VII

ORDER OF SINGULARITIES[1]

32. The fractional derivative $D_x^\alpha f(x)$ is defined as follows:

$$D_x^0 f(x) = f(x),$$

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_k^x (x - z)^{-1-\alpha} f(z) dz, \alpha < 0,$$

$$D_x^\alpha f(x) = \frac{d^p}{dx^p} D_x^{\alpha-p} f(x), \quad \alpha > 0,$$

where p is the smallest integer not exceeded by α.

The derivative thus defined for $\alpha < 0$ differs from the function

$$\frac{1}{\Gamma(-\alpha)} \int_0^x (x - z)^{-1-\alpha} f(z) dz$$

by a function which is regular except perhaps at 0 and k.

The Taylor expansion of $D_x^\alpha f(x)$ is obtained from the second of the above formulas. The series

$$(x - z)^{-1-\alpha} f(z) = (x - z)^{-1-\alpha} \sum a_n z^n$$

converges uniformly along the curve $C_{0,x}$;[2] integrating termwise, we have

[1] Hadamard, *loc. cit.* See pp. 154 ff.

[2] Provided $\alpha \leq -1$; if $-1 < \alpha < 0$, the termwise integration can be justified by a theorem of Osgood: *Lehrbuch der Funktionentheorie*, Leipzig (1912), vol. 1, p. 593; the S_{n_m} of that theorem is taken as

$$\int_0^{x-\frac{1}{n}} (x - z)^{-1-\alpha} \sum_{k=1}^{m} a_k z^k dz.$$ [EDITOR]

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-z)^{-1-\alpha} \sum_{n=1}^{\infty} a_n z^n dz$$
$$= \frac{1}{\Gamma(-\alpha)} \sum_{n=1}^{\infty} a_n \int_0^x (x-z)^{-1-\alpha} z^n dz.$$

If we make the substitution $z = tx$, the integral

$$\int_0^x (x-z)^{-1-\alpha} z^n dz$$

becomes

$$x^{-\alpha} x^n \int_0^1 (1-t)^{-1-\alpha} t^n dt$$
$$= x^{-\alpha} x^n B(-\alpha, n+1)$$
$$= x^{-\alpha} x^n \frac{\Gamma(-\alpha)\,\Gamma(n+1)}{\Gamma(n+1-\alpha)},$$

from which we obtain

$$D_x^\alpha f(x) = x^{-\alpha} \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n x^n, \quad \alpha < 0.$$

If α is positive,

$$D_x^\alpha f(x) = \frac{d^p}{dx^p} \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha+p)} a_n x^{n-\alpha+p}$$
$$= \sum \frac{[\Gamma(n+1)](n+p-\alpha)(n+p-\alpha-1)\cdots(n-\alpha+1)}{\Gamma(n+1-\alpha+p)} a_n x^{n-\alpha}$$
$$= x^{-\alpha} \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n x^n.$$

33. The Hadamard operator H_x^α is defined as follows:

$$H_x^0 f(x) = f(x)$$

$$H_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_k^x (\log x - \log z)^{-1-\alpha} f(z)\, d\log z, \quad \alpha < 0,$$

$$H_x^\alpha f(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^p H_x^{\alpha-p} f(x), \quad \alpha > 0,$$

where p is defined as above.

If α is negative,

$$H_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)}\sum_{n=1}^{\infty} a_n \int_0^x \left(\log\frac{x}{z}\right)^{-1-\alpha} z^{n-1}dz.$$

In the integral

$$\int_0^x \left(\log\frac{x}{z}\right)^{-1-\alpha} z^{n-1}dz$$

place $z = tx$. The integral becomes

$$x^n \int_0^1 \left(\log\frac{1}{t}\right)^{-1-\alpha} t^{n-1}dt$$

$$= n^\alpha x^n\,\Gamma(-\alpha).^1$$

We have, then,

$$H_x^\alpha f(x) = \sum_{n=1}^{\infty} a_n n^\alpha x^n.$$

For $\alpha > 0$,

$$H_x^\alpha f(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^p H_x^{\alpha-p} f(x)$$

$$= \left(\frac{1}{x}\frac{d}{dx}\right)^p \sum_{n=1}^{\infty} a_n n^{\alpha-p} x^n$$

$$= \sum_{n=1}^{\infty} a_n n^\alpha x^n.$$

34. Let $f(x)$ be a function, real or complex, of the real variable x, $a \leq x \leq b$. If

$$\left| v\int_{a'}^{b'} f(x)\cos vx\,dx \right| < I, \left| v\int_{a'}^{b'} f(x)\sin vx\,dx \right| < I,$$

[1] For $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. Let $v = e^{-\frac{x}{m}}$, $x = m\log\frac{1}{v}$. Then $\Gamma(\alpha) = m^\alpha \int_0^1 \left(\log\frac{1}{v}\right)^{\alpha-1} v^{m-1}dv.$ [Editor]

for all v, where $a \leq a' < b' \leq b$, and where I is independent of a', b' and v, then $f(x)$ is said to be *of finite deviation* [1] in (a, b).

