NOTE

ON THE DETERMINATION OF PERIODIC FUNCTIONS BY MEANS OF THEIR INITIAL VALUES

1. *Purpose of this Note*

Consider a trigonometric series containing (for greater simplicity) only cosine terms,

$$\Sigma a_n \cos nx.$$

We know, by what has preceded, that if the coefficients a_n satisfy a law of decrease which is sufficiently rapid, the sum of the series is an analytic function of x. Suppose that we assign *a priori* the initial values C_0, C_2, C_4, . . . of a function $f(x)$ and its successive even derivatives for $x = 0$; it will in general be impossible to satisfy these conditions with a periodic function whose Fourier coefficients decrease rapidly enough to make it analytic. In fact, an analytic function is defined at once in terms of the initial values by means of its Taylor development, and this power series represents a periodic function only exceptionally.

On the other hand, if we do not assign a special law of decrease for the Fourier coefficients, it is always possible to construct a periodic function which, for $x = 0$, takes on, with its successive derivatives of even order, the sequence of given values C_0, C_2, . . ., whatever they may be.

It is our plan to prove this statement, and to outline a method of constructing such a function. We shall add some remarks which relate naturally to this question.

2. *Theorem I*

Whatever initial values C_0, C_2, ..., C_{2n}, ... are given, it is always possible to form an even periodic function $f(x)$, such that the differences

$$f^{(2n)}(0) - C_{2n} \quad (n = 1, 2, 3, \ldots)$$

are bounded for all n.

Let $m_1, m_2, \ldots m_n, \ldots$ be an increasing sequence of integers and $a_1, a_2, \ldots a_n, \ldots$ an arbitrary sequence of coefficients. We wish to determine them one after another. We write

$$f(x) = C_0 + \sum_{n=1}^{\infty} a_n \frac{\cos m_n x}{m_n{}^{2n-1}}.$$

From this, we have

$$-f''(0) = m_1 a_1 + \frac{a_2}{m_2} + \frac{a_3}{m_3{}^3} + \ldots$$

$$f^{\text{iv}}(0) = m_1{}^3 a_1 + m_2 a_2 + \frac{a_3}{m_3} + \ldots$$

and so on.

Let us form the system of successive equations:

$$\begin{aligned} -C_2 &= m_1 a_1 \\ C_4 &= m_1{}^3 a_1 + m_2 a_2 \\ -C_6 &= m_1{}^5 a_1 + m_2{}^3 a_2 + m_3 a_3, \end{aligned}$$

and so forth. This is a recurrent system which determines successively the products $m_1 a_1$, $m_2 a_2$, ... $m_n a_n$, ... When these products are known it is obvious that we can always take for the first factors $m_1, m_2, \ldots m_n, \ldots$ an increasing sequence of integers, and moreover one that increases fast enough so that the second factors $a_1, a_2, \ldots a_n, \ldots$ are all contained between -1 and $+1$. The function $f(x)$ is then completely determined by the above trigonometric series, and it is evident that this series satisfies the conditions of the theorem.

Not only are the differences $f^{(2n)}(0) - C_{2n}$ bounded for all n, but in our case they also approach 0 as n becomes infinite.

The preceding theorem reduces the determination of a function which is even and periodic and with its successive derivatives takes on a sequence of given initial values to the same problem where the initial values $C_0, C_2, \ldots, C_{2n}, \ldots$ are bounded. This problem will be solved in the proof of the following theorem.

3. *Theorem II*

Let $m_1, m_2, \ldots m_n, \ldots$ be a sequence of positive integers, increasing rapidly enough so that for all n,

$$\frac{m_{n+1}}{m_n} > \lambda > 1$$

where λ is a fixed positive number. Let then $C_0, C_2, \ldots, C_{2k}, \ldots$ be a sequence of bounded numbers. It is always possible to determine the coefficients of the trigonometric series

$$f(x) = \alpha_1 \cos m_1 x + \alpha_2 \cos m_2 x + \ldots + \alpha_n \cos m_n x + \ldots$$

in such a way that this series and all its derivatives will converge, and the sum function and its derivatives of even order will take on respectively the given values $C_0, C_2, \ldots$ for $x = 0$.

