IV

MEROMORPHIC FUNCTIONS[1]

16. If, in a region D, the function $f(x)$ defined by the series $\Sigma a_n x^n$ can be represented as

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ has only poles in the finite plane, and $f_2(x)$ is regular in D, then $f(x)$ is said to be *meromorphic* in D.

A series having only poles as singularities on its circle of convergence is said to be meromorphic on the circle of convergence.

We shall obtain necessary and sufficient conditions, due to Hadamard, that a series be meromorphic on the circle of convergence.

Denote by $D_{n,m}$ the symmetric determinant

$$\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+m} & \cdots & \cdots & a_{n+2m} \end{vmatrix},$$

where the a_i are the coefficients of the series $\Sigma a_n x^n$. If R is the radius of convergence, then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m}|} \leq \frac{1}{R^{m+1}}. \quad (1)$$

The proof of this statement depends on two lemmas which we proceed to establish.

[1] For the theorems in this chapter, see Hadamard, Journal de Liouville, ser. 4, t. 8 (1892), p. 101.

LEMMA 1: *Given k sequences of positive numbers,*

$$\alpha_n^{(1)}, \alpha_n^{(2)}, \cdots, \alpha_n^{(k)}.$$

Then

$$\overline{\lim_{n\to\infty}}\left[\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)}\right] \leq \overline{\lim_{n\to\infty}}\,\alpha_n^{(1)}\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(2)}\cdots\overline{\lim_{n\to\infty}}\,\alpha_n^{(k)}.$$

Let $\overline{\lim\limits_{n\to\infty}}\,\alpha_n^{(i)} = \rho_i, \quad i = 1, 2, \cdots, k.$ By definition there exists a number n_0 such that, for $n > n_0$, we have

$$\alpha_n^{(i)} < \rho_i\,(1 + \epsilon),$$

$$\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)} < \rho_1\rho_2\cdots\rho_k(1 + \epsilon)^k, \qquad i = 1, 2, \cdots, k.$$

For an arbitrary $\eta > 0$, we may choose ϵ so small that $(1 + \epsilon)^k < 1 + \eta$. Then

$$\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)} < \left[\overline{\lim_{n\to\infty}}\,\alpha_n^{(1)}\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(2)}\cdots\overline{\lim_{n\to\infty}}\,\alpha_n^{(k)}\right](1 + \eta),$$

and therefore

$$\overline{\lim_{n\to\infty}}\left[\alpha_n^{(1)}\alpha_n^{(2)}\cdots\alpha_n^{(k)}\right] \leq \overline{\lim_{n\to\infty}}\,\alpha_n^{(1)}\,\overline{\lim_{n\to\infty}}\,\alpha_n^{(2)}\cdots\overline{\lim_{n\to\infty}}\,\alpha_n^{(k)}.$$

A consequence of this lemma is that, for an arbitrary positive integer m,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_{m+1}}|} \leq \frac{1}{R^{m+1}},$$

where $\alpha_j = n + i_j$, i_j being a fixed number depending on j, and such that $0 \leq i_j \leq 2\,m$. The inequality holds for each value of j.

In fact, for a given i,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_{n+i_j}|} = \overline{\lim_{n\to\infty}}\sqrt[n+i_j]{|a_{n+i_j}|},$$

since

$$\overline{\lim_{n\to\infty}}\,|a_{n+i_j}|^{\frac{1}{n+i_j}} = \overline{\lim_{n\to\infty}}\,\{|a_{n+i_j}|^{\frac{1}{n}}\}^{\frac{n}{n+i_j}}$$

$$= \overline{\lim_{n\to\infty}}\,|a_{n+i_j}|^{\frac{1}{n}}.$$

Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{n+i_j}|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{R},$$

so that for a given m,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_{m+1}}|} \leq \frac{1}{R^{m+1}}.$$

LEMMA 2: *Given p Taylor's series, with coefficients*

$$\{A_n^{(1)}\},\ \{A_n^{(2)}\},\cdots,\ \{A_n^{(p)}\},$$

respectively. If

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(k)}|} = \frac{1}{\rho_k},\quad k = 1,2,\cdots, p,$$

then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(p)}|} \leq \frac{1}{\rho},$$

where ρ is the smallest of the numbers ρ_k.