THEOREM 1: *A function $f(x)$ of limited variation in (a, b) is of finite deviation in (a, b).*

Let $Q = v\int_{a'}^{b'} f(x) \cos vx\, dx = \int_{a'}^{b'} f(x) d \sin vx$.

For a given v, divide (a', b') into subintervals by the points $x = \frac{k\pi}{v}$, using such integral values of k as give points in (a', b').

Consider a subinterval $\left(\frac{k}{v}\pi, \frac{k+1}{v}\pi\right)$, interior to (a', b'). Denote by M_k, m_k, respectively, the upper and the lower bound of $f(x)$ in this interval, and let $p_k = M_k - m_k$. Then, in this interval,

$$f(x) = m_k + \theta(x) p_k, \ 0 \leq \theta \leq 1.$$

$$\int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} f(x) d \sin vx = \int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} m_k d \sin vx + \int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} \theta p_k d \sin vx.$$

The first integral on the right vanishes. Moreover,

$$\left| \int_{\frac{k}{v}\pi}^{\frac{k+1}{v}\pi} \theta\, p_k d \sin\ vx \right| < 2\, p_k.$$

Consider the left-hand interval, $\left(a', \frac{k_0}{v}\pi\right)$. If η and ϵ are positive constants, the latter being arbitrarily small, we have

$$\left| \int_{a'}^{\frac{k_0}{v}\pi} f(x)\, d \sin vx \right| \leq \int_{\frac{k_0-1}{v}\pi}^{\frac{k_0}{v}\pi} [\,|f(a')\,| + \eta] \left| d \sin vx \right|$$
$$= 2\,|\,f(a')\,| + 2\,\eta$$
$$< 2\,|\,f(a')\,| + \epsilon,$$

[1] Hadamard uses the term *écart fini.*

the last inequality being valid for $v > v_0$, since η can approach zero as n becomes infinite. Noting that a similar result is obtained for the right-hand interval we have, finally,

$$|Q| < 2\sum_{i=1}^{r} p_i + 2|f(a')| + 2|f(b')| + 2\,\epsilon,$$

where r is the number of subintervals interior to (a', b'). By hypothesis, Σp_i is bounded for all v, hence the same is true of Q.

A similar process, choosing properly the points of division of (a', b'), shows that the second integral appearing in the definition is also uniformly bounded.[1]

The converse is not true; in particular, an example has been given by H. E. Bray, namely, the function $x \sin \frac{1}{x}$. This function, which is not of limited variation, is of finite deviation. The proof of this statement depends on an investigation of the integrals:

$$\int_c^d \frac{\sin t}{\sqrt{t^2 - 4\,n}}\,dt, \int_c^d \frac{\cos t}{\sqrt{t^2 - 4\,n}}\,dt, \quad 2\sqrt{n} \leq c < d,$$

$$\int_c^d \frac{\sin t}{\sqrt{t^2 + 4\,n}}\,dt, \int_c^d \frac{\cos t}{\sqrt{t^2 + 4\,n}}\,dt.$$

In particular, every function which satisfies the Lipschitz condition is of finite deviation, since every function having a bounded derivative is of limited variation.

[1] A simpler proof of this theorem can be given on the basis of Stieltjes's theorem on integration by parts: *If $\phi(x)$ is continuous, and $\alpha(x)$ of limited variation, $a \leq x \leq b$, then*

$$\int_a^b \alpha(x)\,d\phi(x) = \alpha(x)\,\phi(x)\Big|_a^b - \int_a^b \phi(x)\,d\alpha(x).$$ [EDITOR]

35. Theorem 2: *Let* $f(x) = \Sigma a_n x^n$, *with* $R = 1$. *If* $\Sigma |a_n|$ *converges, and if* $\lim_{n\to\infty} n\,|a_n| \log n = 0$, *then* $f(x) = f(e^{i\phi}) = F(\phi)$ *is continuous and of finite deviation on the unit circle.*

The continuity of $F(\phi)$ is evident, since $\Sigma a_n e^{ni\phi}$ is dominated by the convergent series $\Sigma |a_n|$, and is accordingly uniformly convergent in every interval.

In order to show that I exists, independent of v, such that

$$\left| v \int_{a'}^{b'} F(\phi) \cos v\phi \, d\phi \right| < I, \quad \left| v \int_{a'}^{b'} F(\phi) \sin v\phi \, d\phi \right| < I,$$

it suffices to show that

$$J \equiv \left| v \int_{a'}^{b'} F(\phi) e^{vi\phi} d\phi \right| < I_0,$$

$$L \equiv \left| v \int_{a'}^{b'} F(\phi) e^{-vi\phi} d\phi \right| < I_0.$$

For a given v, we may write

$$J = \left| v \sum_{m=0}^{\infty} a_m \int_{a'}^{b'} e^{(m+v)i\phi} d\phi \right|$$

$$= \left| \sum_{m=0}^{\infty} \int_{a'}^{b'} a_m \frac{v}{m+v} [e^{(m+v)ib'} - e^{(m+v)ia'}] \right|$$

$$\leq \sum_{m=0}^{\infty} |a_m| \frac{v}{m+v} \left| e^{(m+v)ib'} - e^{(m+v)ia'} \right|$$

$$\leq 2 \sum_{m=0}^{\infty} |a_m|$$

$$= 2\,S,$$

and consequently J is uniformly bounded.

$$L = \left| v \sum_{m=0}^{\infty} a_m \int_{a'}^{b'} e^{(m-v)i\phi} d\phi \right| \tag{1}$$

$$= \left| \sum_{m=0}^{\infty} a_m \frac{v}{m - v} [e^{(m-v)ib'} - e^{(m-v)ia'}] \right|,$$

if v is not an integer; otherwise

$$L = \left| \sum_{m=0}^{\infty}{}' a_m \frac{v}{m - v} [e^{(m-v)ib'} - e^{(m-v)ia'}] + va_v(b' - a') \right|,$$

where the prime indicates that the $(v + 1)$-th term is omitted in the summation.