We determine first n coefficients $a_1, a_2, \ldots a_n$, by the system of n equations

$$(1)\qquad \begin{cases} C_0 = a_1 + a_2 + \ldots + a_n \\ -C_2 = m_1^2 a_1 + m_2^2 a_2 + \ldots + m_n^2 a_n \\ \pm C_{2n-2} = m_1^{2n-2} a_1 + m_2^{2n-2} a_2 + \ldots + m_n^{2n-2} a_n \end{cases}$$

and write

$$f_n(x) = a_1 \cos m_1 x + a_2 \cos m_2 x + \ldots + a_n \cos m_n x.$$

Evidently, it suffices to show that as n becomes infinite, $f_n(x)$ approaches a limit function $f(x)$ which satisfies the theorem. For this purpose, we shall show that a_1, a_2,

$a_3, \ldots$ approach determinate limits $\alpha_1, \alpha_2, \alpha_3, \ldots$ respectively, and are contained respectively within limits (whatever the value of n) which insure the convergence of the sum (which changes with n)

$$a_1 \cos m_1 x + a_2 \cos m_2 x + \ldots, \tag{2}$$

and its successive derivatives.

We solve first the system (1). Its determinant Δ is a Vandermonde determinant which may be resolved into a product of factors

$$\Delta = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ m_1^2 & m_2^2 & \ldots & m_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ m_1^{2n-2} & m_2^{2n-2} & \ldots & m_n^{2n-2} \end{vmatrix}$$

$$\begin{aligned} = \; & (m_2^2 - m_1^2)(m_3^2 - m_1^2)\ldots(m_n^2 - m_1^2) \\ & (m_3^2 - m_2^2)\ldots(m_n^2 - m_2^2) \\ & \ldots \quad \ldots \\ & (m_n^2 - m^2_{n-1}). \end{aligned}$$

Denote by $\Delta_1(z)$ the polynomial in z obtained by replacing the letter m_1 in Δ by z; we shall have

$$\frac{\Delta_1(z)}{\Delta} = \frac{(m_2^2 - z^2)(m_3^2 - z^2)\ldots(m_n^2 - z^2)}{(m_2^2 - m_1^2)(m_3^2 - m_1^2)\ldots(m_n^2 - m_1^2)}.$$

According to the theory of equations, the value of the unknown a_1 is equal to a fraction of which the denominator is the determinant Δ and the numerator is obtained by replacing in this determinant the elements of the first column $1, m_1^2, \ldots$ by $C_0, -C_2, \ldots$. Evidently we arrive at the same result by ordering the numerator of the fraction written above according to powers of z^2 and replacing z^0, $z^2, \ldots z^{2n-2}$ by $C_0, -C_2, \ldots, \pm C_{2n-2}$ respectively. The result of this substitution may be given explicitly with the help of a complex definite integral. This will now be shown.

Let M be a positive quantity greater than the absolute values of all the quantities C_{2k}. We describe a circle (C) of radius M around the origin of the z's. Outside and on the boundary of this circle we may define

$$\psi(z) = \frac{C_0}{z} - \frac{C_2}{z^3} + \frac{C_4}{z^5} - \dots,$$

since this series is convergent in that region. If now we integrate along the circle (C) we have (according to the theory of residues)

$$a_1 = \frac{1}{2\pi i}\int_c \frac{\Delta_1(z)}{\Delta}\,\psi(z)dz.$$

In order to study this expression and its analogs, we write

$$\phi_n(z) = \left(1 - \frac{z}{m_1^2}\right)\left(1 - \frac{z}{m_2^2}\right)\dots\left(1 - \frac{z}{m_n^2}\right);$$

whence we have, without difficulty, ϕ' being the derivative of ϕ,

$$\frac{\Delta_1(z)}{\Delta} = \frac{\phi_n(z^2)}{(z^2 - m_1^2)\phi'_n(m_1^2)},$$

and accordingly

$$a_1 = \frac{1}{2\pi i}\int_c \frac{\phi_n(z^2)\ \psi(z)dz}{(z^2 - m_1^2)\phi_n'(m_1^2)}.$$

