X

COMPLEMENTS AND APPLICATIONS OF THE THEOREMS OF HADAMARD AND HURWITZ

45. We have seen that if $f(x)$, $\phi(x)$ have isolated singularities α, β, respectively, the functions remaining uniform about these points, we can write

$$f(x) = \sum_{i=1}^{n} f_i(x) + F_n(x),$$

$f_i(x)$ having only α_i as a singularity, and $F_n(x)$ having no singularities within a circle C_R of arbitrary radius R, if n is sufficiently large. Similarly

$$\phi(x) = \sum_{i=1}^{m} \phi_i(x) + \Phi_m(x),$$

with a corresponding statement for C_{R_1}. We have also seen that

$$H(f, \phi) = \sum_{i,j=1,1}^{n,m} H(f_i, \phi_j)$$

$$+ \sum_{j=1}^{m} H(F_n, \phi_j) + \sum_{i=1}^{n} H(\Phi_m, f_i). \qquad (1)$$

Denote by r, r_1 the radii of convergence of the series for f, ϕ, respectively. None of the series

$$\sum_{j=1}^{m} H(F_n, \phi_j)$$

has a singularity in the circle of radius Rr_1, and none of the series

$$\sum_{i=1}^{n} H(\Phi_m, f_i)$$

has a singularity in the circle of radius R_1r. Let R' be arbitrarily large. Since R is arbitrary, there exists for each R' a number R such that all the singularities of $H(f,\phi)$ in the circle of radius R' occur in

$$\sum_{i,j=1,1}^{n,m} H(f_i, \phi_j).$$

In fact we need only take R,R_1 so that Rr_1 and R_1r exceed R'. Then in the circle $C_{R'}$ are singularities only of the form $\gamma_{ij} = \alpha_i\beta_j$. On the assumption that γ_{ij} can be obtained in only one manner, it will be a singularity of only one function $H(f_i,\phi_j)$. The remaining functions in (1) will be regular in γ_{ij}. That is, $H(f_i, \phi_j)$ is the principal part of the singularity of $H(f, \phi)$ in γ_{ij}. But f_i, ϕ_j are the principal parts of the singularities of f in α_i and ϕ in β_j, respectively. Hence Borel's remark that the kind of singularity of $H(f,\phi)$ in $\gamma_{ij} = \alpha_i\beta_j$ depends only on the kind of singularity of f in α_i and that of ϕ in β_j. We have verified the statement for isolated non-critical points; the remark applies, however, in certain cases to singularities of other kinds. With Borel, we make precise this remark in the following theorem:

46. THEOREM 1: *If γ_{ij} is obtainable in only one way, and if α_i is a pole of order q of f, and β_j is a pole of order p of ϕ, then γ_{ij} is necessarily a singularity of $H(f,\phi)$, and is a pole of order $p + q - 1$.*

Because of the preceding remarks, it suffices to give the proof for the case in which the point 1 is a pole of order q of f, and of order p of ϕ, the functions f and ϕ being regular elsewhere. In order to justify the assumption that the poles are situated at the point 1, suppose α_0, β_0 are the respective poles of f and ϕ. Then the series

$$\sum a_n\alpha_0^n x^n, \quad \sum b_n\beta_0^n x^n$$

have poles at the point 1. If we can show that the series

$$\sum a_n b_n (\alpha_0\beta_0)^n x^n$$

has a pole of order $p + q - 1$ at the point 1, it will follow that

$$\sum a_n b_n x^n$$

has a pole of order $p + q - 1$ at the point $\alpha_0\beta_0$.

We may write

$$f(x) = A_0 + \frac{A_1}{1 - x} + \cdots + \frac{A_q}{(1 - x)^q},$$

$$\phi(x) = B_0 + \frac{B_1}{1 - x} + \cdots + \frac{B_p}{(1 - x)^p}.$$

Now

$$\frac{A_k}{(1 - x)^k} = \sum_{n=0}^{\infty} \frac{A_k}{(k - 1)\,!}(n + k - 1)(n + k - 2)\cdots(n + 1)x^n$$

$$= \sum Q_{k-1}(n)x^n$$

where $Q_0(n) = A_0$, $Q_{k-1}(n) =$

$$\frac{A_k}{(k - 1)\,!}\,(n + k - 1)\cdots(n + 1).$$

Then $f(x) = \sum U_{q-1}(n)x^n$, and similarly, $\phi(x) = \sum V_{p-1}(n)x^n$, writing $U_{q-1}(n) = \sum_{k=1}^{q} Q_{k-1}(n)$, $V_{p-1} = \sum_{k=1}^{p} P_{k-1}(n)$.

Hence, $\quad H(f, \phi) = \sum R_{p+q-2}(n)x^n$,

where $\quad R_{p+q-2} = U_{q-1}\,V_{p-1}$

Let $\quad R_{p+q-2} = c_0 + c_1 n + \cdots + c_{p+q-2}(n)^{p+q-2}$

Then $\sum R_{p+q-2}(n)x^n = \sum (c_0 + \cdots + c_{p+q-2}n^{p+q-2})x^n$

$$= c_0 \sum x^n + c_1 \sum nx^n + \cdots + c_{p+q-2} \sum n^{p+q-2}x^n$$

Each of the functions in these series $\Sigma n^k x^n$ has a pole of order $k+1$ in the point 1. Consequently $H(f,\phi)$ has in the point 1 a pole of order $p+q-1$.

