed relatively to M* if M contains no limit point of H that does not belong to H. The point P is said to be a *boundary point, relatively to M*, of the subset H of M if P belongs both to M and to the boundary of H. The set of all boundary points of H relatively to M is called the *boundary of H relatively to M.*

THEOREM 15. *If the subset H of the point set M is closed relatively to M and $M-H$ is the sum of two mutually separated point sets K and L then K is a domain with respect to M and its boundary with respect to M, if it exists, is a subset of H.*

THEOREM 16. *If M and N are mutually separated closed point sets and M is compact then there exists a domain D containing M such that N contains no point either of $\overline{D}$ or of its boundary.*

Proof. Let β (N) and β $(\overline{D})$ denote the boundaries of N and of $\overline{D}$ respectively. By hypothesis, M and $N+\beta$ (N) are mutually exclusive and M is closed and compact and, by Theorem 12, $N+\beta$ (N) is closed. It follows, by Theorem 6, that there exists a domain D containing M such that $\overline{D}$ and $N+\beta$ (N) are mutually exclusive. Clearly N contains no point of $\overline{D}+\beta(\overline{D})$.

THEOREM 17. *Suppose H and K are two mutually separated closed point sets and $M_1, M_2, M_3, \cdots$ is a sequence of closed and compact point sets such that* (1) *every M_n contains both a point of H and a point of K,* (2) *for each n, M_n contains M_{n+1} and* (3) *no M_n is the sum of two mutually separated closed point sets one containing a point of H and the other containing a point of K. Then if M denotes the common part of all the point sets of this sequence, the closed point set M is not the sum of two mutually separated closed point sets one containing a point of H and the other containing a point of K.*

Proof. Suppose, on the contrary, that M is the sum of two mutually separated point sets M_H and M_K intersecting H and K respectively. By Theorem 16 there exists an open subset D of M containing M_H and such that M_K contains no point of D or of β, the boundary of D with respect to M. For each n, M_n contains a point of β. Otherwise M_n would be the sum of two mutually separated point sets $D \cdot M_n$ and $M_n-(D+\beta) \cdot M_n$ containing $M_n \cdot M_H$ and $M_n \cdot M_K$ respectively. But, by Theorem 12, β is closed. Hence $\beta \cdot M_1$, $\beta \cdot M_2$, $\beta \cdot M_3$, $\cdots$ is a sequence of closed and compact point sets each containing the next one. Hence, by Theorem 3, the point sets of this sequence have at least one point O in common. The point O belongs to $M \cdot \beta$. This involves a contradiction.

THEOREM 18. *If M is a nondegenerate connected point set every point of M either is a limit point of M or is contiguous to some other point of M.*

Proof. If P is a point of M either one of the sets P and $M-P$ contains a limit point of the other one or P is contiguous to some point of $M-P$. Since no point is a limit point of a single point, the conclusion follows.

THEOREM 19. *If H and K are two mutually separated point sets, every connected subset of $H+K$ is a subset either of H or of K.*

THEOREM 20. *If M is a connected point set and L is a point set consisting of M together with some or all of its boundary points, then L is connected.*

THEOREM 21. *If G is a collection of connected point sets and there exists a point set g of the collection G such that, for every other point set x of G, g contains either a point or a boundary point of x then the sum of all the point sets of the collection G is connected.*

DEFINITIONS. A point set which is both closed and connected is called a *continuum*. If H and K are two mutually exclusive closed point sets, the continuum M is said to be *irreducible from H to K* if M contains both a point of H and a point of K but no proper subcontinuum of M does so.[1]

[1] The notion of an irreducible continuum was introduced by L. Zoretti, "La notion de ligne," *Ec. Norm.*, XXVI (1909), pp. 485–497.

THEOREM 22. *If M is a point set and K is a subset of the set of all points X such that X is contiguous to some point of M then $\overline{M}+K$ is closed and if M is connected then $\overline{M}+K$ is a continuum.*

THEOREM 23. *If H and K are two mutually separated closed point sets and M is a closed point set containing H and K and M is not the sum of two mutually separated closed point sets, one containing a point of H and the other containing a point of K, but every closed proper subset of M that contains both a point of H and a point of K is the sum of two such point sets, then M is an irreducible continuum from H to K.*

THEOREM 24. *If the closed and compact point set M contains a point of each of two mutually separated closed point set H and K and is not the sum of two mutually separated closed point sets one containing a point of H and the other one containing a point of K, then M contains a continuum which is irreducible from H to K.*

THEOREM 25. *If the closed and compact point set M intersects each of the two mutually separated closed point sets H and K but M contains no continuum that intersects both of them, then M is the sum of two mutually separated closed point sets intersecting H and K respectively.*

THEOREM 26. *If H and K are mutually exclusive closed subsets of the compact continuum M there is a subcontinuum of M that is irreducible from H to K.*

Theorems 23 and 24 are worded in precisely the same manner as Theorems 31 and 32 of pages 18 and 19 of P. S. T. except for the substitution of "mutually separated" for "mutually exclusive." They may be established by arguments having much in common with the arguments given to prove these Theorems in P. S. T. Theorems 25 and 26 are almost direct corollaries of Theorem 24.

THEOREM 27. *If H and K are mutually separated closed* *subsets of the compact closed point set M but M contains no continuum containing both a point of H and a point of K, then M is the sum of two mutually separated closed point sets one containing H and the other containing K.*

Theorem 27 is the same as Theorem 35 of Chapter I of P. S. T. except for the substitution of "mutually separated" in place of "mutually exclusive." It may be proved with the help of Theorems 25, 16 and 5.

THEOREM 28. *If H and K are two mutually separated closed point sets and the compact continuum M is irreducible from H to K then $M-M\cdot(H+K)$ is connected, $M-M\cdot H$ is connected and every point of $M\cdot H$ is a boundary point of $M-M\cdot H$ with respect to M.*

Theorem 28 may be established with the help of Theorem 27 by an argument largely similar to that employed to prove Theorem 37 of Chapter I of P. S. T.

DEFINITION. A *maximal connected* subset of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M. A maximal connected subset of a point set is also called a *component* of that point set.

THEOREM 29. *If the open subset D of the continuum M is a proper subset of M and D and its boundary are compact then the boundary with respect to M of D contains at least one boundary point of every maximal connected subset of D.*

Theorem 29 may be established with the help of Theorems 26, 27 and 28.