In the first case,

$$L \leq \sum_{m=0}^{\infty} | a_m | \frac{2 v,}{| m - v |},$$

and in the second case,

$$L \leq \sum_{m=0}^{\infty}{}' | a_m | \frac{2 v}{| m - v |} + 2 \pi | a_v |$$

Choose k and K so that $0 < k < 1 < K$. Consider, in (1), all the subscripts m for which $m \leq kv$. Each corresponding term is less than $\frac{2 | a_m |}{1 - k}$, so that

$$\sum_{m \leq kv} | a_m | \frac{2 v}{| m - v |} < \frac{2 S}{1 - k}.$$

Similarly,

$$\sum_{m \geq Kv} | a_m | \frac{2 v}{| m - v |} < \frac{2 S}{K - 1}.$$

Take next all the m's for which $kv < m < v$.

$$\sum_{kv<m<v} | a_m | \frac{2 v}{| m - v |} \leq 2 v | a_l | \left(1 + \frac{1}{2} + \cdots + \frac{1}{h}\right), \tag{2}$$

where a_l is the coefficient of greatest absolute value for $kv < m \leq Ev$, and h the number of such integers m.

Since l is one of the indices occurring in the sum, the inequalities

$$k < \frac{l}{\upsilon} \leq 1$$

are verified. Hence

$$2\,\upsilon\,|\,a_l\,|\left(1+\frac{1}{2}+\cdots+\frac{1}{h}\right) < \frac{2\,l}{k}\,|\,a_l\,|\left(1+\frac{1}{2}+\cdots+\frac{1}{h}\right).$$

But we have

$$\begin{aligned} h &= E\,[(1-k)\upsilon]-1 \\ &= (1-k)\upsilon - 1 - \epsilon_\upsilon, \quad 0 \leq \epsilon_\upsilon < 1, \\ &= \left(1-k-\frac{1+\epsilon_\upsilon}{\upsilon}\right)\upsilon \\ &= \rho_\upsilon \upsilon, \end{aligned}$$

where $$\rho_\upsilon = 1 - k - \frac{1+\epsilon_\upsilon}{\upsilon}.$$

Moreover,

$$\lim_{\upsilon\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{\upsilon\rho_\upsilon}}{\log\,(\upsilon\rho_\upsilon)} = 1, \tag{3}$$

since $\log \upsilon\rho_\upsilon = \log \upsilon + \log \rho_\upsilon$, and $\lim_{\upsilon\to\infty} \rho_\upsilon = 1 - k \neq 0$, so that $h = \upsilon\rho_\upsilon$ becomes infinite with υ.

On the other hand,

$$1 \leq \frac{\upsilon}{l} < \frac{1}{k} = \delta,$$

so that

$$0 \leq \log \upsilon - \log l < \log \delta.$$

Hence, for all υ, we have

$$|\log \upsilon - \log l| < M, \tag{4}$$

and we may therefore replace $\log v\rho_v$ in (3) by $\log l$:

$$\lim_{v\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{h}}{\log l} = 1.$$

Now, given ϵ, there exists a number v_ϵ such that for $v > v_\epsilon$ we have, from (2),

$$\sum_{kv<m<v} |a_m| \frac{2v}{|m-v|} < \frac{2l}{k}|a_l|(1+\epsilon)\log l. \qquad (5)$$

By hypothesis $l\,|a_l|\,\log l$ approaches 0 as l becomes infinite, hence as v becomes infinite, because of (4). And since the right hand side of (5) is bounded for $v < v_\epsilon$, having only a finite number of terms, it follows that this expression is bounded for all v.

There exists, by hypothesis, a number L_0 such that $L_0\,|a_{L_0}|\log L_0$ exceeds any other $l\,|a_l|\log l$. We may therefore write

$$\sum_{kv<m<v} |a_m| \frac{2v}{|m-v|} < \mu L_0 |a_{L_0}| \log L_0,$$

where μ is a certain constant.

In a similar way it may be shown that

$$\sum_{1+Ev<m<Kv} |a_m| \frac{2v}{|m-v|} < \mu' L_0 |a_{L_0}| \log L_0.$$

In case v is an integer there remains to be examined only the term $va_v(b'-a')$. This expression is bounded, for it approaches zero as v becomes infinite.