The corresponding values of a_2, a_3, ... are obtained by a simple permutation of the letters. Permuting m_1 and m_k, we have

$$a_k = \frac{1}{2\pi i}\int_c \frac{\phi_n(z^2)\ \psi(z)dz}{(z^2 - m_k^2)\ \phi_n'(m_k^2)}. \tag{3}$$

The denominators, only, depend on k.

As n grows indefinitely great, the product denoted by $\phi_n(z^2)$ acquires new factors and is developed as a product with an infinite number of factors, which we may denote by $\phi(z)$,

$$\phi(z) = \left(1 - \frac{z}{m_1^2}\right)\left(1 - \frac{z}{m_2^2}\right)\dots.$$

Since this product converges, a_k approaches a definite limit α_k which is expressed by the integral

$$\alpha_k = \frac{1}{2\pi i}\int_c \frac{\phi(z^2)\psi(z)dz}{(z^2 - m_k^2)\phi'(m_k^2)}.$$

In order to complete the demonstration it remains to show that as n varies the sum (2) and its derivatives are uniformly convergent. For this purpose we shall seek an upper bound of the $|a_k|$, which, when put in place of a_k in this sum, will insure the absolute convergence of this sum (extended to infinity) and its successive derivatives.

We return to the expression (3) for the a_k. In order to obtain an upper bound for this expression we must look for a lower bound of the absolute value of its denominator, or (what amounts to the same thing) of

$$m_k^2\phi_n'(m_k^2).$$

This expression, like $\phi(z^2)$, is a product of factors. We divide them into two groups. The first group, formed from the first $k - 1$ factors, has the form

$$\left(\frac{m_k^2}{m_1^2} - 1\right)\left(\frac{m_k^2}{m_2^2} - 1\right)\cdots\left(\frac{m_k^2}{m_{k-1}^2} - 1\right).$$

It can be put in the form

$$\left(\frac{m_k^{k-1}}{m_1m_2\ldots m_{k-1}}\right)^2\left(1 - \frac{m_{k-1}^2}{m_k^2}\right)\left(1 - \frac{m_{k-2}^2}{m_k^2}\right)\cdots,$$

and consequently exceeds in value the expression

$$\left(\frac{m_k^{k-1}}{m_1m_2\ldots m_{k-1}}\right)^2\left(1 - \frac{1}{\lambda^2}\right)\left(1 - \frac{1}{\lambda^4}\right)\cdots.$$

The second group, made up of the remaining factors, has the form

$$\left(1 - \frac{m_k^2}{m_{k+1}^2}\right)\left(1 - \frac{m_k^2}{m_{k+2}^2}\right)\cdots\left(1 - \frac{m_k^2}{m_n^2}\right),$$

and exceeds the definite quantity defined by the convergent infinite product

$$\left(1-\frac{1}{\lambda^2}\right)\left(1-\frac{1}{\lambda^4}\right)\cdots$$

From this it follows that we can assign a constant h, and then a second constant h' (independent of k and n) so that we have

$$\left| m_k^2 \phi_n'(m_k^2) \right| > h\left(\frac{m_k^{k-1}}{m_1 m_2 \ldots m_{k-1}}\right)^2$$

and consequently

$$\left| a_k \right| < h'\left(\frac{m_1 m_2 \ldots m_{k-1}}{m_k^{k-1}}\right)^2.$$

This upper bound (for k infinite) is an infinitesimal of higher order than any negative power of the m_k; it guarantees therefore the conditions of convergence which we have demanded.

We notice finally that the α_k, which are limits of the a_k, have the same upper bounds.

Theorem II is thus completely proved.

4. *On the law of limitation for the Fourier coefficients*

We shall make the calculation of the last section more definite in a particular case. Let the numbers $m_1, m_2, \ldots, m_k, \ldots$ be successive powers of the same integer $\lambda > 1$.