The series

$$\sum A_n^{(k)} x^n$$

converges for $|x| < \rho_k$, hence for $|x| < \rho$. The series

$$\sum [A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(p)}]\, x^n$$

has, therefore, a radius of convergence R which is at least as great as ρ:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(p)}|} = \frac{1}{R} \leq \frac{1}{\rho}.$$

Let us write $D_{n,m}$ in the form

$$D_{n,m} = A_n^{(1)} + A_n^{(2)} + \cdots + A_n^{(m+1)!}$$

where each term is of the form

$$a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_{m+1}}.$$

Then, as a consequence of the lemmas, we have, for every m,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n^{(k)}|} \leq \frac{1}{R^{m+1}}, \quad k = 1, 2, \cdots, (m+1)!,$$

and

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m}|} \leq \frac{1}{R^{m+1}}.$$

17. Theorem 1: *If a series has, on the circle of convergence, p poles and no other singularities, then*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,p}|} < \frac{1}{R^{p+1}}.$$

By hypothesis there exists a polynomial $P_p(x) = 1 + A_1 x + \cdots + A_p x^p$ such that the series

$$\sum b_n x^n = (1 + A_1 x + \cdots + A_p x^p) \sum a_n x^n,$$

$$b_{k+p} = a_{k+p} + A_1 a_{k+p-1} + \cdots + A_p a_k, \tag{2}$$

has a radius of convergence R_1 which exceeds R. The constant term of $P_p(x)$ has been taken as 1, since by hypothesis the given series is regular at the origin. By means of the relation (2), the determinant $D_{n,p}$ reduces to

$$\begin{vmatrix} a_n & \cdots & a_{n+p-1} & b_{n+p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+p} & \cdots & \cdots & b_{n+2p} \end{vmatrix}.$$

Denote by $\Delta_{n,i}$ the minor of b_{n+i}. Then

$$D_{n,p} = \pm \sum_{i=p}^{2p} (-1)^{i-p} b_{n+i} \Delta_{n,i}.$$

Noting that $\Delta_{n,i}$ is a determinant of order p, we obtain

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\Delta_{n,i}|} \leq \frac{1}{R^p}$$

in the same way as for (1). Moreover, we have, by the choice of $P_p(x)$,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} = \frac{1}{R_1} < \frac{1}{R}.$$

Hence, finally,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,p}|} \leq \frac{1}{R^p}\frac{1}{R_1} < \frac{1}{R^{p+1}}.$$

Corollary: *The preceding inequality holds if p is replaced by any $m > p$.*

18. Theorem 2: *If there exists an integer m such that*

$$i) \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m-1}|} = \frac{1}{R^m},$$

and

$$ii) \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m}|} = \frac{1}{R^m R'} < \frac{1}{R^{m+1}},$$

then, for that m,

$$\lim_{n\to\infty} \sqrt[n]{|D_{n,m-1}|} = \frac{1}{R^m}.$$

We wish to show that given $\epsilon > 0$, no matter how small, there exists a number n_0 such that

$$|D_{n,m-1}| > \left(\frac{1-\epsilon}{R^m}\right)^n \text{ for } n > n_0.$$

Consider the determinant

$$D_{n-1,m} = \begin{vmatrix} a_{n-1} & a_n & \cdots & a_{n+m-1} \\ a_n & a_{n+1} & \cdots & a_{n+m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+m-1} & \cdots & \cdots & a_{n+2m-1} \end{vmatrix}.$$

Among the minors $D_{n+1,m-1}$, $D_{n-1,m-1}$, $D_{n,m-1}$ of this determinant we have the relation

$$D_{n+1,m-1}\, D_{n-1,m-1} - D^2_{n,m-1} = D_{n-1,m}\, D_{n+1,m-2}.\text{[1]} \qquad (3)$$

[1] Bôcher: *Introduction to Higher Algebra*, New York (1907), p. 33.