THEOREM 30. *If K is a closed proper subset of the continuum M and $M-K$ and its boundary are compact then K contains a boundary point of every maximal connected subset of $M-K$.*

Theorem 30 may be easily established with the help of Theorem 29.

Definition. The point set M is said to be connected *in the strong sense*, or *strongly connected*, if for every two points X and Y of M there exists a continuum which contains both X and Y and which is a subset of M. A maximal strongly connected subset of a point set M is a strongly connected subset of M which is not a proper subset of any other such subset of M.

Theorem 31. *If K is a closed proper subset of the compact continuum M, then K contains a boundary point of every maximal strongly connected subset of $M-K$.*

Proof. Let A denote a point of $M-K$ and let W denote the maximal strongly connected subset of $M-K$ that contains A. Suppose K contains no boundary point of W. Since K and W are mutually separated, closed and compact point sets therefore, by Theorem 16, there exists a domain D_W containing W and such that K contains no point either of $\overline{D}_W$ or of its boundary. By Theorem 29, the boundary of D_W contains at least one boundary point P of L, the component of $M \cdot D_W$ that contains A. The point set consisting of L plus its boundary is a continuum lying in $M-K$ and containing both A and the point P. Thus P belongs to W. But this is impossible since W is a subset of D_W and P does not belong to D_W.

Theorem 32. *If D and D_1 are open subsets of the continuum M and both D and β, its boundary with respect to M, are compact and $M-D$ is non-vacuous and $\overline{D}_1$ is a subset of D then there exists a subcontinuum of M lying wholly in $D+\beta-D_1$ and containing both a point of β and a point of β_1, the boundary of D_1.*

Proof. If β and β_1 have a point in common, let N denote one such point. If some point X of β is contiguous to some point X_1 of β_1, let N denote the point set $X+X_1$. If β and β_1 are mutually separated then, by Theorem 26, there exists a subcontinuum K of M which is irreducible from β to β_1

and, by Theorem 28, $K-K\cdot(\beta+\beta_1)$ is connected. Let N denote the point set consisting of $K-K\cdot(\beta+\beta_1)$ together with its boundary with respect to M. In each case the continuum N lies wholly in $D+\beta-D_1$ and contains both a point of β and a point of β_1.

There is a theorem of Sierpinski's to the effect that if the closed and compact point set M is the sum of a countable number (more than one) of mutually exclusive closed sets then M is not connected. This, of course does not hold as a theorem here, for a point set consisting of two contiguous points is a compact continuum. As may be seen from the same example, the proposition that no non-degenerate locally compact continuum is the sum of a countable number of closed and totally disconnected[1] point sets also fails to be a theorem here. If, in the statement of Sierpinski's proposition, the word "separated" is substituted for the word "exclusive" the resulting proposition does hold here.

Definition. If H, K and T are proper subsets of the connected point set M then T is said to separate H from K in M if $M-T$ is the sum of two mutually separated point sets containing H and K respectively.

With the use of this definition, intervals and segments of connected point sets and relations of order between points of such intervals may be defined just as in P. S. T. There is a very large group of propositions concerning these notions of order which hold true here just as in ordinary point set theory. All the numbered theorems from 45 to 69 of Chapter I of P. S. T. hold true here with the exception of Theorems 60 and 63.[2]

[1] A point set is said to be totally disconnected if it contains no non-degenerate connected point set.

[2] If "vacuous or connected" is substituted for "connected" in the statement of the first of these two theorems and "distinct from its endpoints" is substituted for "distinct from P" in the statement of the other one, the resulting propositions hold true here.

THEOREM 33. *If M is a closed and compact point set and α is a convergent sequence of mutually exclusive continua all lying in M then the sequential limiting set of α is either a single point or a non-degenerate perfect continuum which is not the sum of any two mutually exclusive closed point sets.*

Proof. Suppose, on the contrary, that K, the limiting set of α is the sum of two mutually exclusive closed point sets H and L. By Theorem 6, there exists an open subset D of M containing H and such that $\overline{D}$ contains no point of L. For each n, let K_n denote the nth term of the sequence α. There exists a number m such that, for every n greater than m, K_n contains a point of D and a point of $M-D$ and therefore a point P_n belonging to β, the boundary of D. The point set $P_1+P_2+P_3+\cdots$ has a limit point O. The point O belongs both to K and to $\overline{D}-D$. But this involves a contradiction.

It is to be noted that the conclusion of Theorem 33 gives much more information than would be given by the statement that the limiting set of α is either a single point or a non-degenerate perfect continuum. For the Example 3 described shortly after the statement of the Axioms of Σ_c is an example of a space in which there is a perfect continuum (in this case the set of all points) which is the sum of two mutually exclusive closed point sets. Theorem 33 would not, however, continue to be a true theorem here if its conclusion were so strengthened as to require that no *subcontinuum* of the limiting set of α be the sum of two mutually exclusive closed point sets. For there are examples in which the limiting set of such a sequence contains two points which are contiguous to each other.

THEOREM 34. *If the open subset D of the continuum M is a proper subset of M and $\overline{D}$ is compact and H denotes the set of all points X of D such that X is contiguous to some point of*

$M-D$, then $H+(\overline{D}-D)$ contains at least one point or limit point of every component of D and at least one point of every component of $\overline{D}$.

Proof. Since M is connected and D is a proper subset of M therefore $H+(\overline{D}-D)$ exists. Let O denote a point of D and let T denote the component of D that contains O. If O either belongs to H or is contiguous to some point P of $H+(\overline{D}-D)$, then either O or $O+P$ is a connected subset of D containing a point of $H+(\overline{D}-D)$ and therefore $H+(\overline{D}-D)$ contains either the point O or the point P of T.

Suppose O neither belongs to H nor is contiguous to any point of $H+(\overline{D}-D)$. Since every point of H is contiguous to some point of the closed point set $M-D$ therefore, by Axiom C, $M-D$, and therefore $\overline{D}-D$ contains every limit point of H. Hence $H+(\overline{D}-D)$ is closed and it and O are mutually separated. The closed point set $\overline{D}$ is not the sum of two mutually separated point sets M_O and M_H containing O and $H+(\overline{D}-D)$ respectively. For if it were then M would be the sum of the two mutually separated point sets M_O and $M_H+(M-\overline{D})$ contrary to the hypothesis that M is connected. But $\overline{D}$ is compact. Hence, by Theorem 27, $\overline{D}$ contains a continuum containing both O and a point of $H+(\overline{D}-D)$. Hence, by Theorem 26, $\overline{D}$ contains a continuum N which is irreducible from O to $H+(\overline{D}-D)$. By Theorem 28, $N-N\cdot[H+(\overline{D}-D)]$ is connected and every point of $N\cdot[H+(\overline{D}-D)]$ is a boundary point of it. Hence $N-N\cdot[H+(\overline{D}-D)]$ is a subset of T and every point of $N\cdot[H+(\overline{D}-D)]$ either belongs to T or is a limit point of it.