If v is not an integer, consider

$$a_{Ev}\frac{v}{Ev-v}\left[e^{(Ev-v)ib'} - e^{(Ev-v)ia'}\right] \qquad (6)$$

and

$$a_{Ev+1}\frac{v}{Ev+1-v}\left[e^{(Ev+1-v)ib'} - e^{(Ev+1-v)ia'}\right]. \qquad (7)$$

We have

$$\left|e^{(Ev-v)ib'} - e^{(Ev-v)ia'}\right| = \left|2\,ie^{(Ev-v)i\frac{a'+b'}{2}}\sin\frac{(Ev-v)(b'-a')}{2}\right|$$
$$< (Ev-v)(a'+b').$$

Hence the expression (6) is, in absolute value, less than

$$|a_{Ev}|\,v(b'+a') \leq 2\,\pi v\,|a_{Ev}|$$

if we take $b'+a' \leq 2\,\pi$. The last expression approaches zero as v becomes infinite.

Similarly the quantity $2\,\pi v\,|a_{Ev+1}|$ is an upper bound for the expression (7).

Combining the foregoing results, we obtain

$$L < \frac{2\,S}{1-k} + \frac{2\,S}{K-1} + \mu L_0|a_{L_0}|\log L_0 + \mu' L_0|a_{L_0}|\log L_0 + 2M, \quad (8)$$

where M exceeds the largest of the numbers of the form $v\,|\,a_v\,|\,(b'-a')$, $2\,\pi v\,|\,a_{E_v}\,|$, $2\,\pi v\,|\,a_{E_{v+1}}\,|$. Since each of these expressions approaches zero as v becomes infinite, the number M exists.

COROLLARY: *The deviation of $f(x)$ satisfies the inequality*

$$L < k_1 S + k_2 L_0|a_{L_0}|\log L_0 + k_3 l_0|a_{l_0}|, \quad (8')$$

where k_1, k_2, k_3 are numbers not depending on the particular series, and $l_0\,|\,a_{l_0}\,|$ is the maximum value of $k\,|\,a_k\,|$ for all k, the numbers S and L_0 having their previous significance.

On the other hand, we have also the following theorem:

THEOREM 3: *Given $f(x) = \Sigma a_n x^n$, $R = 1$. If $f(e^{i\phi}) = F(\phi)$ is continuous and of finite deviation on the unit circle, then for every $\epsilon > 0$,*

(*a*) $\sum \frac{|a_n|}{n^\epsilon}$ *converges, and*

(*b*) $\lim_{n\to\infty} n^{1-\epsilon}|a_n|\log n = 0.$[1]

[1] Or, what is the same thing,

$$\lim_{n\to\infty} n^{1-\epsilon}|a_n| = 0 \text{ for every } \epsilon > 0.$$ [EDITOR]

Construct a circle C_r with center at the origin and radius $r < 1$. Then

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(x)dx}{x^{n+1}}.$$

On the other hand, since

$$\lim_{r \to 1} f(re^{i\phi}) = f(e^{i\phi})$$

uniformly for all ϕ, we have

$$\lim_{r \to 1} \int_{C_r} \frac{f(x)}{x^{n+1}} dx = \int_{C} \frac{f(x)}{x_{n+1}} dx,$$

where C is the unit circle. Consequently

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) e^{-ni\phi} d\phi.$$

Since $f(e^{i\phi})$ is of finite deviation, L_0 exists such that

$$\left| n \int_0^{2\pi} f(e^{i\phi})\, e^{-ni\phi} d\phi \right| < L_0,$$

and therefore

$$n \mid a_n \mid < L, \qquad L = \frac{L_0}{2\pi},$$

$$\frac{\mid a_n \mid}{n^{\epsilon}} < \frac{L}{n^{1+\epsilon}},$$

so that $$\sum \frac{\mid a_n \mid}{n_{\epsilon}} \text{ converges.}$$

In order to prove the second part of the theorem, we need only note that

$$n \frac{\mid a_n \mid}{n^{\epsilon}} \log n < \frac{L \log n}{n^{\epsilon}},$$

and that the limit of the right-hand side is zero.

36. The *order* of $f(x)$ on an arc of the unit circle is the number ω such that $H^{-(\omega+\epsilon)} f(e^{i\phi})$ is continuous and of finite deviation on the arc, whereas, for $H^{-(\omega-\epsilon)} f(e^{i\phi})$, at least one of these properties fails, that is, $H^{-(\omega-\epsilon)} f$ is either discontinuous or not of finite deviation, or both discontinuous and not of finite deviation.

The order of $f(e^{i\phi})$ on any arc, $a \leq \phi \leq b$, of the unit circle is clearly not greater than the order of $f(e^{i\phi})$ in an arc containing (a, b). The *order of $f(e^{i\phi})$ in the point* ϕ_0 is defined as the lower bound of the orders of $f(e^{i\phi})$ in all the intervals containing ϕ_0. If $f(e^{i\phi})$ is of order ω on the circumference, there is at least one point on the circumference at which the order is ω; there is no point on the circumference at which the order exceeds ω.