We consider in this case the trigonometric series

$$f(x) = \alpha_1 \cos \lambda x + \alpha_2 \cos \lambda^2 x + \ldots + \alpha_k \cos \lambda^k x + \ldots$$

in which the upper bound of the α_k, found at the end of the preceding section, takes the form

$$\left| \alpha_k \right| < h'\left(\frac{\lambda \cdot \lambda^2 \ldots \lambda^{k-1}}{\lambda^{k(k-1)}}\right) = \frac{h'}{\lambda^{k(k-1)}} = h' e^{-k(k-1)\log \lambda}.$$

The series which we have just considered comes under the general type of trigonometric series

$$a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots,$$

where all the coefficients vanish except when $n = \lambda^k$, in which case $a_n = \alpha_k$. Let us express the law of limitation of the a_n as a function of n. We have $k = \log n/\log \lambda$; consequently

$$| a_n | < h' e^{-\log n \left(\frac{\log n}{\log \lambda} - 1\right)}.$$

We can accordingly fix two constants A and a so that we shall have for any n

$$(4) \qquad | a_n | < A e^{-a(\log n)^2}.$$

This gives us the following theorem:

There are always an infinity of periodic functions which with their successive derivatives take on for $x = 0$ an arbitrarily given bounded sequence of values, and such that their Fourier coefficients a_n obey a law of limitation of the form (4).

5. *On certain classes of functions determined by the system of initial values*

It is possible to define, in terms of the form of their trigonometric development, certain classes of functions which are neither analytic nor quasi-analytic, and yet such that a function of the class is determined by its value and that of its successive derivatives at $x = 0$.

We have, in fact, the following theorem:

Let $m_1, m_2, \ldots m_n \ldots$ be a sequence of positive integers, increasing sufficiently fast so that the quotient

$$\frac{m_1 m_2 \ldots m_n}{m_{n+1}}$$

approaches 0 *as n becomes infinite. The functions which are representable by the trigonometric series*

$$a_1 \cos m_1 x + a_2 \cos m_2 x + \ldots + a_n \cos m_n x + \ldots$$

with the condition of limitation

$$|a_n| < h'\left(\frac{m_1 m_2 \ldots m_{n-1}}{m_n{}^{n-1}}\right)^2 \qquad (h' \text{ const.})$$

form a class in which every function is determined by its value and those of its derivatives at $x = 0$. *Moreover we find that these values remain arbitrary, provided they are bounded.*

We notice first that the sum of two functions of the class considered belongs to the same class. With this fact, it is enough to prove that a function of this class vanishes identically if all its initial values are zero.

Suppose then that all the initial values are zero. The n first coefficients $a_1, a_2, \ldots, a_n$ must satisfy the equations (1) of section 3, provided we consider there the first members $C_0, -C_2, \ldots$ as representing, with opposite signs, the portions of infinite series discarded from the second members (or the remainders of those series). The calculation made in section 3 shows (on letting n become infinite) that $a_1, a_2, \ldots$ are all zero, if these remainders themselves approach zero.

The two conditions in the statement of the theorem allow us to satisfy ourselves easily that the first term of the remainder in the last equation of the system approaches zero. The whole remainder also approaches zero on account of the rapidity of decrease of the coefficients. All the remainders approach zero *a fortiori* in the other equations of the system. The theorem is therefore proved.

For example, the class of functions which may be given by trigonometric series

$$a_1 \cos x + a_2 \cos 2^2 x + a_3 \cos 3^3 x + \ldots + a_n \cos n^n x + \ldots$$

with the limitation condition

$$|a_n| < h\left(\frac{1 \cdot 2^2 \cdot 3^3 \ldots (n-1)^{n-1}}{n^{(n-1)n}}\right)^2 \qquad (h \text{ const.})$$

is of the preceding kind: a function of this class is completely determined by its value and that of its successive derivatives for $x = 0$.

The functions of this class are neither analytic nor quasianalytic.

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