THEOREM 35. *If D and D_1 are open subsets of the continuum M, $\overline{D}$ is compact, $M-D$ is non-vacuous and $\overline{D}_1$ is a subset of D then there exists a subcontinuum of M lying wholly in $\overline{D}-D_1$ and containing both a point of the boundary of D_1 and a point of the boundary of $M-D$.*

THEOREM 36. *If D and D_1 are open subsets of the closed point set M, $\overline{D}$ is compact, $\overline{D}_1$ is a subset of D and $M_1, M_2, M_3, \cdots$ is a sequence of mutually exclusive continua lying in M and, for each n, M_n contains both a point of D_1 and a point of $M-D$, then (1) there exists a number m such that, for every n greater than m, M_n contains a non-degenerate continuum H_n lying in $\overline{D}-D_1$ and containing both a point of $\beta(D_1)$, the boundary of D_1 with respect to M, and a point of $\beta(M-D)$, the boundary of $M-D$ with respect to M and (2) some subsequence of $H_1, H_2, H_3, \cdots$ converges to a continuum lying in $\overline{D}-D_1$ and containing a perfect continuum lying in $D-\overline{D}_1$.*

Proof. If $\beta(\overline{D}_1)$ and $\beta(M-D)$ denote the boundaries with respect to M of $\overline{D}_1$ and of $M-D$ respectively, there do not exist infinitely many points X such that X belongs to one of the point sets $\beta(\overline{D}_1)$ and $\beta(M-D)$ and either belongs to the other one or is contiguous to some point of it. Hence there exists a number m such that if $n>m$ then $M_n\cdot[\overline{D}_1+\beta(\overline{D}_1)]$ and $M_n\cdot[(M-D)+\beta(M-D)]$ are mutually separated.

Let K_n denote a component of $M_n\cdot\overline{D}$ containing some point of D_1. By Theorem 34, K_n contains a point of $\beta(M-D)$. Since it is a subset of $\overline{D}$, the continuum K_n is compact and it intersects both $\beta(M-D)$ and $\overline{D}_1+\beta(\overline{D}_1)$. Therefore, by Theorem 26, there exists a subcontinuum H_n of K_n which is irreducible from $\beta(M-D)$ to $\overline{D}_1+\beta(\overline{D}_1)$. By Theorem 28, if $n>m$, then $H_n-H_n\cdot[\overline{D}_1+\beta(\overline{D}_1)+\beta(M-D)]$ is a connected point set L_n and $\overline{D}_1+\beta(\overline{D}_1)$ contains a point P such that either P is a limit point of L_n or P is contiguous to some point X of L_n. In the first case, P belongs to the boundary of D_1. In the second case, X or P belongs to the boundary of $\overline{D}_1$ according as P does or does not belong to D_1. So, in either case, H_n contains a point of the boundary of D_1. The continuum H_n is either $\overline{L}_n$ or the point set obtained by adding to $\overline{L}_n$ one or more points that are contiguous to L_n. But L_n contains no point of $\overline{D}_1+\beta$ $(\overline{D}_1)$. Hence H_n contains no point of D_1. Hence it is a subset of $\overline{D}-D_1$.

There exists an ascending sequence of numbers $n_1, n_2, n_3, \cdots$ such that $H_{n_1}, H_{n_2}, H_{n_3}, \cdots$ is convergent. Since, for every n greater than m, H_n intersects each of the two mutually exclusive closed point sets $\overline{D}_1+\beta(\overline{D}_1)$ and $\beta(M-D)$, the sequential limiting set of the sequence $H_{n_1}, H_{n_2}, H_{n_3}, \cdots$ contains at least two distinct points, one belonging to $\overline{D}_1$ and one to $\overline{D}-D$. Hence, by Theorem 33, it is a non-degenerate perfect continuum T which is not the sum of two mutually exclusive closed point sets. The continuum T intersects both $\overline{D}-D$ and $\overline{D}_1$ and it is a subset of $\overline{D}-D_1$. Since $T\cdot(\overline{D}_1)$ and $T\cdot(\overline{D}-D)$ are non-vacuous, closed and mutually exclusive, T contains a point O belonging to neither of them and therefore belonging to the point set $D-\overline{D}_1$. There exist open subsets D_2 and D_3, of M, containing O such that $\overline{D}_2$ is a subset of $D-\overline{D}_1$ and $\overline{D}_3$ is a subset of D_2. There exists a number k such that each continuum of the sequence H_{n_k}, $H_{n_{k+1}}, H_{n_{k+2}}, \cdots$ contains a point of D_3. But each of them contains a point of $\overline{D}_1$ and therefore of $M-D_2$. Hence there exist an infinite subsequence $m_1, m_2, m_3, \cdots$ of the sequence $n_1, n_2, n_3, \cdots$ and a sequence $Q_{n_1}, Q_{n_2}, Q_{n_3}, \cdots$ such that (1) for each i, Q_{n_i} is a subset of H_{n_i} lying wholly in $\overline{D}_2-D_3$ and containing both a point of $\beta(D_3)$ and a point of $\beta(M-D_2)$, (2) the sequence $Q_{n_1}, Q_{n_2}, Q_{n_3}, \cdots$ converges to a non-degenerate continuum Q. The sequence $H_{n_1}, H_{n_2}, H_{n_3}, \cdots$ converges to T, and Q is a perfect subcontinuum of T which lies in $D-\overline{D}_1$ and intersects $\overline{D}_3$ and $\overline{D}_2-D_2$ and which is not the sum of two mutually exclusive closed point sets.

Definition. If A and B are two points, by a simple chain of domains from A to B is meant a finite sequence $R_1, R_2, \cdots, R_m$ of domains such that (1) R_1 contains A and

R_m contains B, (2) no domain of this sequence except R_1 contains A and none except R_m contains B, (3) if $i>j+1$, no point of R_i is identical with, or contiguous to, any point of R_j, (4) if $i=j+1$, some point of R_i is identical with, or contiguous to, some point of R_j.