47. THEOREM 2: *If γ_{ij} is determined in only one way, and if α_i, β_j are essential singularities of f,ϕ, respectively, then γ_{ij} is an essential singularity of $H(f,\phi)$.*

Assume that the point 1 is an essential singularity of f and ϕ, and that these functions have no other singularities. If

$$f(x) = \sum a_n x^n, \quad \phi(x) = \sum b_n x^n,$$

then, from Faber's theorem, $a_n = g(n)$, where $g(n)$ is an integral function such that

$$|\, g(z) \,| < e^{\epsilon r} \text{ for } r > r_\epsilon. \quad r = |\, z \,|.$$

Similarly $b_n = g_1(n)$, where $|\, g_1(z) \,| < e^{\epsilon r}$ for $r > r'_\epsilon$. Hence $G = gg_1$, an integral function (not a polynomial, since g, g_1 are not polynomials) such that

$$|\, G(z) \,| < e^{\epsilon r}, \quad r > r''_\epsilon,$$

r''_ϵ being the greater of $r_{\frac{\epsilon}{2}}$ and $r'_{\frac{\epsilon}{2}}$.

By Faber's theorem, the series

$$\sum G(n)x^n = \sum a_n b_n x^n = H(f,\phi)$$

has the point 1 as its only singularity, and from the remarks in § 42 it follows that this point is surely singular for H, and likewise essential.

It is to be noted that the point 1 is an essential singularity of $H(f,\phi)$ if this point is an essential singularity merely of one of the functions f, ϕ, provided that for the other function it is an isolated, non-critical singular point.[1]

In general, if the α, β are not isolated, it may happen, as shown by an example of Soula,[2] that even if γ_{ij} is determined

[1] Faber, Jahresb. der deutsche Math. Verein., bd. xvi (1907), p. 285.

[2] Jour. de Math. pures et appliquées, t. 86 (1921), p. 197.

in only one way, none of the points γ is a singularity of $H(f,\phi)$. Another example is the following. Suppose

$$f(x) = \sum a_{\lambda_n} x^{\lambda_n}, \quad \overline{\lim_{n\to\infty}}\, (\lambda_{n+1} - \lambda_n) = \infty,$$

the series having the point 1 as its only singularity. The existence of such a series was proved in Chapter V. Let $\{\lambda'_n\}$ denote a sequence chosen from the sequence complementary to $\{\lambda_n\}$ and having the property that

$$\lim_{n\to\infty} (\lambda'_{n+1} - \lambda'_n) = \infty.$$

Form the function

$$\phi(x) = \sum_{n=0}^{\infty} b_{\lambda'_n} x^{\lambda'_n},$$

the series having unit radius of convergence. Then $H(f,\phi) = 0$. By Hadamard's theorem, the series for $\phi(x)$ has its circle of convergence as a cut. Its singularities therefore consist of all points whose absolute value is at least 1. Since $\alpha = 1$, we have $|\alpha\beta| = |\beta| \geq 1$. But none of the points $\alpha\beta$ is a singularity of $H(f,\phi)$ which vanishes identically.

48. We now show that the theorem of Hurwitz may be applied to give the angle of at least one singular point on the circle of convergence.

Theorem 3 (Mandelbrojt): *Let $g(z)$ be an integral function such that*

$$|g(z)| < e^{\epsilon r} \text{ for } r > r_\epsilon,\ r = |z|.$$

Let $f(x) = \Sigma a_n x^n$, a series with unit radius of convergence, and having all its singularities on the circle of convergence. Let

$$\gamma_n = a_1 g(n+1) - C_n^1 a_2 g(n) + \cdots + (-1)^n a_{n+1} g(1),$$

$$k = \overline{\lim}\sqrt[n]{|\gamma_n|}.$$

Then one of the angles $\cos^{-1}\left(1 - \dfrac{k^2}{2}\right)$ is the argument of at least one singularity of $f(x)$.

Proof: Consider the series

$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{z^{n+1}}, \tag{2}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n g(n+1)}{z^{n+1}}. \tag{3}$$

Denote by α the singularities of $f(x)$. Then $|\alpha| = 1$. The series (2) has singularities only of the form $\alpha' = \frac{1}{\alpha}$. The series (3) has the point -1 as its only singularity, since, by Faber's theorem, the point 1 is the only singularity of $\Sigma g(n+1)x^{n+1}$. The operation of Hurwitz applied to (2) and (3) gives

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{z^{n+1}}. \tag{4}$$

Every singularity γ of (4) is of the form $\alpha' - 1$. The radius of convergence of (4) is given by

$$k = \overline{\lim} \sqrt[n]{|\gamma_n|}.$$

Hence there exists a singularity of (4), say $ke^{i\phi_0}$, $0 \leq \psi_0 \leq 2\pi$, and α_0' such that

$$\begin{aligned} ke^{i\psi_0} &= \alpha_0' - 1 \\ &= e^{i\phi_0} - 1, \quad \alpha_0' = e^{i\phi_0}. \end{aligned}$$

That is,

$$k(\cos\psi_0 + i\sin\psi_0) = \cos\phi_0 - 1 + i\sin\phi_0.$$

Equating the real and the imaginary parts, we obtain

$$\cos\phi_0 = 1 - \frac{k^2}{2}.$$

At least one of the ϕ_0 thus determined is the angle of a singularity α' of (2). But since $\alpha = \frac{1}{\alpha'}$, the same is true for $f(x)$.

By means of this theorem it is possible to determine the positions of the singularities of certain functions all of whose singularities are on the circle of convergence.