THEOREM 4: *The order of $f(x)$ on the circle of convergence is given by*

$$\omega = \overline{\lim_{n\to\infty}} \frac{\log |a_n|}{\log n} + 1\cdot$$

We have, by definition, that for all except a finite number of values of n,

$$1 + \frac{\log |a_n|}{\log n^1} = \frac{\log (n |a_n|)}{\log n} < \omega + \epsilon,$$

with $\epsilon > 0$ arbitrary. Hence

$$n |a_n| < n^{\omega+\epsilon},$$

$$\frac{|a_n|}{n^{\omega+2\epsilon}} < \frac{1}{n^{1+\epsilon}}.$$

The series $\sum \frac{|a_n|}{n^{\omega+2\epsilon}}$ accordingly converges.

Moreover, from the preceding inequality, we have, for n sufficiently large,

$$\frac{n |a_n|}{n^{\omega+2\epsilon}} \log n < \frac{1}{n^{\frac{\epsilon}{2}}},$$

$$\lim_{n\to\infty} \frac{n |a_n|}{n^{\omega+2\epsilon}} \log n = 0.$$

Hence the series $\sum b_n x^n$, where $b_n = \frac{a_n}{n^{\omega+2\epsilon}}$, satisfies the hypothesis of Theorem 2. Consequently $H^{-(\omega+2\epsilon)} f(e^{i\phi})$ is continuous and of finite deviation on the unit circle.

On the other hand, there is an infinity of values of n such that

$$\frac{\log (n \mid a_n \mid)}{\log n} > \omega - \epsilon,$$

$$\frac{\mid a_n \mid}{n^{\omega-\epsilon}} > \frac{1}{n},$$

$$\frac{n \mid c_n \mid}{n^{\epsilon}} \log n > \log n,$$

where $c_n = \frac{a_n}{n^{\omega-2\epsilon}}$. The last inequality shows that $H^{-(\omega-2\epsilon)}f(x) = \Sigma c_n x^n$ is not both continuous and of finite deviation; otherwise we should have, by Theorem 3,

$$\overline{\lim_{n\to\infty}}\, n^{1-\epsilon} \mid c_n \mid \log n = 0.$$

We have seen that the expression

$$\overline{\lim_{n\to\infty}} \frac{\log \mid a_n \mid}{n} = -\log R$$

gives the absolute value of at least one singular point. It is natural to inquire whether similar formulas yield information regarding the kind of singularities on the circle of convergence. The answer is affirmative. In fact, the preceding theorem shows that the value of the expression

$$\overline{\lim_{n\to\infty}} \frac{\log \mid a_n \mid}{a_n}$$

determines the order of the function on its circle of convergence.

37. We have often employed the following fact: If x_0 is an isolated essential singularity of the function $f(x)$ about which the function remains uniform, then we may write

$$f(x) = f_1(x) + f_2(x),$$

where x_0 is the only singularity of $f_1(x)$, and $f_2(x)$ is holomorphic in x_0. We may state a similar fact for the separation of the arcs on which the function has a given order.

THEOREM 5: *Let $f(x)$ be a function having the order ω on the closed arc Γ of the unit circle (of convergence). Then, given $\epsilon > 0$, we may write*

$$f(x) = f_1(x) + f_2(x),$$

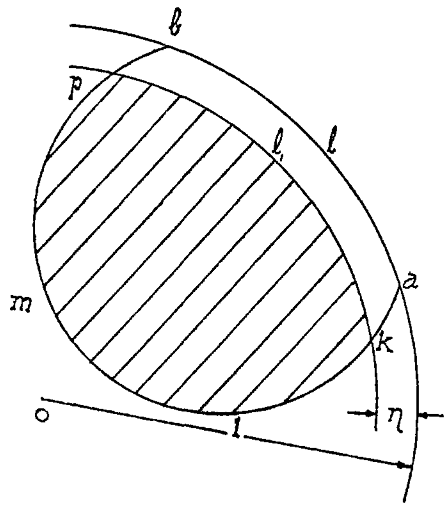
where $f_1(x)$ is of order not greater than $\omega + \epsilon$ on the entire unit circle, and $f_2(x)$ is holomorphic on Γ, except perhaps at a and b.

By hypothesis, the function

$$\phi(x) = H_x^{-\omega-\epsilon} f(x)$$

is continuous and of finite deviation on Γ.

In the figure, Γ is the arc ab. Draw a circle passing through a and b, and having its center within the unit circle. With O as center, draw a circle of radius $1 - \eta$. Let T denote the shaded region.



For every point of T, we may write $\phi(x) = \phi_1(x) + \phi_2(x)$, where

$$\phi_1(x) = \frac{1}{2\pi i}\int_{klp} \frac{\phi(z)}{z-x}\,dz,$$

$$\phi_2(x) = \frac{1}{2\pi i}\int_{pmk} \frac{\phi(z)}{z-x}\,dz.$$

If we let η approach zero, $\phi_1(x)$ and $\phi_2(x)$ vary continuously, since $\phi(x)$ is continuous. We may therefore write

$$\phi_1(x) = \frac{1}{2\pi i}\int_{alb} \frac{\phi(z)}{z-x}\,dz,$$

$$\phi_2(x) = \frac{1}{2\pi i}\int_{bma} \frac{\phi(z)}{z-x}\,dz.$$

By writing

$$\frac{1}{z-x} = \frac{1}{z}\,\frac{1}{1-\dfrac{x}{z}} = \frac{1}{z}\left\{1 + \frac{x}{z} + \frac{x^2}{z^2} + \cdots\right\},$$