THEOREM 37. *If M is a connected point set, A and B are two distinct points of M and G is a set of domains covering M then there exists a simple chain from A to B such that every link of this chain is a domain of the set G.*

DEFINITIONS.[1] The point set M is said to be *connected* im kleinen at the point O if O belongs to M and for every open subset D of M that contains O there exists an open subset of M which contains O and which is a subset of a component of D. The point set M is said to be *locally connected* at the point O if O belongs to M and every open subset of M that contains O contains a connected open subset of M containing O. If a point set is locally connected at every one of its points it is said to be *locally connected* and if it is connected im kleinen at every one of its points it is said to be connected im kleinen.

A connected im kleinen continuum is called a *continuous curve.*

If a point set M is connected im kleinen at every point of some open subset of M that contains the point O of M then M is locally connected at O.

Theorems 1–37 are consequences of the axioms of Σ_c exclusive of Axiom 2. With the help of Axiom 2 and Theorem 37 the following theorem may be established by an argument having much in common with that used to establish the first theorem of Chapter II of P. S. T.

[1] Cf. Hans Hahn, *Jahresbericht der D. Math. Vereinigung*, XXIII (1914), pp. 318–322, and S. Mazurkiewicz, *Fundamenta Mathematicae*, 1 (1920), and earlier papers in Polish, referred to therein.

THEOREM 38. *If A and B are distinct points of a connected domain D there exists a simple continuous arc from A to B that lies wholly in D.*

Indeed the first nineteen numbered theorems of Chapter II of P. S. T. hold here without any change whatever in their wording. Each of the following four theorems, 39–42, either belongs to this group or is a modification[1] of one belonging to it.

THEOREM 39. *If the continuous curve M is regarded as a space and the term "region" is interpreted to mean a connected open subset of M, then, with respect to this interpretation of "point" and "region," the axioms of Σ_c are satisfied and "limit point" is invariant under this change.*

THEOREM 40. *If D is a point set, M is a point set, α is a sequence of distinct components of $M \cdot D$, and O is a point belonging to the limiting set of α and lying in some region which is a subset of D then M is not connected im kleinen at O.*

THEOREM 41. *If A, B and O are three distinct points of a subset M of a locally connected and connected point set T and M is closed with respect to T and there exists at least one point of M that separates O from both A and B in T and neither of the points A and B separates the other one from O in T, then there exists a point of M which separates O from $A+B$ in T and which is not separated from $A+B$ in T by any point of M.*

THEOREM 42. *If O and A are two points of a subset M of a locally connected and connected point set T and M is closed with respect to T, then the set of all points of M that separate O from A in T is compact.*

THEOREM 43. *If O is a point of the locally compact con-*

[1] Theorems 41 and 42 correspond to propositions which may be obtained from Theorems 12 and 13 of P. S. T. by weakening their hypotheses in the manner indicated in the appendix of the book. Cf. the references made there to the work of O. H. Hamilton and F. B. Jones in this connection.

tinuum M and M is not connected im kleinen at O then, if R is a region containing O, there exist a connected domain D containing O and lying in R, an infinite sequence of points $O_1, O_2, O_3, \cdots$ of D converging to O and an infinite sequence of mutually separated continua $M_1, M_2, M_3, \cdots$ such that (1) $M \cdot \overline{D}$ is compact, (2) for each n, M_n is a component of $M \cdot \overline{D} - O$ containing O_n and either a point of $\overline{D} - D$ or a point of D which is contiguous to some point of $M - D \cdot M$, (3) the sequence $M_1, M_2, M_3, \cdots$ converges to a perfect subcontinuum K of M containing O and a point of $\overline{D} - D$ and such that there is a perfect subcontinuum of K lying wholly in D.

Theorem 43 may be established with the help of Theorems 34 and 36 and an argument having much in common with the proof of Theorem 8 of Chapter II of P. S. T.

THEOREM 44. *Under the hypothesis of Theorem 43, if D is a domain and $K, M_1, M_2, M_3, \cdots$ are continua fulfilling the conditions described in the statement of that theorem, then there exists a sequence of points $O_1', O_2', O_3', \cdots$ converging to a point O' such that, for each n, O_n' belongs to M_n and O' is a point of $K \cdot D$ which is not contiguous to any point of $M - D$.*

Proof. Let T denote a perfect subcontinuum of K lying in D. Every point of T is a limit point of T. Since T and the closed point set $M - D$ are mutually exclusive it follows, by Axiom C, that there is a point O' of T which is not contiguous to any point of $M - D$. Since O' belongs to $K \cdot D$ there exists a sequence of points $O_1', O_2', O_3', \cdots$ converging to O' and such that, for each n, O_n' belongs to $D \cdot M_n$.

THEOREM 45. *If the continuum M is locally compact at the point O then in order that M should be connected im kleinen at O it is sufficient that for every subcontinuum K of M not containing O there should exist a finite collection of continua filling up M such that no one of them contains both O and a point of K.*

Proof. Suppose D is an open subset of M containing O and such that $\overline{D}$ is compact. If D is M then it is connected. Suppose it is not M. Then $M-D$ has a boundary V with respect to M. If O is not contiguous to any point of $M-D$, let L denote V. If O is contiguous to at least one point of $M-D$, let L denote $V-O$. By Axiom C, the closed point set $M-D$ contains every limit point of the set of all points X such that X is contiguous to a point of $M-D$. Therefore O is not a limit point of $V-O$. Hence, whether or not O belongs to V, L is closed. For each point X of L there exists a finite collection G_X of continua filling up M such that no one of them contains both O and X. Let T_X denote the sum of all the continua of G_X that contain X. Let G denote the collection of all T_X''s for all points X of L. For each X, T_X contains an open subset of M containing X. Hence, since it is closed and compact, L is covered by a finite subset H of G. Let N denote the sum of all the continua of the collection H. The closed point set N does not contain O.