Here the m is of course fixed. We shall first use the identity (3) to show that for n large enough we can obtain an inequality of the form

$$|D_{n-1,m}\,D_{n+1,m-2}| < \left[k\alpha\,\frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2n}, \tag{4}$$

where, by taking ϵ small enough the k will be a positive constant < 1. For this purpose we introduce temporarily quantities ϵ', ϵ'', η', η'', all of which may be made arbitrarily small with n large and ϵ small enough. Indeed, we may take ϵ as small as we like.

In fact, given ϵ', we have from *ii*), by taking n large enough,

$$|D_{n-1,m}| < \left[\frac{1+\epsilon'}{R^m R'}\right]^{n-1},$$

and since $|D_{n+1,m-2}| < [(1+\epsilon'')/R^{m-1}]^{n+1}$ for n large enough, we have

$$|D_{n-1,m}\,D_{n+1,m-2}| < \frac{(1+\epsilon')^{n-1}(1+\epsilon'')^{n+1}}{R'^{n-1}\,R^{2mn-n-1}}$$

$$< \left[\frac{R^{\frac{1}{2}+\frac{1}{2n}}}{R'^{\frac{1}{2}-\frac{1}{2n}}}\;\frac{1+\eta'}{R^m}\right]^{2n},$$

where $1+\eta' > (1+\epsilon')^{\frac{1}{2}-\frac{1}{2n}}(1+\epsilon'')^{\frac{1}{2}+\frac{1}{2n}}$,

$$< \left[\sqrt{\frac{R}{R'}}\;\frac{1+\eta''}{R^m}\right]^{2n}, \quad \text{where } 1+\eta'' > R^{\frac{1}{2n}}R'^{\frac{1}{2n}}(1+\eta'),$$

$$= \left[\sqrt{\frac{R}{R'}}\;\frac{1-\frac{\epsilon}{2}}{R^m}\;\frac{1+\eta''}{1-\frac{\epsilon}{2}}\right]^{2n}$$

$$= \left[(1+\eta)\sqrt{\frac{R}{R'}}\;\frac{1-\frac{\epsilon}{2}}{R^m}\;\frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2n}, \text{where } 1+\eta = \frac{1+\eta''}{1-\epsilon}.$$

By writing then $k = (1 + \eta)\sqrt{\frac{R}{R'}}$, $\alpha = \left(1 - \frac{\epsilon}{2}\right)/R^m$ the desired inequality is proved.

From (3) and (4), for all n sufficiently large, we have therefore the inequality

$$| D_{n+1, m-1} D_{n-1, m-1} - D^2_{n, m-1} | < \left[k\alpha \frac{1 - \epsilon}{1 - \frac{\epsilon}{2}} \right]^{2n}. \tag{5}$$

Given ϵ, no matter how small, there will be infinitely many n for which $| D_{n, m-1} | > \alpha^n$, by i). If this is true for all n, n sufficiently large, the theorem is proved; otherwise there will be an ϵ, and it may be taken as small as we please, for which there will be a $D_{n_0, m-1}$ exceeding α^{n_0} in absolute value, preceded by a $D_{n_0-1, m-1}$ which in absolute value is at most as great as α^{n_0-1}, and this for n_0 arbitrarily large.

But from (5), taking $n = n_0$, we have

$$\left| \frac{D_{n_0+1, m-1} D_{n_0-1, m-1}}{D^2_{n_0, m-1}} \right| = \left| \frac{D_{n_0-1, m} D_{n_0+1, m-2}}{D^2_{n_0, m-1}} \right|$$

$$< \left[k \frac{1 - \epsilon}{1 - \frac{\epsilon}{2}} \right]^{2n_0} < k^{2n_0}.$$

Hence we have

$$\left| \frac{D_{n_0+1, m-1} D_{n_0-1, m-1}}{D^2_{n_0, m-1}} \right| > 1 - k^{2n_0}$$