we have

$$\phi_1(z) = \sum k_n x^n,$$

where

$$k_n = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(x)dx}{x^{n+1}}.$$

Since $\phi(x)$ is of finite deviation on Γ,

$$\left|k_n\right| = \frac{1}{2\pi}\left|\int_\Gamma \phi(x)[\cos(n+1)\theta - i\sin(n+1)\theta]\right.$$
$$\left.[-\sin\theta + i\cos\theta]\,d\theta\right| < \frac{L}{n}, \qquad (9)$$

where $x = \cos\theta + i\sin\theta$, and L is a constant.

On the other hand, we have, by definition of ϕ, ϕ_1, and ϕ_2,

$$\begin{aligned} f(x) &= H_x^{\omega+\epsilon}\phi(x) \\ &= H_x^{\omega+\epsilon}\phi_1(x) + H_x^{\omega+\epsilon}\phi_2(x). \end{aligned} \qquad (10)$$

We may write

$$H_x^{\omega+\epsilon}\phi_1(x) = \sum l_n x^n,$$

where $l_n = k_n n^{\omega+\epsilon}$. Hence, from (9),

$$|l_n| < Ln^{-1+\omega+\epsilon},$$

and the function $H_x^{\omega+\epsilon}\phi_1(x)$ is of order not greater than $\omega + \epsilon$ on Γ, by Theorem 4.

On the other hand, $\phi_2(x)$ is developable in a Taylor's series, since it is the difference of two Taylor's series. Moreover, $\phi_2(x)$ is holomorphic on Γ, except perhaps at a and b, since it is holomorphic in every point which is not on the arc amb. Hence $H_x^{\omega+\epsilon}\phi_2(x)$ is also holomorphic on Γ.

We see therefore from (10) that $f(x)$ is represented in the desired form.

38. We may now prove two theorems which are useful in investigating the behavior of a function in the neighborhood of a singular point.

THEOREM 6: *Suppose $f(x)$ has on the closed arc Γ of the circle of convergence the order ω_1 less than a positive number ω. Let ϕ_1, ϕ_2 be two quantities such that $0 \leq \phi_1 < \phi_2 < 2\pi$, and such that Γ contains the arc $\phi_1\phi_2$ in its interior. Then, as $r \to 1$,*

(*a*) *the quantity*

$$(1 - r)^\omega f(re^{i\phi})$$

approaches zero uniformly with respect to ϕ, provided $\phi_1 \leq \phi \leq \phi_2$; and

(*b*) *the quantity*

$$(1 - r)^\omega I_r,$$

where I_r is the deviation of $f(re^{i\phi})$ on the arc $\phi_1\phi_2$ of the circle of radius r, also approaches zero.

We may assume that $f(x)$ is of order ω on the whole circle, which involves no loss of generality. In fact, from the preceding theorem, we may write $f(x) = f_1(x) + f_2(x)$, where

$f_1(x)$ is of order $< \omega$ on the entire circle, and where $f_2(x)$ is holomorphic on Γ, except at the end points; the theorem is evidently true for $f_2(x)$.

We have, by Theorem 3,

$$\lim_{n\to\infty} \frac{a_n}{n^{\omega-1}} = 0\cdot$$

Now, since

$$\lim_{n\to\infty} \frac{\Gamma(n+\omega)}{\Gamma(n+1)n^{\omega-1}} = 1,\text{[1]}$$

and since

$$\frac{1}{(1-x)^\omega} = \sum d_n x^n,$$

where

$$d_n = \frac{\Gamma(n+\omega)}{\Gamma(\omega)\Gamma(n+1)},$$

it follows that

$$\lim_{n\to\infty} \frac{a_n\Gamma(\omega)\Gamma(n+1)}{\Gamma(n+\omega)} = \lim_{n\to\infty} \frac{a_n}{d^n} = 0.$$

Hence, for an arbitrary $\epsilon > 0$,

$$|\, a_n \,| < \frac{\epsilon}{2}\, d_n, \quad n > n_\epsilon.$$

If we add to $\dfrac{1}{(1-x)^\omega}$ a polynomial,

$$P(x) = \sum_{i=1}^{n_\epsilon} A_i x^i, \quad A_i > |\, a_i \,|,$$

we have, *a fortiori*,

$$|\, a_i \,| < \frac{\epsilon}{2}\, d_i + A_i, \quad i = 1, 2, \cdots, n_\epsilon,$$

$$|\, a_0 + a_1 re^{i\phi} + \cdots + a_{n_\epsilon} r^{n_\epsilon} e^{n_\epsilon i\phi} \,| <$$

$$\frac{\epsilon}{2}\,(d_0 + d_1 r + \cdots + d_{n_\epsilon} r^{n_\epsilon}) + P\,|\,(x)\,|,$$

[1] As may be obtained from the approximation formula:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\, n^n e^{-n}.$$

[Editor]

and

$$| a_{n_\epsilon+1} r^{n_\epsilon+1} e^{(n_\epsilon+1)i\phi} + \cdots | < \frac{\epsilon}{2} d_{n_\epsilon+1} r^{n_\epsilon+1} + \cdots,$$

so that

$$| f(x) | < \frac{\epsilon}{2} \sum_{n=0}^{\infty} d_n r^n + | P(x) |$$

$$= \frac{\epsilon}{2} \frac{1}{(1-r)^\omega} + | P(x) |,$$

$$(1-r)^\omega | f(x) | < \frac{\epsilon}{2} + (1-r)^\omega | P(x) |,$$

where $x = re^{i\phi}$. The first part of the theorem follows at once.