Let Z denote the point set consisting of $M-N$ together with its boundary, let Q denote $\overline{D}\cdot Z$ and let K denote the component of Q that contains O. Suppose that O is a limit point of $M-K$. Then it is a limit point of $Q-K$. Since K is a component of Q, O is not a limit point of any connected subset of $Q-K$. Hence there exist a sequence of points $P_1, P_2, P_3, \cdots$ and a sequence of distinct continua $K_1, K_2, K_3, \cdots$ such that (1) for each n, K_n is a component of Q distinct from K, (2) for each n, P_n belongs to K_n, (3) the point O is the sequential limit point of the sequence $P_1, P_2, P_3, \cdots$. The collection H is finite. It follows, with the help of Theorem 34, that there exist a continuum g of the collection H and an infinite subsequence α of the sequence $K_1, K_2, K_3, \cdots$ such that every continuum of the sequence α has a point in common with g. Let G_1 denote the collection of all continua x of H such that there exists a subcontinuum of M containing both a point of g and a point of x but not containing O. There exists a finite collection G' of continua filling up M such that no continuum of G' contains both O and a point of G_1^*. Let T denote the sum of all the continua of G' that contain O. The continuum T contains an open subset of M containing O. Hence it contains points of two different continua h and k of the sequence α. But h and k are distinct components of Q not containing O. Therefore T contains a point of N belonging to h and therefore a point of G_1^*. Thus the supposition that O is a limit point of $M-K$ has led to a contradiction. Hence K contains an open subset of M containing O. But K is a connected subset of $\overline{D}$. Therefore M is connected im kleinen at O.

THEOREM 46. *If the locally compact continuum M is not connected im kleinen at the point O then there exists a perfect subcontinuum K of M containing O and such that no point of K is separated from O in M by a finite subset of M and such that furthermore there are uncountably many points of K that are not contiguous to O.*

Proof. By Theorem 43 there exist a connected domain D containing O, an infinite sequence of points $O_1, O_2, O_3, \cdots$ converging to O and an infinite sequence of mutually separated continua $M_1, M_2, M_3, \cdots$ such that (1) $M \cdot \overline{D}$ is compact, (2) for each n, M_n is a component of $M \cdot \overline{D} - O$ containing O_n and a point of the boundary of $M-D$, (3) the sequence $M_1, M_2, M_3, \cdots$ converges to a perfect subcontinuum K of M containing O. Suppose X is a point of K and N is a finite subset of M. There exists a number i such that no one of the continua $M_i, M_{i+1}, \cdots$ contains a point of N. Suppose $M-N$ is the sum of two mutually separated point sets H and K containing O and X respectively. By Theorem 19, each continuum of the sequence $M_i, M_{i+1}, M_{i+2}, \cdots$ is a subset either of H or of K. Hence there exists an infinite subsequence α of the sequence $M_i, M_{i+1}, M_{i+2}, \cdots$ such that either H contains every continuum of α or K does so. Let W denote the point set obtained by adding together all the continua of the sequence α and let Q denote H or K according as W is a subset of H or of K. Since O and X are limit points of W belonging to $M-N$ therefore they both belong to Q contrary to the supposition that they belong to H and to K respectively. Since K is uncountable therefore, by Theorem 14, there are uncountably many points of K that are not contiguous to O.

THEOREM 47. *If P is a point of an open subset D of a compact continuum M and L is the set of all points of M that are contiguous to P and T is the common part of L and $\beta(D)$, the boundary of D with respect to M, then there exists an open subset U of M lying in D and containing $P+L\cdot D$, but no point of T, and such that the set of all points of the boundary of U that do not belong to L is either vacuous or closed.*

Proof. By Axiom C, P contains every limit point of L. Hence T and $P+L-T$ are mutually exclusive closed point sets. Hence there exists an open subset W of M containing $P+L-T$ and such that $\overline{W}$ contains no point of T. Let U denote $W\cdot D$. The point set U is an open subset of M containing P and, if $\beta(U)$ denotes its boundary, no point of $\beta(U)-T$ is contiguous to P. By Axiom C, if $C(U)$ denotes the set of all points of $\beta(U)$ that are contiguous to some point of U, $\overline{U}-U$ contains every limit point of $C(U)$ and therefore no limit point of $C(U)$ belongs to T. Since U is a subset of W and $\overline{W}$ contains no point of T, therefore the closed point set $\overline{U}$ contains no point of T. But $\beta(U)=C(U)+(\overline{U}-U)$. Hence $\beta(U)-T$ is closed.

Definition. The point set M is said to be *compactly connected* if every two points of M lie together in some compact continuum which is a subset of M.

Theorem 48. *If P is a point of the compactly connected continuum M, and M is a subset of the point set T and M_P denotes the set of all points X of M such that X is not separated from P in T by any one point, then M_P is a continuum.*

Theorem 48 may be proved with the assistance of Theorem 26.

Theorem 49. *If M is a locally compact and compactly connected subcontinuum of the continuous curve T and O is a point of M at which M is not connected im kleinen then there exists a subcontinuum K of M containing O such that K is not connected im kleinen at O and such that no point of K separates any two points of K from each other in T.*

Proof. By hypothesis and Theorem 43, there exist a domain D containing O, an infinite sequence α of distinct points $P_1, P_2, P_3, \cdots$ of D converging to O and an infinite sequence of mutually separated continua $M_1, M_2, M_3, \cdots$ such that (1) $M \cdot \overline{D}$ is compact, (2) for each n, M_n is a component of $M \cdot \overline{D}$ not containing O but containing P_n and either a point of $\overline{D}-D$ or a point of D which is contiguous to some point of $M-D \cdot M$, (3) the sequence $M_1, M_2, M_3, \cdots$ converges to a perfect subcontinuum L of M which contains O and a point of $\overline{D}-D$ and which contains a perfect continuum lying wholly in D.

Let A denote some definite point of L distinct from O. The point O is not separated from A in M by any one point. Hence it is not separated from A in T by any one point. It follows that if a point of $M-A$ separates a point of M from O in T then it separates it from $A+O$ in T. Therefore, by Theorem 41, if the point P of M is separated from O in T by some point of M and neither of the points A and O separates the other one from P in T, then there exists, in M, a point X_P which separates O from P in T but which is not itself separated from O in T by any point of M. Let M_O denote the set of all points P of M such that no point separates P from O in T. By Theorem 48, M_O is a continuum. Since T is a continuous curve there exist two mutually exclusive and connected open subsets D_O and D_A of T containing O and A respectively and lying wholly in D. There exists a number k such that if $n>k$ then M_n contains both a point of D_O and a point of D_A. Neither of the points A and O separates the other one, in T, from any point of the sequence $P_{k+1}, P_{k+2}, P_{k+3}, \cdots$.