Since, however, $| D_{n_0, m-1} | > \alpha^{n_0}$, $| D_{n_0-1, m-1} | \leq \alpha^{n_0-1}$ this inequality gives the following:

$$| D_{n_0+1, m-1} | > \alpha^{n_0+1} (1 - k)^{2n_0}, \tag{6}$$

and for the same reason also the following:

$$\left| \frac{D_{n_0+1, m-1}}{D_{n_0, m-1}} \right| > \alpha(1 - k^{2n_0}). \tag{7}$$

In particular, for n_0 sufficiently large, k being < 1, we may write (6) in the form

$$\sqrt[n_0+1]{|D_{n_0+1,m-1}|} > \alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}. \tag{6'}$$

We wish to obtain inequalities of the form (6), (7), (6′), holding for $n_0 + i$, where i is arbitrary, provided n_0 is sufficiently large. These inequalities are the following:

$$\left|\frac{D_{n_0+i,m-1}}{D_{n_0+i-1,m-1}}\right| > \alpha\,[1-k^{2n_0}]\,[1-k^{2(n_0+1)}]\cdot\cdots[1-k^{2(n_0+i-1)}], \tag{8}$$

$$|D_{n_0+i,m-1}| > \alpha^{n_0+i}[1-k^{2n_0}]^i\,[1-k^{2(n_0+1)}]^{i-1}\cdots [1-k^{2(n_0+i-1)}], \tag{9}$$

$$\sqrt[n_0+i]{|D_{n_0+i,m-1}|} > \alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}, \tag{10}$$

and we shall prove them by mathematical induction. They hold for $i = 1$; if they hold for i, we shall prove that they hold for $i + 1$.

If we apply (10), in the form

$$|D_{n_0+i,m-1}| > \left[\alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{n_0+i},$$

to (5), we have

$$\left|\frac{D_{n_0+i+1,m-1}D_{n_0+i-1,m-1}}{D^2_{n_0+i,m-1}} - 1\right| < \frac{\left[k\alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2(n_0+i)}}{\left[\alpha \frac{1-\epsilon}{1-\frac{\epsilon}{2}}\right]^{2(n_0+i)}} = k^{2(n_0+i)},$$

Consequently

$$\left|\frac{D_{n_0+i+1,\,m-1}D_{n_0+i-1,\,m-1}}{D^2_{n_0+i,\,m-1}}\right| > 1 - k^{2(n_0+i)},$$

and

$$\left|\frac{D_{n_0+i+1,\,m-1}}{D_{n_0+i,\,m-1}}\right| > \left|\frac{D_{n_0+i,\,m-1}}{D_{n_0+i-1,\,m-1}}\right| [1 - k^{2(n_0+i)}].$$

But this applied to (8) yields immediately the corresponding inequality for $i + 1$, namely,

$$\left|\frac{D_{n_0+i+1,\,m-1}}{D_{n_0+i,\,m-1}}\right| > \alpha\,[1 - k^{2n_0}]\cdot\cdots\cdot[1 - k^{2(n_0+i)}], \tag{11}$$

and if we multiply this inequality by (9), we obtain the second inequality for $(i + 1)$:

$$|D_{n_0+i+1,\,m-1}| > \alpha^{n_0+i+1}\,[1 - k^{2n_0}]^{i+1}\cdot\cdots\cdot[1 - k^{2(n_0+i)}]. \tag{12}$$