For the proof of the second part of the theorem, it suffices, in view of the Corollary of Theorem 2, to show that

$$\lim_{r \to 1} (1-r)^\omega (k_1 S_r + k_2 L_r | a_{L_r} | \log L_r r^{L_r} + k_3 l_r | a_{l_r} | r^{l_r}) = 0.$$

The reasoning of the first part of the theorem shows that

$$\lim_{r \to 1} (1-r)^\omega S_r = 0.$$

If L_r as a function of r remains bounded, say $< m_0$, for all r such that $1 - \eta < r < 1$, where η is fixed, we have

$$\lim_{r \to 1} (1-r)^\omega L_r | a_{L_r} | \log L_r r^{L_r} = 0. \tag{11}$$

Choose ω' so that $\omega_1 < \omega' < \omega$. From Theorem 3, we have, for $m > m_0$,

$$| a_m | < \frac{m^{\omega'-1}}{\log m}, \tag{12}$$

and, placing $r = 1 - \epsilon$,

$$(1-r)^\omega m \log m | a_m | r^m < m^{\omega'} (1-\epsilon)^m \epsilon^\omega, \tag{13}$$

for $m > m_0$.

Consider (11) for the case in which L_r increases indefinitely as $r \to 1$, and in particular when $L_r > m_0$ as may be the case when r is sufficiently near 1.

We note that the expression $m^{\omega'}(1-\epsilon)^m\epsilon^\omega$ does not increase in passing from $m = n$ to $m = n+1$, provided

$$n > \frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1}.$$

For in order that $n^{\omega'}(1-\epsilon)^n\epsilon^\omega < (n+1)^{\omega'}(1-\epsilon)^{n+1}\epsilon^\omega$, it is necessary and sufficient that

$$\left(\frac{n+1}{n}\right)^{\omega'} > \frac{1}{1-\epsilon},$$

$$n < \frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1}.$$

The expression $m^{\omega'}(1-\epsilon)^m\epsilon^\omega$ will therefore attain its greatest value for $m = m_k$, where m_k, or perhaps $m_k + 1$, is given by the expression

$$E\left[\frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1}\right].$$

But we have

$$\frac{1}{z^\alpha - 1} = \frac{1}{\alpha}\,\frac{z}{z-1} + \theta(z),$$

where $\theta(z)$ is bounded in the neighborhood of $z = 1$, *i.e.*,

$$\left|\frac{1}{z^\alpha - 1} - \frac{1}{\alpha}\,\frac{z}{z-1}\right| < M,$$

for $|z-1| < \eta$, where η is sufficiently small.

Let $z = \frac{1}{1-\epsilon}$, $\alpha = \frac{1}{\omega'}$; then

$$\left|\frac{1}{\left(\frac{1}{1-\epsilon}\right)^{\frac{1}{\omega'}} - 1} - \frac{\omega'}{\epsilon}\right| < M_1$$

when $|\epsilon| < \eta_1$. Hence

$$\left| m_k - \frac{\omega'}{\epsilon} \right| < M_2, \quad |\epsilon| < \eta_1,$$

$$\frac{\omega'}{\epsilon} - M_2 < m_k < \frac{\omega'}{\epsilon} + M_2,$$

$$(1-\epsilon)^{m_k} < (1-\epsilon)^{\frac{\omega'}{\epsilon} - M_2} = (1-\epsilon)^{\frac{\omega'}{\epsilon}} C,$$

where $C = (1-\epsilon)^{-M_2} = \text{const.}$ Also

$$m_k^{\omega'} < \left(\frac{\omega'}{\epsilon} + M_2\right)^{\omega'}$$

$$< \left(\frac{\omega'}{\epsilon}\right)^{\omega'} C_1^{\omega'},$$

where $C_1 = \text{const.} > 1 + \dfrac{M_2\epsilon}{\omega'}$. From these inequalities and (13), we have, for $L_r > m_0$,

$$(1-r)^{\omega} L_r \log L_r |a_{L_r}| r^{L_r} < L_r^{\omega'} (1-\epsilon)^{L_r} \epsilon^{\omega}$$

$$\leq m_k^{\omega'} (1-\epsilon)^{m_k} \epsilon^{\omega}.$$

It suffices therefore to show that

$$\lim_{\epsilon \to 0} (1-\epsilon)^{\frac{\omega'}{\epsilon}} \left(\frac{\omega'}{\epsilon}\right)^{\omega'} \epsilon^{\omega} = 0, \tag{14}$$

from which (11) will follow.