Suppose M_O is connected im kleinen at O. Then clearly there exists a natural number m greater than k such that M_O contains no point of the sequence $P_{m+1}, P_{m+2}, P_{m+3}, \cdots$. For each n, let B_n denote P_{m+n} and let O_n denote the point X_{B_n}. For each n, $T-O_n$ is the sum of two mutually separated point sets U_n and V_n containing B_n and O respectively. Let H_n denote the point set $M \cdot U_n + O_n$. For each n, H_n is a continuum containing O_n and B_n. There exists an ascending sequence of natural numbers $n_1, n_2, n_3, \cdots$ such that $O_{n_1}, O_{n_2}, O_{n_3}, \cdots$ converges to some point F and such that $U_{n_1}, U_{n_2}, U_{n_3}, \cdots$ are mutually exclusive. Let E denote a region containing F. There exists a region R containing F and such that $\overline{R} \cdot M$ is compact. Suppose infinitely many of the continua $H_{n_1}, H_{n_2}, H_{n_3}, \cdots$ contain points of $S-R$. Let R_1 denote a region containing F such that $\overline{R}_1$ is a subset of R. Let β and β_1 denote the boundaries of $M \cdot R$ and $M \cdot R_1$ respectively with respect to M. There exists an ascending subsequence $m_1, m_2, m_3, \cdots$ of the sequence $n_1, n_2, n_3, \cdots$ such that, for every i, O_{m_i} belongs to R_1 and H_{m_i} contains a point of $S-R$. By Theorem 36, there exists a number q such that (1) if $i>q$, H_{m_i} contains a non-degenerate continuum H'_{q_i} lying wholly in $\bar{R}-R_1$ and containing both a point of $\beta(R)$, the boundary of R_1 with respect to M, and a point of $\beta(M-R\cdot M)$, the boundary of $M-R\cdot M$ with respect to M, (2) some subsequence $Q_1, Q_2, Q_3, \cdots$ of the sequence $H'_{m_{q+1}}, H'_{m_{q+2}}, \cdots$ converges to a continuum containing a point F' lying in $R-\bar{R}_1$. For each i, let K_i denote the component of $T\cdot(\bar{R}-R_1)$ that contains Q_i. The continua of the sequence $K_1, K_2, K_3, \cdots$ are mutually separated and the limiting set of this sequence contains the point F'. Therefore, by Theorem 40, T is not connected im kleinen at F'. But this is contrary to hypothesis. It follows that each region that contains F contains all but possibly a finite number of the continua $H_{n_1}, H_{n_2}, H_{n_3}, \cdots$. Hence F is O. There exists a region Z containing F and such that $Z\cdot M_O$ is a subset of the component of $D\cdot M_O$, and therefore of W the component of $M\cdot\bar{D}$, that contains F. There exists a natural number δ such that, for every i greater than δ, H_{n_i} is a subset of Z. But H_{n_i} is a continuum containing B_{n_i} and the point O_{n_i} of M_O. Hence $B_{n_{\delta+1}}$ and $B_{n_{\delta+2}}$ both belong to W. But they belong respectively to $M_{m+n_{\delta+1}}$ and $M_{m+n_{\delta+2}}$, two distinct components of $M\cdot\bar{D}$. Thus the supposition that M_O is connected im kleinen at O has led to a contradiction. Since T is a continuous curve, no component of $T-O$ contains a limit point of the sum of the remaining components of $T-O$. It follows that there exists a component U of $T-O$ such that $O+U\cdot M_O$ is a continuum K which is not connected im kleinen at O. The continuum K satisfies all the requirements of Theorem 49.

DEFINITION.[1] The point P of the continuum M is said to be a *regular* point of M, and M is said to be *regular at P*, if

[1] Cf. Karl Menger, "Grundzuge einer Theorie der Kurven," *Math. Ann.*, XCV (1925), p. 279.

every open subset of M containing P contains an open subset of M whose boundary with respect to M is a finite point set. The continuum M is said to be a regular curve if it is regular at every one of its points.

If there are no two contiguous points in space, then, if the continuum M is regular at the point P, every point of M distinct from P is separated from P in M by a finite set of points. But this is not true in every space satisfying the axioms of Σ_c. Suppose, for example, the continuum M consists of the two contiguous points P and X. This continuum is regular at the point P for P is itself an open subset of M which lies in every open subset of M containing P and the boundary of P is the single point X. But since X is contiguous to P it is not separated from P in M by any point set whatsoever. It is to be noted that in this example not every open subset D of M containing P contains an open subset of M whose *boundary* is a subset of D. Thus Theorem 59 of Chapter II of P. S. T. does not hold true here. However it is true that if L is a closed proper subset of the compact continuum M and M is regular at every point of L then every open subset D of M containing L contains an open subset of M containing L and bounded, with respect to L, by a finite subset of M (not necessarily of D). Furthermore, while it is not true that every two mutually exclusive closed subsets of a compact regular curve M are separated in M by a finite subset of M, it is true that every two mutually separated closed ones are. And in order that a compact continuum M should be a continuous curve it is necessary and sufficient that every two mutually separated closed subsets of M should be separated from each other in M by the sum of some finite number of subcontinua of M.[1] This may

[1] Cf. G. T. Whyburn and W. L. Ayres, "On continuous curves in n dimensions," *Bull. Am. Math. Soc.*, XXXIV (1928), pp. 349–360.

be shown with the assistance of Theorem 45. With the help of this same theorem, the proposition of W. A. Wilson stated on page 136 of P. S. T. may also be shown to hold true here.

THEOREM 50. *If P is a point of a compact continuum M and for every closed subset H of M such that P and H are mutually separated there is a finite point set that separates P from H in M then M is regular at P.*

Proof. Suppose D is an open subset of M containing P. Unless M is degenerate, D contains an open subset D_1 of M which contains P and has, with respect to M, a non-vacuous boundary $\beta(D_1)$. If no point of $\beta(D_1)$ is contiguous to P there exists a finite point set N such that $M-N$ is the sum of two mutually separated point sets H_P and H_β containing P and $\beta(D_1)$ respectively. Let I denote the common part of the point sets D_1 and H_P. The point set I is a domain with respect to M, it contains P and its boundary with respect to M is a subset of $N+\beta(D_1)$ and therefore of N since I and $\beta(D_1)$ are mutually separated.