The inequality (10) for $i + 1$ is an immediate consequence of (11) and (12). In fact, from (12),

$$\frac{1}{\alpha}\sqrt[n_0+i+1]{|D_{n_0+i+1,\,m-1}|}$$

$$> [1 - k^{2n_0}]^{\frac{i+1}{n_0+i+1}}\,[1 - k^{2(n_0+1)}]^{\frac{i}{n_0+i+1}}\cdot\cdots\cdot[1 - k^{2(n_0+i)}]^{\frac{1}{n_0+i+1}}$$

$$> [1 - k^{2n_0}]\,[1 - k^{2(n_0+1)}]\cdot\cdots\cdot[1 - k^{2(n_0+i)}]$$

$$> 1 - [k^{2n_0} + k^{2(n_0+1)} + \cdots + k^{2(n_0+i)}]$$

$$> 1 - \frac{k^{2n_0}}{1 - k^2}.$$

If then, finally, we take n_0 great enough so that

$$\frac{k^{2n_0}}{1 - k^2} < \frac{\epsilon}{2 - \epsilon}$$

the inequality (10) will be established as a consequence of (8) and (9), whatever the value of i.

The inequality (10) is the one sought. In fact, replacing α by its value, $\alpha = \left(1 - \frac{\epsilon}{2}\right)/R^m$, we have

$$|D_{n,m-1}| > \left[\frac{1-\epsilon}{R^m}\right]^n, \quad n > n_0,$$

which is what we wished to prove.

It is natural to say that if α is the unique limit of the sequence $\{\alpha_n\}$:

$$\lim_{n\to\infty} \alpha_n = \alpha,$$

the sequence converges *regularly* to α. This terminology is due to Hadamard.

19. THEOREM 3: *Under the hypotheses of Theorem 2, the series has on the circle of convergence exactly m poles and no other singularities.*

In the first place, suppose that the series, having only poles on the circle of convergence, has $m - r$ poles, r being a positive integer. Then, by the corollary to Theorem 1,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,(m-r)+(r-1)}|} < \frac{1}{R^{(m-r)+(r-1)+1}},$$

that is,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{n,m-1}|} < \frac{1}{R^m},$$

which contradicts hypothesis *i*).

We seek next to determine a polynomial of degree m,

$$P_m(x) = 1 + \sum_{i=1}^{m} A_i x^i, \quad A_m \neq 0,$$

such that the series representing the function

$$\phi(x) = P_m(x)f(x) = \sum b_n x^n,$$

where, except for a finite number of terms

$$b_{n+m} = a_{n+m} + a_{n+m-1}A_1 + \cdots + a_n A_m,$$

converges in a circle of radius greater than R.

We have, by Theorem 2,

$$\lim_{n\to\infty} \sqrt[n]{|D_{n,m-1}|} = \frac{1}{R^m},$$

so that $D_{n,m-1}$ is not zero for $n \geq n_0$. Consequently there exist sets of numbers $A_1^{(n)}, A_2^{(n)}, \cdots, A_m^{(n)}$ which satisfy systems of equations of the form

$$\left.\begin{aligned}
&a_{n+m} + a_{n+m-1}A_1^{(n)} + a_{n+m-2}A_2^{(n)} + \cdots + a_n A_m^{(n)} = 0,\\
&a_{n+m+1} + a_{n+m}A_1^{(n)} + \cdots \qquad\qquad + a_{n+1}A_m^{(n)} = 0,\\
&\qquad\qquad \cdots\cdots\\
&a_{n+2m-1} + a_{n+2m-2}A_1^{(n)} + \cdots \qquad + a_{n+m-1}A_m^{(n)} = 0.
\end{aligned}\right\} \quad (14)$$

By means of the substitution

$$\delta_h^{(n)} = A_h^{(n+1)} - A_h^{(n)},\ h = 1, 2, \cdots, m, \quad n = n_o, n_{o+1}, \cdots,$$

the system of equations becomes

$$\left.\begin{aligned}
&a_{n+m}\delta_1^{(n)} + a_{n+m-1}\delta_2^{(n)} + \cdots + a_{n+1}\delta_m^{(n)} = 0,\\
&a_{n+m+1}\delta_1^{(n)} + \cdots \qquad\qquad + a_{n+2}\delta_m^{(n)} = 0,\\
&\qquad\qquad \cdots\cdots\\
&a_{n+2m-2}\delta_1^{(n)} + \cdots \qquad + a_{n+m-1}\delta_m^{(n)} = 0,\\
&a_{n+2m-1}\delta_1^{(n)} + \cdots \qquad + a_{n+m}\delta_m^{(n)} = -H_{n+m},
\end{aligned}\right\} \quad (15)$$

where

$$-H_{n+m} + a_{n+2m} + a_{n+2m-1}A_1^{(n)} + \cdots + a_{n+m}A_m^{(n)} = 0. \qquad (16)$$