The proof of (14) is immediate. For

$$\lim_{\epsilon \to 0} (1-\epsilon)^{\frac{\omega'}{\epsilon}} = e^{\omega'},$$

and, since $\omega > \omega'$,

$$\lim_{\epsilon \to 0} \epsilon^{\omega - \omega'} = 0.$$

It remains to be proved that

$$\lim_{r \to 1} (1-r)^{\omega} l_r |a_{l_r}| r^{l_r} = 0.$$

This statement is verified if l_r as a function of r remains bounded.

If l_r becomes infinite as r approaches 1, then

$$l_r|a_{l_r}| < l_r|a_{l_r}|\log l_r \leq L_r|a_{L_r}|\log L_r.$$

This completes the proof, since it has been shown that

$$\lim_{r\to 1} (1 - r)^{\omega} L_r|a_{L_r}|\log L_r = 0.$$

THEOREM 7: *If, for* $\omega > 0$,

$$\lim_{r\to 1} (1 - r)^{\omega} f(re^{i\theta}) = 0,$$

uniformly, for $\alpha \leq \theta \leq \beta$, *and*

$$\lim_{r\to 1} (1 - r)^{\omega} I_r = 0,$$

where I_r *is the deviation of* $f(re^{i\theta})$ *on the arc* $(\alpha\beta)$, *the function* $f(x)$ *is of order* $\omega' < \omega$ *on the arc* $(\alpha\beta)$.

In determining the order of $f(x)$ we may consider $x^{-\omega}D_x^{-\omega}f(x)$ instead of $H_x^{-\omega}f(x)$, since if one of these functions is continuous and of finite deviation, so is the other.[1]

The expression

$$x^{-\omega''}D_x^{-\omega''}f(x) = F(x) = \frac{1}{\Gamma(\omega'')}\int_0^1 (1 - t)^{\omega''-1} f(tx)\, dt,$$

where $\omega'' > \omega$, is finite and continuous in the region

$$r \leq 1, \quad \alpha \leq \theta \leq \beta, \quad x = re^{i\theta}.$$

For in this region we have, by hypothesis,

$$\left|f(tx)\right| < \frac{A}{(1 - |tx|)^{\omega}} \leq \frac{A}{|1 - t|^{\omega}}.$$

[1] Hadamard, *loc. cit.*, p. 158.

But since $\omega'' - \omega > 0$,

$$\left|\int_0^1 (1-t)^{\omega''-1} f(tx)\,dt\right| < A\int_0^1 (1-t)^{\omega''-\omega-1}\,dt,$$

so that for the values of x in question,

$$|F(x)| < \frac{A}{\Gamma(\omega'')}\int_0^1 (1-t)^{\omega''-\omega-1}\,dt.$$

Thus $F(x)$ is bounded in (α, β). Moreover it is continuous, since the integral converges uniformly.

By definition, the deviation of $F(x)$ is the upper bound of

$$n\left|\int_\alpha^\beta e^{\pm ni\theta} F(e^{i\theta})\,d\theta\right|$$

$$= \frac{n}{\Gamma(\omega'')}\left|\int_\alpha^\beta e^{\pm ni\theta}\int_0^1 (1-t)^{\omega''-1} f(te^{i\theta})\,dt\right|.$$

This integral may be regarded as a double integral, since the integrand is dominated by $A(1-t)^{\omega''-\omega-1}$. We may therefore reverse the order of integration:

$$\frac{1}{\Gamma(\omega'')}\int_0^1 \left\{(1-t)^{\omega''-1}\int_\alpha^\beta n e^{\pm ni\theta} f(te^{i\theta})\,d\theta\right\} dt.$$

But

$$\left|\int_\alpha^\beta n e^{\pm ni\theta} f(te^{i\theta})\,d\theta\right| < I_t,$$

where I_t is the deviation of $f(te^{i\theta})$ in (α, β). By hypothesis

$$I_t < \frac{B}{(1-t)^\omega},$$

where B is fixed. Consequently

$$\frac{n}{\Gamma(\omega'')}\int_\alpha^\beta e^{\pm ni\theta} F(e^{i\theta})\,d\theta < \frac{B}{\Gamma(\omega'')(\omega''-\omega)},$$

and $F(x)$ is of finite deviation on the arc (α, β). Since $\omega'' > \omega$, the theorem is proved.

It appears from the two preceding theorems that the order of $f(x)$ on (α, β), if positive, may be defined as the number ω such that the quantities $(1 - r)^{\omega+\epsilon} f(re^{i\theta})$ and $(1 - r)^{\omega+\epsilon} I_r$ remain bounded for an arbitrary ϵ as r approaches 1, whereas one or both of these expressions fails to remain bounded if ϵ is replaced by $-\epsilon$.

39. For use in a later chapter, we state without proof a theorem of Fabry.[1]

THEOREM 8: *If, for the series* $\Sigma a_{\lambda_n} x^{\lambda_n}$, *we have*

$$\lambda_{n+1} - \lambda_n > k\sqrt{\lambda_n \log \lambda_n},$$

where k is a positive constant, then every point on the circle of convergence has the same order, namely, the order of the function on the circle of convergence.

[1] Comptes Rendus, t. 151 (1910), p. 922.