Suppose at least one point of $\beta(D_1)$ is contiguous to P. Let T denote the set of all such points. By Theorem 9, T is a finite set. If $\beta(D_1)-T$ is vacuous, D_1 is itself a subset of D with a finite boundary. Suppose $\beta(D_1)-T$ is non-vacuous. By Theorem 47, there exists an open subset U of M lying in D_1 and containing P and such that if $\beta(U)$ denotes the boundary of U then T is the set of all points of $\beta(U)$ that are contiguous to P and $\beta(U)-T$ is closed. Since $\beta(U)-T$ is closed and P is not contiguous to any point of it, therefore there exists in M a finite point set N such that $M-N$ is the sum of two mutually separated point sets H_P and $H_{\beta(U)-T}$ containing P and $\beta(U)-T$ respectively. The point set H_P is an open subset of M whose boundary with respect to M is a subset of the finite point set N. The point set $M-T$ is an open subset of M containing P and bounded with respect to M by a subset of the finite point set T. Let Z denote the common part of D_1, H_P and $M-T$. The point set Z is a domain with respect to M. Its boundary $\beta(Z)$ is a subset of $N+T+\beta(U)-T$. But $\beta(U)-T$ and Z are mutually separated. Hence $\beta(Z)$ is a subset of $N+T$. Therefore it is finite.

THEOREM 51. *A locally compact continuum is connected im kleinen at every point at which it is regular.*

Proof. Suppose the locally compact continuum M is regular at the point P but not connected im kleinen there. By Theorem 46, there exists a compact and perfect subcontinuum T of M containing P such that no point of T is separated from P in M by a finite subset of M. Since T is uncountable and compact it contains a point X which is not contiguous to P. By Theorem 16, there exists an open subset D of M containing P and such that X belongs neither to D nor to its boundary with respect to M. There exists an open subset H of M lying in D and containing P and bounded in M by a finite point set N. The point X does not belong to $H+N$. Hence N separates P from X in M. This involves a contradiction.

THEOREM 52. *No compact regular curve contains two mutually exclusive closed point sets H and K and infinitely many mutually exclusive continua each containing both a point of H and a point of K.*

Proof. Suppose that H and K are two mutually exclusive closed subsets of a compact regular curve M and G is an infinite collection of mutually exclusive subcontinua of M each containing both a point of H and a point of K. There exists an infinite sequence of continua $g_1, g_2, g_3, \cdots$ of the collection G converging to some continuum T. Since each continuum of this sequence contains both a point of H and a point of K, so does T. Hence T is non-degenerate. Therefore, by Theorems 33, 4 and 14, T contains two mutually separated points A and B. That A and B are not separated from each other in M by any finite set of points may be shown by an argument similar to that employed in a similar connection in the proof of Theorem 46. But this is contrary to a previously mentioned proposition concerning compact regular curves.

THEOREM 53. *No compact regular curve contains uncountably many mutually exclusive non-degenerate continua.*

Proof. Suppose, on the contrary, that there exist a compact regular curve M and an uncountable set G of mutually exclusive non-degenerate subcontinua of M. It follows from Theorem 14 that there exists an uncountable subcollection H of G such that every two continua of the collection H are mutually separated. It follows that there exist two mutually separated closed subsets K and L of M such that there are uncountably many continua of the set H that intersect both K and L. But this is contrary to Theorem 52.

With the help of Theorem 33 it may be shown that if a compact continuum M contains two closed and mutually exclusive point sets H and K and an infinite number of mutually exclusive continua each containing both a point of H and a point of K then M has a *continuum of condensation*, that is to say there is a non-degenerate proper subcontinuum T of M such that every point of T is a limit point of $M-T$.

DEFINITION. If M is a continuum, a *composant* of M is a point set K such that, for some point P of M, K is the set of all points X such that there is a proper subcontinuum of M containing both P and X.

THEOREM 54. *If K is a composant of a compact continuum M there is not more than one point of $M-K$ which is not a limit point of K and if there is one such point P then P is contiguous to some point of K and furthermore $K=M-P$.*

Thus either every point of $M-K$ is a limit point of K or no one is.

Proof. Suppose there is a point P of $M-K$ which is not a limit point of K. Then clearly $\overline{K}=K$. There exists an open subset D of M containing P such that $\overline{D}$ contains no point of K. The point set $\overline{D}$ contains a boundary point X of T, the component of $M-\overline{D}$ that contains K. Since X does not belong to the closed point set K, it is not a limit point of K. Hence it is contiguous to some point of K. Hence $K+Y$ is a continuum. Since Y does not belong to K, $K+Y$ cannot be a proper subcontinuum of M. Hence Y is identical with P and $K=M-P$.

Just as in ordinary point set theory, an *indecomposable continuum* may be defined as a continuum which is not the sum of any two proper subcontinua of itself. Such continua have been studied by Brouwer, Mazurkiewicz, Yoneyama, Knaster, Kuratowski and others. Brouwer established the existence of such point sets.

If M is an indecomposable continuum and K is a composant of M, every point of $M-K$ is a limit point of K. For if P is a point of $M-K$ then, by Theorem 54, if P is not a limit point of K then $K=M-P$ and P is contiguous to some point X of K. The point set $P+X$ is a non-degenerate proper subcontinuum of M and therefore, since M is indecomposable, $P+X$ is a continuum of condensation of M. Hence P is a limit point of $M-(P+X)$ and therefore of K.

Every numbered theorem[1] *of* P. S. T. *in whose statement the term indecomposable occurs continues to hold true here.*

THEOREM A. *If M is a compact metric space without contiguous points and $X_1, X_2, X_3, \cdots$ and $Y_1, Y_2, Y_3, \cdots$ are sequences of points of M such that X_n is distinct from Y_n but the distance from X_n to Y_n approaches 0 as n increases in-*

[1] Cf. Theorems 108–113 of Chapter I.

definitely then if the point X is called contiguous to the point Y if and only if, for some n, X is one of the points X_n and Y_n and Y is the other one, then all of the axioms of Σ_c except Axiom 2 hold true in the resulting space.

Proof. It is clear that the truth of Axiom 1 is not affected by the agreement to call certain points contiguous to certain others. Clearly Axioms A and B hold true for this interpretation of contiguity. It will be shown that Axiom C holds true. Suppose K is a closed subset of M and P is a limit point of a point set H of which each point is contiguous to some point of K. There exists an ascending sequence of natural numbers $n_1, n_2, n_3, \cdots$ such that either $X_{n_1}, X_{n_2}, X_{n_3}, \cdots$ are distinct points of H such that P is a limit point of the set of all points of this sequence and, for each i, Y_{n_i} is a point of K or $Y_{n_1}, Y_{n_2}, Y_{n_3}, \cdots$ are distinct points of H such that P is a limit point of the point set $Y_{n_1}+Y_{n_2}+Y_{n_3}+\cdots$ and, for each i, X_{n_i} is a point of K. Suppose the former of these alternatives holds true. There exists a subsequence $m_1, m_2, m_3, \cdots$ of the sequence $n_1, n_2, n_3, \cdots$ such that P is the sequential limit point of the sequence $X_{m_1}, X_{m_2}, X_{m_3}, \cdots$ and therefore $\lim_{i\to\infty} d(X_{m_i}, P)=0$. But $\lim_{i\to\infty} d(X_{m_i}, Y_{m_i})=0$. Therefore $\lim_{i\to\infty} d(Y_{m_i}, P)=0$ and therefore, since K is closed, P belongs to K. Thus K contains every limit point of H and Axiom C holds true for this interpretation of contiguity.