Eliminating $A_1^{(n)}, \cdots, A_m^{(n)}$ from (14) and (16), we obtain

$$H_{n+m} = \frac{D_{n,m}}{D_{n,m-1}},$$

and from (15),

$$\delta_h^{(n)} = -\frac{D_{n+1,m-2}^{(h)}}{D_{n+1,m-1}} H_{n+m}$$

$$= -\frac{D_{n,m}\, D_{n+1,m-2}^{(h)}}{D_{n,m-1}\, D_{n+1,m-1}},$$

where $D^{(h)}_{n+1,m-2}$ is a determinant with $m - 1$ rows, composed of the coefficients of the given series. It follows that

$$\left| D^{(h)}_{n+1,m-2} \right| < \left(\frac{1+\epsilon}{R^{m-1}}\right)^n$$

for n sufficiently large. Then, since

$$\lim_{n\to\infty} \sqrt[n]{| D_{n,m-1} |} = \frac{1}{R^m},$$

and

$$\overline{\lim_{n\to\infty}} \sqrt[n]{| D_{n,m} |} = \frac{1}{R^m R'} < \frac{1}{R^{m+1}},$$

there must exist an integer n', independent of h, such that for $n \geq n'$ we shall have

$$\left| D_{n,m} \right| < \left(\frac{1+\epsilon}{R^m R'}\right)^n,$$

$$\left| D_{n,m-1} \right| > \left(\frac{1-\epsilon}{R^m}\right)^n,$$

$$\left| D_{n+1,m-1} \right| > \left(\frac{1-\epsilon}{R^m}\right)^n,$$

$$\left| D^{(h)}_{n+1,m-2} \right| < \left(\frac{1+\epsilon}{R^{m-1}}\right)^n,$$

Accordingly, for $n \geq n'$,

$$| \delta_h^{(n)} | < \left[\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \frac{R}{R'}\right]^n = \left[(1+\eta)\frac{R}{R'}\right]^n,$$

and we shall suppose that ϵ is so small that the quantity in brackets is < 1. Then the series $\sum_{n=n_0}^{\infty} \delta_h^{(n)}$ converges.

Let $A_h = \lim_{n\to\infty} A_h^{(n)}$, $h = 1, 2, \cdots, m$. This limit exists, since

$$\sum_{j=n_0}^{n_0+k-1} \delta_h^{(j)} = A_h^{(n_0+k)} - A_h^{(n_0)}$$

We now show that the series $\Sigma b_n x^n$ for $\phi(x)$ converges in a circle of radius greater than R. We have

$$| A_h - A_h^{(n_0+k)} | = | \sum_{j=n_0+k}^{\infty} \delta_h^{(j)} |$$
$$\leq \sum_{j=n_0+k}^{\infty} \left[(1+\eta)\frac{R}{R'}\right]^j,$$

and

$$b_{n+m} = a_{n+m-1}(A_1 - A_1^{(n)}) + a_{n+m-2}(A_2 - A_2^{(n)}) + \cdots + a_n(A_m - A_m^{(n)}),$$

since $b_{n+m} - a_{n+m} = a_{n+m-1}A_1 + \cdots + a_nA_m$,

and $a_{n+m} = -a_{n+m-1}A_1^{(n)} - \cdots - a_nA_m^{(n)}$.

The inequality shows that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{| A_h - A_h^{(n)} |} \leq \frac{R}{R'}(1+\eta).$$

Consequently

$$\overline{\lim_{n\to\infty}} \sqrt[n+m]{| b_{n+m} |} \leq \frac{R}{R'}(1+\eta)\frac{1}{R} < \frac{1}{R'},$$

and the proof is complete.