Definition. If S is a space in which there are no contiguous points and certain points of S are defined as being contiguous to other points of S in such a way that all of the Axioms of Σ_c except possibly Axiom 2 hold true for this interpretation of contiguity then the resulting space is said to result from the introduction of contiguity into S.

Theorem B. *If S is a continuous curve in which no point* *is contiguous to any other one and* S_C *is a space obtained from* S *by the introduction of contiguity then* S_C *is also a continuous curve.*

The truth of this theorem is clear in view of the fact that the points of S_C are the points of S and every point set which is connected in the old sense in S is connected in the new sense in S_C though there is at least one point set which is connected in the new sense but not in the old.

It is not however true that if S is an arc and S_C is a space obtained from S by the introduction of contiguity then S_C is an arc. Consider the following example.

EXAMPLE. Let O be the origin of coordinates and let A denote the point (1, 0) in a Cartesian space. Let S denote the space consisting of the points of the straight line interval OA. The space S is a simple continuous arc. For each n, let T_n denote the point set consisting of all points of the interval OA whose abscissas are of the form m/n where m is a positive integer not greater than n. Let S_C denote the space whose points are the points of S but in which two points are contiguous to each other if and only if, for some n, they both belong to T_n and their abscissas differ by $2/n$. The space S_C is obtained from the arc S by the introduction, in a certain manner, of contiguity into the space S but, though all of the points of S, except O and A, are cut points of S, not only does S_C have no cut points but it has no *local* cut points and indeed no two points of S_C are separated from each other in S_C by any finite point set. For suppose K is a finite subset of S_C. Let H denote the point set consisting of the points O and A and all the points of K. The points of this set may be designated $A_1, A_2, A_3, \cdots, A_k$ in such a manner that if $i>j$ the abscissa of A_i is greater than that of A_j. There exists a number n such that no two points of the set H are at a distance apart as little as $2/n$ and no point of H has an abscissa equal to m/n where m is a positive integer. Since, for each i less than k, the distance from A_i to A_{i+1} is greater than $2/n$ therefore there exists, in S, a point of T_n between A_i and A_{i+1}. For each such i let L_i denote the rightmost point of T_n between A_i and A_{i+1} and let R_i denote the point whose abscissa exceeds that of L_i by $2/n$. The point R_i is necessarily between A_{i+1} and A_{i+2}. For each i less than k let K_i denote the set of all points of S_C that belong to the segment A_iA_{i+1} of S. While K_i is not a segment in the new sense it is, nevertheless, connected since every point set which is connected in the old sense is also connected in the new. If i is less than k then, since the point L_i of the connected point set K_i is contiguous to the point R_i of the connected point set K_{i+1}, therefore K_i+K_{i+1} is connected in the new sense. It follows that $K_1+K_2+\cdots+K_n$ is connected. But either this point set or the connected set obtained by adding to it one or both of the points O and A is identical with S_C-K. Therefore S_C is not disconnected by the omission of any finite point set K.

Theorems 1–54 are consequences of the axioms of the set Σ_c. Let Σ_c' denote the set consisting of Σ_c together with the following axiom.

Axiom D. There do not exist three distinct points such that each of them is contiguous to each of the others.

Let us now consider a space satisfying the more restricted set of axioms Σ_c'.

Definitions. A *simple closed curve* is a non-degenerate compact continuum which is irreducible with respect to the property of being the sum of two distinct arcs with common endpoints.

A continuous curve is said to be *acyclic* if it contains no simple closed curve. An acyclic continuous curve is also called a *dendron*.

A point P of a continuous curve M is said to be an endpoint of M if P is an endpoint of every arc lying in M and containing P.

THEOREM. *In order that a point set should be a simple closed curve it is necessary and sufficient that it should be the sum of two distinct arcs having only their endpoints A and B in common and such that no point of either of them, except A and B is contiguous to any point of the other one except A and B.*

THEOREM. *If A and B are two distinct points of an acyclic continuous curve M there exists only one arc lying in M and having A and B as endpoints.*

Indication of proof. By Theorems 38 and 39 there exists an arc α lying wholly in M and with endpoints at A and B. Suppose there exists another arc β lying in M and with the same endpoints. One of these arcs contains a point not belonging to the other one. Suppose β contains a point P not belonging to α. There are several cases.

Case 1. Suppose P is contiguous both to A and to B. Then, by Axiom D, A is not contiguous to B. Let $C(P)$ denote the set of all points of M that are contiguous to P. By Axiom C, $P+C(P)-A$ is closed. Hence there is a point O which is the first point of $P+C(P)-A$ in the order from A to B on the arc α. Let AO denote the interval of α whose endpoints are A and O. The point sets AO and $A+P+O$ are arcs having only their endpoints in common and no point of either of them except A and O is contiguous to any point of the other one with the same exceptions. Hence their sum is a simple closed curve lying in M, contrary to the hypothesis that M is acyclic.

Case 2. Suppose P is contiguous to no point of the arc α. Let T denote the closed point set consisting of α and all points that are contiguous to some point of it. Let A'

denote the first point in the order from P to A that the interval PA of β has in common with T and let B' denote the first one, in the order from P to B, that PB has in common with T. Let A'' denote A' or a point of α contiguous to A' according as A' does or does not belong to α and let B'' be defined in a similar manner with respect to B'. Let PA' and PB' denote intervals of β with endpoints as indicated. Let $A''B''$ denote the point A'' or the interval of α from A'' to B'' according as A'' is or is not identical with B''. The point set $PA'+PB'+A'B'$ is a simple closed curve. Thus in this case also a contradiction is reached.

The remaining cases may be treated with the use of largely similar methods.

Theorems 26–34 of Chapter II of P. S. T. all hold true here though, as may be surmised from the above argument, the proofs given there are not in all cases sufficient here. This group includes a number of theorems concerning acyclic continuous curves. Theorems 43 and 44 of that chapter also relate to such curves. The condition of Theorem 43 is sufficient but not necessary here and that of 44 is necessary but not sufficient.